Uniform approximation of Muckenhoupt weights on fractals

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Abstract

Given an A_p-Muckenhoupt weight on a fractal obtained as the attractor of an iterated function system (IFS) with an aditional property, we construct a sequence of approximating weights, which are simple functions belonging uniformly to the A_p class on the approximating spaces.

Introduction

Let (X, d, μ) be an Ahlfors compact metric space of dimension γ with $\operatorname{diam}(X) = 1$.

Let Φ be a set of contractive similitudes

$$\Phi = \{\phi_i : X \to X, i = 1, 2, \dots, H\}$$

such that
1. $d(\phi_i(x), \phi_i(y)) = \beta d(x, y)$ for every $x, y \in X$ and some constant $0 < \beta < 1$ (IFS).
2. there exists an open set $U \subset X$ with

$$\bigcup_{i=1}^{H} \phi_i(U) \subseteq U,$$

and $\phi_i(U) \cap \phi_j(U) = \emptyset$ if $i \neq j$ (OSC).

Set $\mathfrak{I}^n = \{1, 2, \dots, H\}^n$, and for $\boldsymbol{i} = (i_1, i_2, \dots, i_n) \in \mathfrak{I}^n$, we denote

$$\boldsymbol{\phi}^n_{\boldsymbol{i}}(X) = (\phi_{i_n} \circ \phi_{i_{n-1}} \circ \dots \circ \phi_{i_2} \circ \phi_{i_1})(X)$$

and define $X^n_{\boldsymbol{i}} = \boldsymbol{\phi}^n_{\boldsymbol{i}}(X)$ and $X^n = \bigcup_{\boldsymbol{i} \in \mathfrak{I}^n} X^n_{\boldsymbol{i}}$.

The sequences $(X^n)_n$ converges in the sense of the Hausdorff distance to a non-empty compact set X^∞ (*attractor* of the system Φ) and

$$X^\infty = \bigcup_{i=1}^{H} \phi_i(X^\infty)$$

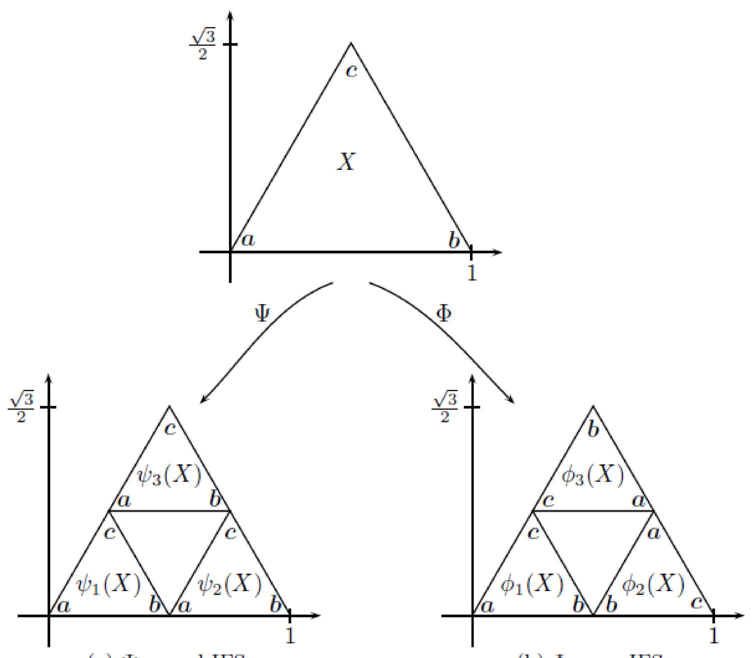
and it is the only set in X satisfying this property. Moreover, since $\phi_i(X) \subseteq X$ for every i, then $X^\infty = \bigcap_{n=1}^\infty X^n$ (see [4] or [3]). So that the system Φ defines or represents the set X^∞.

Assume Φ satisfies the *adjacency property*:

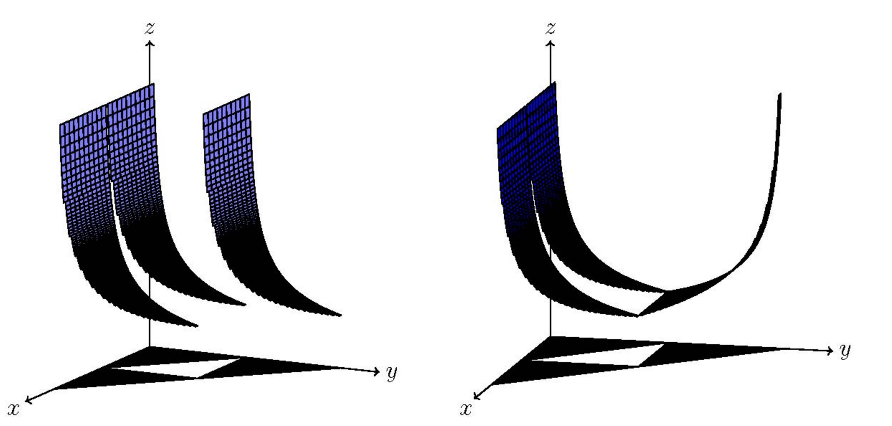
$$B(\boldsymbol{\phi}^n_{\boldsymbol{i}}(z), r) \cap X^n_{\boldsymbol{j}} \subseteq B(\boldsymbol{\phi}^n_{\boldsymbol{j}}(z), cr) \cap X^n_{\boldsymbol{j}}$$

holds for every $n \in \mathbb{N}$, every $\boldsymbol{i}, \boldsymbol{j} \in \mathfrak{I}^n$, every $r > 0$ and every $z \in X$.

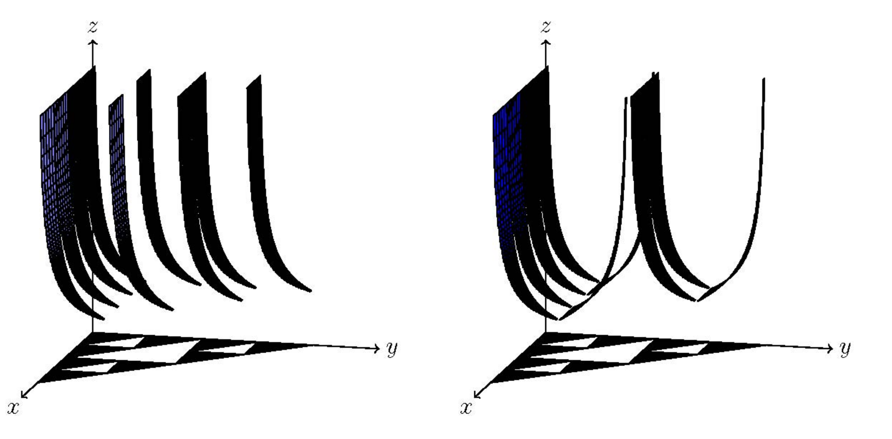
Examples:



(a) Ψ: usual IFS (b) Φ: new IFS



First iteration



Second iteration

Let μ^n be the natural "uniformly distributed" probability measure induced by μ on X^n:

$$\mu^n(E) = \frac{1}{H^n} \sum_{\boldsymbol{j} \in \mathfrak{I}^n} \mu\left((\boldsymbol{\phi}^n_{\boldsymbol{j}})^{-1}(E)\right) = \frac{1}{H^n} \sum_{\boldsymbol{j} \in \mathfrak{I}^n} \mu^n_{\boldsymbol{j}}(E)$$

for E a Borel set in X^n.

The sequence of measures $(\mu^n)_n$ converges in the weak star sense to a Borel probability measure μ^∞ supported on the attractor X^∞. This measure is called *invariant measure* or *self-affine measure* since is the unique satisfying

$$\mu^\infty(A) = \frac{1}{H} \sum_{i=1}^{H} \mu^\infty(\phi_i^{-1}(A))$$

for every Borel set A, and also

$$\int \varphi(x)\, d\mu^\infty(x) = \frac{1}{H} \sum_{i=1}^{H} \int \varphi(\phi_i(x))\, d\mu^\infty(x),$$

for every continuos function φ on X (see [4] or [3]). Moreover, the results in [5] show that $(X^\infty, d, \mu^\infty)$ is an Ahlfors space of dimension $s = -\log_\beta H$.

Finally we shall assume that the system Φ has *null overlapping* if $\mu^n(X^n_{\boldsymbol{i}} \cap X^n_{\boldsymbol{j}}) = \mu^\infty(X^n_{\boldsymbol{i}} \cap X^n_{\boldsymbol{j}}) = 0$ for every n and every $\boldsymbol{i}, \boldsymbol{j} \in \mathfrak{I}^n$, $\boldsymbol{i} \neq \boldsymbol{j}$.

Remark 1. *This property is not strong in the sense that the most of the typical fractals satisfying it. The property is equivalent to that the measures μ^n and μ^∞ are uniformly distributed, in the sense that $\mu^\infty(X^n_{\boldsymbol{j}}) = \mu^n(X^n_{\boldsymbol{j}}) = H^{-n}$ for every $\boldsymbol{j} \in \mathfrak{I}^n$.*

Theorem 1

Let $\nu^n = \sum_{\boldsymbol{i} \in \mathfrak{I}^n} v(\boldsymbol{i})\, \mu^n_{\boldsymbol{i}}$, with $\mu^n_{\boldsymbol{i}}(E) = \mu((\boldsymbol{\phi}^n_{\boldsymbol{i}})^{-1}(E))$. Then

1. *if $v(\boldsymbol{i}) \in A_p(\mathfrak{I}^n, \tilde{d}, card)$ uniformly in n, then $d\nu^n = w_n d\mu^n$, with $w_n \in A_p(X^n, d, \mu^n)$ uniformly in n;*
2. *if also we have that $\nu^n \xrightarrow{*} \nu$, then $d\nu = w d\mu^\infty$, with $w \in A_p(X^\infty, d, \mu^\infty)$.*

For the proof, we need the following resul given in [1]:
There exists a constant C such that

$$M_n f\left(\boldsymbol{\phi}^n_{\boldsymbol{i}}(z)\right) \leq C \mathfrak{M}_n\left(M(f \circ \boldsymbol{\phi}^n)(z)\right)(\boldsymbol{i}) \qquad (1)$$

holds for every $f \in L^1(X^n, \mu^n)$, $z \in X$, $\boldsymbol{i} \in \mathfrak{I}^n$ and $n \in \mathbb{N}$, where $M(f \circ \boldsymbol{\phi}^n)(z)$ denotes the function g on $\mathfrak{I}^n$ defined by $g(\boldsymbol{j}) = M(f \circ \boldsymbol{\phi}^n_{\boldsymbol{j}})(z)$.

Proof: Using (1), the hypothesis 1. and the L^p boundedness of M on (X, d, μ) and $\mathfrak{M}_n$ on $(\mathfrak{I}^n, \tilde{d}, \mathrm{card})$ we obtain

$$\begin{aligned}
\int_{X^n} |M_n f|^p\, d\nu^n &= \sum_{\boldsymbol{i} \in \mathfrak{I}^n} \int_X |M_n f(\boldsymbol{\phi}^n_{\boldsymbol{i}}(z))|^p\, v(\boldsymbol{i})\, d\mu(z) \\
&\leq C \int_X \sum_{\boldsymbol{i} \in \mathfrak{I}^n} |\mathfrak{M}_n\left(M(f \circ \boldsymbol{\phi}^n)(z)\right)(\boldsymbol{i})|^p\, v(\boldsymbol{i})\, d\mu(z) \\
&\leq C \int_X \sum_{\boldsymbol{i} \in \mathfrak{I}^n} |M(f \circ \boldsymbol{\phi}^n_{\boldsymbol{i}})(z)|^p v(\boldsymbol{i})\, d\mu(z) \\
&\leq C \sum_{\boldsymbol{i} \in \mathfrak{I}^n} \int_X |(f \circ \boldsymbol{\phi}^n_{\boldsymbol{i}})(z)|^p\, d\mu(z) v(\boldsymbol{i}) \\
&= C \int_{X^n} |f|^p\, d\nu^n.
\end{aligned}$$

Then, by Theorem 4 in [2] we have that ν^n is absolutely continuous with respect to μ^n, and its Radon-Nikodym derivative is an $A_p(X^n, d, \mu^n)$ weight. Finally, if $\nu^n \xrightarrow{*} \nu$, by Theorem 8 in [2] we have that ν is absolutely continuous with respect to μ^∞, and its Radon-Nikodym derivative is an $A_p(X^\infty, d, \mu^\infty)$ weight. □

Theorem 2

Given $w \in A_p(X^\infty, d, \mu^\infty)$, for each natural number n let us define a measure ν^n on X by

$$\nu^n := \sum_{\boldsymbol{i} \in \mathfrak{I}^n} v(\boldsymbol{i}) d\mu^n_{\boldsymbol{i}},$$

where $v(\boldsymbol{i}) := \frac{1}{H^n} \fint_{X^n_{\boldsymbol{i}}} w(y)\, d\mu^\infty(y)$. Then
1. *$d\nu^n = w_n d\mu^n$, with $w_n \in A_p(X^n, \mu^n)$ uniformly in n,*
2. *$\nu^n \xrightarrow{*} \nu$ where $d\nu := w\, d\mu^\infty$.*

Moreover,

$$w_n(x) = \sum_{\boldsymbol{i} \in \mathfrak{I}^n} \left(\fint_{X^n_{\boldsymbol{i}}} w(y)\, d\mu^\infty(y) \right) \mathcal{X}_{X^n_{\boldsymbol{i}}}(x),$$

where $\mathcal{X}_A$ denotes the indicator function on the set A.

Proof: • We shall prove that $v(\boldsymbol{i}) \in A_p(\mathfrak{I}^n, \tilde{d}, \mathrm{card})$:

$$\left(\sum_{\boldsymbol{j} \in \mathcal{B}} v(\boldsymbol{j})\right) \left(\sum_{\boldsymbol{j} \in \mathcal{B}} v(\boldsymbol{j})^{\frac{1}{1-p}}\right)^{p-1} \leq C\, \mathrm{card}(\mathcal{B})^p,$$

Hence, by Theorem 1 $d\nu^n = w_n d\mu^n$ with $w_n \in A_p(X^n, d, \mu^n)$ uniformly in n.
• Moreover, by definition of ν^n we have

$$\begin{aligned}
&\int_{X^n} f(x)\, w_n(x)\, d\mu^n(x) = \sum_{\boldsymbol{i} \in \mathfrak{I}^n} \int_{X^n_{\boldsymbol{i}}} f(x)\, v(\boldsymbol{i})\, d\mu^n_{\boldsymbol{i}}(x) \\
&= \sum_{\boldsymbol{i} \in \mathfrak{I}^n} H^n \int_{X^n} f(x)\, v(\boldsymbol{i})\, \mathcal{X}_{X^n_{\boldsymbol{i}}}(x)\, d\mu^n(x) \\
&= \int_{X^n} f(x) \left(H^n \sum_{\boldsymbol{i} \in \mathfrak{I}^n} v(\boldsymbol{i})\, \mathcal{X}_{X^n_{\boldsymbol{i}}}(x) \right) d\mu^n(x).
\end{aligned}$$

Then

$$w_n(x) = \sum_{\boldsymbol{i} \in \mathfrak{I}^n} \left(\fint_{X^n_{\boldsymbol{i}}} w(y)\, d\mu^\infty(y) \right) \mathcal{X}_{X^n_{\boldsymbol{i}}}(x).$$

• Finally we prove the weak star convergence:

$$\lim_{n \to \infty} \int_X \varphi(x) w_n(x) d\mu^n = \int_X \varphi(x) w(x) d\mu^\infty.$$

Notice first that

$$\begin{aligned}
&\int_X \varphi(x) w_n(x)\, d\mu^n \\
&= \int_X \varphi(x) \sum_{\boldsymbol{i} \in \mathfrak{I}^n} \left(\frac{1}{\mu^\infty(X^n_{\boldsymbol{i}})} \int_X w(y) \mathcal{X}_{X^n_{\boldsymbol{i}}}(y) d\mu^\infty \right) \mathcal{X}_{X^n_{\boldsymbol{i}}}(x) d\mu^n \\
&= \sum_{\boldsymbol{i} \in \mathfrak{I}^n} \int_X \int_X \varphi(x) \mathcal{X}_{X^n_{\boldsymbol{i}}}(x) \frac{1}{\mu^\infty(X^n_{\boldsymbol{i}})} w(y) \mathcal{X}_{X^n_{\boldsymbol{i}}}(y)\, d\mu^\infty\, d\mu^n \\
&= \int_X \sum_{\boldsymbol{i} \in \mathfrak{I}^n} \left(\frac{1}{\mu^\infty(X^n_{\boldsymbol{i}})} \int_X \varphi(x) \mathcal{X}_{X^n_{\boldsymbol{i}}}(x) d\mu^n \right) \mathcal{X}_{X^n_{\boldsymbol{i}}}(y) w(y)\, d\mu^\infty \\
&= \int_X g_n(y)\, w(y)\, d\mu^\infty(y),
\end{aligned}$$

where from the null overlaping property and the fact that $\mu^\infty(X^n_{\boldsymbol{i}}) = \mu^n(X^n_{\boldsymbol{i}}) = H^{-n}$, for each $y \in X^\infty$

$$g_n(y) = \frac{1}{\mu^\infty(X^n_{\boldsymbol{i}_0})} \int_X \varphi(x) \mathcal{X}_{X^n_{\boldsymbol{i}_0}}(x) d\mu^n(x) = \fint_{X^n_{\boldsymbol{i}_0}} \varphi(x)\, d\mu^n(x).$$

Since X is compact, φ is uniformly continuous on X, so that given $\varepsilon > 0$ there exists $\delta > 0$ such that $|\varphi(x) - \varphi(y)| < \varepsilon$ provided that $d(x, y) < \delta$. Let N_0 be such that $\beta^n < \delta$ if $n \geq N_0$. Hence, since $\operatorname{diam}(X^n_{\boldsymbol{i}_0}) = \beta^n$, for every $n \geq N_0$ we have

$$|g_n(y) - \varphi(y)| \leq \fint_{X^n_{\boldsymbol{i}_0}} |\varphi(x) - \varphi(y)|\, d\mu^n(x) < \varepsilon.$$

So that $\lim_{n \to \infty} g_n(y) = \varphi(y)$, and from the Lebesgue dominated convergence theorem we obtain

$$\lim_{n \to \infty} \int_X g_n(y) w(y)\, d\mu^\infty = \int_X \varphi(x) w(y) d\mu^\infty.$$

□

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