From additive combinatorics to geometric measure theory, Part I

Pablo Shmerkin

Department of Mathematics and Statistics
Universidad T. Di Tella and CONICET

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Outline

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Plan for the course

First lecture: Basic concepts and highlights from Additive Combinatorics.

Second lecture: Applications of Additive Combinatorics to problems in Geometric Measure Theory and Analysis.

Third lecture: Furstenberg's conjecture on the intersection of $\times 2$, $\times 3$ invariant Cantor sets.

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What is additive combinatorics?

We will see it through a sample of some important concepts and results.

One of the main features is that is has many (bi-directional) connections:

- Harmonic Analysis
- Geometric Measure Theory
- Ergodic Theory
- Number Theory
- Combinatorics

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Ambient group

Additive combinatorics takes place in some ambient Abelian group Z. For the purposes of this course, you can think of:

- $\mathbb{R}$.
- $\mathbb{Z}$.
- The circle $\mathbb{R}/\mathbb{Z}$ (written additively as $[0, 1)$).
- $\mathbb{Z}/p\mathbb{Z}$ (with p usually a prime).

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Arithmetic progressions

Definition

A *k-AP* is

$$a, a+v, a+2v, \ldots, a+(k-1)v$$

with $a, v \in \mathbb{Z}$ and $v \neq 0$.

Question

What conditions of size and/or structure ensure that A contains (long) arithmetic progressions?

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Szemerédi's Theorem

Definition

Let $r_k(N)$ be the size of the largest subset of $\{1, \ldots, N\}$ that does not contain a k-AP.

Theorem (Szemerédi 1975)

For any $k \geq 3$,

$$\lim_{N \to \infty} \frac{r_k(N)}{N} = 0.$$

Corollary

A subset of the integers of positive upper density contains arbitrarily long arithmetic progressions.

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Remarks on Szemerédi's Theorem

1. The case $k = 3$ was proved by K. Roth in the 1952 using the Fourier transform. The Fourier transform does not work at all if $k \geq 4$.
2. Very influential proofs of Szemerédi's Theorem were given by H. Furstenberg (Ergodic Theory), T. Gowers (Higher order Fourier analysis), T. Tao (finitary ergodic theory), and others.
3. It is still a problem of current interest to give (upper and lower) bounds on $r_k(N)$. For $k = 3$ there are very good upper bounds due to Bourgain, Sanders, Bloom, for $k \geq 4$ the bounds are poorer and the record is due to very important work by Gowers.
4. There have been many generalizations and extensions, the most famous of which is the Green-Tao Theorem extending Szemerédi's Theorem to the primes.
5. An active area of research concerns Szemerédi-type phenomena in subsets of Euclidean space: Geometric Measure Theory+Harmonic Analysis.

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Generalized Arithmetic Progressions

Definition

A GAP is a set of the form

$$\{a + i_1 v_1 + i_2 v_2 + \ldots + i_d v_d : 0 \le i_j < k_i\} = a + [\mathbf{k}].\mathbf{v},$$

where $\mathbf{k} = (k_1, \ldots, k_d) \subset \mathbb{N}^d$, $a \in Z$, $\mathbf{v} = (v_1, \ldots, v_d) \in Z^d$, $v_i \ne 0$.

A GAP A is proper if

$$|A| = n_1 \cdots n_d,$$

i.e. all of the sums $a + i_1 v_1 + i_2 v_2 + \ldots + i_d v_d$ are different.

The rank of the GAP is d (a GAP of rank 1 is an AP).

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Sumsets and difference sets

Definition

If $A, B \subset \mathbb{Z}$ we define their sumset and difference set as

$$A + B = \{x + y : x \in A, y \in B\},$$
$$A - B = \{x - y : x \in A, y \in B\},$$
$$nA = \underbrace{A + \cdots + A}_{n \text{ times}}.$$

Remark

One of the most fundamental problems of additive combinatorics is to understand the relationship between the sizes (and the structure) of A, B and sets obtained from them via sums and differences.

When the ambient groups is a ring (as in all of our examples), one is also interested in product sets $A.B$.

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Size of sumsets and additive structure

- For any set A,

$$|A| \leq |A+A| \leq \min\left(\frac{1}{2}|A|(|A|+1), |Z|\right).$$

  So, to first order, $|A+A|$ varies between $|A|$ and $|A|^2$ (or $|Z|$ if $|Z| \leq |A|^2$).
- We think of sets A with $|A+A| \sim |A|$ as sets with additive structure or as approximate subgroups.

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Examples of sets with/without additive structure

Examples of sets for which $|A+A| \sim |A|$:

- Subgroups (if they exist).
- Arithmetic progressions: $|A+A| \lesssim 2|A|$.
- Proper GAPs: $|A+A| \leq 2^d|A|$ where d is the rank.
- Dense subsets of a set with $|A+A| \sim |A|$ (such as a GAP).

Examples of sets for which $|A+A| \sim |A|^2$:

- Random sets (pick each element of $\mathbb{Z}/p\mathbb{Z}$ with probability $p^{-\alpha}$).
- Lacunary sets (powers of 2).
- $A \cup B$ where A, B are disjoint of the same size, A is one of the previous examples and B is arbitrary.

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One calculation

Lemma

Let $B \subset A$ where A is a proper GAP *of rank d and $|B| \geq |A|/100$. Then* $|B+B| \leq 100 \cdot 2^d |B|$.

Proof.

Let

$$A = \{a + i_1 v_1 + i_2 v_2 + \ldots + i_d v_d : 0 \leq i_\ell < k_\ell\}.$$

Then

$$\begin{aligned} |B+B| &\leq |A+A| \\ &= |\{2a + j_1 v_1 + j_2 v_2 + \ldots + j_d v_d : 0 \leq j_\ell < 2k_\ell\}| \\ &\leq 2^d k_1 \cdots k_\ell = 2^d |A| \quad (A \text{ is proper}) \\ &\leq 100 \cdot 2^d |B|. \end{aligned}$$

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\end{aligned}$$

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Freiman's Theorem

Theorem (Freiman 1966)

Given $K > 1$ there are $d(K)$ and $S(K)$ such that the following holds.

Suppose $|A + A| \leq K|A|$. Then there is a GAP *P of rank $d(K)$ such that $A \subset P$ and $|P| \leq S(K)|A|$.*

In other words, sets of small doubling are always dense subsets of GAP*s of small rank.*

Remarks on Freiman's Theorem

- Freiman's Theorem can be seen as an inverse or classification theorem: based on qualitative information about A, it returns structural information.
- In applications it is important to have quantitative estimates on $d(K)$ and $S(K)$. Good bounds were obtained by Ruzsa, Chang, Sanders and Schoen, with Schoen's current record being: $d(K) \le K^{1+\varepsilon}$, $S(K) \le \exp(K^{1+\varepsilon})$.
- The theorem does not guarantee that P is proper. But it can be taken to be proper (with worse quantitative bounds).
- The conjecture is that d and S can be both taken polynomial in K.
- At least with the current bounds, Freiman's Theorem says nothing if K grows with $|A|$, in particular if $K = |A|^\delta$. In the next lectures, we will see a result of Bourgain that gives structural information about A when $|A + A| \le |A|^{1+\delta}$.

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Additive energy

Definition

The additive energy $E(A,B)$ between two sets A, B is

$$E(A,B) = |\{(x_1,x_2,y_1,y_2) \in A^2 \times B^2 : x_1 + y_1 = x_2 + y_2|$$

- Trivial lower bound: $|A||B| \le E(A,B)$ since we always have the quadruples (x,x,y,y).
- Trivial upper bound: $E(A,B) \le |A|^2|B|$, since once we have x_1, y_1, x_2, the value of y_2 is completely determined.
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Additive structure through energy

We can think of sets A with $E(A, A) \sim |A|^3$ as sets with "additive structure". Examples:

- APs and GAPs.
- Dense subsets of APs and GAPs.
- Disjoint unions $A \cup B$ where $E(A, A) \sim |A|^3$ and B is arbitrary. If B has large sumset, then so does $A + B$!

Observation

Having small sumset and having large additive energy are indications of additive structure. These notions cannot agree because both the size of the sumset and the additive energy are increasing functions of A.

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Small sumsets ⇒ large energy

Lemma

$$E(A,A) \geq \frac{|A|^4}{|A+A|}.$$

Proof.

$$|A|^4 = \left(\sum_{z \in A+A} |\{(x,y) \in A^2 : x+y=z\}| \right)^2$$

$$\leq |A+A| \sum_{z \in A+A} |\{(x,y) \in A^2 : x+y=z\}|^2 \quad \text{(Cauchy-Schwartz)}$$

$$= |A+A| \sum_{z \in A+A} |\{(x_1,x_2,y_1,y_2) \in A^4 : x_1+y_1 = x_2+y_2 = z\}|$$

$$= |A+A| E(A,A).$$

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Convolutions and L^p norms

Definition

We work with finitely supported functions $f : Z \to \mathbb{R}$.

We define the L^p norms as $\|f\|_\infty = \max_x |f(x)|$ and

$$\|f\|_p^p = \sum_x f(x)^p.$$

The convolution of f and g is

$$f * g(z) = \sum_{(x,y):x+y=z} f(x)g(y) = \sum_x f(x)g(z-x).$$

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Additive energy as the L^2 norm of convolutions

Lemma

$$E(A,B) = \|\mathbf{1}_A * \mathbf{1}_B\|_2^2.$$

Proof.

Note that

$$\begin{aligned}\mathbf{1}_A * \mathbf{1}_B(z) &= \sum_{(x,y):x+y=z} \mathbf{1}_A(x)\mathbf{1}_B(y)\\ &= |\{(x,y)\in A\times B : x+y\in Z\}|,\end{aligned}$$

so

$$E(A,B) = \sum_z |\{(x,y)\in A\times B : x+y\in Z\}|^2 = \|\mathbf{1}_A * \mathbf{1}_B\|_2^2.$$

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- Additive energy is very natural for doing analysis. But it is easier to understand sets of small doubling (e.g. Freiman's Theorem).
- By Young's inequality (in this context, simply the convexity of $t \mapsto t^p$),
$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$
Since $\|\mathbf{1}_A\|_1 = |A|$ and $\|\mathbf{1}_A\|_2 = |A|^{1/2}$, sets with $E(A, A) \sim |A|^3$ are sets for which Young's inequality applied to $\|\mathbf{1}_A * \mathbf{1}_A\|_2$ is "almost" an equality.
- The examples of sets with additive energy $\sim |A|^3$ we have seen are of the form: a set with small doubling $\cup$ an arbitrary set of similar size. Are there any other examples?

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The Balog-Szemerédi-Gowers Theorem

Theorem (Balog-Szemerédi (1994), Gowers (1998), Schoen (2014))

There are constants $c, C > 0$ *such that the following holds. Suppose* $E(A, A) \geq |A|^3/K$.

Then there exists $A' \subset A$ *such that* $|A'| \geq c|A|/K$ *and* $|A' + A'| \leq CK^4|A'|$.

Remarks on BSG

- The proof is an elementary (but far from easy!) argument involving paths on bi-partite graphs.
- Gowers (1998) obtained polynomial bounds in K in his proof of a quantitative version of Szemerédi's Theorem for progressions of length 4.
- There is a very similar statement for two different sets A, B of similar size (for example, $B = -A$), but the bounds become meaningless if one set is much larger than the other. We will see next an asymmetric version of BSG that gives information if $\log|A|$ and $\log|B|$ are comparable.

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Asymmetric BSG

The following is a special case/corollary of the asymmetric version of the BSG theorem:

Theorem (Tao-Vu, based on ideas of Bourgain)

Given $\delta > 0$, there is $\varepsilon > 0$ such that the following holds for large enough N.
Let $A, B \subset \{1, \ldots, N\}$ such that $E(A, B) \geq N^{-\varepsilon}|A||B|^2$.
Then there are sets $X, H \subset \{1, \ldots, N\}$ such that:

- $|H + H| \leq N^{\delta}|H|$,
- $|A \cap (X + H)| \geq N^{-\delta}|A| \geq N^{-2\delta}|X||H|$,
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B is approximately contained in an approximate group H, and A is approximately a union of disjoint translations of H

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BSG, partial sumset formulation

Lemma

- *If $E(A,A) \geq |A|^3/K$, then there exists $G \subset A \times A$ such that $|G| \geq |A|^2/2K$ and the partial sumset $A \overset{G}{+} A := \{x+y : (x,y) \in G\}$ satisfies $A \overset{G}{+} A \leq 2K|A|$.*
- *Conversely, if $G \subset A \times A$, then*

$$E(A,A) \geq \frac{|G|^2}{|A \overset{G}{+} A|}.$$

Corollary (of BSG and lemma)

If there is $G \subset A \times A$ such that $|G| \geq |A|^2/K$ and $|A \overset{G}{+} A| \leq K|A|$, then there is $A' \subset A$ such that $|A'| \geq K^{-C}|A|$ and $|A' + A'| \leq K^C|A'|$.

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This is an important theorem!

Opinion

The Balog-Szemerédi-Gowers is one of the most important theorems from the last 25 years.

Outline

Plünnecke's inequalities

Motivation

Freiman's Theorem says that if $|A + A| \leq K|A|$ *then* A *is a dense subset of a low-rank* GAP*.*

Using this it is easy to show that $|A + A + A| \leq f(K)|A|$ *and so on. In other words, having a small sumset implies having a small n-sumset nA.*

But can we do better than Freiman's Theorem in this direction?

Theorem (Plünnecke Inequalities, 1969)

Suppose $|A + A| \leq K|A|$*. Then* $|nA| \leq K^n|A|$*.*

More generally, if $|A + B| \leq K|A|$*, then* $|nB| \leq K^n|A|$*.*

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G. Petridis' proof of Plünnecke's inequalities

- Suppose $|A+B| \leq K|A|$. We want to show $|nB| \leq K^n|A|$.
- Choose a subset A' of A which minimizes the ratio $|A'+B|/|A'|$, let K' be the ratio (so $K' \leq K$).
- Then by definition we have:

$$|A'+B| = K'|A'|,$$
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Lemma (Petridis)

For every set C,

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Plünnecke's inequality: applying the main lemma

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Proof.

Induction in $|C|$ (clever but short argument) □

Proof of Plünnecke's inequalities, assuming lemma.

We prove by induction that

$$|nB| \leq |A' + nB| \leq (K')^n |A'| \leq K^n |A|$$

For $n = 1$, this is the definition of K'.

For the induction step, apply the lemma to $C = (n-1)B$. □

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Plünnecke inequalities: connections

- The Plünnecke inequalities are a key component of all the quantitative proofs of Freiman's Theorem.
- Usually one uses the contrapositive: in order to prove that $|A + A| \gg |A|$, it is enough to prove that $|A + A + \cdots A| \gg |A|$, which is easier since repeated sumsets have far more structure/smoothness.
- There is a useful version of Plünnecke's inequalities (due to Kaimanovich-Vershik) for entropy, with convolutions of measures in place of sumsets.

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No Plünnecke inequalities for Hausdorff dimension

Theorem (T. Körner, J. Schmeling-P.S.)

For any non-decreasing sequence α_n *of numbers in* $[0,1]$ *there exists a compact set* A *such that*

$$\dim_H\left(\underbrace{A+\cdots+A}_{n \text{ times}}\right)=\alpha_n.$$

Remark

- *Körner proved the result first but we were not aware of it; the constructions are different.*
- *We also consider simultaneously lower and upper box-counting dimensions, and show that there are no Plünnecke inequalities for upper box dimension, but they do hold for lower box dimension.*

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Outline

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Szemerédi's Theorem: Dense subsets of $\mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}$ contain arbitrarily long arithmetic progressions.

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End of part I

Thank you!!