Maximal operators associated with certain geometric configurations.

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A problem recently studied in [1] is the relation between sizes of sets $B, S \subset \mathbb{R}^2$ when B contains the boundary (or the vertices) of a square with center in every point of S and sides parallel to the axis. The n-dimensional case, when $B, S \subset \mathbb{R}^n$ and B contains the k-skeleton of an n-dimensional cube with center in every point of S was studied in [2]. In this work we study the maximal operator associated with this type of problems.

k -Skeleton of an n-cube

- An n-cube will always mean an n-dimensional cube with all sides parallel to the axes, unless otherwise specified. That is, an n-cube is a set of the form $x + \prod_{i=1}^{n}[a,b]$ for some $x, a < b \in \mathbb{R}^n$.
- The expression $\binom{[n]}{k}$ stands for k-element subsets of $\{1,\dots,n\}$. For $x \in \mathbb{R}^n$ and $I \in \binom{[n]}{k}$, x_I is the vector in $\mathbb{R}^k$ formed by taking the entries of x indexed by I. The **k-skeleton** of an n-cube $x + [a,b]^n$ is the set $x + \bigcup_{I \in \binom{[n]}{k}} \prod_{i=1}^{n} A_{I,i}$ where $A_{I,i} = [a,b]$ if $i \in I$ and $\{a,b\}$ otherwise.

Some results about dimension

If $0 \le k < n$ and $B \subset \mathbb{R}^n$ contains a k-skeleton of an n-cube centered at every point $S \subset \mathbb{R}^n$ of dimension s (for some dimension) then the best lower bound for the dimension (for the same dimension) of B is shown in the following table (see [1],[2],[4]). The second and third column refers to the 2-dimensional case and the last column to the n-dimensional case.

Dimension	**Vertices (n=2,k=0)**	**Boundary (n=2,k=1)**	**k-Skeleton of an n-cube**
$\dim_P$	$\frac{3}{4}s$	$1+\frac{3}{8}s$	$k + \frac{(n-k)(2n-1)}{2n^2}s$
$\overline{\dim}_B$	$\frac{3}{4}s$	$\max\left\{1, \frac{7}{8}s\right\}$	$\max\left\{k, \left(1-\frac{n-k}{2n^2}\right)s\right\}$
$\underline{\dim}_B$	$\frac{3}{4}s$	$\max\left\{1, \frac{7}{8}s\right\}$	$\max\left\{k, \left(1-\frac{n-k}{2n^2}\right)s\right\}$
$\dim_H$	$\max\{0, s-1\}$	1	$\max\{k, s-1\}$

k -Skeleton maximal function

Notation

- We denote with $S_k(x,r)$ the k-skeleton of the n-cube with center x and side length $2r$.
  $S_k^j(x,r)$, $j = 1, \dots, \binom{n}{k}2^{n-k}$, are the k-type elements of $S_k(x,r)$. By k-type elements we mean, for example vertices in case $k=0$, edges in case $k=1$, faces in case $k=2$, etc.
  In the next we denote $N = N(k,n) := \binom{n}{k}2^{n-k}$.
- If $0 < \delta < 1$, $S_{k,\delta}(x,r) := \{x' \in \mathbb{R}^n : d(S_k(x,r), x') < \delta\}$ is a δ-neighborhood of $S_k(x,r)$, with d the distance induced by the infinity norm.
  If $j = 1, \dots, \binom{n}{k}2^{n-k}$, $S_{k,\delta}^j(x,r)$ denote the δ-neighborhood of the k-type elements of $S_k(x,r)$.

Definition. *The* **k-skeleton maximal function** *with width δ of $f \in L^1_{loc}(\mathbb{R}^n)$ is the function*

$$M_\delta^k f : \mathbb{R}^n \to [0,\infty)$$

$$M_\delta^k f(x) = \sup_{1\le r\le 2} \min_{j=1}^{N} \frac{1}{\mathcal{L}(S_{k,\delta}^j(x,r))} \int_{S_{k,\delta}^j(x,r)} |f|\, d\mathcal{L}. \tag{1}$$

This operator is not sub-linear and this is the first difference with classical related problems. However, is better taking into consideration every k-type element instead of the whole k-skeleton to avoid trivial and unnatural results.

Our purpose is to study the behavior of (1) when δ tends to 0. Easily, we have the trivial proposition:

Proposition. *For all $f \in L^1_{loc}(\mathbb{R}^n)$,*

1. $\left\|M_\delta^k f\right\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^\infty(\mathbb{R}^n)}$.

2. $\left\|M_\delta^k f\right\|_{L^\infty(\mathbb{R}^n)} \le 2^{-k}\delta^{k-n}\|f\|_{L^1(\mathbb{R}^n)}$.

A negative result

Proposition. *If $p < \infty$, there can be no inequality*

$$\left\|M_\delta^k f\right\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all} \quad 0<\delta<1, f \in L^p(\mathbb{R}^n),$$

with C independent of δ. Even more, $\delta^{(k-n)/2np} \lesssim \left\|M_\delta^k\right\|$ (*).

To prove this we apply the k-skeleton maximal function over particular functions. In this case, we take f as the indicator of a compact set $B \subset \mathbb{R}^n$ that contains the k-skeleton of an n-cube with center in every point of $[0,1]^n$ and $\dim_B B = k + \frac{(n-k)(2n-1)}{2n}$. This set was constructed in [1], for the case $n=2$, and in [2] for the case $n \ge 3$.

Discretization and linearization

For each $z \in \mathbb{Z}^n$, Q_z denote the half-open n-cube with bottom left vertex z and side length 1. If $0 < \delta < 1$, $Q_z^* := Q_z \cap \delta\mathbb{Z}^n$.

Consider the following functions,

$$\psi : Q_z \to Q_z^*$$
$$\rho : Q_z^* \to [1,2] \cap \delta\mathbb{Z}.$$

If $x \in Q_z$, $\psi(x)$ assigns the upper right vertex of the half-open n-cube with vertices in Q_z^* and side length δ containing x. Given $y \in Q_z^*$, $\rho(y)$ determine the side length to the k-skeleton $S_k(y, \rho(y))$.

Definition. *Fix $z \in \mathbb{Z}^n$. Given a function ρ and $0 < \delta < 1$, if $f \in L^1_{loc}(\mathbb{R}^n)$ we define the* ρ, k**-skeleton maximal function** *with width δ,*

$$\tilde{M}_{\rho,\delta}^k f : Q_z \to [0,\infty)$$

$$\tilde{M}_{\rho,\delta}^k f(x) = \frac{1}{\mathcal{L}(l_{x,\delta})}\int_{l_{x,\delta}} |f|\, d\mathcal{L},$$

where l_x is a k-type element of $S_k(\psi(x), \rho(\psi(x)))$ and $l_{x,\delta}$ its respective δ-neighborhood.

Remarks

- By definition is trivial that $\tilde{M}_{\rho,\delta}^k$ is a linear operator.
- There is an appropriate way to choose, for each $x \in Q_z$, the k-type element l_x.
- If $0 < \delta < 1$,
$$\left\|M_\delta^k\right\|_{L^p(9Q_z)\to L^q(Q_z)} \le C \sup_{\rho \ne 0}\left\|\tilde{M}_{\rho,4\delta}^k\right\|_{L^p(9Q_z)\to L^q(Q_z)}, \tag{2}$$
  where $C = C(k,n)$ is a constant.
- To bound this linear operator we use an argument of duality an some ideas from [3, Chapter 22] related with Kakeya maximal function.

Results

Theorem. *For all $0 < \delta < 1$ and $f \in L^2(9Q_z)$,*

$$\left\|\tilde{M}_{\rho,4\delta}^k f\right\|_{L^2(Q_z)} \le C(k,n)\delta^{\frac{k-n}{4n}}\|f\|_{L^2(9Q_z)}.$$

Theorem. *For all $0 < \delta < 1$, $m \in \mathbb{N}$, $p_m = \frac{2m}{2m-1}$ and $f \in L^{p_m}(9Q_z)$,*

$$\left\|\tilde{M}_{\rho,4\delta}^k f\right\|_{L^{p_m}(Q_z)} \le C(k,m,n)\delta^{\frac{k-n}{2np_m}}\|f\|_{L^{p_m}(9Q_z)}.$$

Corollary. *For all $0 < \delta < 1$ and $f \in L^1(9Q_z)$,*

$$\left\|\tilde{M}_{\rho,4\delta}^k f\right\|_{L^1(Q_z)} \le C(k,n)\delta^{\frac{k-n}{2n}}\|f\|_{L^1(9Q_z)}.$$

Using interpolation, we obtain:

$$\left\|\tilde{M}_{\rho,4\delta}^k f\right\|_{L^p(Q_z)} \le \tilde{C}\delta^{\frac{k-n}{2np}}\|f\|_{L^p},$$

with $1 \le p < \infty$ and $\tilde{C} = \tilde{C}(k,n,p)$.

Whit this bounds for the linear operator and since (2) holds, we have

$$\left\|M_\delta^k f\right\|_{L^p(Q_z)} \lesssim_{k,n,p} \delta^{\frac{k-n}{2np}}\|f\|_{L^p(9Q_z)} \quad \text{for all} \quad f \in L^p(9Q_z), \quad 1 \le p < \infty.$$

This result is not depending on the n-cube Q_z selected, so we can extend the result over $\mathbb{R}^n$.

Theorem. *For all $0 < \delta < 1$ and $1 \le p < \infty$,*

$$\left\|M_\delta^k f\right\|_{L^p(\mathbb{R}^n)} \lesssim_{k,n,p} \delta^{\frac{k-n}{2np}}\|f\|_{L^p(\mathbb{R}^n)} \quad \textit{for all} \quad f \in L^p(\mathbb{R}^n). \tag{3}$$

Remarks

- From (*) and (3) we have that the bounds found for $\left\|M_\delta^k\right\|$ are sharp, except for a constant.
- Using the bounds for the k-skeleton maximal function, we recover some results obtained in [2] related with the box-counting dimension of a set B containing a k-skeleton of an n-cube centered at every point $S \subset \mathbb{R}^n$, with $\mathcal{L}(S) > 0$.

References

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