BIFURCATION AT INFINITY FOR A SEMILINEAR WAVE EQUATION WITH NON-MONOTONE NONLINEARITY

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Abstract. We prove bifurcation at infinity for a semilinear wave equation depending on a parameter λ and subject to Dirichlet-periodic boundary conditions. We assume the nonlinear term to be asymptotically linear and not necessarily monotone. We prove the existence of L^∞ solutions tending to $+\infty$ when the bifurcation parameter approaches eigenvalues of finite multiplicity of the wave operator. Further details are presented in cases of simple eigenvalues and odd multiplicity eigenvalues.

1. **Introduction.** The purpose of this paper is to establish bifurcation at infinity for the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \lambda u + h(u) := \Box u + \lambda u + h(u) = 0, \quad x \in (0,\pi), \ t \in \mathbb{R}, \qquad (1)$$

subject to the Dirichlet-periodic boundary conditions

$$u(0,t) = u(\pi,t) = 0, \qquad u(x,t) = u(x, t + 2\pi). \qquad (2)$$

A main feature of this study is that the nonlinearity $\lambda u + h(u)$ need not be monotone. For the monotone case the reader is referred to [12, 11, 1] where, taking advantage of the monotonicity, compactness arguments motivated by elliptic theory may be adapted to the hyperbolic equation in (1). In our case, linearizations of the left hand side of (1) may have infinite dimensional kernel making compactness inapplicable. We overcome this difficulty by using estimates on the measure of pre-images of

neighborghoods of zeros of trigonometric polynomials, see Lemmas 1.4 and 1.5. These lemmas are derived from Theorem 1.3.

We assume that $h \in C^1(\mathbb{R})$ and that there exist $h_0 > 0$ and $\gamma > 1$ such that if $|x| \geq h_0$ then

$$|h'(x)| \leq \frac{1}{|x|^\gamma}. \tag{3}$$

Without loss of generality, we assume that $\gamma < 2$. Finally, we assume that there is $A > 0$ such that

$$\lim_{x \to \infty} h(x) = A \quad \text{and} \quad \lim_{x \to -\infty} h(x) = -A. \tag{4}$$

We let $\sigma(\square)$ be the spectrum of $\square$ subject to (2). It is given by

$$\sigma(\square) = \left\{ k^2 - j^2 \ : \ k = 1, 2, \ldots, j = 0, 1, 2, \ldots \right\}. \tag{5}$$

For $k = 1, 2, \ldots$ and $j = 0, 1, \ldots$, denote by

$$\vartheta_{k0}(x, t) = \frac{1}{\pi} \sin(kx)$$

$$\vartheta_{kj}(x, t) = \frac{\sqrt{2}}{\pi} \sin(kx) \cos(jt), \qquad \varrho_{kj}(x, t) = \frac{\sqrt{2}}{\pi} \sin(kx) \sin(jt), \quad j \geq 1, \tag{6}$$

the normalized eigenfunctions associated to the eigenvalue $k^2 - j^2 \in \sigma(\square)$. Note that 0 is the only eigenvalue of infinite multiplicity, 1 and 4 are the only simple eigenvalues, and the eigenvalues of odd multiplicity are of the form k^2 with $k = 1, 2, \ldots$.

Let $\Omega = (0, \pi) \times (0, 2\pi)$ and H be the Sobolev space of functions $u \in L^2(\Omega)$ with $u_x, u_t \in L^2(\Omega)$ and that satisfy (2). The norm in H is given by $\|u\|_1 = \sqrt{\|u_x\|^2 + \|u_t\|^2}$ where $\|\cdot\|$ is the norm in $L^2(\Omega)$. The kernel of $\square$ is given by

$$N = \overline{\text{span} \left\{ \vartheta_{kk}, \varrho_{kk} \ : \ k = 1, 2, \ldots \right\}}, \tag{7}$$

where the closure is taken in $L^2(\Omega)$. The range of $\square$ subject to (2) is given by the set of elements in H that are $L^2(\Omega)$-orthogonal to N.

We say that $u = v + y \in N \oplus Y$ is a weak solution of (1) subject to (2) if

$$\int_\Omega \left[(y_t \tilde{y}_t - y_x \tilde{y}_x) - (\lambda u + h(u))(\tilde{v} + \tilde{y}) \right] dx dt = 0, \tag{8}$$

for all $\tilde{v} \in N$ and all $\tilde{y} \in Y$.

Our main results are the following.

Theorem 1.1. *Let* $\lambda = \lambda_0 - \epsilon$, $\epsilon > 0$ *and* $h : \mathbb{R} \to \mathbb{R}$ *a* C^1 *bounded function satisfying* (3)-(4). *If* $-\lambda_0 \in \sigma(\square)$ *is an eigenvalue of finite multiplicity, then there exists* $\epsilon_0 \in (0, 1/2)$ *such that if* $\epsilon < \epsilon_0$ *the problem* (1)-(2) *has a non-trivial weak solution* $u_\epsilon = v_\epsilon + y_\epsilon \in (N \cap L^\infty(\Omega)) \oplus (Y \cap L^\infty(\Omega))$. *Furthermore, if* $\epsilon \to 0$ *then* $\|v_\epsilon\| + \|y_\epsilon\|_1 \to \infty$.

If $-\lambda_0$ is an odd multiplicity eigenvalue, we further have the following result.

Theorem 1.2. *If* $-\lambda_0 \in \sigma(\square)$ *is an odd multiplicity eigenvalue,* $h : \mathbb{R} \to \mathbb{R}$ *is a* C^1 *bounded function and* $\lim_{|x| \to \infty} h'(x) = 0$, *then there exists a maximal connected set of weak solutions* (λ, u_λ) *to* (1)-(2) *with* $\lim_{\lambda \to \lambda_0} \|u_\lambda\| = \infty$. *Moreover, if* $-\lambda_0$ *is a simple eigenvalue of* $\square$, *then there is* $\delta_0 > 0$ *such that the maximal continuum of weak solutions* (λ, u_λ) *to the problem* (1)-(2) *is a continuous curve* $(\lambda(s), u(s))$ *when* $\lambda \in (-\delta_0, \delta_0)$.

For related results on (1) with non-monotone nonlinearities see [3, 6, 4, 5, 14, 10, 7, 2]. In [10, 14] the density of the range of $Lu := \Box u + \lambda u + h(u)$ subject to (2) is proven for $-\lambda \notin \sigma(\Box)$. In [7, 6, 5] sufficient conditions for p in the range of L are provided. In [3] sufficient conditions for the nonexistence of continuous solutions are given. In [4] the "imperfect" bifurcation at 0 is studied. Finally in [2] the case where $u(x,t) = u(x, t+T)$ with T an irrational multiple of π is considered.

The following result on trigonometric polynomials plays a central role in the proofs of our main Theorems. For details of its proof see [9] and [13, p. 239-236].

Theorem 1.3 (Nazarov-Turán Lemma)**.** *Let $m_1, \ldots, m_n$ be non-negative integers. If $p : \mathbb{R}^n \to \mathbb{R}$ is defined by*

$$p(x) = \sum_{k_i \leq m_i, i=1,\ldots,n} c_k e^{ik \cdot x}, \quad c_k \in \mathbb{C}, \tag{9}$$

then there exists a constant $C_{NT} > 0$ such that, for any measurable set $E \subset [0, 2\pi]^n$,

$$\mu(E)^{m_1 + \cdots + m_n} \sup_{x \in \mathbb{R}^n} |p(x)| \leq C_{NT} \sup_{x \in E} |p(x)|, \tag{10}$$

where $\mu(E)$ denotes the n-dimensional Lebesgue measure of E.

For future use we deduce the following estimates from Theorem 1.3.

Lemma 1.4. *If $-\lambda_0 \in \sigma(\Box)$ be an eigenvalue of finite multiplicity and Z the corresponding eigenspace, then there exists $C > 0$ such that for each $\beta > 0$ there exists $\alpha > 0$ with*

$$\mu\left(\{(x,t) \in \Omega : |\psi(x,t)| < \epsilon^\beta\}\right) < C\epsilon^\alpha, \tag{11}$$

for any $\psi \in Z$ with $\|\psi\| = 1$.

Proof. Since Z is finite dimensional, there exists $C_1 > 0$ such that

$$\|\psi\| \leq C_1 \max\{|\psi(x,t)| ; (x,t) \in \Omega\} \quad \text{for any } \psi \in Z. \tag{12}$$

Since every ψ in Z may be written as

$$\psi(x,t) = \sum_{k^2 - j^2 = \lambda_0} c_{k,j} \sin(kx)\cos(jt) + d_{k,j}\sin(kx)\sin(jt), \tag{13}$$

every element is Z satisfies (9) with $m_1 = m_2 = |\lambda_0|$. Letting $E := \{(x,t) \in \Omega ; |\psi(x,t)| < \epsilon^\beta\}$, by Theorem 1.3, we have

$$\mu(E) \leq \left(C_{NT} C_1 \epsilon^\beta\right)^{\frac{1}{2|\lambda_0|}}. \tag{14}$$

This proves the Lemma with $\alpha = \beta/|2\lambda_0|$. $\square$

Lemma 1.5. *If Z is as in Lemma 1.4, then there exists $K > 0$ such that for each $\beta > 0$ there exists $\alpha > 0$ with*

$$\mu\left(\{x \in [0,\pi]; |\psi(x, r \pm x)| < \epsilon^\beta\}\right) < K\epsilon^\alpha, \tag{15}$$

for any $\psi \in Z$ with $\|\psi\| = 1$, and any $r \in [0, 2\pi]$. Here μ denotes the one-dimensional Lebesgue measure.

Proof. Let $\psi \in Z$ be such that $\|\psi\| = 1$. Writing ψ in Fourier series we see that there exists positive constants k_1, k_2 such that

$$k_1 \leq \|\psi(\cdot, r + \cdot)\|_{L^2[0,\pi]} \leq k_2 \quad \text{for all} \ \ r \in [0, 2\pi]. \tag{16}$$

Since

$$\psi(x, r \pm x) = \sum_{k^2 - j^2 = \lambda_0} \alpha_{k,j}(r)\sin(kx)\sin(jx) + \beta_{k,j}(r)\sin(kx)\cos(jx), \qquad (17)$$

every element is Z satisfies (9) with $m_1 = |\lambda_0|$. Letting $E := \{x \in [0, \pi]; |\psi(x, r \pm x)| < \epsilon^\beta\}$, by Theorem 1.3, we have

$$\mu(E) \le \left(C_{NT}C_2\epsilon^\beta\right)^{1/|\lambda_0|}, \qquad (18)$$

where C_2 is such that $\|\psi(\cdot, r \pm \cdot)\|_{L^2[0,\pi]} \le C_2 \max\{|\psi(x, r \pm x); x \in [0, \pi]\}$. This proves the Lemma with $\alpha = \beta/|\lambda_0|$. $\square$

2. **Lyapunov-Schmidt reduction.** Let Z be as above, and $d := \dim Z$. Let W be the closure of the subspace of Y spanned by eigenfunctions corresponding to eigenvalues in $\sigma(\square) \setminus \{-\lambda_0\}$. Projecting (8) onto the subspaces N, Z and W one sees that $u = v + z + w \in N \oplus (Z \cap Y) \oplus (W \cap Y)$ is a weak solution to (1)-(2) if and only if

$$v = -\frac{1}{\lambda}P_N h(u), \qquad (19)$$

$$w = -(\square + \lambda I)^{-1}P_W h(u), \qquad (20)$$

$$z = \frac{1}{\epsilon}P_Z h(u), \qquad (21)$$

where P_N, P_W and P_Z are the L^2-orthogonal projections onto N, W and Z, respectively.

Next we establish the existence of approximate solutions to (21).

Lemma 2.1. *If h satisfies (3) and (4), then there exists $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$ there exists $\varphi_\star \in Z$ such that*

$$-\epsilon\varphi_\star + P_Z h(\varphi_\star) = 0. \qquad (22)$$

Furthermore, there exist $c_1 > c_0 > 0$ depending only on h such that

$$c_0\epsilon^{-1} \le \|\varphi_\star\| \le c_1\epsilon^{-1}. \qquad (23)$$

Proof. Let $\epsilon > 0$. For $z \in Z$ we define the functional $J_\epsilon : Z \to \mathbb{R}$ by

$$J(z) := J_\epsilon(z) := \frac{-\epsilon}{2}\int_\Omega |z|^2 dx + \int_\Omega H(z)dx, \qquad (24)$$

where $H(x) = \int_0^x h(s)\,ds$. Therefore

$$\begin{aligned} J[z] &\le \frac{-\epsilon}{2}\int_\Omega |z|^2 + |h|_\infty \int_\Omega |z| \\ &\le -\frac{\epsilon}{2}\|z\|^2 + |h|_\infty \|z\|\sqrt{2}\pi \\ &\to -\infty \quad \text{as} \quad \|z\| \to +\infty. \end{aligned} \qquad (25)$$

Since Z is finite dimensional and there exists $\varphi_\star$ such that $J[\varphi_\star] = \max_Z J$. Hence for all $z \in Z$ $\langle J'[\varphi_\star], z \rangle = 0$, which implies (22).

From (25) we see that $J(z) < 0$ for $\|z\| > 2\sqrt{2}\pi|h|_\infty\epsilon^{-1}$. This and $J(\varphi_\star) \ge J(0) = 0$, imply

$$\|\varphi_\star\| \le 2\sqrt{2}\pi|h|_\infty\epsilon^{-1} := c_1\epsilon^{-1}. \qquad (26)$$

By (4), there exists $M_1 > 0$ such that $H(x) \geq A|x|/2 - M_1$ for all $x \in \mathbb{R}$. Since Z if finite dimensional there exists a real number $C > 0$ such that $\int_\Omega |z|dx \geq C\|z\|$ for all $z \in Z$. Therefore, for $2\epsilon\|z\| = AC$,

$$
\begin{aligned}
J(z) &\geq -\frac{\epsilon}{2}\|z\|^2 + \frac{CA\|z\|}{2} - 2\pi^2 M_1 \\
&\geq \frac{A^2 C^2}{8\epsilon} - 2\pi^2 M_1.
\end{aligned}
\tag{27}
$$

Hence $J(\varphi_\star) \geq \frac{A^2 C^2}{8\epsilon} - 2\pi^2 M_1$. Taking $\epsilon_0 = A^2 C^2/(32\pi^2 M_1)$, we have $J(\varphi_\star) \geq \frac{AC}{16\epsilon}$ for $\epsilon \in (0, \epsilon_0)$. This and (25) yield

$$
\|\varphi_\star\| \geq \frac{A^2 C^2}{16\sqrt{2}\pi|h|_\infty \epsilon} := c_0 \epsilon^{-1}.
\tag{28}
$$

This and (26) complete the proof of the lemma. $\square$

3. **Kernel equation.** In this section we establish the solvability of (19), given $w \in W$, and $z \in Z$.

Let B_1 be the set of all 2π-periodic measurable functions $p : \mathbb{R} \to \mathbb{R}$ such that $\int_0^{2\pi} p = 0$ and $\|p\|_\infty \leq r_1 := 4|h|_\infty$. Let $R_1 = 8(1 + |\lambda_0|)|h|_\infty$, $B_2 := \{w \in W, : \|w\|_\infty \leq R_1\}$, and $B_3 := \left\{z \in Z : \|z\| \leq \frac{c_0}{4\epsilon}\right\}$. For $w \in B_2$, $z \in B_3$, and $p \in B_1$ let

$$
F(\epsilon, w, z, p) = \frac{1}{2\pi\lambda} \int_0^\pi [h(u(x, r + x)) - h(u(x, r - x))]\, dx,
\tag{29}
$$

where $\lambda = \lambda_0 - \epsilon$, $u(x, t) = w(x, t) + \varphi_\star(x, t) + z(x, t) + v(x, t)$ and $v(x, t) = p(x, t) - p(x - t)$. Recall that v is a solution to (19) if and only if p is a fixed point of F (see [6, Lemma 5.2]).

Lemma 3.1. *Let $\epsilon_0 > 0$ be as in Lemma 2.1. There exists $\epsilon_1 \in (0, \epsilon_0)$ such that for each $\epsilon \in (0, \epsilon_1)$, F defines a contraction in the variable p. In particular, equation (19) has a unique solution and such a solution depends continuously on (ϵ, w, z).*

Proof. From the definition of r_1, we have $F(\epsilon, w, z, p) \in B_1$ for any $(\epsilon, w, z, p) \in (0, 1/2) \times B_2 \times B_3 \times B_1$.

Let $\epsilon_1 := \min\{1/2, \pi(2|\lambda_0| - 1)/(4(|h'|_\infty + \pi)(1 + \pi))\}$, and $M_2 > 0$ such that if $|x| \geq M_2$ then $|h'(x)| < \epsilon_1$. For $i = 1, 2$, let $p_i \in B_1$ and $v_i(x, t) = p_i(t + x) - p_i(t - x)$. Let $w \in B_2$, $z \in B_3$, and $u_i(x, t) = (\varphi_\star + z + w + v_i)(x, t)$. From (23), $\|\varphi_\star + z\| \geq 3c_0\epsilon^{-1}/4$. Let $D = M_2 + r_1 + R_1$. By Lemma 1.5, for any $r \in [0, 2\pi]$,

$$
\begin{aligned}
\mu(G) &:= \mu(\{x; |(\varphi_\star + z)|(x, r \pm x) \leq D\} \\
&= \mu(\{x; |(\varphi_\star + z)|(x, r \pm x)/\|\varphi_\star + z\| \leq D/\|\varphi_\star + z\|\} \\
&\leq \mu(\{x; |(\varphi_\star + z)|(x, r \pm x)/\|\varphi_\star + z\| \leq 4D\epsilon/(3c_0)\} \\
&\leq c_2 \epsilon^{1/|\lambda_0|},
\end{aligned}
\tag{30}
$$

where $c_2 > 0$ is a constant independent of (ϵ, r).

From the definition of B_1, B_2, B_3 and D we have $|h(u_2(x, r \pm x) - h(u_1(x, r \pm x)| \leq \epsilon_1 |p_1(x) - p_2(x)|$ for all $x \in [0, \pi] \setminus G := G^c$. Therefore,

$$\int_0^\pi |h(u_2(x, r \pm x)) - h(u_1(x, r \pm x))| \, dx$$

$$\leq \int_G |h'(\zeta)||v_2(x, r \pm x) - v_1(x, r \pm x)| \, dx ds$$

$$+ \int_{G^c} \epsilon_1 |v_2(x, r \pm x) - v_1(x, r \pm x)| \, dx ds \tag{31}$$

$$\leq (|h'|_\infty c_2 \epsilon^{1/|\lambda_0|} + \epsilon_1) \|v_2 - v_1\|_\infty$$

$$\leq 2(|h'|_\infty c_2 \epsilon^{1/|\lambda_0|} + \epsilon_1) \|v_2 - v_1\|_\infty.$$

This prove that F is a contraction and hence the lemma.   □

4. **Range equation.** In this section we establish a priori estimates for solutions to (20). We will use that, for $w \in W$ and $|\lambda - \lambda_0| < 1/2$,

$$\left\|(\Box + \lambda I)^{-1} w\right\|_\infty \leq 8(|\lambda_0| + 1)|h|_\infty. \tag{32}$$

The proof of (32) follows by writing w in . Fourier series

Lemma 4.1. *Let* $u = (\varphi_\star + z) + w + v(z, w, \epsilon) \in Z \oplus W \oplus N$ *and* $v(z, w, \epsilon)$ *is as in Lemma 3.1. If* $w \in B_2$, $z \in B_3$, *and* $\epsilon \in (0, \epsilon_1)$, *then*

$$\left\|(\Box + \lambda I)^{-1} P_Z(h(\varphi_\star) - h(u))\right\| \leq \frac{c_0}{4\epsilon}. \tag{33}$$

Proof. Let γ be as in (3), $\beta \in (0, (\gamma - 1)/\gamma)$, and C, α as in (11). For $s \in [0, 1]$, let $\psi_s := \frac{\varphi_\star + sz}{\|\varphi_\star + sz\|}$, $\Omega_s' := \{(x, t) \in \Omega : |\psi_s(x, t)| < \epsilon^\beta\}$, and

$$\Omega_s := \left\{(x, t) \in \Omega : |\varphi_\star(x, t) + sz(x, t)| < \frac{3c_0}{4}\epsilon^{\beta-1}\right\}. \tag{34}$$

By (23) and the definition of B_2, for $z \in B_2$ we have $\Omega_s \subset \Omega_s'$. For $(x, t) \notin \Omega_s$ define the number

$$\xi(x, t, s) := \varphi_\star(x, t) + s(v(x, t) + w(x, t) + z(x, t)). \tag{35}$$

Then, if $(x, t) \notin \Omega_s$ and $\zeta \in Z$ is such that $\|\zeta\| = 1$, we have

$$|((\Box + \lambda I)^{-1} P_Z(h(\varphi_\star) - h(u)) \mid \zeta)|$$

$$\leq \frac{1}{\epsilon} \int_\Omega |h(\varphi_\star(x, t)) - h(\varphi_\star(x, t) + v(x, t) + w(x, t) + z(x, t)||\zeta(x, t)| \, dx dt$$

$$\leq \frac{d}{\epsilon}\left[\int_0^1 \int_{\Omega_s^c} |h'(\xi(x, t, s))||v(x, t) + z(x, t) + w(x, t)| \, dx dt ds + \int_{\Omega_s} 2|h|_\infty\right]$$

$$\leq \frac{d}{\epsilon}\left[\frac{2\pi^2}{c_0^\gamma 2^\gamma}\epsilon^{\gamma(1-\beta)}\left(r_1 + R_1 + \frac{c_0}{4\epsilon}\right) + 2|h|_\infty\epsilon^\alpha\right] \tag{36}$$

$$\leq \frac{c_0}{4\epsilon}.$$

Since that ζ was arbitrary with $\|\zeta\| = 1$, we have that (36) implies (33).   □

5. **Proof of Theorem 1.1.** Let $D := B_2 \times B_3$ and

$$G(w, z) := ((\Box + \lambda I)^{-1} P_W h(u), (\Box + \lambda I)^{-1} P_Z (h(\varphi_\star) - h(u))). \qquad (37)$$

Due to the compactness of $(\Box + \lambda I)^{-1} P_W$ and $(\Box + \lambda I)^{-1} P_Z$, G is compact. By (32), Lemma 4.1, and the Schuader Fixed Point Theorem, G has a fixed point in $(w, z) \in D$. Hence it is a solution to (20) and (21). This and (3.1), prove that $u = \varphi_\star + v(\epsilon, w, z) + w + z$ is a weak solution to (1) subject to (2). From (23), and the definition of B_1, B_2, B_3, $\lim_{\epsilon \to 0} \|\varphi_\star + v(\epsilon, w, z) + w + z\| = \infty$. This proves Theorem 1.1.

6. **Proof of Theorem 1.2.** We assume now that $-\lambda_0$ is an odd multiplicity eigenvalue of $\Box$. Let N, Y, W and Z as before. Take $R_0 := \max\{4|h|_\infty, 8(|\lambda_0| + 1)\}$ Choose $M_0 > 0$ such that $|x| > M_0$ for

$$|h'(x)| < \min\left\{\frac{1}{16}, \frac{1}{1024(|\lambda_0| + 1)\pi}\right\}. \qquad (38)$$

For $\beta = 1$ take $\alpha_0 = 0$ and $K = K_0$ as in Lemma 1.4 and $\alpha_1 = \alpha$ and $C_0 = C$ as in Lemma 1.5. Recall that $\alpha_0, \alpha_1, K_0, C_0$ depend only on λ_0. Let ρ_0 large enough such that

$$\rho_0 > \max\left\{\left(\frac{16 K_0 |h'|_\infty}{\pi}\right)^{1/\alpha_0} (M_0 + 4R_0), \qquad (39)\right.$$

$$\left. (1024 C_0 (|\lambda_0| + 1)\pi |h'|_\infty)^{1/\alpha_1} (M_0 + 4R_0)\right\}.$$

Let B_1 and B_2 as in Section 3. Define $B_3 := \{z \in Z : \|z\| \geq \rho_0\}$, $B_4 := [-\lambda_0 - 1/2, -\lambda_0 + 1/2]$ and define de function $\Gamma : B_1 \times B_2 \times B_3 \times B_4 \to B_1 \times B_2$ by the formula

$$\Gamma(p, w, z, \lambda) = \left(\frac{1}{2\pi\lambda} \int_0^\pi h(u(x, \cdot + x)) - h(u(x, \cdot - x))\, dx, \qquad (40)\right.$$

$$\left. - (\Box + \lambda I)^{-1} P_W h(u(x, t))\right)$$

where $u = v + w + z$. By the same arguments used in the proof of Theorem 1.1, Γ is well defined and is easy to verify that

$$d(\Gamma(p_1, w_1, z, \lambda); \Gamma(p_2, w_2, z, \lambda)) \leq \frac{1}{2}(\|p_1 - p_2\|_\infty + \|w_1 - w_2\|_\infty). \qquad (41)$$

where $d(p_1, w_1; p_2, w_2) = \|p_1 - p_2\|_\infty + \|w_1 - w_2\|_\infty$. By the Contraction Principle with Parameters, for $(z, \lambda) \in B_3 \times B_4$ fixed, Γ has a unique fixed point $(p(z, \lambda), w(z, \lambda)) \in B_1 \times B_2$. Even more, p and w depend continuously on (z, λ).

In order to prove Theorem 1.2 we appeal to the following results of M.A. Krasnoselskii, M. G. Crandall and P.H. Rabinowitz in their local forms ([12, p. 491] and [8, p. 383]).

Theorem 6.1 (Krasnoselskii-Rabinowitz)**.** *Let $\mathbb{E}$ a real Banach space, $\mathcal{E} = \mathbb{R} \times \mathbb{E}$ and $G : \mathcal{E} \to \mathbb{E}$ a compact application. Suppose that G can be written in the form $G(\lambda, u) = \lambda L u + H(\lambda, u)$ with $H(\lambda, u) = o(\|u\|)$ at 0 uniformly in bounded λ intervals and $L : \mathbb{E} \to \mathbb{E}$ is a compact linear map. Assume that O is a bounded set in $\mathcal{E}$ containing $(\lambda', 0)$ such that $G : \overline{O} \to \mathbb{E}$ is continuous and bounded. Also assume that $\frac{1}{\lambda'} \in \sigma(L)$ has odd multiplicity. Then there is a continuum of solutions $C_{\lambda'}$ (a closed connected set) of $G(\lambda, u) = u$ such that or $\partial O \cap C_{\lambda'} \neq \emptyset$ or $(\lambda'', 0) \in C_{\lambda'}$ where $\frac{1}{\lambda''} \in \sigma(L)$ and $\lambda'' \neq \lambda'$.*

Theorem 6.2 (Crandall-Rabinowitz). *Let X be a real Banach space, $K \in L(X)$, $\Omega \subset \mathbb{R} \times X$ a neighborhood of $(\lambda_0, 0)$ and $G : \Omega \to X$ such that G_x, G_λ, $G_{\lambda x}$ are continuous on Ω. Suppose also that (a) $H(\lambda, x) = o(\|x\|)$ as $x \to 0$, uniformly in λ near λ_0. (b) $I - \lambda K$ is Fredholm of index zero and λ_0 is a simple characteristic value of K.*

Then $(\lambda_0, 0)$ is a bifurcation point for $F(\lambda, x) = x - \lambda K x + H(\lambda, x) = 0$ and there is a neighborhood U of $(\lambda_0, 0)$ such that

$$F^{-1}(0) \cap U = \{(\lambda_0 + \sigma(t), tv + tz(t)) \; : \; |t| < \delta\} \cup \{(\lambda, 0) \; : \; (\lambda, 0) \in U\}$$

for some $\delta > 0$, with continuous functions σ and z such that $\sigma(0) = 0$, $z(0) = 0$ and the range of z is contained in $\ker(I - \lambda_0 K)^\perp = \mathrm{span}\,\{v\}^\perp$.

Proof of Theorem 1.2. For $z \in Z$, take $\psi = z/\|z\|$. Then (21) is equivalent to

$$(\lambda - \lambda_0)\psi - \|\psi\|^2 P_Z h(v(\lambda, \psi) + w(\lambda, \psi) + \|\psi\|^{-2}\psi) = 0, \tag{42}$$

applying Theorem 6.1 with $\mathbb{E} = Z$, $\overline{O} = B_3 \times B_4$, $\lambda' = -\lambda_0$ and $H(\lambda, \psi) = -\|\psi\|^2 P_Z h(v(\lambda, \psi) + w(\lambda, \psi) + \|\psi\|^{-2}\psi)$ we see that there is continuum of solutions to (42) accumulating to $(-\lambda_0, 0)$. Due to the equivalence between (1)-(2) and (42)-(2) this continuum of solutions yields a continuum of solutions to (1)-(2) accumulating in (λ_0, ∞).

Similarly, applying Theorem 6.2 with $Y = X$ and H as before we see that there is a parametrized continuous curve of solutions to (42)-(2)

$$\{(-\lambda_0 + \sigma(t), t\sin(\sqrt{-\lambda_0}\cdot) + tz(t)) \; : \; |t| < \delta\} \cup \{(\lambda, 0) \; : \; (\lambda, 0) \in U\} \tag{43}$$

in Y. Arguing as before, the proof is complete.

7. **Final comments.** Consider the problem of finding weak solutions $u : \mathbb{R}^2 \to \mathbb{R}$ to the problem

$$\Box u + \lambda u + h(u) = 0 \tag{44}$$

subject to the double-periodic conditions

$$u(x, t) = u(x + 2\pi, t) = u(x, t + 2\pi). \tag{45}$$

For the problem (44)-(45) we have a the same results with the same hypothesis for h as in Theorem 1.1 and Theorem 1.2 respectively. The argument of the proof is the same. The only technical difference is that a function v in the kernel N is characterized by $v(x, t) = \bar{v} + p(t - x) + q(t - x)$ where $\int_0^{2\pi} p = \int_0^{2\pi} q = 0$ and

$$\bar{v} = \frac{-1}{4\pi^2 \lambda} \int_\Omega h(u(x, t)) \tag{46}$$

$$2\pi\lambda(p(r) + \bar{v}) + \int_0^{2\pi} h(u(x, r - x)) = 0 \tag{47}$$

$$2\pi\lambda(q(r) + \bar{v}) + \int_0^{2\pi} h(u(x, r + x)) = 0. \tag{48}$$

Even more, if in addition to hypothesis made on h in Theorem 1.1, we assume that $h(x) > 0$ for $x \geq 0$ and $\liminf_{x \to \infty} h(x) > 0$ we can find bifurcation in the eigenvalue of infinite multiplicity ($-\lambda_0 = 0$) just taking $\varphi_\star \in N$ as a constant function satisfying $\lambda\varphi_\star + h(\varphi_\star) = 0$ and searching for solutions of the form $(\varphi_\star + v) + y \in N \oplus Y$ in a similar way we did as in proof of Theorem 1.1.

The bifurcation at infinity in the eigenvalue of infinite multiplicity for the problem (1)-(2) seems to be more difficult and is still open.

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