Course: Weighted inequalities and dyadic harmonic analysis

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CIMPA2017 Research School - IX Escuela Santaló
Harmonic Analysis, Geometric Measure Theory and Applications
Buenos Aires, July 31 - August 11, 2017

Outline

- **Lecture 1.**
  Weighted Inequalities and Dyadic Harmonic Analysis.
  Model cases: Hilbert transform and Maximal function.
- **Lecture 2.**
  Brief Excursion into Spaces of Homogeneous Type.
  Simple Dyadic Operators and Weighted Inequalities à la Bellman.
- **Lecture 3.**
  Case Study: Commutators.
  Sparse Revolution.

Outline Lecture 1

1. Weighted Norm Inequalities
   - Hilbert transform
   - One weight inequalities
   - Maximal function
   - Sharp extrapolation
   - Hytönen's A_p Theorem
   - Two weight problem for H and M
2. Dyadic harmonic analysis on $\mathbb{R}^d$
   - Dyadic Maximal Function
   - $1/3$ Trick
3. Spaces of Homogeneous Type (SHT)
   - Dyadic cubes in Spaces of Homogeneous Type
   - Haar basis in Spaces of Homogeneous Type
   - Some history and further results

Weighted norm inequalities

Question (Two-weights L^p-inequalities for operator T)

Given a pair of weights (u, v), is there a constant $C_p(u, v, T) > 0$ such that

$$\|Tf\|_{L^p(v)} \leq C_p(u, v, T)\, \|f\|_{L^p(u)} \text{ for all } f \in L^p(u)?$$

Goals

1. *Given operator T (or family of operators), identify and classify pairs of weights (u, v) for which the operator(s) T is(are) bounded from $L^p(u)$ to $L^p(v)$.*
2. *Understand nature of constant $C_p(u, v, T)$.*

Some notation

- The weights u, v are $L^1_{loc}(\mathbb{R}^d)$ positive a.e. functions.
- $f \in L^p(u)$ iff $\|f\|_{L^p(u)} := (\int_{\mathbb{R}^d} |f(x)|^p u(x)\,dx)^{1/p} < \infty$.
- Consider linear or sublinear operators $T : L^p(u) \to L^p(v)$.
  - Prototypical Calderón-Zygmund singular integral operator (linear): Hilbert transform on $\mathbb{R}$ given by convolution with kernel p.v. $\frac{1}{\pi x}$

$$Hf(x) := \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\,dy.$$

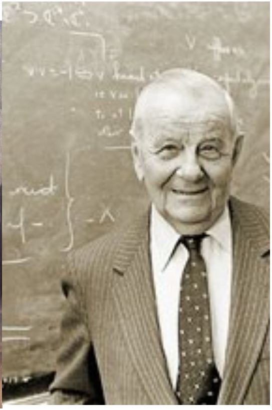
    Naturally appears in Complex Analysis, L^p-convergence of partial Fourier Sums/Integrals, etc.
  - Prototypical sublinear operator:
    Hardy-Littlewood Maximal function

$$Mf(x) := \sup_{Q: x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\,dy.$$

    $Q \subset \mathbb{R}^d$ are cubes with sides parallel to the axis, $|Q|$ =volume of Q.

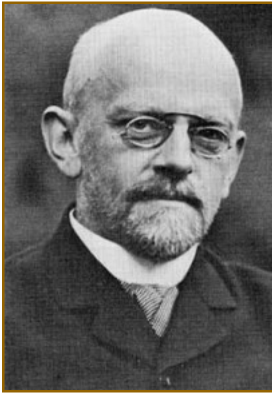
Calderón's family photo

Zygmund

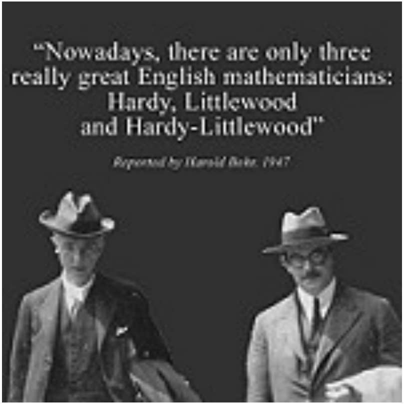


Calderon's 75th birthday conference held in Chicago (1996).
1st Row: M. Christ, C. Sadosky, A. P. Calderon, M. A. Muschietti.
2nd Row: C. E. Kenig, J. Alvarez, C. Gutierrez, E. Berkson, J. Neuwirth.
3rd Row: A. Torchinsky, J. Polking, S. Vagi, R. R. Reitano, E. Gatto, R. Seeley.

Hilbert   Hardy and Littlewood



David Hilbert (1862-1943)



Family photo appeared in the Selected Papers of Alberto P. Calder'on with Commentary Edited by: A. Bellow, C. E. Kenig, P. Malliavin. AMS 2008, and in Volumes 1-2 Celebrating Cora Sadosky's Life, AWM-Springer 2016 and 2017, co-edited with S. Marcantognini, A. Stokolos, W. Urbina.

Others came from the internet: Wikipedia, etc.

We concentrate on *one-weight L^p inequalities*, $u = v = w$, for

- Maximal function.
- CZ operators T, such as the Hilbert transform H.
- Dyadic analogues: dyadic maximal function, martingale transform, square function, Haar shift multipliers, dyadic paraproducts, and sparse operators.
- Their commutators $[T, b] := Tb - bT$ with functions $b \in BMO$.

Recall: A locally integrable function $b \in BMO$ iff

$$\|b\|_{BMO} := \sup_Q \frac{1}{|Q|} \int_Q |b(x) - m_Q b| \, dx < \infty, \text{ where } m_Q b = \frac{1}{|Q|} \int_Q b(t) \, dt.$$

Note that $L^\infty \subsetneq BMO$ (e.g. $\log |x| \in BMO \setminus L^\infty$).

Question (One-weight L^p inequality for operator T, $1 < p < \infty$)

Given weight w, is there $C_p(w, T) > 0$ such that $\forall f \in L^p(w)$

$$\|Tf\|_{L^p(w)} \leq C_p(w, T) \, \|f\|_{L^p(w)} \,?$$

The Hilbert transform H

Definition (On space side)

$$Hf(x) := \text{p.v.}\frac{1}{\pi}\int \frac{f(y)}{x-y}\,dy := \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|x-y|>\epsilon}\frac{f(y)}{x-y}\,dy.$$

Definition (On frequency or Fourier side)

$$\widehat{Hf}(\xi) = m_H(\xi)\,\widehat{f}(\xi), \quad \text{where} \quad m_H(\xi) = -i\,\text{sgn}(\xi).$$

Multiplication on Fourier side corresponds to convolution on space

$$Hf(x) = K_H * f(x), \quad K_H(x) := (m_H)^{\vee}(x) = \text{p.v.}\frac{1}{\pi x}.$$

Recall: The *Fourier transform* and *convolution* of Schwartz functions are defined by

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x)\,e^{-2\pi i\xi x}dx, \qquad f*g(x) = \int_{\mathbb{R}} f(y)g(x-y)dy = g*f(x).$$

Fourier transform can be extended to be an isometry in $L^2(\mathbb{R})$: $\|\widehat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$.

L^p Boundedness of H

- Fourier theory ensures boundedness in $L^2(\mathbb{R})$ (isometry)

$$\|Hf\|_2 = \|\widehat{Hf}\|_2 = \|\widehat{f}\|_2 = \|f\|_2$$

- Hausdorff-Young's inequality for $p \geq 1$: if $g \in L^1(\mathbb{R}), f \in L^p(\mathbb{R})$ then $\|g * f\|_p \leq \|g\|_1 \|f\|_p$.

But K_H is not in $L^1(\mathbb{R})$, despite this fact:

Properties (shared by all CZ singular integral operators)

- *H is bounded on $L^p(\mathbb{R})$ for all $1 < p < \infty$ (M. Riesz '27):*

$$\|Hf\|_p \leq C_p \|f\|_p \quad \textit{(best constant Pichorides '72)}.$$

- *H is not bounded on L^1, is of weak-type (1,1) (Kolmogorov '27).*
- *H is not bounded on L^∞, is bounded on BMO (C. Fefferman '71).*

Example (Hilbert transform of indicator $\mathbb{1}_{[a,b]}$)

$H\mathbb{1}_{[a,b]}(x) = (1/\pi) \log \big(|x-a|/|x-b|\big)$, and $\log|x|$ is in BMO but not in L^∞.

Boundedness of H on $L^p(w)$

Theorem (Hunt, Muckenhoupt, Wheeden 1973)

$$w \in A_p \Leftrightarrow \|Hf\|_{L^p(w)} \leq C_p(w)\|f\|_{L^p(w)}.$$

(*Same holds for maximal function M*, Muckenhoupt '72.)
A weight w is in the Muckenhoupt A_p class iff $[w]_{A_p} < \infty$, where

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{\frac{-1}{p-1}}(x)\,dx\right)^{p-1}, \quad 1 < p < \infty,$$

the supremum is over all cubes in $\mathbb{R}^d$ with sides parallel to the axes.
Dependence of the constant on $[w]_{A_p}$ was found 30 years later.

Theorem (Petermichl, JAMS '07)

$$\|Hf\|_{L^p(w)} \leq C_p[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(w)}.$$

Theorem (Petermichl, AJM '07)

$$\|Hf\|_{L^p(w)} \leq C_p [w]_{A_p}^{\max\{1,\frac{1}{p-1}\}} \|f\|_{L^p(w)}.$$

Cartoon of the proof.

- Write H as an *average* of *dyadic shift operators* (Petermichl '00).
- Find uniform (on the dyadic grids) *linear estimates* for *dyadic shift operators* on $L^2(w)$.
- Use *sharp extrapolation theorem* for $p \neq 2$ from linear $L^2(w)$ estimate.

□

Same holds for ALL Calderón-Zygmund singular integral operators (solving the famous A_2 conjecture, Hytönen '12).

Note: *estimate is linear for* $p \geq 2$, *and of power* $\frac{1}{p-1}$ *for* $1 < p < 2$.

Maximal function bounded on $L^p(w) \Rightarrow w \in A_p$

IF maximal function is bounded on $L^p(w)$: $\|Mf\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$.
For all $\lambda > 0$, $E_\lambda^{Mf} := \{x \in \mathbb{R}^d : Mf(x) \geq \lambda\}$, then, by Chebychev's:

$$w(E_\lambda^{Mf}) = \int_{E_\lambda^{Mf}} w(x)\,dx \leq \frac{1}{\lambda^p}\int_{\mathbb{R}^d} |Mf(x)|^p w(x)\,dx \leq \frac{C^p}{\lambda^p}\|f\|^p_{L^p(w)}.$$

Consider $f \geq 0$, supported on cube Q, let $\lambda = \frac{1}{|Q|}\int_Q f(y)\,dy$. Then $Mf(x) \geq \lambda$ for all $x \in Q$ and $Q \subset E_\lambda^{Mf}$ hence

$$\Big(\frac{1}{|Q|}\int_Q f(x)\,dx\Big)^p w(Q) \leq \lambda^p\, w(E_\lambda^{Mf}) \leq C^p \int_Q f^p(x) w(x)\,dx.$$

"Choose" $f = \mathbb{1}_Q w^{\frac{-1}{p-1}}$ so both integrands coincide ($f = f^p w$),

$$\frac{1}{|Q|^p}\Big(\int_Q w^{\frac{-1}{p-1}}(x)\,dx\Big)^{p-1} w(Q) \leq C.$$

Distribute $|Q|$ and take sup over cubes Q: $[w]_{A_p} \leq C^p$, then $w \in A_p$. $\square$

Boundedness properties of Maximal Function

We just showed that IF M is bounded on $L^p(w)$ then it is of weighted weak-type (p,p), moreover $[w]_{A_p}^{1/p} \leq \|M\|_{L^p(w)\to L^{p,\infty}(w)}$, where the weak-$L^p(w)$ norm is the smallest constant C such that for all $\lambda > 0$

$$w(E_\lambda^{Mf}) \leq \frac{C^p}{\lambda^p}\|f\|_{L^p(w)}^p.$$

- Maximal function is bounded on $L^\infty(\mathbb{R}^d)$ with norm 1.
- Maximal function is not bounded on $L^1(\mathbb{R}^d)$ (compute $M\mathbb{1}_{[0,1]}$!).
- Maximal function is of weak-type (1,1) (Hardy, Littlewood '30).
- *Interpolation* gives boundedness on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$.
- Weak-$L^p(w)$ iff $w \in A_p$ with norm $\sim [w]_{A_p}^{1/p}$ (Muckenhoupt '72).
- Bounded on $L^p(w)$ with norm $\sim_p [w]_{A_p}^{1/(p-1)}$ (Buckley '93).

  In particular $p = 2$: $\|Mf\|_{L^2(w)} \lesssim [w]_{A_2}\|f\|_{L^2(w)}$

Why are we interested in these estimates?

- FOURIER ANALYSIS: Boundedness of "periodic" H on $L^p(\mathbb{T})$ implies convergence on $L^p(\mathbb{T})$ of the partial Fourier sums.
- COMPLEX ANALYSIS: Hf is the boundary value of the harmonic conjugate of the Poisson extension of a function $f \in L^p(\mathbb{R})$.
- APPROXIMATION THEORY: To show that wavelets are unconditional bases on several functional spaces.
- PDES: Boundedness properties of *Riesz transforms* (SIO on $\mathbb{R}^d$) have deep connections to partial differential equations.
- QUASICONFORMAL THEORY: Boundedness of the *Beurling transform* (SIO on $\mathbb{C}$) on $L^p(w)$ for $p > 2$ and with linear estimates on $[w]_{A_p}$ implies borderline regularity result (Astala, Iwaniecz, Saksman - Duke '01, Petermichl, Volberg - Duke '02).
- OPERATOR THEORY: Weighted inequalities appear naturally in the theory of *Hankel and Toeplitz operators*, perturbation theory, etc (Cotlar, Sadosky 80's-90's).

First Linear Estimates: $\|Tf\|_{L^2(w)} \leq C[w]_{A_2}\|f\|_{L^2(w)}$

- *Maximal function* (Buckley '93).
- *Martingale transform* (Wittwer '00).
- *Dyadic square function* (Hukovic, Treil, Volberg '00; Wittwer '02).
- *Beurling transform* (Petermichl, Volberg '02).
- *Hilbert transform* (Petermichl '07).
- *Riesz transforms* (Petermichl '08).
- *Dyadic paraproduct* (Beznosova '08).

Estimates based on Bellman functions and (bilinear) Carleson estimates (except for maximal function). Bellman function method introduced in the 90's to harmonic analysis by Nazarov, Treil, Volberg (NTV) .

How about estimates on $L^p(w)$?

Rubio de Francia Extrapolation Theorem

Theorem (Rubio de Francia '82)

T sublinear, $1 < r < \infty$. If for all $w \in A_r$ $C_{T,r,d,w} > 0$ such that

$$\|Tf\|_{L^r(w)} \leq C_{T,r,d,w} \|f\|_{L^r(w)} \text{ for all } f \in L^r(w).$$

then for each $1 < p < \infty$ and for all $w \in A_p$, there is $C_{T,p,r,d,w} > 0$

$$\|Tf\|_{L^p(w)} \leq C_{T,p,r,d,w} \|f\|_{L^p(w)} \text{ for all } f \in L^p(w).$$

Choose $r = 2$, paraphrasing Antonio Córdoba[1]

There is no L^p just weighted L^2,

(since $w \equiv 1 \in A_p$ for all p).

Classic book García-Cuerva, Rubio de Francia '85.

Modern take Cruz-Uribe, Martell, Pérez '11.

[1] See page 8 in José García-Cuerva's eulogy for *José Luis Rubio de Francia* '87.

Sharp extrapolation

Theorem (Dragičević, Grafakos, P. , Petermichl '05)

T sublinear, $1 < r < \infty$. If for all $w \in A_r$ $\exists \alpha, C_{T,r,d} > 0$ such that

$$\|Tf\|_{L^r(w)} \leq C_{T,r,d}[w]_{A_r}^{\alpha} \|f\|_{L^r(w)} \text{ for all } f \in L^r(w).$$

then for each $1 < p < \infty$ and for all $w \in A_p$, there is $C_{T,p,r,d} > 0$

$$\|Tf\|_{L^p(w)} \leq C_{T,p,r,d}[w]_{A_p}^{\alpha \max\{1, \frac{r-1}{p-1}\}} \|f\|_{L^p(w)} \text{ for all } f \in L^p(w).$$

We follow Rubio de Francia and García-Cuerva's proof.
Key are Buckley's sharp bounds for the maximal function

$$\|Mf\|_{L^p(w)} \leq C_p[w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)}, \qquad 1 < p < \infty.$$

Alternative/streamlined proof (Duoandikoetxea '11). Can replace pair (Tf, f) by pair of functions (g, f) (Cruz-Uribe, Martell, Pérez '11).

Sharp extrapolation is not sharp

Example

Start with Buckley's sharp estimate on $L^r(w)$, $\alpha = \frac{1}{r-1}$, for the maximal function, extrapolation will give sharp bounds only for $p \le r$.

Example

Sharp extrapolation from $r = 2$, $\alpha = 1$, is sharp for the Hilbert, Beurling, Riesz transforms for all $1 < p < \infty$ (for $p > 2$ Petermichl, Volberg '02, '07, '08; $1 \le p < 2$ DGPPet '05).

Example

Extrapolation from linear bound in $L^2(w)$ is sharp for the dyadic square function only when $1 < p \le 2$ ("sharp" DGPPet '05, "only" Lerner '07). However, extrapolation from square root bound on $L^3(w)$ is sharp (Cruz-Uribe, Martell, Pérez '12)

Hytönen's A_p Theorem

Theorem (Hytönen '12)

Let $1 < p < \infty$ *and let* T *be any Calderón-Zygmund singular integral operator in* $\mathbb{R}^d$, *then there is a constant* $c_{T,d,p} > 0$ *such that*

$$\|Tf\|_{L^p(w)} \leq c_{T,d,p}\, [w]_{A_p}^{\max\{1,\frac{1}{p-1}\}} \|f\|_{L^p(w)}.$$

Cartoon of the proof.

- Enough to show $p = 2$ thanks to sharp extrapolation.
- Prove a *representation theorem* in terms of Haar shift operators of arbitrary complexity and paraproducts on *random dyadic grids*.
- Prove *linear estimates* on $L^2(w)$ with respect to the A_2 characteristic for paraproducts and Haar shift operators with polynomial dependence on the complexity (independent of the dyadic grid).

Two weight problem for Hilbert transform

- Cotlar, Sadosky '80s: à la Helson-Szegö.
- Various sets of sufficient conditions in between à la Muckenhoupt.
- Necessary and sufficient conditions for (uniform and individual) *martingale transform* and *well-localized dyadic operators* were found by Nazarov, Treil, Volberg '99, '08.
  (Using Bellman function techniques.)
- Long-time seeked necessary and sufficient conditions for two-weight boundedness of the Hilbert transform where found very recently by Lacey, Sawyer, Shen, Uriarte-Tuero '14 . (Quantitative estimates. Using delicate stopping time arguments.)

Two weight estimates for Maximal function

- (Sawyer '82) $M : L^2(u) \to L^2(v)$ bounded if and only if testing conditions hold: there is $C_{u,v} > 0$ such that for all cubes Q

$$\int_Q \big(M(\mathbb{1}_Q u^{-1})(x)\big)^2 v(x)\,dx \le C_{u,v} u^{-1}(Q) \quad (+\text{dual } \; u^{-1} \leftrightarrow v).$$

- (Moen '09) Two weight operator norm of M is comparable to $C_{u,v}$. Note that Sawyer's testing conditions imply joint $\mathcal{A}_2$:

$$[u,v]_{\mathcal{A}_2} := \sup_Q \langle u^{-1}\rangle_Q \langle v\rangle_Q < \infty, \quad \text{where} \quad \langle v\rangle_Q := v(Q)/|Q|$$

- (Pérez, Rela '15) When $(u,v) \in \mathcal{A}_2$ and $u^{-1} \in A_\infty := \cup_{p>1} A_p$ then

$$\|M\|_{L^2(u)\to L^2(v)} \lesssim [u,v]_{\mathcal{A}_2}^{\frac12}[u^{-1}]_{A_\infty}^{\frac12}.$$

When $u = v = w$ get the improved mixed-type estimate

$$\|M\|_{L^2(w)} \lesssim [w]_{A_2}^{\frac12}[w^{-1}]_{A_\infty}^{\frac12} \le [w]_{A_2}.$$

Little Intermission

Figure: LUIS SANTALÓ 1911-2001

http://www.cienciaenlavidriera.com.ar/wp-content/santalo-luis-7-sonrie.jpg

Dyadic vs Continuous Harmonic Analysis

- Dyadic maximal function controls maximal function M.
- Martingale transform a dyadic toy model for CZ operators.
- Hilbert transform H *commutes with translations, dilations and anticommutes with reflections.* A linear and bounded operator T on $L^2(\mathbb{R})$ with those properties must be a constant multiple of the Hilbert transform: $T = cH$.
  Using this principle Stephanie Petermichl '00 showed that one can write H as an "average of dyadic shift operators" over *random dyadic grids.*
- Similarly for Beurling and Riesz transforms, and ultimately all CZ operators can be written as averages of suitable dyadic operators.
- Current Fashion: "pointwise domination by finitely many sparse positive dyadic operators". Identifying the sparse collections involves using *stopping-time techniques* and *adjacent dyadic grids.*

Dyadic intervals

Definition

The *standard dyadic intervals* $\mathcal{D}$ is the collection of intervals of the form $[k2^{-j}, (k+1)2^{-j})$, for all integers $k, j \in \mathbb{Z}$.

They are organized by generations: $\mathcal{D} = \cup_{j\in\mathbb{Z}} \mathcal{D}_j$, where $I \in \mathcal{D}_j$ iff $|I| = 2^{-j}$. Each generation is a partition of $\mathbb{R}$. They satisfy

Properties

- *Nested:* $I, J \in \mathcal{D}$ *then* $I \cap J = \emptyset, \quad I \subseteq J,$ *or* $J \subset I$.
- *One parent: if* $I \in \mathcal{D}_j$ *then there is a unique interval* $\tilde{I} \in \mathcal{D}_{j-1}$ *(the parent) such that* $I \subset \tilde{I}$, *and* $|\tilde{I}| = 2|I|$.
- *Two children: There are exactly two disjoint intervals* $I_r, I_l \in \mathcal{D}_{j+1}$ *(the right and left children), with* $I = I_r \cup I_l$, $|I_r| = |I_l| = |I|/2$.

Note: 0 *separates positive and negative dyadic interval,* 2 *quadrants.*

The (weighted) dyadic maximal function

Definition

The *weighted dyadic maximal function* w.r.t weight u is

$$M_u^{\mathcal{D}} f(x) := \sup_{I \in \mathcal{D}, I \ni x} \frac{1}{u(I)} \int_I |f(y)|\, u(y)\, dy.$$

Here $u(I) := \int_I u(x)\, dx$. When $u \equiv 1$ then $M_1 = M$.

- $M_u^{\mathcal{D}}$ is of weak-$L^1(u)$ type, with constant 1 (independent of dimension). Corollary of CZ lemma (stopping time).
- $M_u^{\mathcal{D}}$ is bounded on $L^\infty(u)$ with constant 1.
- $M_u^{\mathcal{D}}$ is bounded on $L^p(u)$, $1 < p$: $\|M_u^{\mathcal{D}} f\|_{L^p(u)} \leq p' \|f\|_{L^p(u)}$.
- $M^{\mathcal{D}} f(x) \leq Mf(x) \leq 6\big(M^{\mathcal{D}^0} f(x) + M^{\mathcal{D}^1} f(x)\big)$,

  where $\mathcal{D}^i := \cup_{j \in \mathbb{Z}} \mathcal{D}^i_j$, $i = 0, 1$, are 1/3 shifted grids

$$\mathcal{D}^i_j := \{2^{-j}\big([0,1) + m + (-1)^j \tfrac{i}{3}\big) : m \in \mathbb{Z}\}.$$

1/3 Shifted dyadic grids

The families $\mathcal{D}^i := \cup_{j\in\mathbb{Z}}\mathcal{D}^i_j$, for $i = 0, 1, 2$, where

$$\mathcal{D}^i_j := \{2^{-j}\big([0,1) + m + (-1)^j \frac{i}{3}\big) : m \in \mathbb{Z}\}.$$

- $\mathcal{D}^0 = \mathcal{D}$. Grids $\mathcal{D}^i$ are as "far away" as possible from each other.
- The grids $\mathcal{D}^1$ and $\mathcal{D}^2$ are nested but there is only one quadrant ($\mathbb{R}$).
- $I \in \mathbb{R}$ any finite interval. For *at least two values* of $i = 0, 1, 2$, there are $J \in \mathcal{D}^i$ such that $I \subset J$, $3|I| \leq |J| \leq 6|I|$. Hence for $i \neq k$,

$$\begin{aligned} Mf(x) &= \sup_{I\ni x} \frac{1}{|I|}\int_I |f(y)|\,dy \leq 6 \sup_{J\in\mathcal{D}^i\cup\mathcal{D}^k, J\ni x} \frac{1}{|J|}\int_J |f(y)|\,dy \\ &\leq 6\max\{M^{\mathcal{D}^i}f(x), M^{\mathcal{D}^k}f(x)\} \leq 6\Big[M^{\mathcal{D}^i}f(x) + M^{\mathcal{D}^k}f(x)\Big]. \end{aligned}$$

- In $\mathbb{R}^d$ need 2^d or $(d+1)$ cleverly chosen grids (Tao Mei '12).

End of Lecture 1

Here is where I stopped the first lecture. Below you will find the slides I had prepared regarding Spaces of Homegeneous Type (cubes, Haar basis, and wavelets in this context). I had to drastically cut them but some may find bits of information that I excluded useful.

This construction seems very rigid, very dependent on the geometry of the cubes and on the group structure of Euclidean space $\mathbb{R}^d$.

Question

Can we do dyadic analysis in other settings?

Answer: YES!!!!

Spaces of Homogeneous Type

introduced by Coifman and Weiss '71.

One is amazed by the dramatic changes that occurred in analysis during the twentieth century. In the 1930s complex methods and Fourier series played a seminal role. After many improvements, mostly achieved by the Calderón-Zygmund school, the action takes place today on spaces of homogeneous type. No group structure is available, the Fourier transform is missing, but a version of harmonic analysis is still present. Indeed the geometry is conducting the analysis.

Yves Meyer[2] in his preface to [DH][3].

Figure: RAPHY COIFMAN, YVES MEYER GUIDO WEISS[3]

[2]Recipient of the 2017 Abel Prize. [3]Oberwolfach Photo Collection

[3]D.G. Deng and Y. Han, *Harmonic analysis on SHT*, Springer-Verlag 2009.

Spaces of Homogeneous Type (SHT)

Definition (Coifman, Weiss '71)

(X, ρ, μ) is a space of homogeneous type in Coifman-Weiss's sense if

- $\rho : X \times X \longrightarrow [0, \infty)$ is a quasi-metric on X:
  - $\rho(x, y) = \rho(y, x) \geq 0$ for all $x, y \in X$; $\rho(x, y) = 0$ iff $x = y$;
  - *quasi-triangle inequality*: $\exists A_0 \in [1, \infty)$ such that

$$\rho(x, y) \leq A_0\big(\rho(x, z) + \rho(z, y)\big), \quad \forall x, y, z \in X.$$

- μ is a nonzero Borel regular doubling measure[a]: $\exists D_\mu \geq 1$ s.t. $\mu(B(x, 2r)) \leq D_\mu\, \mu(B(x, r)) < \infty$ $\forall x \in X$ and $r > 0$.
  It implies that there are $\omega > 0$ (known as the upper dimension of μ) and $C > 0$ such that for all $x \in X$, $\lambda \geq 1$ and $r > 0$

$$\mu(B(x, \lambda r)) \leq C\lambda^{\omega} \mu(B(x, r)).$$

[a] μ-msble quasimetric ball $B(x, r) = \{y \in X : \rho(x, y) < r\}$ for $x \in X, r > 0$.

Notes

- A quasi-metric ρ may NOT be *Hölder regular*. A quasi-metric ρ is Hölder regular if there are $0 < \theta < 1$ and $C_0 > 0$ such that

  $$|\rho(x,y) - \rho(x',y)| \leq C_0 \rho(x,x')^{\theta} \left[\rho(x,y) + \rho(x,y')\right]^{1-\theta} \quad \forall x, x', y \in X.$$

  Metrics are Hölder regular for any $0 < \theta \leq 1$, $C_0 = 1$.
- Macías-Segovia '79 showed given SHT (X, ρ, μ) such that quasi-balls are open sets, there is an *equivalent* Hölder regular quasi-metric ρ' on X for some $\theta \in (0,1)$, for which all balls are *1-Ahlfors regular* (AR), i.e:

  $$\mu\big(B_{\rho'}(x,r)\big) \sim r^1.$$

- Quasi-metric balls may NOT be open in the topology induced by the quasi-metric[4].

[4] Largest topology $\mathcal{T}$ such that for each $x \in X$ the quasi-metric balls centered at x form a fundamental system of neighborhoods of equivalently $\Omega \in \mathcal{T}$ iff for each $x \in \Omega$ there exists $r > 0$ such that $B(x,r) \subset \Omega$.

Examples of SHT

- $\mathbb{R}^n$, Euclidean metric, and Lebesgue measure.
- $\mathbb{R}^n$, Euclidean metric, $d\mu = w\,dx$ where w is a doubling weight (e.g. $w \in A_\infty$ or A_p or RH_q weights).
- Quasi-metric spaces with d-Ahlfors regular measure: $\mu(B(x,r)) \sim r^d$ (e.g. Lipschitz surfaces, fractal sets, n-thick subsets of $\mathbb{R}^n$).
- Compact Lie groups.
- C^∞ manifolds with doubling volume measure for geodesic balls.
- Carnot-Caratheodory spaces.
- Nilpotent Lie groups (e.g. Heisenberg group).

The recent book *Hardy Spaces on Ahlfors-Regular Quasi Metric Spaces A Sharp Theory* by Ryan Alvarado, Marius Mitrea '15 uses the Segovia-Macías philosophy heavily.

Informal: dyadic cubes on SHT

Definition

A geometrically doubling quasi-metric space (X, d) is one such that every quasi-metric ball of radius r can be covered with at most N quasi-metric balls of radius $r/2$.

E.g.: SHTs are geometrically doubling (Coifman-Weiss 70's).

Sawyer-Wheeden, David 80's, Christ '90, Hytönen-Kairema '12 built *systems of dyadic cubes* on SHT and on geometrically doubling quasi-metric spaces. Families of "cubes" organized in generations $\mathcal{D}_k$, $k \in \mathbb{Z}$, such that for a $\delta \in (0, 1)$:

- nested generations $\mathcal{D}_k$ that partition X,
- no partial overlap across generations,
- unique ancestors in earlier generation,
- at most M children (this is a consequence of the geometric doubling),
- inner and outer balls of radius roughly δ^k the "sidelength" of a cube in $\mathcal{D}_k$,
- child's outer ball is inside parent's outer ball.

Dyadic cubes on SHT

Theorem (Hytönen, Kairema '12, Theorem 2.2)

Given (X, d) *a geometrically doubling quasi-metric space. Suppose that constants* $0 < c_0 \leq C_0 < \infty$ *and* $\delta \in (0,1)$ *satisfy* $12A_0^3 C_0 \delta \leq c_0$. *Given a maximal set in* X *of* δ^k*-separated points* $\{z_\alpha^k\}$, $\alpha \in \mathcal{A}_k$, *for every* $k \in \mathbb{Z}$, *we can construct families of sets* $\widetilde{Q}_\alpha^k \subseteq Q_\alpha^k \subseteq \overline{Q}_\alpha^k$ *such that:*

- $\widetilde{Q}_\alpha^k$ *and* $\overline{Q}_\alpha^k$ *are the interior and closure of* Q_α^k, *respectively;*
- *(nested) if* $\ell \geq k$, *then either* $Q_\beta^\ell \subseteq Q_\alpha^k$ *or* $Q_\alpha^k \cap Q_\beta^\ell = \emptyset$;
- *(partition)* $X = \bigcup_{\alpha \in \mathcal{A}_k} Q_\alpha^k$ *(disjoint union)* *for all* $k \in \mathbb{Z}$;
- $B(z_\alpha^k, c_1\delta^k) \subseteq Q_\alpha^k \subseteq B(z_\alpha^k, C_1\delta^k)$, *where* $c_1 := (3A_0^2)^{-1}c_0$ *and* $C_1 := 2A_0C_0$;
- *if* $\ell \geq k$ *and* $Q_\beta^\ell \subseteq Q_\alpha^k$, *then* $B(z_\beta^\ell, C_1\delta^\ell) \subseteq B(z_\alpha^k, C_1\delta^k)$.

The open and closed cubes $\widetilde{Q}_\alpha^k$ *and* $\overline{Q}_\alpha^k$ *depend only on the points* z_β^ℓ *for* $\ell \geq k$. *The half-open cubes* Q_α^k *depend on* z_β^ℓ *for* $\ell \geq \min(k, k_0)$, *where* $k_0 \in \mathbb{Z}$ *is a preassigned number entering the construction.*

Haar in Spaces of Homogeneous Type

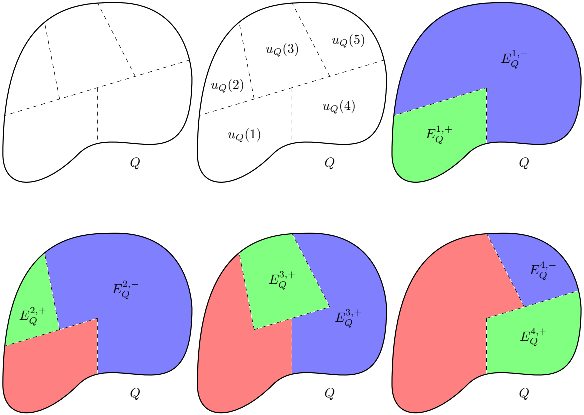


Figure: The 4 Haar functions for a cube with 5 children in SHT

Figures by David Weirich, PhD Dissertation, UNM 2017

Haar in Spaces of Homogeneous Type (X, ρ, μ), $Q \in \mathcal{D}$

Consider finite dimensional subspace with $\dim(S_Q) = \#ch(Q) - 1$.

$$S_Q := \{f \in L^2(Q) : \text{ constant on } ch(Q) \text{ with } \int_Q f(x)\,dx = 0\}.$$

The Haar basis described in previous slide is given by

$$h_Q^i(x) = a\mathbb{1}_{E_Q^{i,+}}(x) - b\mathbb{1}_{E_Q^{i,-}}(x), \quad 1 \le i < \#ch(Q).$$

Let $E_Q^i := E_Q^{i,+} \cup E_Q^{i,-}$, $E_Q^0 := Q$, to find a, b solve system:

$$\begin{aligned} \int_Q |h_Q^i(x)|^2 d\mu = a^2\mu(E_Q^{i,+}) + b^2\mu(E_Q^{i,-}) &= 1 \\ \int_Q h_Q^i(x)\,d\mu = a\,\mu(E_Q^{i,+}) - b\,\mu(E_Q^{i,-}) &= 0. \end{aligned}$$

$$a = \sqrt{\frac{\mu(E_Q^{i,-})}{\mu(E_Q^i)\mu(E_Q^{i,+})}}, \quad b = \sqrt{\frac{\mu(E_Q^{i,+})}{\mu(E_Q^i)\mu(E_Q^{i,-})}}.$$

By construction for each $Q \in \mathcal{D}$ the collection $\{h_Q^i\}_{i=1}^{\dim(S_Q)}$ is normalized in $L^2(d\mu)$, each has mean zero, and by nestedness properties is an o.n. family. No matter what enumeration for $\operatorname{ch}(Q)$ we use we get an o.n. basis of S_Q. The orthogonal projection onto S_Q of $f \in L^2(X, d\mu)$ is independent of the o.n. basis on S_Q. Given $x \in Q$ choose enumeration so that $x \in u_Q(1) = R \in ch(Q)$ then

$$\operatorname{Proj}_{S_Q} f(x) = \langle f, h_Q^1\rangle_\mu h_Q^1(x) = \langle f\rangle_R^\mu - \langle f\rangle_Q^\mu .$$

Using telescoping sum, completeness of Haar basis in $L^2(\mu)$ hinges on

$$\lim_{j\to\infty} E_j^\mu f \overset{L^2(\mu)}{=} f,$$

$$\lim_{j\to\infty} E_j^\mu f \overset{L^2(\mu)}{=} 0,$$

where $E_j f := \langle f\rangle_Q^\mu$, with $x \in Q \in \mathcal{D}_j$, or $E_j f = \sum_{Q\in\mathcal{D}_j} \langle f\rangle_Q^\mu \mathbb{1}_Q$.

Some history

Haar-type bases for $L^2(X, \mu)$ have been constructed in general metric spaces, and the construction is well known to experts.

- Haar-type wavelets associated to nested partitions in abstract measure spaces were constructed by Girardi, Sweldens '97.
- Such Haar functions are also used in geometrically doubling metric spaces, Nazarov, Reznikov, Volberg '13.
- For the case of spaces of homogeneous type there is local expertise, see Aimar, Gorosito '00, Aimar '02, Aimar, Bernadis, Jaffei '07, and Aimar, Bernadis, Nowak '11.
- For the case of geometrically doubling quasi-metric space (X, ρ), with a positive Borel measure μ, see Kairema, Li, P., Ward '16.

Random dyadic grids and adjacent dyadic grids in SHT

- Can introduce notion of *random dyadic grids* in geometrically doubling quasi-metric spaces X context by randomizing the order relations in the construction of the HK cubes (Hytönen, Martikainen '11, Hytönen, Kairema '12).
- Hytönen, Tapiola '14 modified the randomization to improve upon Auscher-Hytönen wavelets in metric spaces.
- Can find finitely many *adjacent families* of HK dyadic cubes with same parameters, $\mathcal{D}^t$, $t = 1, \dots, T$ that play the role of the $1/3$ shifted dyadic grids in $\mathbb{R}$. Main property they have: given any ball $B(x,r) \in X$, with $r \sim \delta^k$, then there is $t \in \{1, 2, \dots, T\}$ and $Q \in \mathcal{D}^t_k$ such that $B(x,r) \subset Q \subset B(x, Cr)$ (Hytönen, Kairema '12).
- Given a σ-finite measure μ on X, the adjacent dyadic systems can be chosen so that all cubes have small boundaries: $\mu(\partial Q) = 0$ for all $Q \in \cup_{t=1}^{T} \mathcal{D}^t$ (Hytönen, Kairema '12).

Informal: wavelets on SHT

Auscher, Hytönen '13 constructed a remarkable o.n. basis of $L^2(X)$.
Given nested maximal sets $\mathcal{X}^k$ of δ^k-separated points in X for $k \in \mathbb{Z}$.
Let $\mathcal{Y}^k := \mathcal{X}^{k+1} \setminus \mathcal{X}^k$, relabel points in $\mathcal{Y}^k$ by y^k_α.
To each point y^k_α, AH associate a wavelet function ψ^k_α (a linear spline) of regularity $0 < \eta < 1$ that is morally

- supported near y^k_α at scale δ^k, with mean zero,
- these functions are not compactly supported, but have exponential decay.
- of Hölder regularity $\eta = \frac{\log\left(1-(M(L+1))^{-1}\right)}{\log \delta}$ where M, L are finite quantities needed for extra labeling of *random dyadic neighbor grids* used in the construction.

Note: to each KH-cube Q^k_α there corresponds "center" x^k_α and to each child Q^{k+1}_β, there corresponds a center x^{k+1}_β, one of them equal to x^k_α. The number[5] of indexes α in $\mathcal{Y}_k$ for each Q^k_α is exactly $N(Q^k_\alpha) - 1$.
Hytönen, Tapiola '14 can build them for all $0 < \eta < 1$ in metric spaces.

[5] $N(Q^k_\alpha)$ the number of children in Q^k_α.