Multivalued Extended Best Φ-Polynomial Approximation Operator

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CIMPA 2017 Research School - IX Escuela Santaló, Buenos Aires (Argentina), July 31 - August 11, 2017

Setting

Let $\Im = \{\varphi : [0,\infty) \to [0,\infty),$ continuous, non decreasing such that $\varphi(t) > 0\ \forall t > 0$ and $\varphi \in \Delta_2\}$.
Recall that $\varphi \in \Delta_2$ iff $\exists \Lambda_\varphi > 0$ such that $\varphi(2a) \le \Lambda_\varphi \varphi(a)$ for all $a \ge 0$.
If $\varphi \in \Im$, we consider $\Phi(x) = \int_0^x \varphi(t)\,dt \Longrightarrow \Phi : [0,\infty) \to [0,\infty)$ is convex and $\Phi(a) = 0$ iff $a = 0$.
If $\frac{\Phi(x)}{x} \to 0$ as $x \to 0$ and $\frac{\Phi(x)}{x} \to \infty$ as $x \to \infty \Longrightarrow \Phi$ is an N-function and $\Phi' = \varphi$ satisfies $\varphi(0^+) = 0$ and $\varphi(x) \to \infty$ as $x \to \infty$.

Why Δ_2 condition on $\varphi \in \Im$?

- $\varphi \in \Delta_2 \Longleftrightarrow \Phi \in \Delta_2$
- $\varphi \in \Delta_2 \Longrightarrow \frac{1}{2}(\varphi(a) + \varphi(b)) \le \varphi(a+b) \le \Lambda_\varphi(\varphi(a) + \varphi(b)), \forall a, b \ge 0$.
- $\Phi \in \Delta_2 \Longrightarrow \frac{x}{2\Lambda_\varphi}\varphi(x) \le \Phi(x) \le x\varphi(x),\ \forall x \ge 0$.

Let B be a bounded measurable set in $\mathbb{R}^s$. If $\varphi \in \Im$, then

$$L^\varphi(B) = \left\{ f : B \to \mathbb{R}, \text{Lebesgue measurable function such that} \int_B \varphi(|f|)\,dx < \infty \right\}.$$

For the convex function Φ, $L^\Phi(B)$ is the classical Orlicz space treated in [1] and [2].

Best polynomial approximation operator

Let Π^m be the space of algebraic polynomials, defined on $\mathbb{R}^s$, of degree at most m.
A polynomial $P \in \Pi^m$ is said to be a best approximation of $f \in L^\Phi(B)$ iff

$$\int_B \Phi(|f - P|)\,dx = \inf_{Q \in \Pi^m} \int_B \Phi(|f - Q|)\,dx. \qquad (1)$$

Definition 1. *For $f \in L^\Phi(B)$ we denote by $\mu_\Phi(f)$ the set of all polynomials P that satisfy (1) and we refer to this set as the multivalued best polynomial approximation operator (b.p.a.o).*

Theorem 1 (Existence). *Let $\varphi \in \Im$ and $f \in L^\Phi(B)$. Then, there exists $P \in \Pi^m$ such that*

$$\int_B \Phi(|f - P|)\,dx = \inf_{Q \in \Pi^m} \int_B \Phi(|f - Q|)\,dx.$$

Theorem 2 (Characterization). *Let $\varphi \in \Im$ and $f \in L^\Phi(B)$. Then, $P \in \Pi^m$ belongs to $\mu_\Phi(f)$ iff*

$$\left| \int_B \varphi(|f - P|)\text{sgn}(f - P)Q\,dx \right| \le \varphi(0^+) \int_{\{f=P\}} |Q|\,dx,\ \forall Q \in \Pi^m.$$

Remark 3. Let $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\,dt$ and $f \in L^\Phi(B)$.
If $\varphi(0^+) = 0$, then $P \in \Pi^m$ is in $\mu_\Phi(f)$ if and only if

$$\int_B \varphi(|f - P|)\text{sgn}(f - P)Q\,dx = 0,\ \forall Q \in \Pi^m. \qquad (2)$$

Theorem 4. *Let $\varphi \in \Im$ and let $f \in L^\varphi(B)$. Suppose the polynomial $P \in \Pi^m$ satisfies*

$$\left| \int_B \varphi(|f - P|)\text{sgn}(f - P)Q\,dx \right| \le K \int_{\{f=P\}} |Q|\,dx,$$

with $K \ge 0$ and for every $Q \in \Pi^m$. Then

$$\int_B \varphi(|P|)|Q|\,dx \le 5\Lambda_\varphi \int_B \varphi(|f|)|Q|\,dx + K\Lambda_\varphi \int_{\{f=P\}} |Q|\,dx,$$

for every $Q \in \Pi^m$ satisfying $\text{sgn}(Q(t)P(t)) = (-1)^\eta$ at any $t \in B$ such that $Q(t)P(t) \ne 0$ and where $\eta = 0$ or $\eta = 1$.

Corollary 5. *Let $\varphi \in \Im$ and $f \in L^\Phi(B)$.*
If $\varphi(0^+) > 0$ and P is a best polynomial approximation of $f \in L^\Phi(B)$, then

$$\int_B \varphi(|P|)|P|\,dx \le \tilde{K}\|P\|_\infty.$$

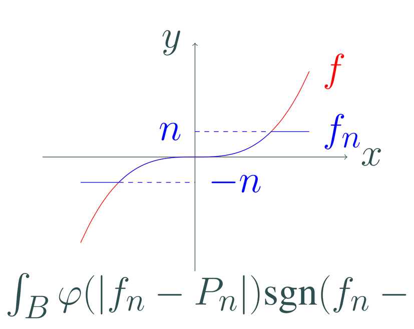
Extension of the best polynomial approximation operator

Prior knowledge

[3]: Extension of the b.p.a.o. from $L^p(B)$ to $L^{p-1}(B)$ for $p > 1$.
[4]: Extension of the b.p.a.o. from $L^\Phi(B)$ to $L^\varphi(B)$ where Φ is an N-function, i.e. $\varphi(x) \to 0$ as $x \to 0$ and $\varphi(x) \to \infty$ as $x \to \infty$.

Extension technique

Let $f \in L^\varphi(B) \Longrightarrow$
$f_n = \min(\max(f, -n), n) \in L^\Phi(B) \subset L^\varphi(B)$.



Now, $\exists \{P_n\} \subset \Pi^m : P_n \in \mu_\Phi(f_n)\ \forall n \in \mathbb{N} \Longleftrightarrow 0 = \int_B \varphi(|f_n - P_n|)\text{sgn}(f_n - P_n)Q\,dx\ \forall n \in \mathbb{N}$.
Then

- Uniformly boundedness: $\exists M > 0 : \|P_n\|_\infty \le M\ \forall n \in \mathbb{N} \Rightarrow \exists \{P_{n_k}\} \subseteq \{P_n\}$ and $P \in \Pi^m$ such that $P_{n_k} \xrightarrow{u} P$ on B. Essential $\varphi(x) \to \infty$ as $x \to \infty$.
- Convergence: $\int_B \varphi(|f_{n_k} - P_{n_k}|)\text{sgn}(f_{n_k} - P_{n_k})Q\,dx \to \int_B \varphi(|f - P|)\text{sgn}(f - P)Q\,dx = 0 \Longrightarrow P$ is b.p.a.o extended for $f \in L^\varphi(B)$. Essential $\varphi(x) \to 0$ as $x \to 0$.

Objective

We want to extend the b.p.a.o from $L^\Phi(B)$ to $L^\varphi(B)$ where Φ is not an N-function, i.e., $\Phi' = \varphi$ does not satisfy $\varphi(x) \to 0$ as $x \to 0$ or $\varphi(x) \to \infty$ as $x \to \infty$.

Result

Theorem 6. *Let $\varphi \in \Im$. If $f \in L^\varphi(B)$, then there exists $P \in \Pi^m$ such that*

$$\left| \int_B \varphi(|f - P|)\text{sgn}(f - P)Q\,dx \right| \le \varphi(0^+) \int_{\{f=P\}} |Q|\,dx,\ \forall Q \in \Pi^m.$$

And

$$\int_B \Phi(|P|)\,dx \le C\|P\|_\infty \left(\int_B \varphi(|f|)\,dx + 1 \right),$$

for a suitable constant C.

Definition 2. *Let $\varphi \in \Im$.*
For $f \in L^\varphi(B)$ we denote by $\mu_\varphi(f)$ the set of polynomials $P \in \Pi^m$ that satisfy

$$\left| \int_B \varphi(|f - P|)\text{sgn}(f - P)Q\,dx \right| \le \varphi(0^+) \int_{\{f=P\}} |Q|\,dx,\ \forall Q \in \Pi^m. \qquad (3)$$

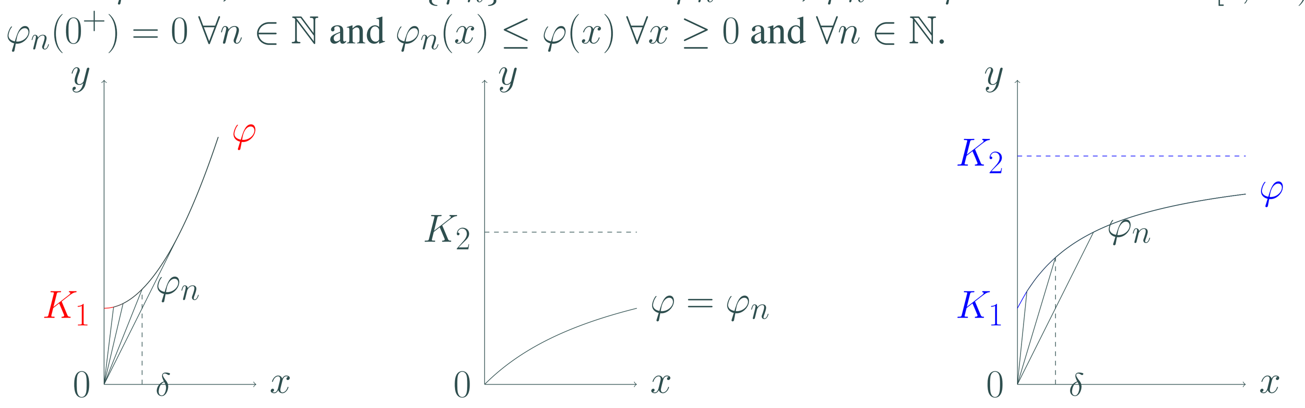
We refer to $\mu_\varphi(f)$ as the multivalued extended best polinomial approximation operator (e.b.p.a.o).

Remark 7. We get an extension of $\mu_\Phi(f)$ without requesting the function φ to satisfy $\varphi(0^+) = 0$ or $\varphi(x) \to \infty$ as $x \to \infty$

Outline of the proof

Uniformly boundedness

Given $\varphi \in \Im$, we choose $\{\varphi_n\}$ such that $\varphi_n \in \Im$, $\varphi_n \xrightarrow{u} \varphi$ as $n \to \infty$ on $[\delta, \infty)$ for every $\delta > 0$, $\varphi_n(0^+) = 0\ \forall n \in \mathbb{N}$ and $\varphi_n(x) \le \varphi(x)\ \forall x \ge 0$ and $\forall n \in \mathbb{N}$.



Let $f_n \in L^{\Phi_n}(B)$ for every $n \in \mathbb{N}$ where $\Phi_n(x) = \int_0^x \varphi_n(t)\,dt$.

Lemma 8. *Let $\varphi \in \Im$ be an upper unbounded function such that $\varphi(0^+) > 0$. If the sequence Λ_{φ_n} is bounded and there exists $C > 0$ that satisfies $\int_B \varphi_n(|f_n|)\,dx \le C$, then $\{\|P_n\|_\infty : P_n \in \mu_{\Phi_n}(f_n), n = 1, 2, \ldots\}$ is bounded.*

Lemma 9. *Let $\varphi \in \Im$ be an upper bounded function such that $\varphi(0^+) \ge 0$. If $P_n \in \mu_{\Phi_n}(f_n)$ for each $n \in \mathbb{N}$, then $\{P_n\}$ is uniformly bounded.*

Convergence+Boundedness

$\exists \{P_{n_k}\} \subset \{P_n\}$ and $\exists P \in \Pi^m$ such that $P_{n_k} \xrightarrow[k\to\infty]{u} P$ on B and $P_{n_k} \in \mu_{\Phi_n}(f_{n_k})$. Then

$$\left| \int_B \varphi(|f - P|)\text{sgn}(f - P)Q\,dx \right| =$$

$$\left| \int_B \varphi(|f - P|)\text{sgn}(f - P)Q\,dx - \int_B \varphi_{n_k}(|f_{n_k} - P_{n_k}|)\text{sgn}(f_{n_k} - P_{n_k})Q\,dx \right| \underset{0 \le \varphi_{n_k} \le \varphi}{\le}$$

$$\left| \int_{B-\{f=P\}} \varphi(|f - P|)\text{sgn}(f - P)Q\,dx - \int_{B-\{f=P\}} \varphi_{n_k}(|f_{n_k} - P_{n_k}|)\text{sgn}(f_{n_k} - P_{n_k})Q\,dx \right| +$$

$$\int_{\{f=P\}} \varphi(|f_{n_k} - P_{n_k}|)|Q|\,dx \underset{\text{L.D.C.T.}}{<}$$

$$2\varepsilon + \varphi(0^+) \int_{\{f=P\}} |Q|\,dx,\ \ \forall Q \in \Pi^m \text{ and } \forall \varepsilon > 0.$$

Final remarks

- If $\varphi(0^+) = 0$, uniqueness of the e.b.p.a.o can be obtained working as in [4].
- If $\varphi(x) \equiv 1$ on $[0, \infty)$, (3) gives the extension of the b.p.a.o. from $L^1(B)$ to $L^0(B)$, being $L^0(B)$ the set of all measurable functions. This problem was studied in [3] for functions belonging to a proper subset of $L^0(B)$.
- We obtain $\mu_\varphi(f) \ne \emptyset$ in a different way to that to that developed in [3].

Forthcoming Research

We are exploring the possibility of extending the b.p.a.o. using norms in Orlicz spaces instead of the modular.

References

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