Gabor Frames

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CIMPA, Buenos Aires, August 2017

Outline

von Neumann and Gabor

J. von Neumann (1932), *Foundations of quantum mechanics*
The set of functions $\{e^{2\pi ilt}e^{-\pi(t-k)^2} : k, l \in \mathbb{Z}\}$ *spans a dense subspace of* $L^2(\mathbb{R})$
Answer: YES (Bargmann, Perelomov, etc. 1970s)

D. Gabor (1946), *information theory*
Every function $f \in L^2(\mathbb{R})$ *can be expanded into a series*

$$f(t) = \sum_{k,l\in\mathbb{Z}} c_{kl} e^{2\pi ilt} e^{-\pi(t-k)^2}$$

Discrete expansion with respect to coherent states.
Answer: YES and NO (Bastiaans, Janssen, etc. 1980s)
Series is unstable and converges only in $\mathcal{S}'(\mathbb{R})$

$\boxed{\text{Gabor expansions}}$

Local Fourier Analysis

Expand f into local Fourier series by segmentation

$$f(t)\chi_{[k,k+1]}(t) = \sum_{l\in\mathbb{Z}} c_{kl} e^{2\pi ilt}$$

with

$$c_{kl} = \int_k^{k+1} f(t)e^{-2\pi ilt}\,dt$$

so that

$$f(t) = \sum_{k,l\in\mathbb{Z}} c_{kl} e^{2\pi ilt}\chi_{[k,k+1]}(t)$$

in $L^2(\mathbb{R}$.
Not interesting, because $|c_{kl}| = \mathcal{O}\big(|l|^{-1}\big)$
Representation of f is not sparse
Improvement: smooth cut-off

Time-Frequency Shifts

Translation operator: $T_x f(t) = f(t-x)$

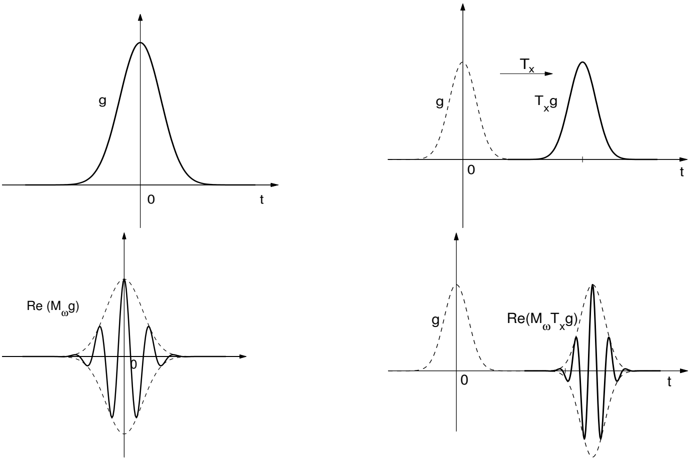
Modulation operator $M_\xi f(t) = e^{2\pi i \xi \cdot t} f(t)$

Time-frequency shift (phase-space shift): $z = (x, \xi) \in \mathbb{R}^{2d}$, $t \in \mathbb{R}^d$

$$\pi(z) f(t) = \underbrace{e^{2\pi i \xi \cdot t}}_{M_\xi} \underbrace{f(t-x)}_{T_x f\ (t)}$$

$\pi(z)$ is unitary on $L^2(\mathbb{R}^d)$ and an isometry on $L^p(\mathbb{R}^d)$

Time-Frequency Shifts



Filter banks

Fix “filter” g with supp $\hat{g} \subseteq [-L/2, L/2]$ (low pass filter)
Idea: decompose a signal f into frequency bands and then sample.

$$\text{supp}\, \widehat{M_{\beta l} g} = \text{supp}\ T_{\beta l}\hat{g} \subseteq [-L/2 + \beta l, L/2 + \beta l]$$

Then

$$\begin{aligned}(f * M_{\beta l} g)(\alpha k) &= \int_{\mathbb{R}^d} f(t) e^{2\pi i \beta \lambda(\alpha k - t)} g(\alpha k - t)\, dt \\ &= \langle f, M_{\beta l} T_{\alpha k} \tilde{g} \rangle e^{2\pi i \alpha \beta k l}\end{aligned}$$

Issues:
- reconstruction of f
- interpretation
- choice of g, α, β

Transmission of Information by OFDM

Transmission of "digital word" $(c_k), c_k \in \mathbb{C}$ via pulse g
Transmitted signal is

$$f(t) = \sum_{k=0}^{\infty} c_k g(t - \alpha k)$$

Multiplexing
Transmission of several "words" ($\Longleftrightarrow$ simultaneous transmission of a symbol group) by distribution to different frequency bands with modulations
Partial signal for ℓ-th word $\mathbf{c}^{(\ell)} = (c_{k\ell})_{k \in \mathbb{Z}}$ is

$$f_\ell = {\color{red} M_{\beta\ell}} \Big(\sum_k c_{k\ell} T_{\alpha k} g \Big)$$

OFDM

Total signal is the Gabor series (Gabor expansion)

$$f = \sum_{k,\ell} c_{k\ell} M_{\beta\ell} T_{\alpha k} g$$

Requirements: • supp $g \subseteq [-\alpha/2, \alpha/2]$ and supp $\hat{g} \subseteq [-\beta/2, \beta/2]$.

pulse shaping

- Conditions so that the coefficients c_{kl} are uniquely determined:
- $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is orthonormal sequence, then $c_{kl} = \langle f, M_{\beta l} T_{\alpha k} g \rangle$

OFDM (orthogonal frequency division multiplexing)

- $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is Riesz sequence, then $c_{kl} = \langle f, M_{\beta l} T_{\alpha k} \gamma \rangle$ for some "dual" window.

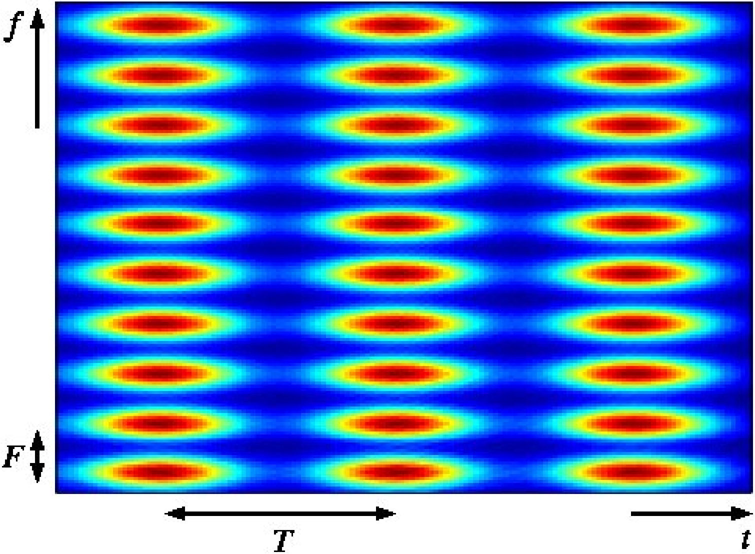


Figure: Each cell carries a coefficient $c_{k\ell}$.

Gabor Systems — the Objects

Time for definitions, $x, \xi, t \in \mathbb{R}^d$

- Point $z = (x, \xi) \in \mathbb{R}^{2d}$ time-frequency space (phase space)
- time-frequency shift, phase-space shift

$$\pi(z)g(t) = e^{2\pi i \xi \cdot t} g(t - x)$$

- Lattice $\Lambda = A\mathbb{Z}^{2d}$ for $2d \times 2d$-matrix A with $\det A \neq 0$,

  $\mathrm{vol}(\Lambda) = |\det A|$ (more generally, $\Lambda \subseteq \mathbb{R}^{2d}$ arbitrary countable set)
- "Window" $g \in L^2(\mathbb{R}^d)$, $g \neq 0$

- Gabor family

$$\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$$

Rectangular lattice $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$

Separable lattice $\Lambda = P\mathbb{Z}^d \times Q\mathbb{Z}^d$, $P, Q \in \mathrm{GL}(d, \mathbb{R})$

Gabor Frames

Definition

(i) $\mathcal{G}(g, \Lambda)$ is a Gabor frame, if for some $A, B > 0$

$$A\|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B\|f\|_2^2 \qquad \forall f \in L^2(\mathbb{R}^d)$$

Equivalently, the frame operator

$$Sf = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g$$

is invertible on $L^2(\mathbb{R}^d)$, since

$$A\|f\|_2^2 = \langle Sf, f \rangle = \sum_{\lambda} \langle f, \pi(\lambda)g \rangle \langle \pi(\lambda)g, f \rangle \leq B\|f\|_2^2$$

Gabor Riesz Sequences

Definition

(ii) $\mathcal{G}(g,\Lambda)$ is a (Gabor) Riesz sequence, if for some $A, B > 0$

$$A\|\mathbf{c}\|_2^2 \leq \|\sum_{\lambda\in\Lambda}|c_\lambda \pi(\lambda)g\|_2^2 \leq B\|\mathbf{c}\|_2^2 \qquad \forall \mathbf{c} \in \ell^2(\Lambda)$$

Equivalently, the Gramian

$$(G\mathbf{c})_\lambda = \sum_{\mu\in\Lambda}\langle \pi(\mu)g, \pi(\lambda)g\rangle c_\mu$$

is invertible on $\ell^2(\Lambda)$, since

$$A\|\mathbf{c}\|_2^2 \leq \|\sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)g\|_2^2 = \langle G\,\mathbf{c}, \mathbf{c}\rangle \leq B\|\mathbf{c}\|_2^2$$

Gabor Expansions — Solution to Reconstruction Problem

Lemma

If $\mathcal{G}(g,\Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ is a frame, then there exists a $\gamma \in L^2(\mathbb{R}^d)$ (dual window), e.g., $\gamma = S^{-1}g$, such that

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)\gamma = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma\rangle \pi(\lambda)g$$

with unconditional convergence of the series in $L^2(\mathbb{R}^d)$.

Proof:

$$S\pi(\lambda) = \pi(\lambda)S \qquad \forall\lambda \in \Lambda$$

$$f = S^{-1}Sf = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda) S^{-1} g$$

Riesz Sequences and Wireless Communication

Assume that $\mathcal{G}(g, \Lambda)$ is a Riesz sequence.
Transmit signal $f = \sum_{\mu \in \Lambda} c_\mu \pi(\lambda) g$.
At receiver compute correlations

$$y_\lambda = \langle f, \pi(\lambda) g \rangle = \sum_{\mu \in \Lambda} c_\mu \langle \pi(\mu) g, \pi(\lambda) g \rangle = (G\mathbf{c})_\lambda$$

$$\boxed{\text{so } \mathbf{y} = G\mathbf{c}.}$$

Consequently

$$\mathbf{c} = G^{-1} G\mathbf{c} = G^{-1}\mathbf{y}$$

Mathematical Problems

- Find conditions on g and Λ, such that $\mathcal{G}(g, \Lambda)$ is a frame or a Riesz sequence.
- Find characterizations of Gabor frames
- Find (classes of) examples
- Given g, characterize all lattices Λ, such that $\mathcal{G}(g, \Lambda)$ is a frame.
- Relevance and relations to other fields?

Gabor analysis

$\boxed{\text{Coarse Structure of Gabor Frames}}$

Duality of Gabor Systems

Definition: Let $\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. If $\Lambda = A\mathbb{Z}^{2d}$ is a lattice, the lattice $\Lambda^\circ = \mathcal{J}(A^T)^{-1}\mathbb{Z}^{2d}$ is called the *adjoint* lattice.

Theorem (Janssen, Ron-Shen, Feichtinger-Kozek-Zimmermann)

Let $g \in L^2(\mathbb{R}^d)$, $g \neq 0$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. TFAE:

(i) *$\mathcal{G}(g, \Lambda)$ is a frame.*

(ii) *$\mathcal{G}(g, \Lambda^\circ)$ is a Riesz sequence.*

(iii) *$\mathcal{G}(g, \Lambda)$ is a Bessel sequence and there exists a dual window $\gamma \in L^2(\mathbb{R}^d)$, such that $\mathcal{G}(\gamma, \Lambda)$ is Bessel and γ satisfies the biorthogonality condition*

$$(\mathrm{vol}(\Lambda))^{-1}\langle \gamma, \pi(\mu)g\rangle = \delta_{\mu,0} \qquad \forall \mu \in \Lambda^\circ .$$

Duality II

$$A\|f\|_2^2 \le \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \le B\|f\|_2^2 \qquad \forall f \in L^2(\mathbb{R}^d)$$

if and only if

$$A'\|\mathbf{c}\|_2^2 \le \|\sum_{\mu\in\Lambda^\circ} c_\mu \pi(\mu)g\|_2^2 \le B'\|\mathbf{c}\|_2^2 \qquad \forall \mathbf{c} \in \ell^2(\Lambda^\circ)$$

Keywords for proof: gymnastics of time-frequency shifts, orthogonality relations for short-time Fourier transform, Poisson summation formula applied to spectrogram.

Characterization of Gabor Frames for Rectangular Lattices

Lemma

Let $g \in L^2(\mathbb{R}^d)$ *and* $\alpha, \beta > 0$. *TFAE:*

(i) $\mathcal{G}(g, \alpha, \beta)$ *is a frame.*

(ii) *There exist* $A, B > 0$, *such that for all* $\mathbf{c} \in \ell^2(\mathbb{Z}^d)$ *and almost all* $x \in \mathbb{R}^d$

$$A\|\mathbf{c}\|_2^2 \leq \sum_{j\in\mathbb{Z}^d} \Big| \sum_{k\in\mathbb{Z}^d} c_k g(x + \alpha j - \tfrac{k}{\beta})\Big|^2 \leq B\|\mathbf{c}\|_2^2 .$$

Frame Set

Definition

Given $g \in L^2(\mathbb{R}^d)$ fixed. Then

$$\mathcal{F}_{\text{full}}(g) = \{\Lambda \text{ lattice} : \mathcal{G}(g, \Lambda) \ \text{ is frame}\}$$

is called the full frame set of g, and

$$\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}^2_+ : \mathcal{G}(g, \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d) \ \text{ is frame}\}$$

is called the reduced frame set of g

Likewise

$$\mathcal{R}_{\text{full}}(g) = \{\Lambda \text{ lattice} : \mathcal{G}(g, \Lambda) \ \text{ is Riesz sequence }\}$$

and

$$\mathcal{R}(g) = \{(\alpha, \beta) \in \mathbb{R}^2_+ : \mathcal{G}(g, \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d) \ \text{ is Riesz sequence }\}$$

Modulation Spaces

A function g belongs to the modulation space $M^1(\mathbb{R}^d)$ (Feichtinger's algebra), if

$$\int_{\mathbb{R}^{2d}} |\langle g, \pi(z)g\rangle| \, dz < \infty \, .$$

Lemma

For $f \in M^1(\mathbb{R}^d)$ the Poisson summation formula is valid.

$$\sum_{k\in\mathbb{Z}^d} f(k) = \sum_{k\in\mathbb{Z}^d} \hat{f}(k) \qquad f \in M^1 \, .$$

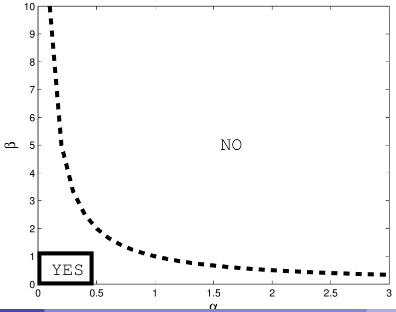
Note: If $h(z) = f(Az)$, then $\hat{h}(\zeta) = |\det A|^{-1}\hat{f}((A^T)^{-1}\zeta)$.

Coarse Structure — Main Theorem

Theorem

Assume that $g \in M^1(\mathbb{R}^d)$. Then $\mathcal{F}_{\mathrm{full}}(g)$ is an open subset of $\{\Lambda \text{ lattice} : \mathrm{vol}(\Lambda) < 1\}$ and contains a neighborhood of $\mathbf{0}$.

Likewise, $\mathcal{F}(g)$ is an open subset of $\{(\alpha, \beta) \in \mathbb{R}^2_+ : \alpha\beta < 1\}$ and contains a neighborhood of $(0, 0)$.



Fine Structure of Gabor Frames

$\boxed{\text{How can we test when } \mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z}) \text{ is a frame?}}$

Philosophical answer: apply one of two dozen characterizations.
Successful in certain cases:
(i) construct a dual window (Janssen, Christensen, KG-Stöckler)
(ii) Apply Lemma 4.

$\boxed{\text{BUT}}$

Examples/Questions

- Let $g(t) = te^{-\pi t^2}$ (first Hermite function)

Is $\mathcal{G}(g, \frac{2}{3}\mathbb{Z} \times \mathbb{Z})$ a frame?
Is $\mathcal{G}(g, 0.666666\mathbb{Z} \times \mathbb{Z})$ a frame?
Is $\mathcal{G}(g, 0, 4\mathbb{Z} \times \mathbb{Z})$ a frame?
[Are the points $(2/3, 1)$ and $(0.66666, 1)$ in $\mathcal{F}(g)$?]

- Let $g(t) = \chi_{[-1/2,1/2]} * \chi_{[-1/2,1/2]} = (1 - |x|)_+$.

Is $\mathcal{G}(g, \frac{2}{3}\mathbb{Z} \times \mathbb{Z})$ a frame?
Is $\mathcal{G}(g, \frac{1}{7}\mathbb{Z} \times 2\mathbb{Z})$ a frame?
Is $\mathcal{G}(g, \frac{1}{7}\mathbb{Z} \times 2.0001\mathbb{Z})$ a frame?
[Are $(2/3, 1), (1/7, 2), (1/7, 2.0001) \in \mathcal{F}(g)$?]

??

Examples/Questions

- Let $g(t) = te^{-\pi t^2}$ (first Hermite function)

Is $\mathcal{G}(g, \frac{2}{3}\mathbb{Z} \times \mathbb{Z})$ a frame? *NO*
Is $\mathcal{G}(g, 0.666666\mathbb{Z} \times \mathbb{Z})$ a frame? ??
Is $\mathcal{G}(g, 0, 4\mathbb{Z} \times \mathbb{Z})$ a frame? *YES*
[Are the points $(2/3, 1)$ and $(0.66666, 1)$ in $\mathcal{F}(g)$?]

- Let $g(t) = \chi_{[-1/2,1/2]} * \chi_{[-1/2,1/2]} = (1 - |x|)_+$.

Is $\mathcal{G}(g, \frac{2}{3}\mathbb{Z} \times \mathbb{Z})$ a frame? *YES*
Is $\mathcal{G}(g, \frac{1}{7}\mathbb{Z} \times 2\mathbb{Z})$ a frame? *NO*
Is $\mathcal{G}(g, \frac{1}{7}\mathbb{Z} \times 2.0001\mathbb{Z})$ a frame? ??
[Are $(2/3, 1), (1/7, 2), (1/7, 2.0001) \in \mathcal{F}(g)$?]

Precise Results about Gabor Frames in $1-D$

1. Lyubarski-Seip (1992) for Gaussian $g(t) = e^{-at^2}$
   $\mathcal{G}(g, \Lambda)$ is frame $\Leftrightarrow$ $\mathrm{vol}(\Lambda) < 1$
2. Janssen-Strohmer (2002) for hyperbolic cosine $g(t) = (\cosh at)^{-1}$
   $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is frame $\Leftrightarrow$ $\alpha\beta < 1$
3. Janssen (2003) for exponential $g(t) = e^{-a|t|}$
   $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is frame $\Leftrightarrow$ $\alpha\beta < 1$
4. Janssen (1996) for one-sided exponential function
   $g(t) = e^{-at}\chi_{\mathbb{R}^+}(t)$
   $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is frame $\Leftrightarrow$ $\alpha\beta \leq 1$
5. $g(t) = (1 + at^2)^{-1}$
   $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is frame $\Leftrightarrow$ $\alpha\beta < 1$
6. $g(t) = (1 - iat)^{-1}$ for $a > 0$
   $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is frame $\Leftrightarrow$ $\alpha\beta \leq 1$

Some (False) Conjectures

- I. Daubechies (1990): If $g > 0$ and $\hat{g} > 0$, then $\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$
  Disproved (1996) disproved by Janssen (1996)
  Updated conjecture with different concept of positivity by KG. and Stöckler (2013) (almost completed)

- Gröchenig (2014): frame set for Hermite functions and B-splines: *In the absence of additional obstructions, or, more precisely, as long as we do not discover other types of obstructions, the next best conjecture . . . is as follows.*
  Disproved in two papers by Lemvig (2016) (after numerical simulations)

Totally positive functions

ϕ is totally positive, if for all finite sequences $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 \cdots < y_n$

$$\det \Big(\phi(x_j - y_k) \Big)_{j,k=1,\dots,n} \geq 0$$

Schoenberg: $\phi \in L^1(\mathbb{R})$ is totally positive, if and only if

$$\hat{\phi}(\xi) = c e^{-\gamma \xi^2} e^{2\pi i \nu \xi} \prod_{j=1}^{N} (1 + 2\pi i \nu_j \xi)^{-1}$$

with $\nu, \nu_j \in \mathbb{R}$, $\gamma \geq 0$, $N \in \mathbb{N} \cup \{\infty\}$ and $0 < \gamma + \sum_j \nu_j^2 < \infty$.

- finite type: $\gamma = 0$ and $N \in \mathbb{N}$
- Gaussian type: $\gamma > 0$ and $N \in \mathbb{N}$
- infinite type: $N = \infty$.

Totally positive functions II

Examples of finite type

- $\phi(x) = \nu^{-1} e^{-x/\nu} \chi_{[0,\infty)}(\nu x)$ (one-sided exponential)
- $e^{-\nu|x|}$ (symmetric exponential)
- $x^n e^{-\nu x} \chi_{[0,\infty)}(x)$
- general formula (by partial fraction decomposition)

$$\phi(x) = \sum_{j=1}^{N} \left(\frac{1}{\nu_j} e^{-\frac{x}{\nu_j}} \chi_{[0,\infty)}(\nu_j x) \prod_{k=1,\, k\neq j}^{N} \left(1 - \frac{\nu_k}{\nu_j}\right)^{-1} \right).$$

Gaussian type: $\phi(x) = e^{-\gamma x^2}$

Infinite type: $\phi(x) = \cosh(\beta x)^{-1} = (e^{\beta x} + e^{-\beta x})^{-1}$

Gabor Frames and Totally Positive Functions

Theorem (G., Stöckler (2013))

Assume that g is a totally positive function of finite type $M \geq 2$*. Then* $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ *is a frame, if and only if* $\alpha\beta < 1$*.*

New

Theorem (G., Romero, Stöckler (2016))

Assume that g is a totally positive function of Gaussian type and $\Lambda \subseteq \mathbb{R}$ *separated.*
Then $\mathcal{G}(g, \Lambda \times \beta\mathbb{Z})$ *is a frame for* $L^2(\mathbb{R})$ *if and only if* $0 < \beta < D^-(\Lambda)$*.*

Corollary

Assume g totally positive function of Gaussian type. Then $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ *is a frame, if and only if* $\alpha\beta < 1$*.*

Proof uses ideas from

- complex analysis (counting density of zeros),
- spectral invariance
- connection to Gabor frames
- Beurling technique of weak limits of sets

Zero sets for Gaussian generator

Proposition

Let $f = \sum_{k\in\mathbb{Z}} c_k e^{-\pi(x-k)^2}$ with $c \in \ell^2(\mathbb{Z})$ or $c \in \ell^\infty$ and $N_f = \{x \in \mathbb{R} : f(x) = 0\}$. Then $D^-(N_f) \le 1$.

Proof.

$$e^{-\pi(x+iy-k)^2} = e^{-\pi(x-k)^2} e^{\pi y^2} e^{-2\pi ixy} e^{2\pi iky}$$

leads to

- Observation 1: $|f(x+iy)| \le Ce^{\pi y^2}$
- Observation 2: If $f(x) = 0$, then $f(x+il) = 0$ for all $l \in \mathbb{Z}$.

$$f(x+iy) = \sum_{k\in\mathbb{Z}} c_k e^{-\pi(x+iy-k)^2} = e^{\pi y^2} e^{-2\pi ixy} \sum_{k\in\mathbb{Z}} c_k e^{2\pi iky} e^{-\pi(x-k)^2}$$

Zero sets for Gaussian generator

Jensen's formula for $n(r) = \#\{z \in \mathbb{C} : |z| \leq r, f(z) = 0\}$

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{it})| \, dt = \log |f(0)| + \int_0^R \frac{n(r)}{r} \, dr$$

Obs. 1 implies

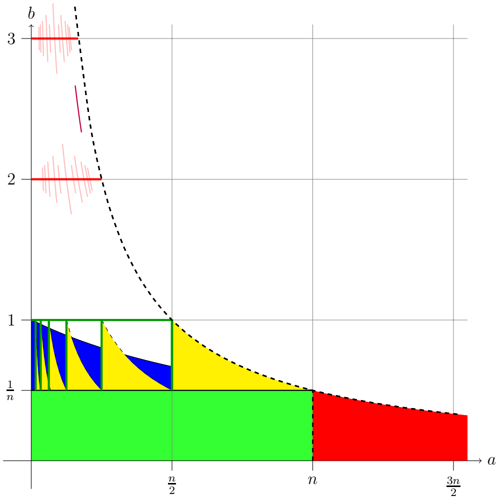
$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{it})| \, dt = \frac{1}{2\pi} \int_0^{2\pi} (\log C + \pi R^2 \sin^2 t) \, dt \asymp \frac{\pi R^2}{2}$$

Obs. 2 leads to

$$n(r) \geq (D^-(N_f) - \epsilon)\pi r^2$$
$$\int_{R_0}^R \frac{n(r)}{r} \, dr \geq (D^-(N_f) - \epsilon)\pi \frac{R^2}{2}$$

Splines

$g = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$ ($n+1$-times)



Hermite Functions

$$h_n = c_n e^{\pi x^2} \frac{d^n}{dx^n}(e^{-2\pi x^2})$$

Theorem (KG, Lyubarski)

If $\mathrm{vol}(\Lambda) < \frac{1}{n+1}$, *then* $\mathcal{G}(h_n, \Lambda)$ *is a frame.*

However

Proposition (Lyubarski, Nes)

If $g \in L^2(\mathbb{R})$ *is odd and* $\alpha\beta = 1 - \frac{1}{N}$ *for* $N = 2, 3, \dots$, *then* $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ *is NOT a frame.*

Hermite Functions

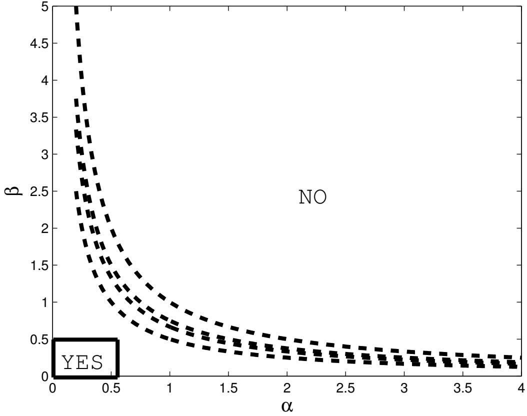


Figure: Possible frame set of odd function

Frame Bounds

Estimates for frame bounds for $\Lambda = \alpha\mathbb{Z}^2$

$$A(\alpha)\|f\|_2^2 \leq \sum_{\lambda \in \alpha\mathbb{Z}^2} |\langle f, \pi(\lambda)\varphi\rangle|^2 \leq B(\alpha)\|f\|_2^2 \qquad \forall f \in L^2(\mathbb{R}^d)$$

Theorem (Borichev, KG, Lyubarski.)

For $1/2 \leq \alpha < 1$

$$\begin{aligned} c &\leq B(\alpha) \leq C \\ c(1-\alpha^2) &\leq A(\alpha) \leq C(1-\alpha^2) \end{aligned}$$

- Can be extended to other windows.

Frame Bounds II

Estimates for frame bounds for $\Lambda = \alpha\mathbb{Z}^2$
Let $\phi(t) = e^{-\pi t^2}$ and $A(\Lambda) = \|S^{-1}\|^{-1}$ and $B(\Lambda) = \|S\|$ be the optimal frame bounds in

$$A\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\|f\|_2^2 \qquad \forall f \in L^2(\mathbb{R}^d)$$

Conjecture (Strohmer 2001):
(i) Among all rectangular lattices with $\alpha\beta = \sigma < 1$, the condition number $B(\Lambda)/A(\Lambda)$ is minimized by the square lattice $\sqrt{\sigma}\mathbb{Z}^2$.
(Proved by Faulhuber/Steinerberger for $\sigma = (2N)^{-1}, N \in \mathbb{N}$)
(ii) Among *all* lattices with $\mathrm{vol}(\Lambda) = \sigma < 1$, the condition number $B(\Lambda)/A(\Lambda)$ is minimized by the hexagonal lattice. (open)

Further Directions

- Zak transform methods
- Gabor frames and function spaces (characterizations of modulation spaces)
- Gabor frames and pseudodifferential operators (almost diagonalization of pseudodifferential operators with Gabor frames)
- Gabor frames on finite Abelian groups
- Deformation results
- Gabor frames and Schrödinger equation
- …

References

K. Gröchenig.
The mystery of Gabor frames.
J. Fourier Anal. Appl., 20(4):865–895, 2014.

H. G. Feichtinger and G. Zimmermann.
A Banach space of test functions for Gabor analysis.
In *Gabor analysis and algorithms*, pages 123–170. Birkhäuser Boston, Boston, MA, 1998.

H. G. Feichtinger and F. Luef.
Wiener amalgam spaces for the fundamental identity of Gabor analysis.
Collect. Math., (Vol. Extra):233–253, 2006.

K. Gröchenig.
Foundations of time-frequency analysis.
Birkhäuser Boston Inc., Boston, MA, 2001.

K. Gröchenig and Y. Lyubarskii.
Gabor (super)frames with Hermite functions.
Math. Ann., 345(2):267–286, 2009.

K. Gröchenig and J. Stöckler.
Gabor frames and totally positive functions.
Duke Math. J, 162(6):1003-1031, 2013.

K. Gröchenig, J.-L. Romero, and J. Stöckler.
Sampling theorems for shift-invariant spaces, Gabor frames, and totally positive functions.
https://arxiv.org/pdf/1612.00651v1.pdf

C. Heil.

History and evolution of the density theorem for Gabor frames.
J. Fourier Anal. Appl., 13(2):113–166, 2007.

A. J. E. M. Janssen.

Duality and biorthogonality for Weyl-Heisenberg frames.
J. Fourier Anal. Appl., 1(4):403–436, 1995.

A. J. E. M. Janssen.

Some Weyl-Heisenberg frame bound calculations.
Indag. Math., 7:165–182, 1996.

A. J. E. M. Janssen.

Zak transforms with few zeros and the tie.
In *Advances in Gabor Analysis*. Birkhäuser Boston, Boston, MA, 2002.

J. Lemvig

On some Hermite series identities and their applications to Gabor analysis, and Counterexamples to the B-spline conjecture for Gabor frames.
arXiv:1511.09470 and arXiv:1507.03982

A. Ron and Z. Shen.

Weyl–Heisenberg frames and Riesz bases in $L_2(\mathbb{R}^d)$.
Duke Math. J., 89(2):237–282, 1997.

Links

- My homepage:

http://homepage.univie.ac.at/karlheinz.groechenig/

- Numerical Harmonic Analysis Group:

www.nuhag.eu <http://www.nuhag.eu>
http://www.univie.ac.at/nuhag-php/bibtex/index.php (contains most/all papers related to Gabor Analysis)

- The Large Time-Frequency Analysis Toolbox

http://ltfat.sourceforge.net/