From additive combinatorics to geometric measure theory, Part III

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Outline

1. Times 2, times 3: Furstenberg's conjectures

2. An inverse theorem for the L^q norm of convolutions

A random looking sequence

22022221201010210101020101201101000202020000220101010122220
20000211022012102212212011120121020001010212220221000001212
22100011100210211002000200121021111222200222210012221221022
01110002111121202101020100200211121102200000010112002100122
10221210002200111102012101120220002210221211101102221212000
12002222120120202021102221210200120212021021202220221011101
11200001212211100001221200101222121101021111200211212211201
10211202020020222121200021100220101001011201222102222100212
02102200121000120200121202211021202202001121002120220221020
00122001201100021122221012102120012200210110010222220102202
1221010202111121122110021120212012001222 1

... that is far from random

The sequence is the base 3 expansion of 2^{1000}

Definition

The base p expansion of $n \in \mathbb{N}$ is given by

$$n = \sum_{i=1}^{k} x_i p^i \qquad x_i \in \{0, 1, \ldots, p-1\}$$

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Furstenberg's principle

Principle (Furstenberg)

Expansions in bases 2 and 3 have no common structure.
More generally, this holds for bases p and q which are not powers of a common integer or, equivalently, $\log p / \log q$ is irrational.

Ternary expansions of powers of 2

Principle (Folklore, Furstenberg)

The ternary expansion of large of powers of 2 should "look random".

Open problem

With finitely many exceptions, the base 3 expansion of 2^n contains the digit 1.

We cannot even establish some properties which are far weaker than "pseudo-randomness".

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Base p expansions

Let $p \in \mathbb{N}_{\geq 2}$. Every point $x \in [0,1]$ has an expansion to base p:

$$x = 0.x_1 x_2 \ldots = \sum_{n=1}^{\infty} x_n p^{-n}, \quad x_i \in \{0, 1, \ldots, p-1\}.$$

Basic facts:

1. All but countably many (rational) points have a unique expansion; the remaining ones have two expansions.
2. A point is rational if and only if the expansion is eventually periodic.
3. Expansions in bases p^n and p^k are “almost the same” (look at base p in blocks of length n and k).

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Multiplication by p

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For $p \in \mathbb{N}_{\geq 2}$, let

$$T_p(x) = px \bmod 1 = \text{fractional part of } px$$

be multiplication by p on the circle $[0, 1)$.

Observation

The map T_p shifts the base p expansion by one position:

$$T_p(0.x_1 x_2 \ldots) = (0.x_2 x_3 \ldots).$$

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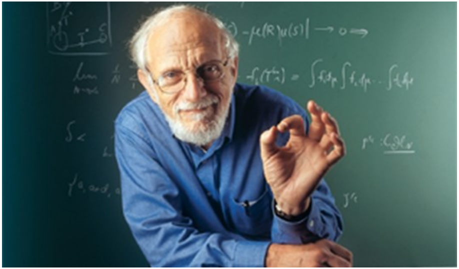
Multiplying by 2 and by 3: the founding father



Some of the areas that Furstenberg initiated

1. Ergodic theoretic methods in combinatorics (ergodic proof of Szemerédi's Theorem,...).
2. Products of random matrices, non-commutative ergodic theory (simplicity of Lyapunov exponents, ...).
3. Unique ergodicity of horocycle flow, toral maps, ...
4. Disjointness of dynamical systems.
5. Structure theorems (distal systems, general systems).
6. $\times 2 \times 3$, rigidity of higher order actions.
7. Fractal geometry $\cap$ ergodic theory (CP-processes, ...).

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Invariant sets

Definition

A set $A \subset [0, 1)$ is T_p-invariant if $T_p(A) \subset A$. That is, shifting the p-ary expansion of a point in A gives another point in A.

- $[0, 1)$ is T_p-invariant.
- $\{0, 1/q, \ldots, (q-1)/q\}$ is T_p-invariant.
- Let $D \subset \{0, 1, \ldots, p-1\}$. The set $A = A_{p,D}$ of points whose base p-expansion has only digits from D is T_p-invariant. We call it a p-Cantor set. Example: the middle-thirds Cantor set: the set of points whose base 3 expansion omits the digit 1.
- There is a wild abundance of invariant sets and no classification or description is possible.

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Some p-Cantor sets

$A_{3,\{0,2\}}$, aka the middle-thirds Cantor set:

$A_{4,\{0,3\}}$, aka the middle-one quarter Cantor set:

$A_{4,\{0,1,3\}}$:

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Invariant sets and shared structure

Principle (Furstenberg, slightly more concrete version)

If A, B are closed and invariant under T_2, T_3 respectively, then A and B have no common structure.

Theorem (Furstenberg (1967))

If A is closed and invariant under T_2 and T_3, then A is either finite or the whole circle $[0, 1)$.

Remarks

- *The theorem is a weak confirmation of the principle since the set A and itself certainly have a lot of common structure!*
- *One should think of finite sets and the whole circle as sets "without structure".*

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A corollary in terms of orbits

Observation

- *If x is rational, then the orbit $\{T_2^n T_3^m x\}_{n,m=1}^{\infty}$ is finite.*
- *If x is irrational, then the orbit $\{T_2^n T_3^m x\}_{n,m=1}^{\infty}$ is infinite (and its closure is invariant under T_2 and T_3).*

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“The” $\times 2$, $\times 3$ Furstenberg conjecture

Definition

A Borel probability measure μ on $[0,1)$ is T_p-invariant if

$$\mu(B) = \mu(T_p^{-1}B) \quad \text{for all Borel sets } B.$$

Heuristically, μ is the distribution of a random point in $[0,1]$ which is invariant under shifting the p-ary expansion.

Conjecture (Furstenberg 1967)

If μ is T_2 and T_3 invariant, then μ is a convex combination of Lebesgue measure and an atomic measure supported on rationals.

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How to quantify “shared structure”

1. Furstenberg’s Theorem says that non-trivial T_2 and T_3 invariant sets do not have too much shared structure in the most basic sense: they cannot be equal.
2. How can we quantify shared structure in finer/more quantitative ways? The sets we are interested in are fractal: they are uncountable but of zero Lebesgue measure, and have some form of (sub)-self-similarity.
3. Geometry helps quantify common structure. For example, if two sets $A, B \subset \mathbb{R}$ have no shared structure one expects the sumset

$$A + B = \{a + b : a \in A, b \in B\}$$

to be “as large as possible” and the intersections $A \cap B$ and $A \cap (\lambda B + t)$ to be “as small as possible”. If this does not happen, then there are “resonances” between A and B!

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Hausdorff dimension of p-Cantor sets

$A_{3,\{0,2\}}$:

Dimension$=\log 2/\log 3 \approx 0.631$

$A_{4,\{0,3\}}$:

Dimension$=\log 2/\log 4 = 0.5$

$A_{4,\{0,1,3\}}$:

Dimension$=\log 3/\log 4 \approx 0.792$

Furstenberg's sumset conjecture

Conjecture 1

Let A, B be closed and T_p, T_q invariant (with $\log p / \log q \notin \mathbb{Q}$). Then

$$\dim_{\mathrm{H}}(A+B) = \max(\dim_{\mathrm{H}}(A) + \dim_{\mathrm{H}}(B), 1).$$

Motivation

- *By Marstrand's Projection Theorem applied to $A \times B$,*

  $$\dim_{\mathrm{H}}(A+\lambda B) = \max(\dim_{\mathrm{H}}(A) + \dim_{\mathrm{H}}(B), 1) \text{ for almost all } \lambda \in \mathbb{R}.$$

  The goal is to prove that there are no exceptions at all (outside of the trivial case $\lambda = 0$).
- *Moreover, the right-hand side is always a (trivial) upper bound.*
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Solution to Furstenberg's sumset conjecture

Theorem (Y.Peres-P.S. 2009, F. Nazarov-Y.Peres-P.S. 2012)

If A, B are a p-Cantor set and a q-Cantor set, then

$$\dim_H(A + \lambda B) = \min(\dim_H(A) + \dim_H(B), 1) \text{ for all } \lambda \in \mathbb{R} \setminus \{0\}.$$

Solution to Furstenberg's sumset conjecture

Theorem (M.Hochman-P.S. 2012)

If A, B are closed and T_p, T_q-invariant, then

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No exceptions in the projection theorems

Theorem (Marstrand 1954)

For any Borel set $A \subset \mathbb{R}^2$,

$$\dim_H(P_\theta A) = \dim_H(A)$$

for almost all θ.

Remark

The methods introduced to solve Furstenberg's sumset conjecture allow to show that for large classes of dynamically defined fractal sets and measures, there are no exceptions in Marstrand's projection theorem (and variants), other than the trivial ones.

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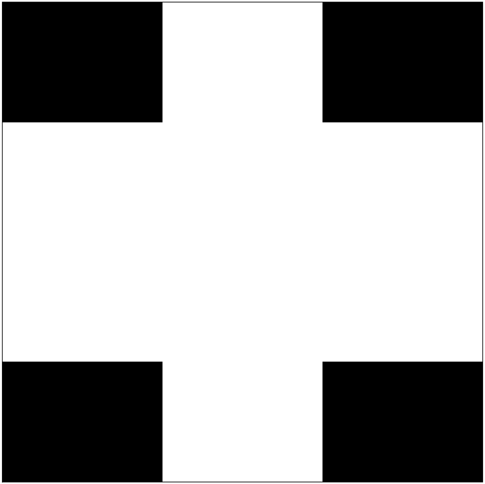
Product, projection, fiber



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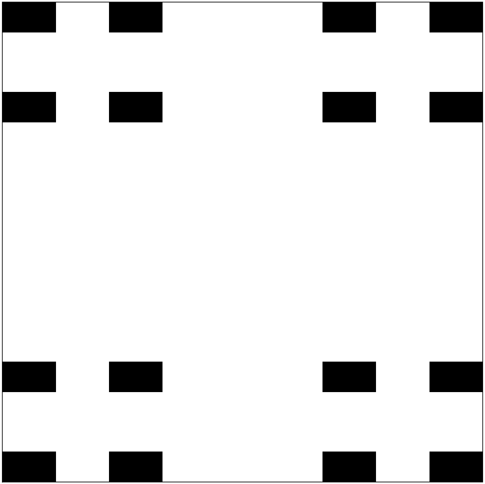

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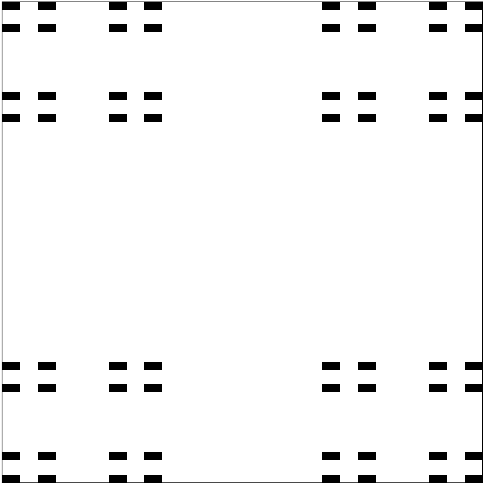

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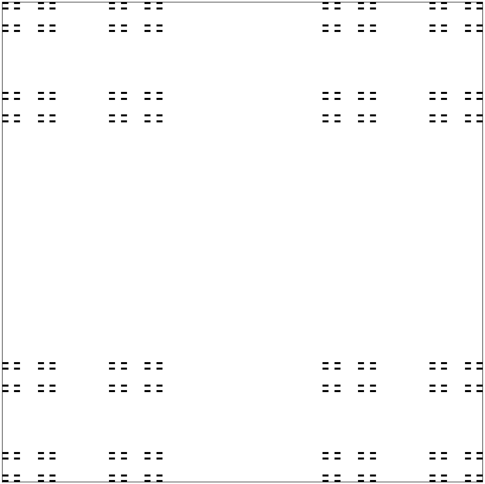

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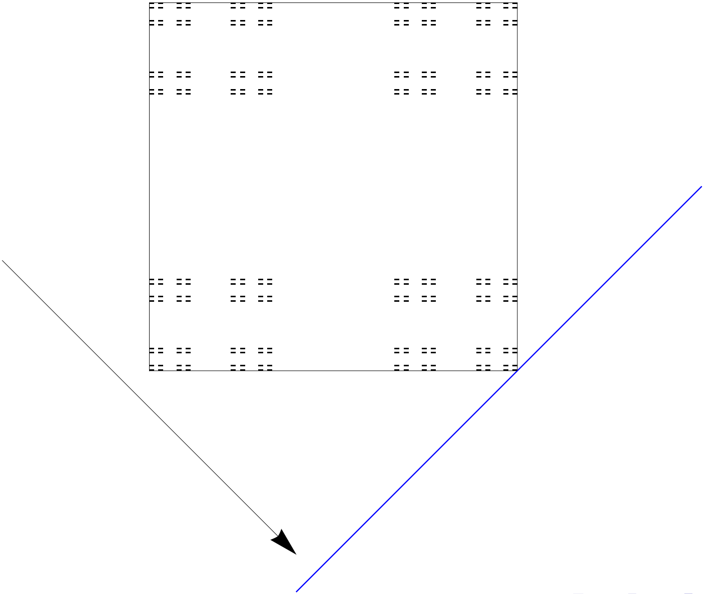

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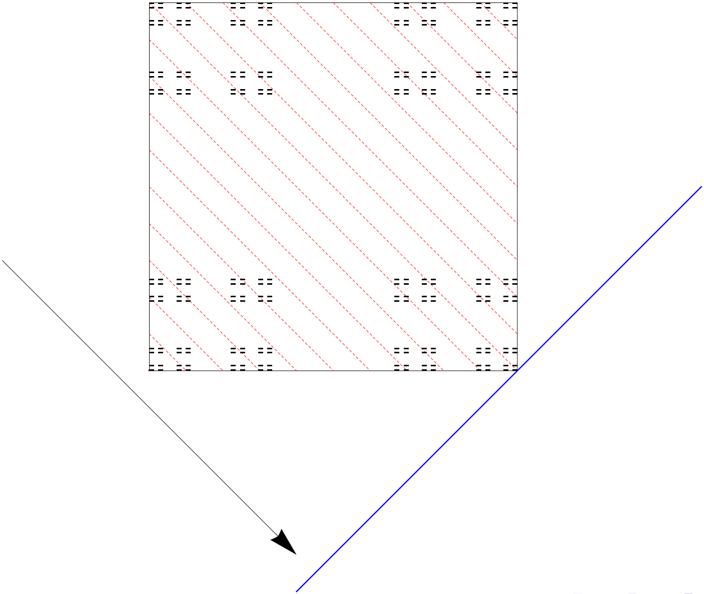

More general notions of shared structure?

- I argued that if

$$\dim_H(A+B) < \min(\dim_H(A)+\dim_H(B),1),$$

then A and B have "common structure" at many scales.

- But the opposite is far from true! For many ("most") sets A, even of dimension $\leq 1/2$, even T_p-invariant ones,

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Marstrand's slice theorem

Theorem (Marstrand 1954)

Let $A \subset \mathbb{R}^2$ be a Borel set.

- *For almost all lines ℓ in $\mathbb{R}^2$,*

$$\dim_H(A \cap \ell) \leq \max(\dim_H(A) - 1, 0).$$

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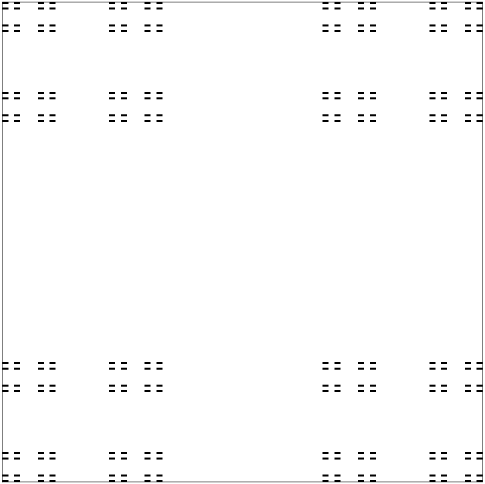
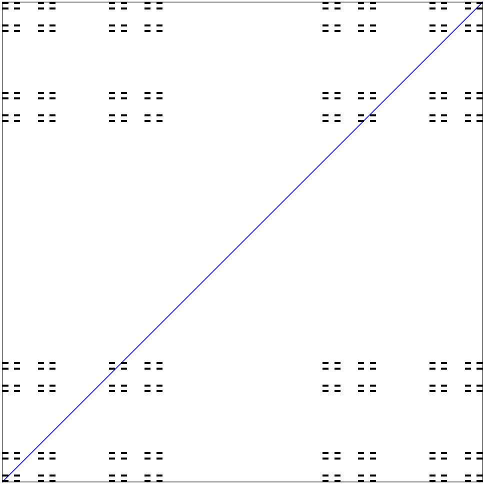


Our old friend again: $A \times B$.

More pictures!



$A \times B \cap \text{diagonal} = A \cap B.$

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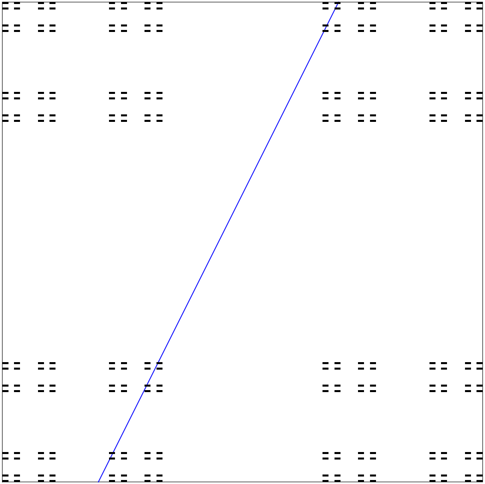

$A \times B \cap$ any line $= A \cap$ affine image of B.

Furstenberg's intersection conjecture

Conjecture 2 (Furstenberg 1969)

Let A, B be closed and invariant under T_p, T_q (seen as subsets of $\mathbb{R}$), with $\log p/\log q \notin \mathbb{Q}$.

Then for every affine bijection $f : \mathbb{R} \to \mathbb{R}$,

$$\dim_{\mathrm{H}}(A \cap f(B)) \leq \max(\dim_{\mathrm{H}}(A) + \dim_{\mathrm{H}}(B) - 1, 0).$$

Motivation

- *The conjecture says that for $A \times B$ there are no exceptional lines in the slicing theorem (other than horizontal/vertical ones).*
- *Conjecture 2 is stronger than Conjecture 1. Heuristically, the sumset $A + B$ is "large" if "many" fibers are "small". The conjecture asserts that all fibers are small.*

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The conjecture holds if $\dim_H(A) + \dim_H(B) \leq 1/2$*. More generally, one always has*

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A corollary on subsets of integers

Definition

Fix $M \gg 1$, and let A_M be the set of natural numbers whose base 2 expansion has at least M zeros between any two ones. Note that:

$$\lim_{n\to\infty} \frac{\log |A_M \cap \{1,\ldots,n\}|}{\log n} > 0.$$

Corollary (of Furstenberg's Intersection Conjecture)

For any block $u = (u_1 \ldots u_k)$ of ternary digits, if $M \gg 1$ and $E_{M,u}$ is the set of numbers in A_M whose base 3 expansion misses the block u, then

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- The corollary says that nearly all natural numbers whose binary expansion has a sparse number of 1s has a "fairly dense" ternary expansion.
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Tools involved in the proof

1. Additive combinatorics: an inverse theorem for the L^q norm of the convolution of two finitely supported measures(Balog-Szemerédi-Gowers Theorem, Bourgain's additive part of discretized sum-product results).
2. Ergodic theory: key role played by subadditive cocycle over a uniquely ergodic transformation (cocycle borrowed from Nazarov-Peres-S. 2012, uses the proof of the subadditive ergodic theorem given by Katznelson-Weiss).
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Outline

1. Times 2, times 3: Furstenberg's conjectures

2. An inverse theorem for the L^q norm of convolutions

L^q norms of discrete measures

- From now on a measure is a probability measure supported on $2^{-m}\mathbb{Z} \cap [0,1) = \{j2^{-m} : 0 \le j < 2^m\}$ for some large m.
- The L^q norm of μ ($q \ge 1$) is

$$\|\mu\|_q^q = \sum_x \mu(x)^q, \quad \|\mu\|_\infty = \max_x \mu(x).$$

- $$2^{-m/q'} \le \|\mu\|_q \le 1,$$

  with a "small" L^q norm corresponding to "uniform" measures and a large L^q norm to "localized" measures.

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L^q norms of convolutions

- The convolution of μ, ν is

$$(\mu * \nu)(x) = \sum_{a+b=x} \mu(a)\nu(b).$$

(Addition modulo 1, although it makes no difference)

- Young's inequality (just convexity of $t \mapsto t^q$)

$$\|\mu * \nu\|_q \leq \|\mu\|_q \|\nu\|_1 = \|\mu\|_q.$$

- When is there (almost) equality in Young's inequality? (for $1 < q < \infty$). Two easy situations:
  1. μ is (almost) uniform.
  2. ν is (almost) an atom.
- There are less trivial examples: let A be a set that is "uniform" on some scales and "an atom" at the complementary scales. Then $\mu = \mathbf{1}_A/|A|$ satisfies $\|\mu * \mu\|_q \sim \|\mu\|_q$.

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An inverse theorem for the flattening of L^q norms

Theorem (Informal version)

Let μ, ν be measures such that

$$\|\mu * \nu\|_q \geq 2^{-\varepsilon m}\|\mu\|_q.$$

Then there are "regular" sets A, B of "large" μ, ν-measure such that in a "multiscale decomposition", on each scale either "A is almost uniform" or "B is an atom".

Trees, branching, regular sets

Definition

Suppose $m = \ell m'$ for some (large) ℓ, m'. Given a set $A \subset m\mathbb{Z} \cap [0,1)$, we consider the associated base-2^ℓ tree T_A: its vertices of level j are those dyadic intervals I of length $(2^{-\ell})^j$ that intersect A.

Definition

Given a sequence $k = (k_1, \ldots, k_{m'})$ with $k_i \in \{1, \ldots, \ell\}$, we say that A is k-regular if the following holds:
For each dyadic interval of I of length $2^{-j\ell}$ that intersects A, there are exactly k_{j+1} intervals J of length $2^{-(j+1)\ell}$ that intersect $A \cap I$.
In other words, for the tree T_A, each vertex of level j has exactly k_{j+1} children.

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For each dyadic interval of I of length $2^{-j\ell}$ that intersects A, there are exactly k_{j+1} intervals J of length $2^{-(j+1)\ell}$ that intersect $A \cap I$.
In other words, for the tree T_A, each vertex of level j has exactly k_{j+1} children.

The inverse theorem with more details

Theorem (P.S. 2016)

Given $\delta > 0$, there is $\varepsilon > 0$ such that the following holds for ℓ, m' large enough. Let $m = \ell m'$. If

$$\|\mu * \nu\|_q \geq 2^{-\varepsilon m}\|\mu\|_q,$$

then there are sets A, B such that:

- $\|\mu|_A\|_q \geq 2^{-\delta m}\|\mu\|_q$, $\nu(B) \geq 2^{-\delta m}\|\nu\|_1$.
- *$\mu(x) \leq 2\mu(y)$ for all $x, y \in A$, same for ν and B.*
- *A and B are k-regular and k' regular respectively for some sequences $(k_1, \ldots, k_{m'})$, $(k'_1, \ldots, k'_{m'})$.*
- *For each j,*

$$\text{Either} \quad k_j \geq 2^{(1-\delta)\ell} \quad \text{or} \quad k'_j = 1.$$

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A corollary

Definition

A set $B \subset [0,1]$ is η-porous if for every interval $I \subset [0,1]$ there is an interval $J \subset I \cap [0,1] \setminus B$ with $|J| \geq \eta |I|$.

If $B \subset 2^{-m}\mathbb{Z} \cap [0,1]$, then we only require this for $|I| \geq 2^{-m}/\eta$.

Corollary

If supp(μ) is η-porous, then either

$$\|\nu\|_q \geq 2^{-\delta m},$$

or

$$\|\mu * \nu\|_q \leq 2^{-\varepsilon m}\|\mu\|_q,$$

where $\varepsilon = \varepsilon(\eta, \delta, q) > 0$.

In particular, this holds if μ is a (discretization of) an Ahlfors-regular measure, generalizing a result of Dyatlov-Zahl.

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Main tool I: Asymmetric Balog-Szemerédi-Bowers

Theorem (Tao-Vu, using ideas of Bourgain)

Given $\varepsilon > 0$, there is $\delta > 0$ such that the following holds:

If $A, B \subset 2^{-m}\mathbb{Z} \cap [0,1)$ are such that

$$\|\mathbf{1}_A * \mathbf{1}_B\|_2 \geq 2^{-\delta m}\|\mathbf{1}_A\|_2,$$

then there are subsets $A' \subset A$, $B' \subset B$ such that $|A'| \geq 2^{-\varepsilon}|A|$, $|B'| \geq 2^{-\varepsilon}|B|$, and

$$|A' + B'| \leq 2^{\varepsilon m}|A'|.$$

Main Tool 2: Bourgain's additive part of sum-product machinery

Remark

Recall that Freiman's Theorem *says that if* $|A + A| \leq K|A|$, *then* A *is a large subset of a* GAP. *But the best known bounds in Freiman's Theorem say nothing if* K *grows exponentially (e.g.* $2^{\eta m}$*).*

Theorem (Bourgain (implicitly))

Given $\delta > 0$, *there is* $\eta > 0$ *such that the following holds for large enough* m.

If $|A + A| \leq 2^{\eta m}|A|$, *then* A *contains a* k*-regular subset* A' *such that:*

1. $|A'| \geq 2^{-\delta m}|A|$,
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The end of part III

Thank you!!!