The Riesz transform, rectifiabilty, and harmonic measure

Xavier Tolsa
ICREA and Universitat Autonoma de Barcelona, Catalonia
xtolsa@mat.uab.cat

These lectures will be devoted to the David-Semmes problem and its applications to the study of the geometric properties of harmonic measure. Roughly speaking, the David-Semmes problem consists in proving that, given a set $E \subset \mathbb{R}^d$ with finite Hausdorff n-dimensional measure H^n, the L^2 boundedness of the n-dimensional Riesz transform with respect to the Hausdorff measure $H^n|_E$ implies the n-rectifiability of E. In the late 1990's, in the case $n = 1$ this problem was solved for AD-regular sets by Mattila, Melnikov and Verdera, and in full generality by David and Léger, by using the connection between Menger curvature and the Cauchy kernel. The case of codimension 1 ($n = d-1$) was solved more recently by Nazarov, Tolsa and Volberg, by combining quasiorthogonality and variational arguments. The David-Semmes problem is still open for n different from 1 and $d - 1$.

The second part of the lectures will deal with the applications to the study of harmonic measure and rectifiability. In particular, we will see a recent result by Azzam, Hofmann, Martell, Mayboroda, Mourgoglou, Tolsa and Volberg which asserts that if the harmonic measure is absolutely continuous with respect to the Hausdorff measure H^n on some subset F of the boundary of an open set in $\mathbb{R}^{n+1}$, with $H^n(F) < \infty$, then F is n-rectifiable. This result can be considered as a converse of the famous theorem of the Riesz brothers on harmonic measure for simply connected domains in the plane, with no topological assumptions in $\mathbb{R}^{n+1}$.

Further, we will also review another recent application to a two phase problem for harmonic measure by Azzam, Mourgoglou and Tolsa, whose proof uses the connection between Riesz transforms and rectifiability and some blowup techniques inspired by work of Kenig, Preiss and Toro. If time permits, we will see an easy application of these blowup methods to the proof of Tsirelson's theorem about triple points for harmonic measure.