LECTURE NOTES ON THE FOURIER TRANSFORM AND HAUSDORFF DIMENSION

PERTTI MATTILA

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Most of these lectures are based on the book P. Mattila: The Fourier transform and Hausdorff dimension, Cambridge University Press, 2015.

During the lectures I shall give more proofs or sketches of proofs than are here. On the other hand, these notes probably contain several things I don't have time to discuss dring the lectures.

1. Lecture 1: Hausdorff dimension, Fourier transform and applications

The s-dimensional *Hausdorff measure* $\mathcal{H}^s$, $s \geq 0$, is defined by

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A),$$

where, for $0 < \delta \leq \infty$,

$$\mathcal{H}^s_\delta(A) = \inf\{\sum_j d(E_j)^s : A \subset \bigcup_j E_j, d(E_j) < \delta\}.$$

Here $d(E)$ denotes the diameter of the set E.

The *Hausdorff dimension* of $A \subset \mathbb{R}^n$ is

$$\dim A = \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}.$$

For $A \subset \mathbb{R}^n$, let $\mathcal{M}(A)$ be the set of Borel measures μ such that $0 < \mu(A) < \infty$ and μ has compact support spt$\mu \subset A$. The following is a useful tool for lower bounds for the Hausdorff dimension:

Theorem 1.1 (Frostman's lemma)**.** Let $0 \leq s \leq n$. For a Borel set $A \subset \mathbb{R}^n$, $\mathcal{H}^s(A) > 0$ if and only there is $\mu \in \mathcal{M}(A)$ such that

$$\mu(B(x,r)) \leq r^s \quad \text{for all } x \in \mathbb{R}^n, r > 0. \tag{1.1}$$

In particular,

$$\dim A = \sup\{s : \text{there is } \mu \in \mathcal{M}(A) \text{ such that (1.1) holds}\}.$$

The *s-energy*, $s > 0$, of a Borel measure μ is

$$I_s(\mu) = \iint |x-y|^{-s}\, d\mu x\, d\mu y = \int k_s * \mu\, d\mu,$$

where k_s is the *Riesz kernel*:

$$k_s(x) = |x|^{-s}, \quad x \in \mathbb{R}^n.$$

Integration of Frostman's lemma gives

Theorem 1.2. For a closed set $A \subset \mathbb{R}^n$,

$$\dim A = \sup\{s : \text{there is } \mu \in \mathcal{M}(A) \text{ such that } I_s(\mu) < \infty\}.$$

The *Fourier transform* of $\mu \in \mathcal{M}(\mathbb{R}^n)$ is

$$\widehat{\mu}(\xi) = \int e^{-2\pi i \xi \cdot x} \, d\mu x, \quad \xi \in \mathbb{R}^n.$$

The s-energy of $\mu \in \mathcal{M}(\mathbb{R}^n)$ can be written in terms of the Fourier transform:

$$I_s(\mu) = c(n,s) \int |\widehat{\mu}(x)|^2 |x|^{s-n} \, dx.$$

Thus we have

$$\dim A = \sup\{s < n : \exists \mu \in \mathcal{M}(A) \text{ such that } \int |\widehat{\mu}(x)|^2 |x|^{s-n} \, dx < \infty\}. \tag{1.2}$$

Notice that if $I_s(\mu) < \infty$, then $|\widehat{\mu}(x)|^2 < |x|^{-s}$ for most x with large norm. However, this need not hold for all x with large norm. The Fourier dimension of a set A captures some information on how one can put measures on A with good Fourier decay:

The *Fourier dimension* of a set $A \subset \mathbb{R}^n$ is

$$\dim_F A = \sup\{s \leq n : \exists \mu \in \mathcal{M}(A) \text{ such that } |\widehat{\mu}(x)| \leq |x|^{-s/2} \, \forall x \in \mathbb{R}^n\}.$$

Then

$$\dim_F A \leq \dim A.$$

A is called a Salem set if $\dim_F A = \dim A$.

Examples of Salem sets are smooth planar curves with non-zero curvature and trajectories of Brownian motion. But line segments in $\mathbb{R}^n, n \geq 2$, have zero Fourier dimension. The Fourier dimension of the classical $1/3$ Cantor set is 0, but many other Cantor sets have positive Fourier dimension and many random Cantor sets are Salem sets.

Fraser, Orponen and Sahlsten proved in [FOS] the following on the Fourier dimension of graphs:

Theorem 1.3. For any function $f : A \to \mathbb{R}^{n-m}, A \subset \mathbb{R}^m$, we have for the graph $G_f = \{(x, f(x)) : x \in A\}$,

$$\dim_F G_f \leq m.$$

The Hausdorff dimension of one-dimensional Brownian graphs is almost surely $3/2$, so they are not Salem sets. Fraser and Sahlsten proved in [FS] that their Fourier dimension is almost surely 1.

We shall now discuss applications of the Fourier transform on some geometric problems on Hausdorff dimension.

How do the projections

$$p_e(x) = e \cdot x, \quad x \in \mathbb{R}^n, e \in S^{n-1},$$

affect the Hausdorff dimension? Notice that p_e is essentially the orthogonal projection onto the line with direction e. For simplicity, I only consider projection onto lines although analogous results hold for projections onto m-planes, $0 < m < n$.

The first two items of the following theorem were proved by Marstrand [M] in 1954 and the third by Falconer and O'Neil [FO] in 1999 and by Peres and Schlag [PS] in 2000, ($\mathcal{L}^m$ denotes the Lebesgue measure in $\mathbb{R}^m$):

Theorem 1.4. Let $A \subset \mathbb{R}^n$ be a Borel set.

(1) If $\dim A \leq 1$, then

$$\dim p_e(A) = \dim A \quad \text{for almost all } e \in S^{n-1}.$$

(2) If $\dim A > 1$, then

$$\mathcal{L}^{n-1}(p_e(A)) > 0 \quad \text{for almost all } e \in S^{n-1}.$$

(3) If $\dim A > 2$, then $p_e(A)$ has non-empty interior for almost all $e \in S^{n-1}$.

Let us prove (2) using a proof due to Kaufman [Ka]. Choose by (1.2) a measure $\mu \in \mathcal{M}(A)$ such that $\int |x|^{-1}|\widehat{\mu}(x)|^2\,dx < \infty$. Let $\mu_e \in \mathcal{M}(p_e(A))$ be the push-forward of μ under p_e: $\mu_e(B) = \mu(p_e^{-1}(B))$. Directly from the definition of the Fourier transform we see that $\widehat{\mu_e}(t) = \widehat{\mu}(te)$ for $t \in \mathbb{R}, e \in S^{n-1}$. Integrating in polar coordinates we obtain

$$\int_{S^{n-1}} \int_{-\infty}^{\infty} |\widehat{\mu_e}(t)|^2\,dt\,de = 2\int_{S^{n-1}} \int_0^{\infty} |\widehat{\mu}(te)|^2\,dt\,de = 2\int |x|^{-1}|\widehat{\mu}(x)|^2\,dx < \infty.$$

Thus for almost all $e \in S^{n-}1$, $\widehat{\mu_e} \in L^2(\mathbb{R})$ which means that μ_e is absolutely continuous with L^2 density and hence $\mathcal{L}^1(p_e(A)) > 0$.

For the proof of (3) one shows that for almost all $e \in S^{n-}1$, $\widehat{\mu_e} \in L^1(\mathbb{R})$ which implies that μ_e is absolutely continuous with continuos density.

The following theorem was proved by Orponen and myself in [MO]:

Theorem 1.5. Let A and B be Borel subsets of $\mathbb{R}^n$.

(i) If $\dim A > 1$ and $\dim B > 1$, then

$$\mathcal{H}^{n-1}\left(\{e \in S^{n-1} : \mathcal{L}^{n-1}(p_e(A) \cap p_e(B)) > 0\}\right) > 0.$$

(ii) If $\dim A > 2$ and $\dim B > 2$, then

$$\mathcal{H}^{n-1}\left(\{e \in S^{n-1} : \operatorname{Int}(p_e(A) \cap p_e(B)) \neq \varnothing\}\right) > 0.$$

(iii) If $\dim A > 1, \dim B \leq 1$ and $\dim A + \dim B > 2$, then for every $\varepsilon > 0$,

$$\mathcal{H}^{n-1}\left(\{e \in S^{n-1} : \dim(p_e(A) \cap p_e(B)) > \dim B - \varepsilon\}\right) > 0.$$

This was applied to exceptional set estimates for dimensions of slices. Further, Orponen proved in [O4] the following sharp estimate on radial projections. For $x \in \mathbb{R}^n$ define

$$\pi_x : \mathbb{R}^n \setminus \{x\} \to S^{n-1}, \quad \pi_x(y) = \frac{y-x}{|y-x|}.$$

Theorem 1.6. Let $A \subset \mathbb{R}^n$ be a Borel set with $\dim A > n-1$. Then there is a Borel set $B \subset \mathbb{R}^n$ with $\dim B \leq 2(n-1) - \dim A$ such that for every $x \in \mathbb{R}^n \setminus B$, $\mathcal{H}^{n-1}(\pi_x(A)) > 0$.

One can improve Theorem 1.4 by more precise information on the size of the exceptional sets of directions. Kaufman [Ka] proved in 1968 the first item of the following theorem, Falconer [F1] in 1982 the second and Peres and Schlag [PS] in 2000 the third:

Theorem 1.7. Let $A \subset \mathbb{R}^n$ be a Borel set.

(1) If $\dim A \leq 1$, then

$$\dim\{e \in S^{n-1} : \dim p_e(A) < \dim A\} \leq \dim A. \tag{1.3}$$

(2) If $\dim A > 1$, then

$$\dim\{e \in S^{n-1} : \mathcal{L}^{n-1}(p_e(A)) = 0\} \leq n - \dim A. \tag{1.4}$$

(3) If $\dim A > 2$, then

$$\dim\{e \in S^{n-1} : \mathrm{Int}(p_e(A)) \neq \varnothing\} \leq n + 1 - \dim A. \tag{1.5}$$

Theorem 1.7 and much more, for instance exceptional set estimates for Benoulli convolutions, is included in the setting of generalized projections developed by Peres and Schlag in [PS]. Later these general estimates have been improved in some special cases, for example for Bernoulli convolutions by Shmerkin and Solomyak in [SS].

The bounds $\dim A$ and $n - \dim A$ in (1) and (2) are sharp by the examples which Kaufman and I [KM] constructed in 1975. It is not known if the bound in (3) is sharp. Another, seemingly very difficult problem, is estimating the dimension of the set in (1) when $\dim A$ is replaced by some $t < \dim A$. For example, it might be true, and has been conjectured by D.M. Oberlin [Ob1], that

$$\dim\{e \in S^1 : \dim p_e(A) < t\} \leq 2t - \dim A.$$

This would be sharp, as the constructions in [KM] show. This estimate is known only when $t = \dim A/2$ and due to Bourgain, [B3], [B4]. Orponen considered the case $\dim A = 1$ in [O5] and [O6] but with $\dim p_e(A)$ replaced by the packing dimension of $p_e(A)$.

There are also exceptional set results related to the dimension of the slices $A \cap V$ with generic planes V and to the intersections $A \cap (g(B) + z)$ with generic rotations g and translations by z. Some recent exceptional estimates for these can be found in [O1], [MO] and [M3].

Another problem where the Fourier transform has been extremely useful is the distance set problem. For $A \subset \mathbb{R}^n$ define the distance set

$$D(A) = \{|x - y| : x, y \in A\} \subset [0, \infty).$$

The following Falconer's conjecture seems plausible:

Conjecture 1.8. If $n \geq 2$ and $A \subset \mathbb{R}^n$ is a Borel set with $\dim A > n/2$, then $\mathcal{L}^1(D(A)) > 0$, or even $\mathrm{Int}(D(A)) \neq \varnothing$.

Falconer [F2] proved in 1985 that $\dim A > (n+1)/2$ implies $\mathcal{L}^1(D(A)) > 0$, and we also have then $\mathrm{Int}(D(A)) \neq \varnothing$ by Sjölin and myself [MS].

The best known result is due to Wolff [W1] for $n = 2$ and to Erdogan [E] for $n \geq 3$:

Theorem 1.9. If $n \geq 2$ and $A \subset \mathbb{R}^n$ is a Borel set with $\dim A > n/2 + 1/3$, then $\mathcal{L}^1(D(A)) > 0$.

The proof uses restriction and Kakeya methods and results, which will be discussed in the last lecture. In particular, the case $n \geq 3$ relies on Tao's bilinear restriction theorem.

Various partial results on distance sets have recently been proved, among others, by Iosevich and Liu [IL1], [IL2], Luca and Rogers [LR], Orponen [O3] and Shmerkin [S1], [S2].

2. Lecture 2: Besicovitch sets and Kakeya problems

We say that a Borel set in $\mathbb{R}^n, n \geq 2$, is a *Besicovitch set*, or a Kakeya set, if it has zero Lebesgue measure and it contains a line segment of unit length in every direction. This means that for every $e \in S^{n-1}$ there is $b \in \mathbb{R}^n$ such that $\{te + b : 0 < t < 1\} \subset B$. It is not obvious that Besicovitch sets exist but they do in every $\mathbb{R}^n, n \geq 2$:

Theorem 2.1. 1919, 1964] For any $n \geq 2$ there exists a Borel set $B \subset \mathbb{R}^n$ such that $\mathcal{L}^n(B) = 0$ and B contains a whole line in every direction. Moreover, there exist compact Besicovitch sets in $\mathbb{R}^n$.

It is enough to prove this in the plane, then $B \times \mathbb{R}^{n-2}$ is fine in $\mathbb{R}^n$. The proof of Besicovitch from 1964 uses projections and duality between points and lines. More precisely, parametrize the lines, except those parallel to the y-axis, by $(a, b) \in \mathbb{R}^2$:

$$l(a, b) = \{(x, ax + b) : x \in \mathbb{R}\}.$$

Then if $C \subset \mathbb{R}^2$ is some parameter set and $B = \cup_{(a,b)\in C} l(a, b)$, one checks that

$$B \cap \{(t, y) : y \in \mathbb{R}\} = \{t\} \times \pi_t(C)$$

where

$$\pi_t : \mathbb{R}^2 \to \mathbb{R}^2, \quad \pi_t(a, b) = ta + b,$$

is essentially an orthogonal projection. Suppose that we can find C such that $\pi(C) = [0, 1]$, where $\pi(a, b) = a$, and $\mathcal{L}^1(\pi_t(C)) = 0$ for almost all t. Then $\mathcal{L}^2(B) = 0$ by Fubini's theorem and taking the union of four rotated copies of B gives the desired set. It is not trivial that such sets C exist but they do. For example, a suitably rotated copy of the product of a standard Cantor set with dissection ratio $1/4$ with itself is such.

In this lecture we shall mainly be interested in what can be said about the Hausdorff dimension of Besicovitch sets.

Conjecture 2.2 (Kakeya conjecture). All Besicovitch sets in $\mathbb{R}^n$ have Hausdorff dimension n.

Theorem 2.3 (Davies 1971)**.** For every Besicovitch set $B \subset \mathbb{R}^2$, $\dim B = 2$. In particular, the Kakeya conjecture is true in the plane.

The proof of this is, up to some technicalities, reversing the above argument for the proof of Theorem 2.1 and using Marstrand's projection Theorem 1.4(1).

Córdoba proved in 1977 that $\dim B \geq 2$ for every Besicovitch set in $\mathbb{R}^n$. Oberlin proved in [Ob1] that this true even for the Fourier dimension. Recall that $\dim_F \leq \dim$.

Theorem 2.4. For every Besicovitch set $B \subset \mathbb{R}^2$, $\dim_F B \geq 2$.

The Kakeya conjecture is open for $n \geq 3$. I shall discuss partial results later but let us look first at what one could do with above argument using projections, for example in $\mathbb{R}^3$. Now we parametrize the lines, except those parallel to the (y, z)-plane by $(a, b) \in \mathbb{R}^2 \times \mathbb{R}^2$:

$$l(a, b) = \{(x, ax + b) : x \in \mathbb{R}\}.$$

Then again if $C \subset \mathbb{R}^4$ is parameter set and $B = \cup_{(a,b)\in C} l(a, b)$ we have for $t \in \mathbb{R}$,

$$B \cap \{(t, y) : y \in \mathbb{R}^2\} = \{t\} \times \pi_t(C)$$

where

$$\pi_t : \mathbb{R}^4 \to \mathbb{R}^2, \quad \pi_t(a, b) = ta + b.$$

Suppose now that $\pi(C) = [0, 1]^2$, where $\pi(a, b) = a$. Then in particular, $\dim C \geq 2$. The projection theorem we would need should tell us that $\dim \pi_t(C) = 2$ for almost all t. However, we don't know of any such projection theorem since we now only have a one-dimensional family of projections. Notice that the space of all orthogonal projections from $\mathbb{R}^4$ onto 2-planes is 4-dimensional.

There are theorems for small restricted families of projections, for example by Fässler and Orponen, [FOr] and [O2], E. and M. Järvenpää and Keleti [JJK], and D.M. and R. Oberlin [Ob3], [OO] but they are too weak for the problem on Besicovitch sets.

Let us now look at some relations between unions of lines and line segments. Keleti made the following conjecture in [Ke]:

Conjecture 2.5. If A is the union of a family of line segments in $\mathbb{R}^n$ and B is the union of the corresponding lines, then $\dim A = \dim B$.

This is true in the plane, as proved by Keleti:

Theorem 2.6. Conjecture 2.5 is true in $\mathbb{R}^2$.

If Keleti's conjecture is true in $\mathbb{R}^n, n \geq 3$, it gives a lot of new information on the dimension of Besicovitch sets:

Theorem 2.7 (Keleti [Ke]). (1) If Conjecture 2.5 is true for some n, then, for this n, every Besicovitch set in $\mathbb{R}^n$ has Hausdorff dimension at least $n-1$.

(2) If Conjecture 2.5 is true for all n, then every Besicovitch set in $\mathbb{R}^n$ has upper Minkowski dimension n for all n.

The upper Minkowski (or box counting) dimension $\dim_M A$ of $A \subset \mathbb{R}^n$ is defined by

$$\dim_M A = \inf\{s \geq 0 : \lim_{\delta \to 0} \delta^{s-n} \mathcal{L}^n(\{x : dist(x, A) < \delta\}) = 0\}.$$

Then $\dim A \leq \dim_M A$.

Using projection theorems Falconer and I proved in [FM] that in Theorem 2.6 line segments can be replaced by sets of positive one-dimensional measure. Later Héra, Keleti and Máthé in [HKM] proved that sets of dimension one are enough. These methods and results extend to subsets of hyperplanes in $\mathbb{R}^n$, but they do not extend to lower dimensional planes. In particular they do not apply to Besicovitch sets in higher dimensions. Héra, Keleti and Máthé studied the principle 'k-dimensional family of s-dimensional sets should have dimension $s+k$' more systematically with many interesting results.

Now I discuss one of the basic tools for better estimates for the Hausdorff dimension of Besicovitch sets.

For $a \in \mathbb{R}^n, e \in S^{n-1}$ and $\delta > 0$, define the tube $T_e^\delta(a)$ with center a, direction e, length 1 and radius δ:

$$T_e^\delta(a) = \{x \in \mathbb{R}^n : |(x-a) \cdot e| \leq 1/2, |x - a - ((x-a) \cdot e)e| \leq \delta\}.$$

Then $\mathcal{L}^n(T_e^\delta(a)) = \alpha(n-1)\delta^{n-1}$, where $\alpha(n-1)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^{n-1}$.

Definition 2.8. The *Kakeya maximal function* with width δ of $f \in L^1_{loc}(\mathbb{R}^n)$ is

$$\mathcal{K}_\delta f : S^{n-1} \to [0, \infty],$$

$$\mathcal{K}_\delta f(e) = \sup_{a \in \mathbb{R}^n} \frac{1}{\mathcal{L}^n(T_e^\delta(a))} \int_{T_e^\delta(a)} |f| \, d\mathcal{L}^n.$$

We have the trivial but sharp proposition:

Proposition 2.9. For all $0 < \delta < 1$ and $f \in L^1_{loc}(\mathbb{R}^n)$,

$$\|\mathcal{K}_\delta f\|_{L^\infty(S^{n-1})} \leq \|f\|_{L^\infty(\mathbb{R}^n)},$$

$$\|\mathcal{K}_\delta f\|_{L^\infty(S^{n-1})} \leq \alpha(n-1)^{1-n} \delta^{1-n} \|f\|_{L^1(\mathbb{R}^n)}.$$

Conjecture 2.10. [Kakeya maximal conjecture]

$$\|\mathcal{K}_\delta f\|_{L^n(S^{n-1})} \leq C(n, \varepsilon)\delta^{-\varepsilon} \|f\|_{L^n(\mathbb{R}^n)}$$

for all $\varepsilon > 0, 0 < \delta < 1, f \in L^n(\mathbb{R}^n)$.

This is true in the plane:

Theorem 2.11 (Córdoba 1977).

$$\|\mathcal{K}_\delta f\|_{L^2(S^1)} \leq C\sqrt{\log(1/\delta)}\|f\|_{L^2(\mathbb{R}^2)}$$

for all $0 < \delta < 1, f \in L^2(\mathbb{R}^2)$.

The following, rather easy, theorem is a key to the the Hausdorff dimension of Besicovitch sets:

Theorem 2.12. [B1]
Suppose that $1 < p < \infty, \beta > 0$ and $n - \beta p > 0$. If

$$\|\mathcal{K}_\delta f\|_{L^p(S^{n-1})} \leq C(n,p,\beta)\delta^{-\beta}\|f\|_p \quad \text{for } 0 < \delta < 1, f \in L^p(\mathbb{R}^n),$$

then the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is at least $n - \beta p$. In particular, if for some $p, 1 < p < \infty$,

$$\|\mathcal{K}_\delta f\|_{L^p(S^{n-1})} \leq C(n,p,\varepsilon)\delta^{-\varepsilon}\|f\|_{L^p(\mathbb{R}^n)}$$

holds for all $\varepsilon > 0, 0 < \delta < 1, f \in L^p(\mathbb{R}^n)$, then the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is n. Thus the Kakeya maximal conjecture implies the Kakeya conjecture.

It often helps to discretize the L^p-estimates for the Kakeya maximal function:

Proposition 2.13. Let $1 < p < \infty, q = \frac{p}{p-1}, 0 < \delta < 1$ and $0 < M < \infty$. Suppose that

$$\|\sum_{k=1}^m t_k \chi_{T_k}\|_{L^q(\mathbb{R}^n)} \leq M$$

whenever $T_1, \dots, T_m$ are δ-separated (in directions) δ-tubes and $t_1, \dots, t_m$ are positive numbers with

$$\delta^{n-1}\sum_{k=1}^m t_k^q \leq 1.$$

Then

$$\|\mathcal{K}_\delta f\|_{L^p(S^{n-1})} \leq C(n)M\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all } f \in L^p(\mathbb{R}^n).$$

A fairly easy argument of Bourgain [B1] based on this gives

Theorem 2.14. For all Lebesgue measurable sets $E \subset \mathbb{R}^n$,

$$\sigma^{n-1}(\{e \in S^{n-1} : \mathcal{K}_\delta(\chi_E)(e) > \lambda\}) \leq C(n)\delta^{1-n}\lambda^{-n-1}\mathcal{L}^n(E)^2$$

for all $0 < \delta < 1$ and $\lambda > 0$.

The above restricted weak type inequality is very close to,

$$\|\mathcal{K}_\delta f\|_{L^q(S^{n-1})} \leq C(n,p,\varepsilon)\delta^{-(n/p-1+\varepsilon)}\|f\|_p$$

for all $\varepsilon > 0$ with $p = (n+1)/2, q = n+1$. So Theorem 2.14 yields that the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is at least $(n+1)/2$. But this was proved by Drury already in 1983.

Bourgain's proof used bushes; many tubes containing some point. Wolff [W1] replaced this with hairbrushes; many tubes intersecting some tube, to prove

Theorem 2.15. Let $0 < \delta < 1$. Then for $f \in L^{\frac{n+2}{2}}(\mathbb{R}^n)$,

$$\|\mathcal{K}_\delta f\|_{L^{\frac{n+2}{2}}(S^{n-1})} \leq C(n,\varepsilon)\delta^{\frac{2-n}{2+n}-\varepsilon}\|f\|_{L^{\frac{n+2}{2}}(\mathbb{R}^n)} \tag{2.1}$$

for all $\varepsilon > 0$. In particular, the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is at least $(n+2)/2$.

Wolff's estimate $\dim B \geq 3$ is still the best known in $\mathbb{R}^4$. Now I discuss briefly improvements in other dimensions.

Bourgain introduced in [B2] a combinatorial method, further developed by Katz and Tao [KT1], which is better than the above geometric method in high dimensions:

Theorem 2.16. Let $\varepsilon_0 = 1/6$. Suppose that A and B are finite subsets of $\lambda\mathbb{Z}^m$ for some $m \in \mathbb{N}$ and $\lambda > 0$, $\#A \leq N$ and $\#B \leq N$. Suppose also that $G \subset A \times B$ and

$$\#\{x + y \in G : (x,y) \in G\} \leq N. \tag{2.2}$$

Then

$$\#\{x - y \in G : (x,y) \in G\} \leq N^{2-\varepsilon_0}.$$

The proof is quite elementary but tricky. The best value of ε_0 is not known, but it cannot be taken bigger than $\log 6/\log 3 = 0.39907\ldots$. By some Fubini-type arguments this leads to

Theorem 2.17. For any Besicovitch set B in $\mathbb{R}^n$, $\dim B \geq 6n/11 + 5/11$.

Once Theorem 2.16 is available this result is easy for the Minkowski dimension $\dim_M B$, somewhat more difficult for the Hausdorff dimension. Later Katz and Tao [KT2] improved the arguments considerably to prove

Theorem 2.18. For any Besicovitch set B in $\mathbb{R}^n$, $\dim B \geq (2-\sqrt{2})(n-4)+3$.

This theorem improves Wolff's $(n+2)/2$ bound for all $n \geq 5$. Quite recently Katz and Zahl [KZ] were able to establish an epsilon improvement on Wolff's bound $5/2$ in $\mathbb{R}^3$ with very involved and complicated arguments:

Theorem 2.19. For any Besicovitch set B in $\mathbb{R}^3$, $\dim B \geq 5/2+\varepsilon$ where ε is a small constant.

Many other versions of Besicovitch type sets have been studied; curves replaced by line segments, even by rectifiable sets by Chang and Csörnyei [CC], planes in place of lines, etc.

Furstenberg sets are kind of fractal versions of Besicovitch sets: $F \subset \mathbb{R}^2$ is a Furstenberg s-set, $0 < s \leq 1$, if for every $e \in S^1$ there is a line L_e in direction e such that $\dim F \cap L_e \geq s$. What can be said about the dimension of F? Wolff [W2], Section 11.1, showed that $\dim F \geq \max\{2s, s+1/2\}$ and that there is such an F with $\dim F = 3s/2+1/2$. The lower bound $2s$ is easier and its proof resembles the proof of Theorem 2.12. When $s = 1/2$ Bourgain [B3] improved the lower bound 1 to $\dim F \geq 1+c$ for some absolute constant $c > 0$. Other recent results are due to Molter and Rela [MR1], [MR3] and [MR2], Oberlin [Ob4], Orponen [O6] and Venieri [V]. Rela has a survey in [R].

3. Lecture 3: Restriction and Kakeya

When does $\widehat{f}|S^{n-1}$ make sense? If $f \in L^1(\mathbb{R}^n)$ it obviously does, if $f \in L^2(\mathbb{R}^n)$ it obviously does not.

$\widehat{f}|S^{n-1}$ makes sense for $f \in L^p(\mathbb{R}^n)$ if we have for some $q < \infty$ an inequality

$$\|\widehat{f}\|_{L^q(S^{n-1})} \leq C(n,p,q)\|f\|_{L^p(\mathbb{R}^n)} \tag{3.1}$$

valid for all $f \in \mathcal{S}(\mathbb{R}^n)$.

The restriction problems ask for which p and q (3.1) holds.

By duality (3.1) is equivalent, with the same constant $C(n,p,q)$, to

$$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^n)} \leq C(n,p,q)\|f\|_{L^{q'}(S^{n-1})}. \tag{3.2}$$

Here p' and q' are conjugate exponents of p and q and $\widehat{f}$ means the Fourier transform of the measure $f\sigma^{n-1}$ where σ^{n-1} is the surface measure on the sphere S^{n-1}.

The following theorem was proved by Tomas in 1975 for $q > 2(n+1)/(n-1)$ and by Stein in 1986 for the end-point:

Theorem 3.1. We have for $f \in L^2(S^{n-1})$,

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq C(n,q)\|f\|_{L^2(S^{n-1})}$$

for $q \geq 2(n+1)/(n-1)$. The lower bound $2(n+1)/(n-1)$ is the best possible.

The sharpness of the range of q follows using the so-called Knapp example; f is the characteristic function of a spherical cap.

Conjecture 3.2 (Restriction conjecture). $\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq C(n,q)\|f\|_{L^p(S^{n-1})}$ for $q > 2n/(n-1)$ and $q = \frac{n+1}{n-1}p'$.

This is equivalent to

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq C(n,q)\|f\|_{L^\infty(S^{n-1})} \quad \text{for } q > 2n/(n-1),$$

and to

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq C(n,q)\|f\|_{L^q(S^{n-1})} \quad \text{for } q > 2n/(n-1).$$

The range $q > 2n/(n-1)$ would be optimal. Stein-Tomas theorem implies that these inequalities are true when $q \geq 2(n+1)/(n-1)$.

Fefferman 1970 and Zygmund 1974 proved in the plane

$$\|\widehat{f}\|_{L^q(\mathbb{R}^2)} \leq C(q)\|f\|_{L^p(S^1)} \text{ for } q > 4 \text{ and } q = 4p'$$

.

Thus the restriction conjecture is true in the plane.

The following result of Bourgain [B1] (although it is already almost present in Fefferman's 1971 paper on ball multipliers) tells us that the restriction conjecture implies Kakeya maximal and Kakeya conjectures:

Theorem 3.3. Suppose that $2n/(n-1) < q < \infty$ and

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \lesssim_{n,q} \|f\|_{L^q(S^{n-1})} \quad \text{for } f \in L^q(S^{n-1}). \tag{3.3}$$

Then with $p = q/(q-2)$,

$$\|\mathcal{K}_\delta f\|_{L^p(S^{n-1})} \lesssim_{n,q} \delta^{4n/q-2(n-1)}\|f\|_p$$

for all $0 < \delta < 1, f \in L^p(\mathbb{R}^n)$. In particular, the restriction conjecture implies the Kakeya maximal conjecture.

The proof uses Khintchine's inequalities and the Knapp example.

Altough the restriction conjecture is open in higher dimension, Tao proved in [T] the following sharp result on bilinear restriction:

Theorem 3.4 (Tao 2003). Let $c > 0$ and let $S_j \subset \{x \in S^{n-1} : x_n > c\}, j = 1, 2$, with $d(S_1, S_2) \geq c > 0$. Then

$$\|\widehat{f_1}\widehat{f_2}\|_{L^q(\mathbb{R}^n)} \leq C(n,q,c)\|f_1\|_{L^2(S_1)}\|f_2\|_{L^2(S_2)}$$

for $q > (n+2)/n$ and for all $f_j \in L^2(S^{n-1})$ with spt $f_j \subset S_j, j = 1, 2$.

The lower bound $(n+2)/n$ is the best possible due to the Knapp example.

Powerful recent multilinear Kakeya methods and polynomial methods have lead to improvements on restriction. Here is a summary on some of the main steps in the progress on restriction conjecture:

Conjecture: $\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^\infty(S^{n-1})}$ for $q > 2n/(n-1), \quad q > 3$ for $n = 3$.
Tomas 1975: $q > (2n+2)/(n-1), \quad q > 4$ for $n = 3$.
Stein 1986: $q = (2n+2)/(n-1), \quad q = 4$ for $n = 3$.
Bourgain 1991: $q > (2n+2)/(n-1) - \varepsilon_n, \quad q > 31/8 = 4 - 1/8$ for $n = 3$.
Tao, Vargas and Vega 1998, Tao 2003 by bilinear restriction: $q > (2n+4)/n$, $q > 10/3 = 31/8 - 13/24$ for $n = 3$.
Bennett, Carbery and Tao 2006, Bourgain and Guth 2011 by multilinear restriction: $q > 33/10 = 10/3 - 1/30$ for $n = 3$.
(Dvir 2009), Guth 2014 by polynomial method: $q > 13/4 = 33/10 - 3/40 = 3 + 1/4$ for $n = 3$.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. BOX 68, FI-00014 UNIVERSITY OF HELSINKI, FINLAND,
E-mail address: pertti.mattila@helsinki.fi