Fourier decay of self-similar measures

Carolina Mosquera [1] - Pablo Shmerkin [2]

[1] Universidad de Buenos Aires, IMAS - CONICET

[2] Universidad Torcuato Di Tella, CONICET

Given a finite Borel measure on $\mathbb{R}$, its **Fourier transform** is defined as

$$\widehat{\mu}(\xi) := \int e^{2\pi i x\xi}\, d\mu(x).$$

The decay properties of $\widehat{\mu}(\xi)$ as $|\xi| \to +\infty$ give crucial information about μ.

We say that $\widehat{\mu}(\xi)$ has **polynomial decay** if there exist $C_\sigma, \sigma > 0$ such that

$$|\widehat{\mu}(\xi)| \leq C_\sigma |\xi|^{-\sigma/2}.$$

The supreme of these σ is called **Fourier dimension** of the measure μ : $\dim_F(\mu)$.
Then

$$\dim_F(\mu) > 0 \text{ if and only if } \widehat{\mu} \text{ has polynomial decay.}$$

Self-similar measures

Given a finite set of contracting similarity maps of $\mathbb{R}^d$ $S_1, \dots, S_m$, and weights $p_1, \dots, p_m$ such that $p_1 + \dots + p_m = 1$, there exists a unique probability Borel measure μ on $\mathbb{R}^d$ such that

$$\mu(A) = \sum_{i=1}^{m} p_i \mu(S_i^{-1}A), \quad \text{for all Borel sets } A \subseteq \mathbb{R}^d.$$

$(S_1, \dots, S_m), (p_1, \dots, p_m)$ is an **iterated function system of similarities with weights or IFSw** and μ is the **invariant measure or attractor of the IFSw.**

The simplest class of self-similar measures is

Bernoulli convolutions: $d = 1, m = 2, \lambda \in (0, 1)$ and

$$S_1(x) = \lambda x - 1,\ S_2(x) = \lambda x + 1,\ p_1 = p_2 = 1/2.$$

The attractor of this IFSw is called **Bernoulli convolution** μ_λ. The special case $\lambda = 1/3$ yields the Cantor-Lebesgue measure.

Goal

Give **explicit** bounds for the polynomial decay of the Fourier transform of self-similar measures outside a small set of exceptions.

Previous work

- Erdős [3, 4], for Bernoulli convolutions, proved that $\dim_F(\mu_\lambda) > 0$ for almost all λ but there is an infinite numerable set of λ's such that $\widehat{\mu}_\lambda(\xi)$ does not even tends to zero when $|\xi| \to +\infty$.
- Kahane [5] used the Erdős argument to show that $\dim_F(\mu_\lambda) > 0$ for all λ outside of zero Hausdorff-dimensional set of exceptions.
- In [1, 2], for d=1, the authors showed that certain Bernoulli convolutions associated to algebraic numbers have at least logarithmic decay.
- Kaufman [6] proved that if F is any C^2 diffeomorphism of $\mathbb{R}$ such that $F'' > 0$ then $\dim_F(F_\mu) > 0$ where $F_\mu(A) := \mu(F^{-1}A)$ for all Borel sets $A \subset \mathbb{R}$. He proved his result for Bernoulli convolutions with $\lambda \in (0, 1/2)$.
- The study of the Fourier decay of self-similar measures has become relevant since it is a key component of a method developed in [7, 8] to show that certain self-similar measures are absolutely continuous.

Results

Let $\mu_{a,t}^p$ be the self-similar measure for the IFSw $\{ax + t_i\}_{i=1}^m$ with weights $p = (p_1, \dots, p_m)$, where $t = (t_1, \dots, t_m)$, $a \in (0, 1)$. In this case

$$\widehat{\mu}_{a,t}^p(u) = \prod_{n=1}^{\infty} \Phi(a^n u),$$

where $\Phi(u) = \Phi_{p,t}(u) = \sum_{j=1}^m p_j \exp(2\pi i t_j u)$.

Lemma. *The following holds for all $y \in \mathbb{R}$ and $c \in (0, 1)$: if $d(y, \mathbb{Z}) > \frac{c}{2}$, then $|\Phi(y)| < 1 - \eta(c, p)$, where*

$$\eta(c, p) = p_1 + p_2 - \sqrt{p_1^2 + 2p_1p_2\cos(\pi c) + p_2^2}.$$

In the special case of Bernoulli convolutions, $\Phi(u) = \cos(2\pi u)$ and $\eta(c, p) = 1 - \cos(\pi c)$.

Following Kaufman [6], we use the Erdös-Kahane argument to establish quantitative power decay outside of a sparse set of frequencies:

Proposition. $(\star)$ *Given $a \in (0, 1)$ and a probability vector $p = (p_1, \dots, p_m)$ there is a constant $C = C_a > 0$ such that the following holds: for each $\varepsilon > 0$ small enough (depending continuously on a) the following holds for all T large enough: the set of frequencies $u \in [-T, T]$ such that $|\widehat{\mu}_{a,t}^p(u)| \geq T^{-\varepsilon}$ can be covered by $C_a T^\delta$ intervals of length 1, where $C_a > 0$ depends only on a,*

$$\delta = \frac{\log\left(\lceil 1 + 1/a \rceil\right)\tilde{\varepsilon} + h(\tilde{\varepsilon})}{\log(1/a)},$$

$$\tilde{\varepsilon} = \frac{\log(a)}{\log(1 - \eta(\frac{a}{a+1}, p))}\varepsilon,$$

and $h(\tilde{\varepsilon}) = -\tilde{\varepsilon}\log(\tilde{\varepsilon}) - (1 - \tilde{\varepsilon})\log(1 - \tilde{\varepsilon})$ is the entropy function.

In [9] Tsujii proved this result but did not give any explicit estimates.

Using the above proposition we obtain a generalization of the result of Kaufman [6]:

Theorem. *Let $F \in C^2(\mathbb{R})$ such that $F'' > 0$ and let $\mu = \mu_{a,t}^p$ be a (homogeneous) self-similar measure on $\mathbb{R}$ which is not a single atom. Then there exist $\sigma = \sigma(\mu) > 0$ (independent of F) and $C = C(F, \mu) > 0$ such that*

$$|\widehat{F\mu}(u)| \leq C|u|^{-\sigma}.$$

We underline that the value of σ is effective.

As an example, we obtain that if μ is the Cantor-Lebesgue measure on the middle-thirds Cantor set, then even though $\widehat{\mu}(u)$ does not decay as $u \to \infty$, for $\widehat{F\mu}$ we have a uniform explicit decay:

Corollary. *Let μ be the Cantor-Lebesgue measure. Then for every C^2 function $F : \mathbb{R} \to \mathbb{R}$ such that $F'' > 0$ there exists a constant $C_F > 0$ such that*

$$|\widehat{F\mu}(u)| \leq C_F |u|^{-\sigma}.$$

Using the above results we can obtain estimates for the dimension of Bernoulli convolutions.

L^2 dimension of convolutions

Definition. *Let $q \in (1, +\infty)$. The* $\mathbf{L^q}$ **dimension** *of the measure μ is defined as*

$$\dim_q(\mu) := \lim_{r \to 0} \frac{\log \int \mu([x - r, x + r])^q\, dx}{(q - 1)\log(r)} - \frac{1}{q - 1}.$$

For $q = \infty$, we define

$$\dim_\infty(\mu) := \lim_{r \to 0} \frac{\log \sup_x \mu([x - r, x + r])}{\log r}.$$

The function $q \mapsto \dim_q(\mu)$ is non-increasing for all probability measures.

Theorem. *Let $\mu = \mu_{a,p}^t$ be as above. Given any $\kappa > 0$, there is $\sigma = \sigma(a, p, \kappa) > 0$ such that the following holds: let ν be any Borel probability measure with $\underline{\dim}_2(\nu) \leq 1 - \kappa$. Then*

$$\underline{\dim}_2(\mu * \nu) > \underline{\dim}_2(\nu) + \sigma.$$

More precisely, one can take $\sigma = 2\varepsilon$, where $\varepsilon = \varepsilon(a, p, \kappa)$ is such that the value of $\delta = \delta(\varepsilon, a, p)$ given in Proposition $(\star)$ satisfies

$$\kappa - 2\varepsilon = \delta.$$

Dimension of Bernoulli convolutions

Theorem. *Let $\mu_{\lambda,p}$ be the biased Bernoulli convolution of parameter $\lambda \in (0, 1)$ and weight $p \in (0, 1)$. Then, for every $p_0 \in (0, 1/2)$ there is $C = C(p_0) > 0$ such that*

$$\inf_{p \in [p_0, 1 - p_0]} \dim_2(\mu_{\lambda,p}) \geq 1 - C(1 - \lambda)\log(1/(1 - \lambda)).$$

We present two corollaries. The first is a corollary of the proof rather than the statement. For the case of unbiased Bernoulli convolutions we are able to obtain an improved lower bound:

Corollary. *There is an absolute constant $C > 0$ such that*

$$\dim_2(\mu_\lambda) \geq 1 - C(1 - \lambda)^2 \log(1/(1 - \lambda)).$$

For $\dim_\infty(\mu_{\lambda,p})$ we obtain the same lower bound as in the above theorem and we can conclude that

Corollary.

$$\lim_{\lambda \uparrow 1} \dim_\infty(\mu_{\lambda,p}) = 1$$

with a quantitative rate.

Future work

- Consider the attractor $\mu_{\lambda,O}$ of the IFS $(\lambda O x - I, \lambda O x + I)$ with weights $(1/2, 1/2)$, where $\lambda \in (0, 1)$, O is an orthogonal map on $\mathbb{R}^d$ and I is the identity (that is, a generalization of Bernoulli convolutions for dimension $d > 1$) and study the Fourier decay of $\mu_{\lambda,O}$.
- Given $\lambda_1, \lambda_2 \in (0, 1)$, consider the attractor $\mu_{\lambda_1,\lambda_2}$ of the IFS $(\lambda_1 x, \lambda_2 x + 1)$ with weights $(1/2, 1/2)$ (that is, a non-homogeneous version of Bernoulli convolution) and study the Fourier decay of $\mu_{\lambda_1,\lambda_2}$.

References

[1] A. Bufetov and B. Solomyak. On the modulus of continuity for spectral measures in substitution dynamics. *Adv. Math.*, 260:84–129, 2014.

[2] X. Dai. When does a Bernoulli convolution admit a spectrum? *Adv. Math.*, 231(3–4):1681–1693, 2012.

[3] P. Erdős. On a family of symmetric Bernoulli convolutions. *Amer. J. Math.*, 61:974–976, 1939.

[4] P. Erdős. On the smoothness properties of a family of Bernoulli convolutions. *Amer. J. Math.*, 62:180–186, 1940.

[5] J.P. Kahane. Sur la distribution de certaines séries aléatoires. *Bull. Soc. Math. France*, 25:119–122, 1971.

[6] R. Kaufman. On Bernoulli convolutions. *Contemp. Math.*, 26:217–222, 1984.

[7] P. Shmerkin. On the Exceptional Set for Absolute Continuity of Bernoulli Convolutions. *Geom. Funct. Anal.*, 24(3):946–958, 2014.

[8] P. Shmerkin. On Furstenberg's intersection conjecture, self-similar measures, and the L^q norms of convolutions. Preprint, arXiv:1609.07802, 2016.

[9] M. Tsujii. On the Fourier transforms of self-similar measures. *Dynamical Systems: An International Journal 30*, 4:468–484, 2015.

1.mosquera@dm.uba.ar