Course: Weighted inequalities and dyadic harmonic analysis

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Outline

- **Lecture 1.**
  Weighted Inequalities and Dyadic Harmonic Analysis.
  Model cases: Hilbert transform and Maximal function.
- **Lecture 2.**
  Brief Excursion into Spaces of Homogeneous Type.
  Simple Dyadic Operators and Weighted Inequalities à la Bellman.
- **Lecture 3.**
  Case Study: Commutators.
  Sparse Revolution.

Outline Lecture 2

1. Lerner's proof of Buckley's estimate
2. Random dyadic grids on $\mathbb{R}$
3. Haar basis and Spaces of Homogeneous Type
4. Dyadic Operators
   - Martingale transform
   - Dyadic square function
   - Petermichl's dyadic shift operator
   - Haar shift operators
   - Dyadic paraproduct
5. A_2 theorem for dyadic paraproduct

Buckley's A_p estimate for M (Lerner's '08 proof)

By the 1/3 trick suffices to check that for all $w \in A_p$, $1 < p < \infty$

$$\|M^{\mathcal{D}} f\|_{L^p(w)} \leq C_p [w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)}.$$

For $Q \in \mathcal{D}$ let $A_p(Q) = w(Q)\big(\sigma(Q)\big)^{p-1}/|Q|^p$, where $\sigma = w^{\frac{-1}{p-1}}$, then

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x)|\,dx &= A_p(Q)^{\frac{1}{p-1}}\left[\frac{|Q|}{w(Q)}\Big(\frac{1}{\sigma(Q)}\int_Q |f(x)|\sigma^{-1}(x)\sigma(x)\,dx\Big)^{p-1}\right]^{\frac{1}{p-1}} \\ &\leq [w]_{A_p}^{\frac{1}{p-1}}\left[\frac{1}{w(Q)}\int_Q \big(M_\sigma^{\mathcal{D}}(f\sigma^{-1})(x)\big)^{p-1} w^{-1}(x)w(x)\,dx\right]^{\frac{1}{p-1}}.\end{aligned}$$

Take supremum over $Q \in \mathcal{D}$ to get

$$M^{\mathcal{D}} f(x) \leq [w]_{A_p}^{\frac{1}{p-1}}\left[M_w^{\mathcal{D}}\Big(\big(M_\sigma^{\mathcal{D}}(f\sigma^{-1})\big)^{p-1} w^{-1}\Big)(x)\right]^{\frac{1}{p-1}}.$$

Lerner's proof (cont.)

$$M^{\mathcal{D}}f(x) \leq [w]_{A_p}^{\frac{1}{p-1}} \Big[M_w^{\mathcal{D}} \Big(\big(M_\sigma^{\mathcal{D}}(f\sigma^{-1}) \big)^{p-1} w^{-1} \Big)(x) \Big]^{\frac{1}{p-1}}.$$

Compute $L^p(w)$ norm, recall that $(p-1)p' = p$ where $\frac{1}{p} + \frac{1}{p'} = 1$, get

$$\begin{aligned}
\|M^{\mathcal{D}}f\|_{L^p(w)} &\leq [w]_{A_p}^{\frac{1}{p-1}} \|M_w^{\mathcal{D}}(M_\sigma^{\mathcal{D}}(f\sigma^{-1})^{p-1} w^{-1})\|_{L^{p'}(w)}^{\frac{1}{p-1}} \\
&\leq [w]_{A_p}^{\frac{1}{p-1}} \|M_w^{\mathcal{D}}\|_{L^{p'}(w)}^{\frac{1}{p-1}} \|M_\sigma^{\mathcal{D}}(f\sigma^{-1})\|_{L^p(\sigma)} \\
&\leq [w]_{A_p}^{\frac{1}{p-1}} \|M_w^{\mathcal{D}}\|_{L^{p'}(w)}^{\frac{1}{p-1}} \|M_\sigma^{\mathcal{D}}\|_{L^p(\sigma)} \|f\sigma^{-1}\|_{L^p(\sigma)} \\
&\leq p^{\frac{1}{p-1}} p' \, [w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)}.
\end{aligned}$$

Using uniform bounds of M_w in $L^{p'}(w)$ and M_σ on $L^p(\sigma)$. $\square$

For extensions to two-weights and *fractional maximal function* see Moen '09, 15.

Random dyadic grids on $\mathbb{R}$

Definition

A dyadic grid in $\mathbb{R}$ is a collection of intervals, organized in generations, each of them being a partition of $\mathbb{R}$, that have the nested, one parent, and two equal-length children per interval properties.

Shifted and scaled regular dyadic grid are dyadic grids. There are other grids. The following parametrization captures ALL dyadic grids in $\mathbb{R}$.

Lemma (Hytönen '08)

For each scaling parameter r with $1 \leq r < 2$, and random parameter β with $\beta = \{\beta_i\}_{i\in\mathbb{Z}}$, $\beta_i = 0, 1$, then $\mathcal{D}^{r,\beta} = \cup_{j\in\mathbb{Z}}\mathcal{D}_j^{r,\beta}$ is a dyadic grid. With $\mathcal{D}_j^{r,\beta} := r\mathcal{D}_j^{\beta}$, and $\mathcal{D}_j^{\beta} := x_j + \mathcal{D}_j$, where $x_j = \sum_{i>j} \beta_i 2^{-i}$.

Example (1/3 shift grids $D^i = D^{1,\beta^i}$ for $i \in \{0, 1, 2\}$.)

Where $\beta_j^0 \equiv 0$ (or $\equiv 1$), $\beta_j^1 = \mathbb{1}_{2\mathbb{Z}}(j)$, and $\beta_j^2 = \mathbb{1}_{2\mathbb{Z}+1}(j)$.

The advantage of this parametrization is that there is a very natural probability space, say $(\Omega, \mathbb{P})$ associated to the parameters, $\Omega = [1,2) \times \{0,1\}^{\mathbb{Z}}$. Averaging here means calculating the expectation in this probability space, that is $\mathbb{E}_\Omega f = \int_\Omega f(\omega)\, d\mathbb{P}(\omega)$.

Random dyadic grids have been used for example on:

- Study of $T(b)$ theorems on metric spaces with non-doubling measures, NTV '97, '03, also Hytönen, Martikainen '12.
- Hytönen's representation theorem, Hytönen '12.
- Generalizations to spaces of homogeneous type (SHT) Hytönen, Kairema '10, also Hytönen, Tapiola '15, following pioneering work Sawyer, David, Christ 80s-90s.
- Two-weight problem for Hilbert transform Lacey, Sawyer, Shen, Uriarte-Tuero '14.
- BMO from dyadic BMO on the bidisc and product spaces of SHT Pipher, Ward '08, Chen, Li, Ward '13, inspired by celebrated Garnett, Jones '82.

Haar basis in $\mathbb{R}$

Definition

Given an interval I, its associated *Haar function* is defined to be

$$h_I(x) := |I|^{-1/2}\big(\mathbb{1}_{I_r}(x) - \mathbb{1}_{I_l}(x)\big),$$

where $\mathbb{1}_I(x) = 1$ if $x \in I$, zero otherwise. Note $\int h_I = 0$.

- $\{h_I\}_{I\in\mathcal{D}}$ is a *complete orthonormal system* in $L^2(\mathbb{R})$ (Haar 1910). In particular for all $f \in L^2(\mathbb{R})$, with $\langle f, g\rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx$,

$$f = \sum_{I\in\mathcal{D}} \langle f, h_I\rangle h_I.$$

- The Haar basis is an *unconditional basis* in $L^p(\mathbb{R})$ and in $L^p(w)$ if $w \in A_p$ (Treil, Volberg '96) for $1 < p < \infty$. Deduced from boundedness of the *martingale transform*
- First example of a *wavelet basis* - *Haar multiresolution analysis*.

Dyadic cubes and Haar basis in $\mathbb{R}^d$

Definition

In $\mathbb{R}^d$ the dyadic cubes are cartesian products of dyadic interval of the same generation. A cube $Q \in \mathcal{D}_j(\mathbb{R}^d)$ if $Q = I_1 \times \cdots \times I_d$, with each $I_n \in \mathcal{D}_j(\mathbb{R})$. They are nested, one parent, 2^d children of equal volume.

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For $I \in \mathcal{D}(\mathbb{R})$, let $h_I^0 := h_I, \quad h_I^1 := |I|^{-1/2}\mathbb{1}_I$.

Definition (Tensor product Haar functions in $\mathbb{R}^d$)

For $Q \in \mathcal{D}(\mathbb{R}^d)$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_d)$, with $\epsilon_n = 0$ or 1, let

$$h_Q^{\epsilon}(x_1, \ldots, x_d) := h_{I_1}^{\epsilon_1}(x_1) \times \cdots \times h_{I_d}^{\epsilon_d}(x_d),$$

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$$h_Q^\epsilon(x_1, \ldots, x_d) := h_{I_1}^{\epsilon_1}(x_1) \times \cdots \times h_{I_d}^{\epsilon_d}(x_d),$$

Note that $h_Q^1 = |Q|^{-1/2}\mathbb{1}_Q$. The remaining $(2^d - 1)$ functions are the Haar functions associated to the cube Q: mean zero, L^2-norm one, constant on children. The collection $\{h_Q^\epsilon\}$ over $Q \in \mathcal{D}(\mathbb{R}^d)$ and $\epsilon \neq 1$ is the orthonormal Haar basis in $L^2(\mathbb{R}^d)$, unconditional basis in $L^p(\mathbb{R}^d)$,

Haar in $\mathbb{R}^2$

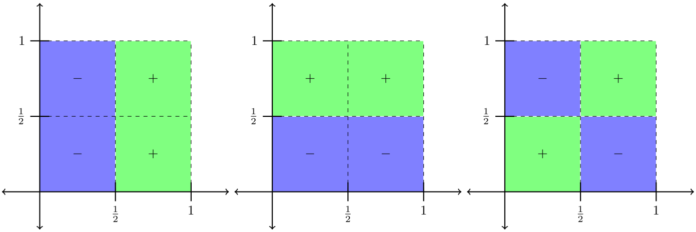

Figure: The three Haar function associated to a square in $\mathbb{R}^2$.

Figures by David Weirich, PhD Dissertation, UNM 2017

Haar in $\mathbb{R}^3$

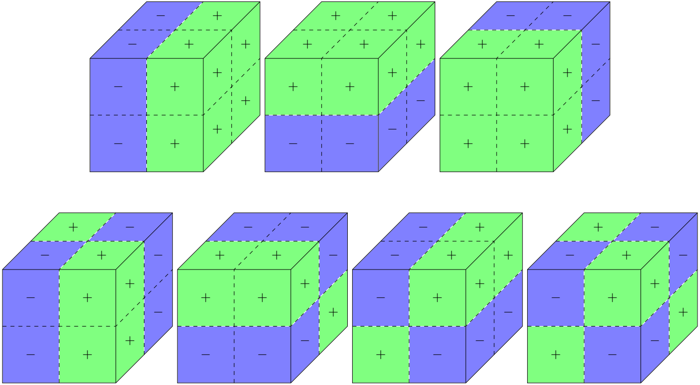

Figure: The seven Haar functions for a cube in $\mathbb{R}^3$

Figures by David Weirich, PhD Dissertation, UNM 2017

This construction seems very rigid, very dependent on the geometry of the cubes and on the group structure of Euclidean space $\mathbb{R}^d$.

Question

CAN WE DO DYADIC ANALYSIS IN OTHER SETTINGS?

Answer: YES!!!!

SPACES OF HOMOGENEOUS TYPE (SHT)

introduced by Coifman and Weiss '71.

Triple (X, ρ, μ) where ρ is a quasi-metric, μ is a doubling measure[1].

AND BEYOND! ask Xavi Tolsa...

There are "dyadic cubes" in SHT (Sawyer, Christ 80-90s, Hytönen-Kairema '12), random and adjacent families of cubes (Hytönen, Kairema, Martikainen, Tapiola '11-14).

[1] $\exists D_\mu \geq 1$ s.t. $\mu(B(x, 2r)) \leq D_\mu\, \mu(B(x, r))\ \forall x \in X$ and $r > 0$.

Examples of SHT

- $\mathbb{R}^n$, Euclidean metric, and Lebesgue measure.
- $\mathbb{R}^n$, Euclidean metric, $d\mu = w\,dx$ where w is a doubling weight (e.g. $w \in A_\infty$ or A_p or RH_q weights).
- Quasi-metric spaces with d-Ahlfors regular measure: $\mu(B(x,r)) \sim r^d$ (e.g. Lipschitz surfaces, fractal sets, n-thick subsets of $\mathbb{R}^n$).
- Compact Lie groups.
- C^∞ manifolds with doubling volume measure for geodesic balls.
- Carnot-Caratheodory spaces.
- Nilpotent Lie groups (e.g. Heisenberg group).

The recent book *Hardy Spaces on Ahlfors-Regular Quasi Metric Spaces A Sharp Theory* by Ryan Alvarado, Marius Mitrea '15 uses the Segovia-Macías philosophy heavily.

Some history

Haar-type bases for $L^2(X, \mu)$ have been constructed in general metric spaces, and the construction is well known to experts.

- Haar-type wavelets associated to nested partitions in abstract measure spaces were constructed by Girardi, Sweldens '97.
- Such Haar functions are also used in geometrically doubling metric spaces, Nazarov, Reznikov, Volberg '13.
- For the case of spaces of homogeneous type there is local expertise, see Aimar, Gorosito '00, Aimar '02, Aimar, Bernadis, Jaffei '07, and Aimar, Bernadis, Nowak '11.
- For the case of geometrically doubling quasi-metric space (X, ρ), with a positive Borel measure μ, see Kairema, Li, P., Ward '16.

Martingale transform

Definition (The Martingale transform)

$$T_\sigma f(x) := \sum_{I\in\mathcal{D}} \sigma_I \langle f, h_I\rangle h_I(x), \qquad \text{where} \quad \sigma_I = \pm 1 \text{ (at random)}.$$

- Martingale transform is a good toy model for CZ singular operators: $\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\,\widehat{f}(\xi)$, and $\langle T_\sigma f, h_I\rangle = \sigma_I \langle f, h_I\rangle$.
- Unconditionality of the Haar basis on $L^p(\mathbb{R})$ follows from uniform (on choice of signs σ) boundedness of T_σ on $L^p(\mathbb{R})$

$$\sup_\sigma \|T_\sigma f\|_p \leq C_p \|f\|_p \quad \text{(Burkholder '84 best } C_p).$$

- Unconditionality on $L^p(w)$ when $w \in A_p$ follows from uniform boundedness of T_σ on $L^p(w)$ (Treil, Volberg '96).
- Sharp linear bounds on $L^2(w)$ when $w \in A_2$ (Wittwer '00).
- Necessary and sufficient conditions on (u, v) are known (NTV '99).

Dyadic square function

Definition (The dyadic square function)

$$(S^{\mathcal{D}}f)^2(x) := \sum_{I\in\mathcal{D}} \frac{|\langle f, h_I\rangle|^2}{|I|} \mathbb{1}_I(x),$$

where $\langle f\rangle_I = (1/|I|)\int_I f(x)\,dx$, $\tilde{I}$ is the parent of I.

- $S^{\mathcal{D}}$ is an isometry on $L^2(\mathbb{R})$ ($\|S^{\mathcal{D}}f\|_2 = \|f\|_2$).
- $S^{\mathcal{D}}$ is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$ furthermore

$$\|S^{\mathcal{D}}f\|_p \sim \|f\|_p.$$

This plays the role of *Plancherel in* L^p (Littlewood-Paley theory). It implies boundedness of T_σ (and Ш) on L^p

$$\|T_\sigma f\|_p \sim \|S^{\mathcal{D}}(T_\sigma f)\|_p = \|S^{\mathcal{D}}f\|_p \sim \|f\|_p.$$

One weight estimates for S^d

- Plancherel in $L^2(w)$: $S^{\mathcal{D}}$ is bounded on $L^2(w)$ if $w \in A_2$ moreover

$$c[w]_{A_2}^{-1/2}\|f\|_{L^2(w)} \leq \|S^{\mathcal{D}}f\|_{L^2(w)} \leq C[w]_{A_2}\|f\|_{L^2(w)}$$

  (Petermichl, Pott '02, Hukovic,Treil, Volberg '00, Wittwer '00). This will give an $L^2(w)$ bound for T_σ (and for Ш) of the form $[w]_{A_2}^{3/2}$. The optimal bound is linear (Wittwer '00).
- Bounded in $L^2(w)$ implies by extrapolation bounded in L^p (and in $L^p(w)$). Buckley '93 has a very simple proof of $L^2(w)$-boundedness.

Note that $\|S^{\mathcal{D}}f\|_{L^2(w)}^2 = \sum_{I\in\mathcal{D}} |\langle f, h_I\rangle|^2 \langle w\rangle_I$.

Petermichl's dyadic shift operator

Definition

Petermichl's dyadic shift operator Ш (pronounced "Sha") associated to the standard dyadic grid $\mathcal{D}$ is defined for functions $f \in L^2(\mathbb{R})$ by

$$Ш f(x) := \sum_{I \in \mathcal{D}} \langle f, h_I \rangle H_I(x),$$

where $H_I = 2^{-1/2}(h_{I_r} - h_{I_l})$.

- Ш is an isometry on $L^2(\mathbb{R})$, i.e. $\|Ш f\|_2 = \|f\|_2$, bounded in $L^p(\mathbb{R})$.
- Ш is a good dyadic model for H: $Ш h_J(x) = H_J(x)$, the functions h_J and H_J can be viewed as localized sine and cosine.
- More evidence comes from the way the family $\{Ш_{r,\beta}\}_{(r,\beta)\in\Omega}$ interacts with translations, dilations and reflections.

Petermichl's representation theorem for H

Each dyadic shift operator does not have symmetries that characterize H, but an average over all random dyadic grids $\mathcal{D}^{r,\beta}$ does.

Theorem (Petermichl 2000)

$$\mathbb{E}_\Omega Ш^{r,\beta} = \int_\Omega Ш^{r,\beta} d\mathbb{P}(r,\beta) = cH.$$

- Result follows once one verifies that $c \neq 0$ (which she did!).
- $Ш^{r,\beta}$ are uniformly bounded on $L^p \Rightarrow H$ is bounded on L^p.
- Similar representation works for the *Beurling* (Petermichl, Volberg '02) and *Riesz* (Petermichl '08) transforms.
- There is a representation valid for ALL Calderón-Zygmund singular integral operators (Hytönen '12).

Haar shift operators of arbitrary complexity

Definition (Lacey, Reguera, Petermichl '10)

A *Haar shift operator of complexity* (m, n) is

$$Ш_{m,n} f(x) := \sum_{L \in \mathcal{D}} \sum_{I \in \mathcal{D}_m(L), J \in \mathcal{D}_n(L)} c^L_{I,J} \langle f, h_I \rangle h_J(x),$$

where the coefficients $|c^L_{I,J}| \leq \frac{\sqrt{|I|\,|J|}}{|L|}$, and $\mathcal{D}_m(L)$ denotes the dyadic subintervals of L with length $2^{-m}|L|$.

- The cancellation property of the Haar functions and the normalization of the coefficients ensures that $\|Ш_{m,n} f\|_2 \leq \|f\|_2$.
- T_σ is a Haar shift operator of complexity $(0, 0)$.
- $Ш$ is a Haar shift operator of complexity $(0, 1)$.
- The dyadic paraproduct π_b is not one of these.

The dyadic paraproduct

Definition

The *dyadic paraproduct* associated to $b \in BMO^d$ is

$$\pi_b f(x) := \sum_{I \in \mathcal{D}} \langle f \rangle_I \langle b, h_I \rangle h_I(x),$$

where $\langle f \rangle_I = \frac{1}{|I|} \int_I f(x)\, dx = \langle f, \mathbb{1}_I / |I| \rangle$.

- Formally, $fb = \pi_b f + \pi_b^* f + \pi_f b$. Product by b is bounded on $L^p(\mathbb{R})$ if and only if $b \in L^\infty(\mathbb{R})$.
- Paraproduct is a bounded operator on $L^2(\mathbb{R})$ if and only if $b \in BMO^d$. By the *Carleson Embedding Lemma.*
- Paraproduct bounded on $L^2(w)$ for all $w \in A_2$, moreover

$$\|\pi_b f\|_{L^2(w)} \leq C[w]_{A_2} \|f\|_{L^2(w)} \qquad \text{(Beznosova '08)}.$$

By extrapolation bounded on $L^p(w)$ for all $w \in A_p$, in particular it is bounded on $L^p(\mathbb{R})$.

Hytönen's Representation theorem

Theorem (Hytönen's Representation Theorem '12)

Let T be a Calderón-Zygmund singular integral operator, then

$$Tf = \mathbb{E}_\Omega \left(\sum_{(m,n)\in\mathbb{N}^2} a_{m,n} \text{Ш}^{r,\beta}_{m,n} f + \pi^{r,\beta}_{T1} f + (\pi^{r,\beta}_{T^*1})^* f \right),$$

with $a_{m,n} = e^{-(m+n)\alpha/2}$, α is the smoothness parameter of T.

- $\text{Ш}^{r,\beta}_{m,n}$ are Haar shift operators of complexity (m,n),
- $\pi^{r,\beta}_{T1}$ a dyadic paraproduct,
- $(\pi^{r,\beta}_{T^*1})^*$ the adjoint of the dyadic paraproduct ,

All defined on random dyadic grid $\mathcal{D}^{r,\beta}$.

A_2 theorem for dyadic paraproduct

Goal is to show Beznosova's linear bound for the paraproduct

$$\|\pi_b f\|_{L^2(w)} \leq C[w]_{A_2}\|f\|_{L^2(w)}.$$

Recall: the *dyadic paraproduct* associated to $b \in BMO^d$ is

$$\pi_b f(x) := \sum_{I \in \mathcal{D}} b_I \langle f \rangle_I h_I(x),$$

where $\langle f \rangle_I = \frac{1}{|I|}\int_I f(x)\,dx$ and $b_I = \langle b, h_I \rangle$.

To start need a few ingredients: (weighted) Carleson sequences and Carleson Embedding Lemma.

Weighted Carleson sequences

Definition

A positive sequence $\{\lambda_I\}_{I\in\mathcal{D}}$ is *w-Carleson* if there is $C>0$ such that

$$\sum_{I\in\mathcal{D}(J)} \lambda_I \leq Cw(J) \quad \text{for all } J\in\mathcal{D}.$$

Smallest $C>0$ is called the *intensity* of the sequence, $w(J)=\int_J w(x)\,dx$.

When $w=1$ a.e. we say that the sequence is Carleson (not 1-Carleson).

Example

If $b\in BMO$ then the sequence $\{b_I^2\}_{I\in\mathcal{D}}$ is Carleson:

$$\sum_{I\in\mathcal{D}(J)} b_I^2 = \sum_{I\in\mathcal{D}(J)} |\langle b,h_I\rangle|^2 = \int_J |b(x)-\langle b\rangle_J|^2 dx \leq \|b\|_{BMO}^2 |J|.$$

(Because $\{h_I\}_{I\in\mathcal{D}(J)}$ is an o.n. basis on $L_0^2(J)=\{f\in L^2(J) : \int_J f(x)\,dx=0\}$.)

Weighted Carleson Lemma ▸ Sigma1 ▸ Sigma2

Lemma (NTV '99)

Given weight v, then $\{\lambda_I\}$ is a v-Carleson sequence with intensity $\mathcal{B}$ iff for all non-negative v-measurable functions F on $\mathbb{R}$

$$\sum_{I\in\mathcal{D}} \lambda_I \inf_{x\in I} F(x) \le \mathcal{B} \int_{\mathbb{R}} F(x)v(x)\,dx.$$

Particular example: $F(x) = (M_v^{\mathcal{D}} f(x))^2$ where $M_v^{\mathcal{D}} f(x) := \sup_{I\in\mathcal{D}:x\in I} \langle |f|\rangle_I^v$, $\langle |f|\rangle_I^v := \frac{\langle |f|v\rangle_I}{\langle v\rangle_I} \le \inf_{x\in I} M_v^{\mathcal{D}} f(x)$ then by Carleson's Lemma

$$\sum_{I\in\mathcal{D}} \lambda_I \,(\langle |f|\rangle_I^v)^2 \le \mathcal{B}\|M_v^{\mathcal{D}} f\|_{L^2(v)}^2 \le 2\mathcal{B}\|f\|_{L^2(v)}^2.$$

In particular, $v=1$, $b\in BMO$, then $\lambda_I = b_I^2$ is Carleson and

$$\|\pi_b f\|_2^2 = \sum_{I\in\mathcal{D}} |\langle \pi_b f, h_I\rangle|^2 \le \sum_{I\in\mathcal{D}} b_I^2\,\langle |f|\rangle_I^2 \le C\|b\|_{BMO}^2\|f\|_2^2.$$

Paraproduct on $L^2(w)$ with bound $[w]_{A_2}^{3/2}\|b\|_{BMO}$

- By duality suffices to show that for all $f \in L^2(w)$, $g \in L^2(w^{-1})$
$$|\langle \pi_b f, g\rangle| \le C\|b\|_{BMO}[w]_{A_2}^{3/2}\|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}.$$
- $|\langle \pi_b f, g\rangle| \le \sum_{I\in\mathcal{D}} \langle |f|\rangle_I\,|b_I|\,|\langle g, h_I\rangle| =: \Sigma_1$
- By Cauchy-Schwarz, weighted Carleson lemma, $\|fw\|_{L^2(w^{-1})} = \|f\|_{L^2(w)}$:

$$\begin{aligned}
\Sigma_1 &\le \left(\sum_{I\in\mathcal{D}} \frac{\langle |f|\rangle_I^2\, b_I^2}{\langle w^{-1}\rangle_I}\right)^{1/2} \left(\sum_{I\in\mathcal{D}} |\langle g, h_I\rangle|^2 \langle w^{-1}\rangle_I\right)^{1/2} \\
&\le \left(\sum_{I\in\mathcal{D}} \left(\frac{\langle |f| w w^{-1}\rangle_I}{\langle w^{-1}\rangle_I}\right)^2 \frac{b_I^2}{\langle w\rangle_I}\langle w\rangle_I\,\langle w^{-1}\rangle_I\right)^{1/2} \|S^d g\|_{L^2(w^{-1})} \\
&\le [w]_{A_2}^{1/2}\left(\sum_{I\in\mathcal{D}} (\langle |f| w\rangle_I^{w^{-1}})^2 \frac{b_I^2}{\langle w\rangle_I}\right)^{1/2} C[w]_{A_2}\|g\|_{L^2(w^{-1})} \\
&\le C[w]_{A_2}^{3/2}\, 4\|b\|_{BMO}\|M_{w^{-1}}(fw)\|_{L^2(w^{-1})}\|g\|_{L^2(w^{-1})} \\
&\le C\|b\|_{BMO}[w]_{A_2}^{3/2}\|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}. \qquad \square
\end{aligned}$$

Beznosova's Little Lemma

To create v-Carleson sequences from a given Carleson sequences we have the following lemma.

Lemma (Beznosova '08)

Let v be a weight, such that v^{-1} is also a weight. Let $\{\lambda_I\}_{I\in\mathcal{D}}$ be a Carleson sequence with intensity $\mathcal{B}$, then for all $J\in\mathcal{D}$

$$\sum_{I\in\mathcal{D}(J)} \frac{\lambda_I}{\langle v^{-1}\rangle_I} \leq 4\mathcal{B}\, v(J).$$

"Sequence $\{\lambda_I/\langle v^{-1}\rangle_I\}_{I\in\mathcal{D}}$ is v-Carleson with intensity $4\mathcal{B}$."

The proof uses a Bellman function argument.

Example ($b\in BMO^d$, $w\in A_2$)

Sequence $\{b_I^2/\langle w\rangle_I\}_{I\in\mathcal{D}}$ is a w^{-1}-Carleson, with intensity $4\|b\|_{BMO}^2$.

Algebra of Carleson sequences

Lemma

Given a weight v. Let $\{\lambda_I\}_{I\in\mathcal{D}}$ and $\{\gamma_I\}_{I\in\mathcal{D}}$ be two v-Carleson sequences with intensities A and B respectively then for any $c, d > 0$

- $\{c\lambda_I + d\gamma_I\}_{I\in\mathcal{D}}$ *is a v-Carleson sequence with intensity at most $cA + dB$.*
- $\{\sqrt{\lambda_I\gamma_I}\}_{I\in\mathcal{D}}$ *is a v-Carleson sequence with intensity at most $\sqrt{AB}$.*

The proof is a simple exercise. Sigma2

Example ($u, v \in A_\infty$, $\Delta_I v := \langle v\rangle_{I_+} - \langle v\rangle_{I_-}$)

- $\left\{ \left|\frac{|\Delta_I v|}{\langle v\rangle_I}\right|^2 |I| \right\}_{I\in\mathcal{D}}$, is a Carleson sequence, with intensity $\log[w]_{A_\infty}$ (Kenig, R. Fefferman, Pipher '91). If $w \in A_2$ then $[w]_{A_\infty} \leq [w]_{A_2}$.
- Let $\alpha_I = \frac{|\Delta_I v|}{\langle v\rangle_I}\frac{|\Delta_I u|}{\langle u\rangle_I}|I|$. Then $\{\alpha_I\}_{I\in\mathcal{D}}$ is a Carleson sequence. When $v \in A_2$, $u = v^{-1}$ (also in A_2) its intensity is $\sim \log[w]_{A_2}$.

The α-Lemma

Lemma (Beznosova '08 for $0 < \alpha < 1/2$, Bellman function proof)

If $w \in A_2$ *and* $0 < \alpha$, *then the sequence*

$$\mu_I := \langle w\rangle_I^\alpha \langle w^{-1}\rangle_I^\alpha |I| \left(\frac{|\Delta_I w|^2}{\langle w\rangle_I^2} + \frac{|\Delta_I w^{-1}|^2}{\langle w^{-1}\rangle_I^2} \right) \quad I \in \mathcal{D}$$

is Carleson with Carleson intensity at most $C_\alpha [w]_{A_2}^\alpha$, *and* $C_\alpha = \frac{72}{\alpha - 2\alpha^2}$.

Algebra + Kenig, Fefferman, Pipher gives worst intensity $[w]_{A_2}^\alpha \log[w]_{A_2}$.

Example ($w \in A_2^d$, $b \in BMO^d$)

By α-Lemma, and algebra of Carleson sequences ‣ Sigma2

- $\{\nu_I := |\Delta_I w|^2 \langle w^{-1}\rangle_I^2 |I|\}_{I\in\mathcal{D}}$ is Carleson with intensity $C_{1/4}[w]_{A_2}^2$.
- Then $\{b_I \sqrt{\nu_I}\}_{I\in\mathcal{D}}$ is Carleson with intensity $C[w]_{A_2}\|b\|_{BMO}$.

Play cards correctly and can get linear bound for paraproduct.

Weighted or disbalanced Haar basis

Definition

Given weight w and interval I, the *weighted Haar function* h_I^w is

$$h_I^w(x) := \frac{1}{\sqrt{w(I)}} \left(\sqrt{\frac{w(I_-)}{w(I_+)}}\, \mathbb{1}_{I_+}(x) - \sqrt{\frac{w(I_+)}{w(I_+)}}\, \mathbb{1}_{I_-}(x) \right).$$

- $\{h_I^w\}_{I \in \mathcal{D}}$ is an orthonormal system in $L^2(w)$.
- There exist sequences α_I^w, β_I^v such that

$$h_I(x) = \alpha_I^w h_I^w(x) + \beta_I^w \frac{\mathbb{1}_I(x)}{\sqrt{|I|}}$$

(i) $|\alpha_I^w| \leq \sqrt{\langle w \rangle_I}$,

(ii) $|\beta_I^w| \leq \frac{|\Delta_I w|}{\langle w \rangle_I}$, and $\Delta_I w := \langle w \rangle_{I_+} - \langle w \rangle_{I_-}$.

Proof of A_2 conjecture for dyadic paraproduct

Suffices by duality to prove:

$$|\langle \pi_b f, g\rangle| \leq C\|b\|_{BMO}[w]_{A_2}\|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}$$

This time introduce weighted Haar functions to obtain two terms

$$|\langle \pi_b f, g\rangle| \leq \sum_{I\in\mathcal{D}} |b_I|\langle |f|ww^{-1}\rangle_I |\langle gw^{-1}w, h_I\rangle| \leq \Sigma_1 + \Sigma_2,$$

where we replace $h_I = \alpha_I^w h_I^w + \beta_I^w \frac{\mathbb{1}_I}{\sqrt{|I|}}$, to get

$$\Sigma_1 := \sum_{I\in\mathcal{D}} |b_I|\langle |f|ww^{-1}\rangle_I |\langle gw^{-1}w, h_I^w\rangle| \sqrt{\langle w\rangle_I}$$

$$\Sigma_2 := \sum_{I\in\mathcal{D}} |b_I|\langle |f|ww^{-1}\rangle_I \langle |g|w^{-1}w\rangle_I \frac{|\Delta_I w|}{\langle w\rangle_I}\sqrt{|I|}$$

Sigma1 Sigma2

First sum

proof

$$\Sigma_1 \leq \sum_{I\in\mathcal{D}} \frac{|b_I|}{\sqrt{\langle w\rangle_I}} \frac{\langle |f|ww^{-1}\rangle_I}{\langle w^{-1}\rangle_I} |\langle gw^{-1}, h_I^w\rangle_w|\, \langle w\rangle_I \langle w^{-1}\rangle_I$$

$$\leq [w]_{A_2} \sum_{I\in\mathcal{D}} \frac{|b_I|}{\sqrt{\langle w\rangle_I}} \inf_{x\in I} M_{w^{-1}}(fw)(x)\, |\langle gw^{-1}, h_I^w\rangle_w|$$

$$\leq [w]_{A_2} \left(\sum_{I\in\mathcal{D}} \frac{|b_I|^2}{\langle w\rangle_I} \inf_{x\in I} M^2_{w^{-1}}(fw)(x) \right)^{\frac{1}{2}} \left(\sum_{I\in\mathcal{D}} \left|\langle gw^{-1}, h_I^w\rangle_w\right|^2 \right)^{\frac{1}{2}}$$

Use Weighted Carleson Lemma with $F(x) = M^2_{w^{-1}}(fw)(x)$ and $v = w^{-1}$, and w^{-1}-Carleson sequence $b_I^2/\langle w\rangle_I$ by Little Lemma.

$$\Sigma_1 \leq [w]_{A_2} \|b\|_{BMO} \left(\int_{\mathbb{R}} M^2_{w^{-1}}(fw)(x) w^{-1}(x) dx \right)^{\frac{1}{2}} \|gw^{-1}\|_{L^2(w)}$$

$$\leq C [w]_{A_2} \|b\|_{BMO} \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})}$$

Second sum

proof Using similar arguments that we used for Σ_1

$$\Sigma_2 \leq \sum_{I\in\mathcal{D}} |b_I| \frac{\langle |f| w w^{-1} \rangle)}{\langle w^{-1}\rangle_I} \frac{\langle |g| w^{-1} w\rangle_I}{\langle w\rangle_I} \sqrt{|\Delta_I w|^2 \langle w^{-1}\rangle_I^2 |I|}$$
$$\leq \sum_{I\in\mathcal{D}} |b_I| \sqrt{\nu_I} \inf_{x\in I} M_{w^{-1}}(fw)(x)\, M_w(gw^{-1})(x),$$

where $|b_I|^2$ and ν_I are Carleson sequences with intensities $\|b\|_{BMO}^2$ and $[w]_{A_2}^2$ Alpha Lemma then by algebra CS the sequence $|b_I|\sqrt{\nu_I}$ is Carleson sequence with intensity $\|b\|_{BMO}[w]_{A_2}$. Using Weighted Carleson Lemma with $v = 1$ and $F(x) = M_{w^{-1}}(fw)(x)\, M_w(gw^{-1})(x)$,

$$\Sigma_2 \leq [w]_{A_2} \|b\|_{BMO} \int_{\mathbb{R}} M_{w^{-1}}(fw)(x) M_w(gw^{-1})(x)dx.$$

To finish use Cauchy-Schwarz and $w^{\frac{1}{2}}(x)\, w^{-\frac{1}{2}}(x) = 1$,

$$\begin{aligned}\Sigma_2 &\leq [w]_{A_2} \|b\|_{BMO} \int_{\mathbb{R}} M_{w^{-1}}(fw)(x)\, M_w(gw^{-1})(x)dx \\ &\leq [w]_{A_2} \|b\|_* \left[\int_{\mathbb{R}} M^2_{w^{-1}}(fw)(x) w^{-1}(x)dx\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}} M^2_w(gw^{-1})(x)w(x)dx\right]^{\frac{1}{2}} \\ &= [w]_{A_2} \|b\|_{BMO} \|M_{w^{-1}}(fw)\|_{L^2(w^{-1})} \|M_w(gw^{-1})\|_{L^2(w)} \\ &\leq C[w]_{A_2} \|b\|_{BMO} \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})}.\end{aligned}$$

We are done!! □

Beznosova's Little Lemma

Lemma (Beznosova '08)

Let w be a weight, such that w^{-1} is a a weight as well. Let $\{\lambda_I\}_{I\in\mathcal{D}}$ be a Carleson sequence with intensity $\mathcal{B}$, then for all $J \in \mathcal{D}$

$$\sum_{I\in\mathcal{D}(J)} \frac{\lambda_I}{m_I w^{-1}} \leq 4\mathcal{B}\, w(J).$$

"The sequence $\{\frac{\lambda_I}{m_I w^{-1}}\}_{I\in\mathcal{D}}$ is w-Carleson with intensity $4\mathcal{B}$."

The proof uses a Bellman function argument, which we now describe.

Proof of the Little Lemma

The first lemma encodes what now is called an *induction on scales argument*. If we can find a Bellman function with certain properties, then we will solve our problem by induction on scales.

Lemma (Induction on scales)

Suppose there exists a real valued function of 3 *variables* $B(x) = B(u,v,l)$, *whose domain* $\mathfrak{D}$ *contains points* $x = (u,v,l)$

$$\mathfrak{D} = \{(u,v,l) \in \mathbb{R}^3 : u,v > 0, \quad uv \geq 1 \quad \textit{and} \quad 0 \leq l \leq 1\},$$

whose range is given by $0 \leq B(x) \leq u$, *and such that the following convexity property holds:* $\forall x, x_\pm \in \mathfrak{D}$ *such that* $x - \frac{x_+ + x_-}{2} = (0,0,\alpha)$ *we have*

$$B(x) - \frac{B(x_+) + B(x_-)}{2} \geq \frac{1}{4v}\alpha.$$

Then the Little Lemma holds.

Induction on scales

Proof. WLOG assume $\mathcal{B} = 1$.

Fix a dyadic interval J. Let $u_J = m_J w$, $v_J = m_J(w^{-1})$ and $l_J = \frac{1}{|J|Q} \sum_{I \in D(J)} \lambda_I$, then $x_J := (u_J, v_J, l_J) \in \mathfrak{D}$. Let $x_\pm := x_{J^\pm} \in \mathfrak{D}$.

$$x_J - \frac{x_{J^+} + x_{J^-}}{2} \;=\; (0, 0, \alpha_J), \;\text{ where }\; \alpha_J := \frac{\lambda_J}{|J|}.$$

Then, by the size and convexity conditions, and $|J^+| = |J^-| = |J|/2$,

$$|J|\, m_J w \;\geq\; |J|\; B(x_J) \geq |J^+| B(x_{J^+}) + |J^-| B(x_{J^-}) + \frac{\lambda_J}{4 m_J(w^{-1})}.$$

Repeat for $|J^+| B(x_{J^+})$ and $|J^-| B(x_{J^-})$, use that $B \geq 0$ on $\mathfrak{D}$ to get:

$$m_J w \geq \frac{1}{4|J|} \sum_{I \in D(J)} \frac{\lambda_I}{m_I(w^{-1})} \;\Rightarrow\; \sum_{I \in \mathcal{D}(J)} \frac{\lambda_I}{m_I v^{-1}} \leq 4\, v(J).$$

□

The Bellman function

Lemma (Beznosova '08)

The function

$$B(u,v,l) := u - \frac{1}{v(1+l)}$$

is defined on $\mathfrak{D}$, $0 \leq B(x) \leq u$ *for all* $x = (u,v,l) \in \mathfrak{D}$ *and on* $\mathfrak{D}$*:*

$$(\partial B/\partial l)(u,v,l) \;\geq\; 1/(4v),$$

$$-\,(du,dv,dl)\, d^2B(u,v,l)\,(du,dv,dl)^t \;\geq\; 0,$$

where $d^2B(u,v,l)$ *denotes the Hessian matrix of the function* B *evaluated at* (u,v,l)*. Moreover, these imply the dyadic convexity condition* $B(x) - \frac{B(x_+)+B(x_-)}{2} \;\geq\; \alpha/(4v)$.

Differential convexity implies dyadic convexity

Proof.

Differential conditions can be check by direct calculation.
By the Mean Value Theorem and some calculus,

$$B(x) - \frac{B(x_+) + B(x_-)}{2} = \frac{\partial B}{\partial l}(u, v, l')\alpha - \frac{1}{2}\int_{-1}^{1}(1 - |t|)b''(t)dt \geq \frac{1}{4v}\alpha.$$

where

$$b(t) := B(x(t)), \;\; x(t) := \frac{1+t}{2}x_+ + \frac{1-t}{2}x_-, \quad -1 \leq t \leq 1.$$

Note that $x(t) \in \mathfrak{D}$ whenever x_+ and x_- do, since $\mathfrak{D}$ is a convex domain and $x(t)$ is a point on the line segment between x_+ and x_-, and l' is a point between l and $\frac{l_+ + l_-}{2}$. □

Sketch proof α-Lemma

Beznosova '08.

- Use the Bellman function method.
- Figure out the domain, range and convexity conditions needed to run an induction on scale arguments that will yield the inequality.
- Verify that the Bellman function $B(u, v) = (uv)^{\alpha}$ satisfies those conditions (or at least a differential version) for $0 < \alpha < 1/2$.

□

Weighted Carleson Lemma

Lemma

Let v be a weight, $\{\alpha_L\}_{L\in\mathcal{D}}$ a v-Carleson sequence with intensity $\mathcal{B}$, and F a positive measurable function on $\mathbb{R}$, then

$$\sum_{L\in\mathcal{D}} \alpha_L \inf_{x\in L} F(x) \leq \mathcal{B} \int_{\mathbb{R}} F(x)v(x)\,dx.$$

Proof.

Assume that $F \in L^1(v)$ otherwise the first statement is automatically true. Setting $\gamma_L = \inf_{x\in L} F(x)$, we can write

$$\sum_{L\in\mathcal{D}} \gamma_L \alpha_L = \sum_{L\in\mathcal{D}} \int_0^\infty \chi(L,t)\,dt\,\alpha_L = \int_0^\infty \Big(\sum_{L\in\mathcal{D}} \chi(L,t)\,\alpha_L \Big) dt,$$

where $\chi(L,t) = 1$ for $t < \gamma_L$ and zero otherwise, and by the MCT.

Proof Weighted Carleson Lemma

Proof (continuation).

Define $E_t = \{x \in \mathbb{R} \,:\, F(x) > t\}$.

- Since F is assumed a v-measurable function then E_t is a v-measurable set for every t.
- Since $F \in L^1(v)$ we have, by Chebychev's inequality, that the v-measure of E_t is finite for all real t.
- Moreover, there is a collection of maximal disjoint dyadic intervals $\mathcal{P}_t$ that will cover E_t except for at most a set of v-measure zero.
- $L \subset E_t$ if and only if $\chi(L,t) = 1$.

All together we can rewrite the integrand in previous page as

$$\sum_{L\in\mathcal{D}} \chi(L,t)\alpha_L = \sum_{L\subset E_t} \alpha_L \le \sum_{L\in\mathcal{P}_t} \sum_{I\in\mathcal{D}(L)} \alpha_I \le \mathcal{B} \sum_{L\in\mathcal{P}_t} v(L) = \mathcal{B}\, v(E_t).$$

Proof Weighted Carleson Lemma

Proof (continuation).

$$\sum_{L\in\mathcal{D}} \chi(L,t)\alpha_L = \sum_{L\subset E_t} \alpha_L \leq \sum_{L\in\mathcal{P}_t} \sum_{I\in\mathcal{D}(L)} \alpha_I \leq \mathcal{B} \sum_{L\in\mathcal{P}_t} v(L) = \mathcal{B}\, v(E_t),$$

we used in the second inequality the fact that $\{\alpha_J\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity $\mathcal{B}$.
Thus we can estimate

$$\sum_{L\in\mathcal{D}} \gamma_L \alpha_L \leq \mathcal{B} \int_0^\infty v(E_t)\, dt = \mathcal{B} \int_{\mathbb{R}} F(x)\, v(x)\, dx.$$

where the last equality follows from the layer cake representation. □