Course: Weighted inequalities and dyadic harmonic analysis

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Outline

- **Lecture 1.**
  Weighted Inequalities and Dyadic Harmonic Analysis.
  Model cases: Hilbert transform and Maximal function.
- **Lecture 2.**
  Brief Excursion into Spaces of Homogeneous Type.
  Simple Dyadic Operators and Weighted Inequalities à la Bellman.
- **Lecture 3.**
  Case Study: Commutators.
  Sparse Revolution.

Outline Lecture 3

1. Case study: Commutator $[H, b]$
   - Dyadic proof of quadratic estimate
   - Transference theorem
   - Coifman-Rochberg-Weiss argument
   - Recent Progress
2. Sparse operators and families of dyadic cubes
   - A_2 theorem for sparse operators
   - Sparse vs Carleson families
   - Domination by Sparse Operators
   - Case study: Sparse operators vs commutators
3. Acknowledgements

Case study: Commutator $[H, b]$

For $b \in BMO$, and H the Hilbert Transform, let

$$[b, H]f := b(Hf) - H(bf).$$

- The commutator is *bounded on* L^p for $1 < p < \infty$ if and only if $b \in BMO$ (Coifman, Rochberg, Weiss '76). Moreover

$$\|[H, b]f\|_p \leq C_p \|b\|_{BMO} \|f\|_p.$$

- Commutator is NOT of weak-type $(1, 1)$ (Pérez '96).
- *Commutator is more singular than H.*
- bH and Hb are NOT necessarily bounded on L^p when $b \in BMO$. *The commutator introduces some key cancellation.* This is very much connected to the celebrated H^1 - BMO duality by Feffferman, Stein '72.

Weighted Inequalities

Theorem (Bloom '85)

If $u, v \in A_2$ *then* $[b, H] : L^p(u) \to L^p(v)$ *is bounded if and only if* $b \in BMO_\mu$ *where* $\mu = u^{-1/p}v^{1/p}$ *and*

$$\|b\|_{BMO_\mu} := \sup_{I \in \mathbb{R}} \frac{1}{\mu(I)} \int_I |b(x) - \langle b \rangle_I| dx < \infty.$$

Theorem (Alvarez, Bagby, Kurtz, Pérez '93)

If $w \in A_p$ *then* $\|[T, b]f\|_{L^p(w)} \le C_p(w)\|b\|_{BMO}\|f\|_{L^p(w)}$.

Result valid for general linear operators T, and two-weight estimates.
Proof used classical Coifman-Rochberg-Weiss '76 argument.

Theorem (Daewon Chung '11)

$$\|[H, b]f\|_{L^2(w)} \le C\|b\|_{BMO}[w]_{A_2}^2\|f\|_{L^2(w)}.$$

Dyadic proof for commutator $[H, b]$

Theorem (Daewon Chung '11)

$$\|[H,b]f\|_{L^2(w)} \leq C\|b\|_{BMO}[w]_{A_2}^2\|f\|_{L^2(w)}.$$

Daewon's "dyadic" proof is based on:

(1) Use Petermichl's dyadic shift operator Ш instead of H, and prove uniform (on grids) quadratic estimates for its commutator $[Ш, b]$.

(2) Decomposition of the product bf in terms of paraproducts

$$bf = \pi_b f + \pi_b^* f + \pi_f b$$

the first two terms are bounded in $L^p(w)$ when $b \in BMO$ and $w \in A_p$, the enemy is the third term. Decomposing commutator

$$[Ш, b]f = [Ш, \pi_b]f + [Ш, \pi_b^*]f + \big[Ш(\pi_f b) - \pi_{Ш f}(b)\big].$$

cont. "dyadic proof" commutator

(3) Linear bounds for paraproducts π_b, π_b^* (Bez '08) and Ш (Pet '07) gives quadratic bounds for first two terms.

$$[Ш, b]f = [Ш, \pi_b]f + [Ш, \pi_b^*]f + [Ш(\pi_f b) - \pi_{Ш f}(b)].$$

(4) Third term is better, it obeys a linear bound, and so do halves of the two commutators (using *Bellman function* techniques):

$$\|Ш(\pi_f b) - \pi_{Ш f}(b)\| + \|Ш\pi_b f\| + \|\pi_b^* Ш f\| \leq C\|b\|_{BMO}[w]_{A_2}\|f\|.$$

Providing uniform quadratic bounds for commutator $[Ш, b]$ hence

$$\|[H, b]\|_{L^2(w)} \leq C\|b\|_{BMO}[w]_{A_2}^2\|f\|_{L^2(w)}.$$

Bad guys non-local terms $\pi_b Ш$, $Ш\pi_b^*$. □

Estimate and extrapolated estimates are sharp! (Chung-P.-Pérez '12).

Afterthoughts

- A posteriori one realizes the pieces that obey linear bounds are generalized Haar Shift operators and hence their linear bounds can be deduced from general results for those operators ...
- As a byproduct of Chung's dyadic proof we get that Beznosova's extrapolated bounds for the paraproduct are optimal:

$$\|\pi_b f\|_{L^p(w)} \leq C_p [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(w)}$$

Proof: by contradiction, if not for some p then $[H, b]$ will have better bound in $L^p(w)$ than the known optimal quadratic bound.

Transference theorem

Theorem (Chung, P., Pérez '12, P. '13)

Given linear operator T and $1 < r < \infty$ if for all $w \in A_r$ there exists a $C_{T,d} > 0$ such that for all $f \in L^r(w)$,

$$\|Tf\|_{L^r(w)} \leq C_{T,d}[w]_{A_r}^{\alpha}\|f\|_{L^r(w)}.$$

then its commutator with $b \in BMO$ obeys the following bound

$$\|[T,b]f\|_{L^r(w)} \leq C_{r,T,d}[w]_{A_r}^{\alpha+\max\{1,\frac{1}{r-1}\}}\|b\|_{BMO}\|f\|_{L^r(w)}.$$

- Proof follows classical Coifman-Rochberg-Weiss '76 argument using (i) Cauchy integral formula; (ii) quantitative Coifman-Fefferman result: $w \in A_r$ implies $w \in RH_q$ with $q = 1 + c_d/[w]_{A_r}$ and $[w]_{RH_q} \leq 2$; (iii) quantitative version: $b \in BMO$ implies $e^{\alpha b} \in A_r$ for α small enough with control on $[e^{\alpha b}]_{A_r}$.

Higher-order-commutator $T_b^k = [b, T_b^{k-1}]$ (powers $\alpha + k\max\{1, \frac{1}{r-1}\}$, k).

A_2 Conjecture (Now Theorem)

Transference theorem for commutators are useless unless there are operators known to obey an initial $L^r(w)$ bound. Do they exist? Yes!

Theorem (Hytönen, Annals '12)

Let T be a Calderón-Zygmund operator, $w \in A_2$. Then there is a constant $C_{T,d} > 0$ such that for all $f \in L^2(w)$,

$$\|Tf\|_{L^2(w)} \leq C_{T,d}[w]_{A_2} \|f\|_{L^2(w)}.$$

We conclude that for all Calderón-Zygmund operators T their commutators obey a quadratic bound in $L^2(w)$.

$$\|[T,b]f\|_{L^2(w)} \leq C_{T,d}[w]_{A_2}^2 \|b\|_{BMO} \|f\|_{L^2(w)}.$$

$$\|[T_b^k f\|_{L^2(w)} \leq C_{T,d}[w]_{A_2}^{1+k} \|b\|_{BMO}^k \|f\|_{L^2(w)}.$$

Some generalizations

- Extensions to commutators with fractional integral operators, two-weight problem Cruz-Uribe, Moen '12
- Extensions using $[w]_{A_1}$, $A_1 \subset \cap_{p>1} A_p$, Ortiz-Caraballo '11 .
- Mixed A_2-A_∞, $A_\infty = \cup_{p>1} A_p$, $[w]_{A_\infty} \leq [w]_{A_2}$, Hytönen, Pérez '13

$$\|[T, b]\|_{L^2(w)} \leq C_n [w]_{A_2}^{\frac{1}{2}} \left([w]_{A_\infty} + [w^{-1}]_{A_\infty}\right)^{\frac{3}{2}} \|b\|_{BMO}.$$

  See also Ortiz-Caraballo, Pérez, Rela '13.
- Matrix valued operators and BMO, Isralowitch, Kwon, Pott '15
- Two weight setting (both weights in A_p, à la Bloom) Holmes, Lacey, Wick '16. Also for biparameter Journé operators Holmes, Petermichl, Wick '17.
- *Pointwise control by sparse operators adapted to commutator, improving weak-type, Orlicz bounds, and quantitative two weight Bloom bounds, Lerner, Ombrosi, Rivera-Ríos, arXiv '17.*

The Coifman-Rochberg-Weiss argument when $r = 2$

"Conjugate" operator as follows: for any $z \in \mathbb{C}$ define

$$T_z(f) = e^{zb}\, T\,(e^{-zb} f).$$

A computation + Cauchy integral theorem give (for "nice" functions),

$$[b, T](f) = \frac{d}{dz} T_z(f)|_{z=0} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{T_z(f)}{z^2}\, dz, \quad \epsilon > 0$$

Now, by Minkowski's inequality

$$\|[b, T](f)\|_{L^2(w)} \leq \frac{1}{2\pi\, \epsilon^2} \int_{|z|=\epsilon} \|T_z(f)\|_{L^2(w)} |dz|, \quad \epsilon > 0.$$

Key point is to find appropriate radius ϵ.
Look at inner norm and try to find bounds depending on z.

$$\|T_z(f)\|_{L^2(w)} = \|T(e^{-zb} f)\|_{L^2(w\, e^{2Re z\, b})}.$$

Use main hypothesis: $\|T\|_{L^2(v)} \leq C[v]_{A_2}$, for $v = w\, e^{2Re z\, b}$.
Must check that if $w \in A_2$ then $v \in A_2$ for $|z|$ small enough.

For $v = w\, e^{2Rez\, b}$. Must check that if $w \in A_2$ then $v \in A_2$ for small $|z|$.

$$[v]_{A_2} = \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\, e^{2Rez\, b(x)}\, dx \right) \left(\frac{1}{|Q|} \int_Q w^{-1}(x)\, e^{-2Rez\, b(x)}\, dx \right).$$

If $w \in A_2 \Rightarrow w \in RH_q$ for some $q > 1$ (Coifman, Fefferman '73).
Quantitative version: if $q = 1 + \frac{1}{2^{d+5}[w]_{A_2}}$ then

$$\left(\frac{1}{|Q|} \int_Q w^q(x)\, dx \right)^{\frac{1}{q}} \leq \frac{2}{|Q|} \int_Q w(x)\, dx,$$

and similarly for $w^{-1} \in A_2$ (since $[w]_{A_2} = [w^{-1}]_{A_2}$),

$$\left(\frac{1}{|Q|} \int_Q w^{-q}(x)\, dx \right)^{\frac{1}{q}} \leq \frac{2}{|Q|} \int_Q w^{-1}(x)\, dx\,.$$

In what follows q is as above.

Using these and Holder's inequality we have for an arbitrary Q

$$[v]_{A_2} = \left(\frac{1}{|Q|}\int_Q w(x)e^{2Rez\, b(x)}\, dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1}e^{-2Rez\, b(x)}\, dx\right)$$

$$\leq \left(\frac{1}{|Q|}\int_Q w^q\right)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q e^{2Rez\, q'\, b}\right)^{\frac{1}{q'}}\left(\frac{1}{|Q|}\int_Q w^{-q}\right)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q e^{-2Rez\, q'\, b}\right)^{\frac{1}{q'}}$$

$$\leq 4\left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-1}\right)\left(\frac{1}{|Q|}\int_Q e^{2Rez\, q'\, b}\right)^{\frac{1}{q'}}\left(\frac{1}{|Q|}\int_Q e^{-2Rez\, q'\, b}\right)^{\frac{1}{q'}}$$

$$\leq 4\,[w]_{A_2}\,[e^{2Rez\, q'\, b}]_{A_2}^{\frac{1}{q'}}$$

Now, since $b \in BMO$ there are $0 < \alpha_d < 1$ and $\beta_d > 1$ such that if $|2Rez\, q'| \leq \frac{\alpha_d}{\|b\|_{BMO}}$ then $[e^{2Rez\, q'\, b}]_{A_2} \leq \beta_d$. Hence for these z,

$$[v]_{A_2} = [w\, e^{2Rez\, b}]_{A_2} \leq 4\,[w]_{A_2}\,\beta_d^{\frac{1}{q'}} \leq 4\,[w]_{A_2}\,\beta_d.$$

If $|z| \leq \frac{\alpha_d}{2q'\|b\|_{BMO}}$ then $[v]_{A_2} \leq 4[w]_{A_2}\,\beta_d$ and

$$\|T_z(f)\|_{L^2(w)} = \|T(e^{-zb}f)\|_{L^2(v)} \lesssim [v]_{A_2}\|f\|_{L^2(w)} \leq 4[w]_{A_2}\,\beta_d\,\|f\|_{L^2(w)}$$

(since $\|e^{-zb}f\|_{L^2(v)} = \|e^{-zb}f\|_{L^2(we^{2Re\,z\,b})} = \|f\|_{L^2(w)}$).

Thus choose the radius $\epsilon := \dfrac{\alpha_d}{2q'\|b\|_{BMO}}$, and get

$$\|[b,T](f)\|_{L^2(w)} \leq \frac{1}{2\pi\,\epsilon^2}\int_{|z|=\epsilon}\|T_z(f)\|_{L^2(w)}|dz|$$

$$\leq \frac{1}{2\pi\,\epsilon^2}\int_{|z|=\epsilon} 4[w]_{A_2}\,\beta_d\,\|f\|_{L^2(w)}|dz| = \frac{1}{\epsilon}\,4[w]_{A_2}\,\beta_d\,\|f\|_{L^2(w)},$$

Note that $\epsilon^{-1} \approx [w]_{A_2}\|b\|_{BMO}$, because $q' = 1 + 2^{d+5}[w]_{A_2} \approx 2^d[w]_{A_2}$,

$$\|[b,T](f)\|_{L^2(w)} \leq C_d\,[w]^2_{A_2}\,\|b\|_{BMO}. \qquad \square$$

Recent progress

Active area of research!

- Extensions to metric spaces with geometric doubling condition and spaces of homogeneous type.
- Generalizations to matrix valued operators (so far 3/2 estimates for paraproducts, linear for square function).
- Pointwise domination by sparse positive dyadic operators:
  - Rough CZ operators and commutators, more next slides.
  - Singular non-integral operators (Bernicot, Frey, Petermichl '15).
  - Multilinear SIO (Culiuc, Di Plinio, Ou; Lerner, Nazarov ; K. Li '16. Benea, Muscalu '17).
  - Non-homogeneous CZ operators (Conde-Alonso, Parcet '16).
  - Uncentered variational operators (Franca Silva, Zorin-Kranich '16).
  - Maximally truncated oscillatory SIO (Krause, Lacey '17).
  - Spherical maximal function (Lacey '17).
  - Radon transform (Oberlin '17).
  - Hilbert transform along curves (Cladek, Ou '17).
  - Convex body domination (Nazarov, Petermichl, Treil, Volberg '17).

Sparse positive dyadic operators

Cruz-Uribe, Martell, Pérez '10 showed the A_2-conjecture in a few lines for *sparse operators* $\mathcal{A}_S$, where S is a sparse collection of dyadic cubes, defined as follows

$$\mathcal{A}_{\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} m_Q f \, \mathbb{1}_Q(x).$$

Definition

A collection of dyadic cubes $\mathcal{S}$ in $\mathbb{R}^d$ is η-sparse, $0 < \eta < 1$ if there are pairwise disjoint measurable sets

$$E_Q \subset Q \quad \text{with} \quad |E_Q| \geq \eta |Q| \quad \forall Q \in \mathcal{S}.$$

(Rough) CZ operators are pointwise dominated by a finite number of sparse operators Lerner '10,'13, Conde-Alonso, Rey '14, Lerner, Nazarov '14, Lacey '15, quantitative form Lerner '15, Hytönen, Roncal, Tapiola '15.

A_2 theorem for $\mathcal{A}_{\mathcal{S}} f(x) = \sum_{Q\in\mathcal{S}} m_Q f\, \mathbb{1}_Q(x)$

For $w \in A_2$, $\mathcal{S}$ sparse family, to show that

$$\|\mathcal{A}_{\mathcal{S}} f\|_{L^2(w)} \leq C[w]_{A_2}\|f\|_{L^2(w)}$$

is equivalent by duality to show $\forall f \in L^2(w)$, $g \in L^2(w^{-1})$

$$|\langle \mathcal{A}_{\mathcal{S}} f, g\rangle| \leq C[w]_{A_2}\|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}.$$

By CS inequality $|E_Q| = \int_{E_Q} w^{\frac{1}{2}} w^{-\frac{1}{2}} \leq (w(E_Q))^{\frac{1}{2}}(w^{-1}(E_Q))^{\frac{1}{2}}$ and

$$\begin{aligned}
|\langle \mathcal{A}_{\mathcal{S}} f, g\rangle| &\leq \sum_{Q\in\mathcal{S}} \langle |f|\rangle_Q \,\langle |g|\rangle_Q\, |Q| \\
&\leq \frac{1}{\eta}\sum_{Q\in\mathcal{S}} \frac{\langle |f|ww^{-1}\rangle_Q}{\langle w^{-1}\rangle_Q}\,\frac{\langle |g|w^{-1}w\rangle_Q}{\langle w\rangle_Q}\,\langle w\rangle_Q\langle w^{-1}\rangle_Q |E_Q| \\
&\leq \frac{[w]_{A_2}}{\eta}\sum_{Q\in\mathcal{S}} \frac{\langle |f|ww^{-1}\rangle_Q}{\langle w^{-1}\rangle_Q}(w^{-1}(E_Q))^{\frac{1}{2}}\,\frac{\langle |g|w^{-1}w\rangle_Q}{\langle w\rangle_Q}(w(E_Q))^{\frac{1}{2}}
\end{aligned}$$

cont. A_2 theorem for sparse operators

$$|\langle \mathcal{A}_{\mathcal{S}} f, g\rangle|$$

$$\leq \frac{[w]_{A_2}}{\eta} \sum_{Q\in\mathcal{S}} \frac{\langle |f|ww^{-1}\rangle_Q}{\langle w^{-1}\rangle_Q} (w^{-1}(E_Q))^{\frac{1}{2}} \frac{\langle |g|w^{-1}w\rangle_Q}{\langle w\rangle_Q} (w(E_Q))^{\frac{1}{2}}$$

$$\leq \frac{[w]_{A_2}}{\eta} \Big[\sum_{Q\in\mathcal{S}} \frac{\langle |f|ww^{-1}\rangle_Q^2}{\langle w^{-1}\rangle_Q^2} w^{-1}(E_Q)\Big]^{\frac{1}{2}} \Big[\sum_{Q\in\mathcal{S}} \frac{\langle |g|w^{-1}w\rangle_Q^2}{\langle w\rangle_Q^2} w(E_Q)\Big]^{\frac{1}{2}}$$

$$\leq \frac{[w]_{A_2}}{\eta} \Big[\sum_{Q\in\mathcal{S}} \int_{E_Q} M_{w^{-1}}^2(fw) w^{-1}\, dx\Big]^{\frac{1}{2}} \Big[\sum_{Q\in\mathcal{S}} \int_{E_Q} M_w^2(gw^{-1}) w\, dx\Big]^{\frac{1}{2}}$$

$$\leq \frac{[w]_{A_2}}{\eta} \|M_{w^{-1}}(fw)\|_{L^2(w^{-1})} \|M_w(gw^{-1})\|_{L^2(w)}$$

$$\leq C[w]_{A_2} \|fw\|_{L^2(w^{-1})} \|gw^{-1}\|_{L^2(w)} = C[w]_{A_2} \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})}. \qquad \square$$

Similar argument yields linear bounds in $L^p(w)$ for $p > 2$ and by duality get $[w]_{A_p}^{\frac{1}{p-1}} = [w^{\frac{-1}{p-1}}]_{A_{p'}}$ when $1 < p < 2$ (Moen '12).

Sparse vs Carleson families of dyadic cubes

Definition

A family of dyadic cubes $\mathcal{S}$ in $\mathbb{R}^d$ is called Λ-Carleson, $\Lambda > 1$ if

$$\sum_{P\in\mathcal{S}, P\subset Q} |P| \leq \Lambda|Q| \quad \forall Q \in \mathcal{D}.$$

Equivalent to: sequence $\{|P|\mathbb{1}_{\mathcal{S}}(P)\}_{P\in\mathcal{D}}$ is Carleson with intensity Λ.

Lemma (Lerner-Nazarov '14 in *Intuitive Dyadic Calculus*)

$\mathcal{S}$ is Λ-Carleson iff $\mathcal{S}$ is $1/\Lambda$-sparse.

Proof ($\Leftarrow$). $\mathcal{S}$ a $1/\Lambda$-sparse means for all $P \in \mathcal{S}$ there are $E_P \subset P$ pairwise disjoint subsets such that $\Lambda|E_P| \geq |P|$. Hence

$$\sum_{P\in\mathcal{S}, P\subset Q} |P| \leq \Lambda \sum_{P\in\mathcal{S}, P\subset Q} |E_P| \leq \Lambda|Q|.$$

Λ-Carleson $\Rightarrow 1/\Lambda$-sparse

Proof ($\Rightarrow$) (Lemma 6.3 in Lerner, Nazarov '14).
IF $\mathcal{S}$ HAD A BOTTOM LAYER $\mathcal{D}_K$, then consider all $Q \in \mathcal{S} \cap \mathcal{D}_K$, choose any sets $E_Q \subset Q$ with $|E_Q| = \frac{1}{\Lambda}|Q|$. Then go up layer by layer, for each $Q \in \mathcal{D}_k$, $k \leq K$, choose any $E_Q \subset Q \setminus \cup_{R\in\mathcal{S}, R\subsetneq Q} E_R$ with $|E_Q| = \frac{1}{\Lambda}|Q|$.
Choice always possible because for every $Q \in \mathcal{S}$ we have

$$\Big| \cup_{R\in\mathcal{S}, R\subsetneq Q} E_R \Big| \leq \frac{1}{\Lambda} \sum_{R\in\mathcal{S}, R\subsetneq Q} |R| \leq \frac{\Lambda - 1}{\Lambda}|Q| = \Big(1 - \frac{1}{\Lambda}\Big)|Q|,$$

Where we used in ($\leq$) the Λ-Carleson hypothesis.
So $|Q \setminus \cup_{R\in\mathcal{S}, R\subsetneq Q} E_R| \geq \frac{1}{\Lambda}|Q|$, and by construction the sets E_Q are pairwise disjoint, and we are done.
BUT, WHAT IF THERE IS NO BOTTOM LAYER? Run construction for each $K \geq 0$ and pass to the limit! Have to be a bit careful!

All we have to do is replace "free choice" with "canonical choice".

from Lerner, Nazarov '14

Λ-Carleson $\Rightarrow 1/\Lambda$-sparse

Cont. proof ($\Rightarrow$) (Lemma 6.3 in Lerner, Nazarov '14).

Fix $K \geq 0$, for $Q \in \mathcal{S} \cap (\cup_{k \leq K} \mathcal{D}_k)$ define $\widehat{E}_Q^{(K)}$ inductively as follows:

- if $Q \in \mathcal{S} \cap \mathcal{D}_K$ then $\widehat{E}_Q^{(K)}$ is cube with same "SW corner" x_Q as Q, and $|\widehat{E}_Q^{(K)}| = \frac{1}{\Lambda}|Q|$, namely $\widehat{E}_Q^{(K)} := x_Q + \Lambda^{-\frac{1}{d}}(Q - x_Q)$.
- if $Q \in \mathcal{S} \cap \mathcal{D}_k$, $k < K$ then $\widehat{E}_R^{(K)}$ are defined for $R \in \mathcal{S}, R \subsetneq Q$, set

$$\widehat{E}_Q^{(K)} := \big(x_Q + t(Q - x_Q)\big) \cup F_Q^{(K)}, \quad F_Q^{(K)} := \cup_{R \in \mathcal{S}, R \subsetneq Q} \widehat{E}_R^{(K)},$$

  and $t \in [0,1]$ is the largest number such that $|E_Q^{(K)}| \leq \frac{1}{\Lambda}|Q|$ where

$$E_Q^{(K)} = \big(x_Q + t(Q - x_Q)\big) \setminus F_Q^{(K)}.$$

  Such $t \in [0,1]$ exists, moreover $|E_Q^{(K)}| = \frac{1}{\Lambda}|Q|$ by monotonicity and continuity of the function $t \to \left|\big(x_Q + t(Q - x_Q)\big) \setminus F_Q^{(K)}\right|$.

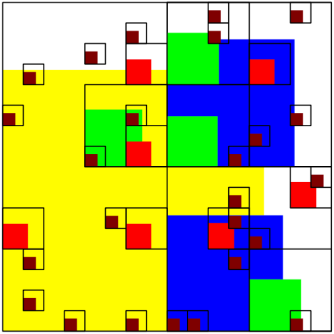

FIGURE 11. The construction from the bottom level (brown) to 4 levels up (yellow). For the largest cube $Q \in \mathcal{S}$ shown, the set $\widehat{E}_Q^{(K)}$ is the total colored area, the set $E_Q^{(K)}$ is the yellow area, and the set $F_Q^{(K)}$ is the area colored with colors other than yellow.

Figure 11 from *Intuitive dyadic calculus: the basics*, by A. K. Lerner, F. Nazarov '14

Λ-Carleson $\Rightarrow 1/\Lambda$-sparse

Cont. proof ($\Rightarrow$) (Lemma 6.3 in Lerner, Nazarov '14).

- Claim: $\widehat{E}_R^{(K)} \subset \widehat{E}_R^{(K+1)}$ for every $Q \in \mathcal{S} \cap \big(\cup_{k \leq K} \mathcal{D}_k \big)$. Proof by backward induction.
- Let $\widehat{E}_Q = \lim_{K \to \infty} \widehat{E}_Q^{(K)} = \cup_{K=0}^{\infty} \widehat{E}_Q^{(K)} \subset Q$.
- Note that $|E_Q^{(K)}| = |\widehat{E}_Q^{(K)} \setminus F_Q^{(K)}| = (1/\Lambda)|Q|$, and $F_Q^{(K)} \subset F_Q^{(K+1)}$.
- $E_Q := \lim_{K \to \infty} E_Q^{(K)} = \widehat{E}_Q \setminus \big(\lim_{K \to \infty} F_Q^{(K)} \big) = \widehat{E}_Q \setminus \big(\cup_{R \in \mathcal{S}, R \subsetneq Q} \widehat{E}_R \big)$ is a well defined subset of Q with $|E_Q| = \frac{1}{\Lambda}|Q|$.
- Sets E_Q with $Q \in \mathcal{S}$ are pairwise disjoint by construction.

□

Lemma (Rey, Reznikov '15)

Let $\{\alpha_Q\}_{I\in\mathcal{D}}$ *be a Carleson sequence, then the positive dyadic operator*

$$T_0 f(x) := \sum_{Q\in\mathcal{D}} \frac{\alpha_Q}{|Q|}\langle f\rangle_Q \mathbb{1}_Q(x)$$

is bounded in $L^2(w)$ *for all* $w\in A_2$*, moreover*

$$\|T_0 f\|_{L^2(w)} \le C[w]_{A_2}\|f\|_{L^2(w)}.$$

Proof. Done if we can dominate T_0 with sparse operators.
Rey, Reznikov '15 showed that localized positive dyadic operators of complexity $m\ge 1$ defined for $\{\alpha_I\}$ Carleson,

$$T_m^{Q_0} f(x) := \sum_{Q\in\mathcal{D}(Q_0)} \sum_{R\in\mathcal{D}_m(Q)} \frac{\alpha_R}{|R|}\langle f\rangle_Q \mathbb{1}_R(x)$$

are pointwise bounded by localized sparse operators.
Lerner, Nazarov '14 removed the localization.
Finally T_0 is a sum of T_1s simply because $\mathbb{1}_Q = \sum_{R\in\mathcal{D}_1(Q)} \mathbb{1}_R$. □

Domination by sparse operators

$\mathcal{S}$, $\mathcal{S}_i$ are sparse families.

- Martingale transform: $|\mathbb{1}_{Q_0} T_\sigma f| \lesssim \mathcal{A}_{\mathcal{S}}|f|$. Same holds for maximal truncations (Lacey '15).
- Paraproduct: $|\mathbb{1}_{Q_0} \pi_b f| \lesssim \mathcal{A}_{\mathcal{S}}|f|$ (Lacey '15).
- CZ operators $|Tf| \leq \sum_{i=1}^{N_d} \mathcal{A}_{\mathcal{S}_i} f$.
- Square function $|S^d f|^2 \leq \sum_{i=1}^{N_d} \sum_{I\in\mathcal{S}_i} \langle |f| \rangle_I^2 \mathbb{1}_I$ (Lacey, K. Li '16).
- Commutator $[b, T]$ for T an ω-CZ operator with ω satisfying a Dini condition, $b \in L^1_{loc}$ can be pointwise dominated by finitely many sparse-like operators and their adjoints (Lerner, Ombrosi, Rivera-Ríos '17).

Case study: Sparse operators vs commutators

- Pérez, Rivera-Ríos ‘17. The following $L\log L$-sparse operator cannot bound pointwise $[T,b]$

$$B_{\mathcal{S}}f(x) = \sum_{Q\in\mathcal{S}} \|f\|_{L\log L,Q}\mathbb{1}_Q(x).$$

($M^2 \sim M_{L\log L}$ is correct maximal function for commutator).

- Lerner, Ombrosi, Rivera-Ríos ’17. Adapted sparse operator and its adjoint provide *pointwise estimates* for $[T,b]$:

$$\mathcal{T}_{\mathcal{S},b}f(x) := \sum_{Q\in\mathcal{S}} |b(x)-\langle b\rangle_Q|\,\langle|f|\rangle_Q\,\mathbb{1}_Q(x),$$

$$\mathcal{T}^*_{\mathcal{S},b}f(x) := \sum_{Q\in\mathcal{S}} \langle|b-\langle b\rangle_Q|\,|f|\rangle_Q\,\mathbb{1}_Q(x).$$

Sparse domination for commutator

Theorem (Lerner, Ombrosi, Rivera-Ríos '17)

Let T an ω-CZ operator with ω satisfying a Dini condition, $b \in L^1_{loc}$. For every compactly supported $f \in L^\infty(\mathbb{R}^n)$, there are 3^n dyadic lattices $\mathcal{D}^{(k)}$ and $\frac{1}{2\cdot 9^n}$-sparse families $\mathcal{S}_k \subset \mathcal{D}^{(k)}$ such that for a.e. $x \in \mathbb{R}^n$

$$|[b,T]f(x)| \leq c_n C_T \sum_{k=1}^{3^n} \big(\mathcal{T}_{\mathcal{S}_k,b}|f|(x) + \mathcal{T}^*_{\mathcal{S}_k,b}|f|(x)\big).$$

- Quadratic bounds on $L^2(w)$ for $[b,T]$ follow from quadratic bounds for this adapted sparse operators.
- Quadratic bounds on $L^2(w)$ for $\mathcal{T}_{\mathcal{S},b}$, $\mathcal{T}^*_{\mathcal{S},b}$,

$$\|\mathcal{T}_{\mathcal{S},b}f\|_{L^2(w)} + \|\mathcal{T}^*_{\mathcal{S},b}f\|_{L^2(w)} \leq C\|b\|_{BMO}[w]^2_{A_2}\|f\|_{L^2(w)},$$

and much more follow from a key lemma.

Key lemma $\mathcal{T}^*_{\widetilde{\mathcal{S}},b} f(x) = \sum_{Q\in\widetilde{\mathcal{S}}} \langle |b-\langle b\rangle_Q|\,|f|\rangle_Q \mathbb{1}_Q(x)$

Lemma (Lerner, Ombrosi, Rivera-Ríos '17)

Given $\mathcal{S}$ η-sparse family in $\mathcal{D}$, $b\in L^1_{loc}$ then $\exists \widetilde{\mathcal{S}}\in\mathcal{D}$ a $\frac{\eta}{2(1+\eta)}$-sparse family, $\mathcal{S}\subset\widetilde{\mathcal{S}}$, such that $\forall Q\in\widetilde{\mathcal{S}}$, with $\Omega(b;R):=\frac{1}{|R|}\int_R |b(x)-\langle b\rangle_R|\,dx$,

$$|b(x)-\langle b\rangle_Q| \le 2^{n+2}\sum_{R\in\widetilde{\mathcal{S}},R\subset Q}\Omega(b;R)\mathbb{1}_R(x),\quad a.e.\ x\in Q,$$

Corollary (Quantitative Bloom, LOR '17)

Let $u,v\in A_p$, $\mu=u^{1/p}v^{-1/p}$, $\|b\|_{BMO_\mu}=\sup_Q |Q|\Omega(b;Q)/\mu(Q)$, then

$$\mathcal{T}^*_{\widetilde{\mathcal{S}},b}|f|(x)\le c_n\|b\|_{BMO_\mu}\mathcal{A}_{\widetilde{\mathcal{S}}}\big(\mathcal{A}_{\widetilde{\mathcal{S}}}(|f|)\mu\big)(x).$$

Hence $\|\mathcal{T}^*_{\mathcal{S},b}|f|\|_{L^p(v)} \le c_{n,p}\|b\|_{BMO_\mu}\|\mathcal{A}_{\widetilde{\mathcal{S}}}\|_{L^p(v)}\|\mathcal{A}_{\widetilde{\mathcal{S}}}\|_{L^p(u)}\|f\|_{L^p(u)}$

$$\le c_{n,p}\|b\|_{BMO_\mu}\big([v]_{A_p}[u]_{A_p}\big)^{\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(u)}.\qquad\square$$

For $u, v \in A_p$, $\mu = u^{1/p}v^{-1/p}$ and $b \in BMO_\mu$ that

$$\|\mathcal{T}^*_{\mathcal{S},b}|f|\|_{L^p(v)} \leq c_{n,p}\|b\|_{BMO_\mu}\left([v]_{A_p}[u]_{A_p}\right)^{\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(u)}.$$

Set now $u = v = w \in A_p$, then $\mu \equiv 1$ and $b \in BMO$

$$\|\mathcal{T}^*_{\mathcal{S},b}|f|\|_{L^p(w)} \leq c_{n,p}\|b\|_{BMO}[w]_{A_p}^{2\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(w)}.$$

;-)

GRACIAS ÚRSULA Y TODO EL COMITÉ ORGANIZADOR POR DARME LA OPORTUNIDAD DE DAR ESTE CURSO!!!! Y POR SUPUESTO GRACIAS A LOS ESTUDIANTES QUE SIN USTEDES NO HAY CURSO!!!

Domination of martingale transform d'après Lacey

Given $I_0 \in \mathcal{D}$, need to find sparse $\mathcal{S} \subset \mathcal{D}$ such that $|\mathbb{1}_{I_0} T_\sigma f| \leq C \mathcal{A}_{\mathcal{S}} |f|$.

- Sharp truncation $T_\sigma^\sharp$ is of weak-type $(1,1)$ (Burkholder '66),

$$\sup_{\lambda>0} \lambda \big|\{x \in \mathbb{R} : T_\sigma^\sharp f(x) > \lambda\}\big| \leq C\|f\|_{L^1(\mathbb{R})}.$$

  Maximal function M is also of weak-type $(1,1)$. So $\exists C_0 > 0$ s.t.

$$F_{I_0} := \{x \in I_0 : \max\{Mf, T_\sigma^\sharp f\}(x) > \tfrac{1}{2} C_0 \langle |f| \rangle_{I_0}\}$$

  satisfies $|F_{I_0}| \leq \frac{1}{2}|I_0|$. Where $T_\sigma^\sharp f = \sup_{I' \in \mathcal{D}} \big| \sum_{I \in \mathcal{D}, I \supset I'} \sigma_I \langle f, h_I \rangle h_I \big|$.

- Let $\mathcal{E}_{I_0} = \{I \in \mathcal{D} : \text{ maximal intervals } I \text{ contained in } F_{I_0}\}$, then

$$|T_\sigma f(x)|\, \mathbb{1}_{I_0}(x) \leq C_0 \langle |f| \rangle_{I_0} + \sum_{I \in \mathcal{E}_{I_0}} |T_\sigma^I f(x)| \tag{1}$$

  where $T_\sigma^I f := \sigma_{\tilde{I}} \langle f \rangle_I \mathbb{1}_I + \sum_{J : J \subset I} \sigma_J \langle f, h_J \rangle h_J$, $\tilde{I}$ is the parent of I.

Domination of martingale transform d'après Lacey

- Repeat for each $I \in \mathcal{E}_{I_0}$, then for each $I' \in \mathcal{E}_I$, etc. Let $\mathcal{S}_0 = \{I_0\}$, and $\mathcal{S}_j := \cup_{I \in \mathcal{S}_{j-1}} \mathcal{E}_I$. Finally let $\mathcal{S} := \cup_{j=0}^{\infty} \mathcal{S}_j$. For each $I \in \mathcal{S}$, let $E_I = I \setminus F_I$, by construction $|E_I| \geq \frac{1}{2}|I|$ and $\mathcal{S}$ is a $\frac{1}{2}$-sparse family.

This is an example of a *stopping time* illustrated below using the house/roof metaphor

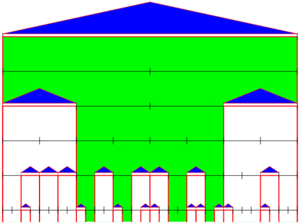

FIGURE 8. The roofs $Q \in \mathcal{S}$ (red intervals under blue triangles) and the houses $H_{\mathcal{S}}(Q)$ (with red walls). The house of the top interval is highlighted in green.

Figure 8 from *Intuitive dyadic calculus: the basics*, by A. K. Lerner, F. Nazarov '14

Domination of martingale transform d'après Lacey

Claim (1): $|T_\sigma f(x)|\,\mathbb{1}_{I_0}(x) \leq C_0 \langle |f| \rangle_{I_0} + \sum_{I \in \mathcal{E}_{I_0}} |T_\sigma^I f(x)|.$

- Note that $|T_\sigma f(x)| \leq T_\sigma^\sharp f(x)$. Thus, if $x \in I_0 \setminus F_{I_0}$ then $|T_\sigma f(x)| \leq \frac{1}{2} C_0 \langle |f| \rangle_{I_0}$, and (1) is satisfied.
- If $x \in F_{I_0}$ then there is unique $I \in \mathcal{S}_1$ with $x \in I$, and

$$\begin{aligned} T_\sigma f(x) &= \sum_{J \supsetneq \tilde{I}} \sigma_J \langle f, h_J \rangle h_J(x) + \sum_{J \subset \tilde{I}} \sigma_J \langle f, h_J \rangle h_J(x) \\ &= \sum_{J \supsetneq \tilde{I}} \sigma_J \langle f, h_J \rangle h_J(x) - \sigma_{\tilde{I}} \langle f \rangle_{\tilde{I}} + T_\sigma^I f(x). \end{aligned}$$

  where $T_\sigma^I f := \sigma_{\tilde{I}} \langle f \rangle_I \mathbb{1}_I + \sum_{J \subset I} \sigma_J \langle f, h_J \rangle h_J$, and $\langle f, h_{\tilde{I}} \rangle h_{\tilde{I}}(x) = \langle f \rangle_I - \langle f \rangle_{\tilde{I}}$.
- $T_\sigma^I - \sigma_{\tilde{I}} \langle f \rangle_I \mathbb{1}_I$ has a similar estimate to (1), we can then recursively get the sparse domination. □