Riesz transforms on variable Lebesgue spaces with Gaussian measure

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Ornstein–Uhlenbeck semigroup

Let $\mathcal{L}$ be the **Ornstein-Uhlenbeck differential operator**

$$\mathcal{L} = -\frac{1}{2}\Delta + x \cdot \nabla,$$

which is the infinitesimal generator of the diffusion semigroup given by

$$e^{-t\mathcal{L}} f(x) = \pi^{-n/2} \int_{\mathbb{R}^n} \frac{e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}}{(1-e^{-2t})^{n/2}} f(y)dy, \; f \in L^2(\mathbb{R}^n, d\gamma),$$

where $d\gamma(x) = e^{-|x|^2}dx$ is the Gaussian measure, that makes $\mathcal{L}$ to be self-adjoint.
For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, the eigenfunctions of $\mathcal{L}$ are the **Hermite polynomials** H_α of degree $|\alpha| = \alpha_1 + \cdots + \alpha_n$, corresponding to the eigenvalue $\lambda = |\alpha|$, which are defined by

$$H_\alpha(x_1, \ldots, x_n) = H_{\alpha_1}(x_1) \ldots H_{\alpha_n}(x_n)$$

where H_{α_i} are the one-dimensional Hermite polynomials given by

$$H_0(x) = 1, \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \; n \geq 1.$$

Gaussian Riesz transforms

Given $1 \leq j \leq n$ and H_α an n-dimensional Hermite polynomial, the j-th **Gaussian Riesz transform of first order** verifies

$$\mathcal{R}_j(H_\alpha)(x) = -\frac{1}{|\alpha|}\frac{\partial}{\partial x_j} H_\alpha(x).$$

More generally, for a given multi-index α, and each multi-index β, the n-dimensional **Gaussian Riesz transforms of order** α verify

$$\mathcal{R}_\alpha(H_\beta)(x) = \frac{(-1)^{|\alpha|}}{|\beta|^{|\alpha|/2}} D^\alpha H_\beta(x) = \frac{(-1)^{|\alpha|}}{|\beta|^{|\alpha|/2}} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} H_\beta(x),$$

and can be written as a principal value with kernel

$$k_\alpha(x,y) = c_n \int_0^1 r^{|\alpha|-1} \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\alpha|}{2}-1} H_\alpha\left(\frac{y-rx}{(1-r^2)^{\frac{1}{2}}}\right) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}} dr.$$

A larger class of singular integrals

We consider $T_F f(x) = p.v. \int_{\mathbb{R}^n} K_F(x,y) f(y)dy$, where

$$K_F(x,y) = c_n \int_0^1 r^{m-1} \left(\frac{-\log r}{1-r^2}\right)^{\frac{m}{2}-1} F\left(\frac{y-rx}{(1-r^2)^{\frac{1}{2}}}\right) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}} dr,$$

for $m \in \mathbb{N}$, with $F \in C^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} F(z) e^{-|z|^2} dz = 0$, and for every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|F(z)| \leq C_\epsilon e^{\epsilon|z|^2}, \qquad |\nabla F(z)| \leq C_\epsilon e^{\epsilon|z|^2}.$$

When $F = H_\alpha$ and $m = |\alpha|$, clearly $T_F = \mathcal{R}_\alpha$.

Variable Lebesgue spaces

Let $p : \mathbb{R}^n \to [1, \infty)$ be a γ-measurable function with

$$1 \leq p^- = \operatorname*{ess\,inf}_{\mathbb{R}^n} p \leq \operatorname*{ess\,sup}_{\mathbb{R}^n} p = p^+ < \infty.$$

The **variable Lebesgue space** associated with p is defined by

$$L^{p(\cdot)}(\mathbb{R}^n, d\gamma) := \left\{ f : \int_{\mathbb{R}^n} (|f(x)|/\lambda)^{p(x)} \, d\gamma(x) < \infty \text{ for some } \lambda > 0 \right\}.$$

A norm for this space is the Luxemburg norm

$$||f||_{p(\cdot),\gamma} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} (|f(x)|/\lambda)^{p(x)} \, d\gamma(x) \leq 1 \right\}.$$

Log-Hölder conditions

The classical conditions required on the exponent p are the **log-Hölder conditions**, the local condition and the decay condition, respectively:

$$p \in LH_0(\mathbb{R}^n) : |p(x) - p(y)| \leq \frac{C_0}{\log(e + 1/|x-y|)} \quad \forall x, y \in \mathbb{R}^n,$$

$$p \in LH_\infty(\mathbb{R}^n) : \exists \, p_\infty \geq 1 / \quad |p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)} \quad \forall x \in \mathbb{R}^n.$$

New decay condition

We require for a stronger decay condition than $LH_\infty(\mathbb{R}^n)$:

$$p \in P_\gamma^\infty(\mathbb{R}^n) : \quad \exists \, p_\infty \geq 1 / \quad |p(x) - p_\infty| \leq \frac{C_\gamma}{|x|^2}, \quad \forall x \in \mathbb{R}^n \setminus \{(0, \ldots, 0)\}.$$

If $1 < p^- \leq p^+ < \infty$, $p \in P_\gamma^\infty(\mathbb{R}^n)$ is equivalent to

$$e^{-p(x)|x|^2} \approx e^{-p_\infty|x|^2} \quad \text{and} \quad e^{-p'(x)|x|^2} \approx e^{-p'_\infty|x|^2}, \tag{1}$$

where $p'(x) = p(x)/(p(x)-1)$ and $p'_\infty = p_\infty/(p_\infty - 1)$.

Main Theorem

Theorem 1. *Let $p \in LH_0(\mathbb{R}^n) \cap \mathcal{P}_\gamma^\infty(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. Then, there exists $C > 0$ such that*

$$\|T_F f\|_{p(\cdot),\gamma} \leq C\|f\|_{p(\cdot),\gamma}, \qquad \forall f \in L^{p(\cdot)}(\mathbb{R}^n, d\gamma).$$

Sketch of the proof

We split T_F into two parts, relative to the hyperbolic balls

$$B(x) = \{y \in \mathbb{R}^n : |x-y| \leq n(1 \wedge 1/|x|)\}, \quad x \in \mathbb{R}^n.$$

Local part:

- We use a covering lemma that decomposes $\mathbb{R}^n$ into balls of hyperbolic type with bounded overlap where, on each of them, γ behaves like a constant;
- T_F over $B(x)$ can be controlled by a Calderón-Zygmund type operator and the Hardy-Littlewood maximal function, both localized on the balls of the covering;
- the boundedness is reduced to the Lebesgue measure case, where $p \in LH_0(\mathbb{R}^n) \cap LH_\infty(\mathbb{R}^n)$ is known to be sufficient for the continuity of the operators mentioned above. We reconstruct γ using (1).

Global part:

- We revisit the estimate of the kernel K_F given in [2]: for $y \in B^c(x)$, if $a = |x|^2 + |y|^2$ and $b = 2\langle x, y\rangle$,
  - i) when $b \leq 0$, for each $0 < \epsilon < 1$, $|K_F(x,y)| \leq C_\epsilon e^{-(1-\epsilon)|y|^2}$;
  - ii) when $b > 0$, for each $0 < \epsilon < 1/n$,

  $$|K_F(x,y)| \leq C_\epsilon e^{-(1-\epsilon)u_0} t_0^{-n/2},$$

  being $t_0 = 2\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}}$ and $u_0 = \frac{1}{2}\left(|y|^2 - |x|^2 + |x+y||x-y|\right)$;
- proceeding as in [2], the kernel $K_F(x,y)$ is controlled by a symmetric and integrable kernel $P(x,y)$, which allows us to use classical arguments for this kind of kernels;
- we apply several results of $L^{p(\cdot)}(\mathbb{R}^n)$ spaces (see [1]) in order to interchange variable and constant exponents, using condition $\mathcal{P}_\gamma^\infty(\mathbb{R}^n)$.

References

[1] Cruz-Uribe D. and Fiorenza A. *Variable Lebesgue spaces. Foundations and Harmonic Analysis.* Birkhäuser/Springer, Heidelberg, 2013.

[2] Pérez S. *The local part and the strong type for operators related to the Gaussian measure.* J. Geom. Anal. 11(3):491–507, 2001.