Hölder coverings of sets of small dimension

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Given a set $A \subset \mathbb{R}^d$, how often does the orthogonal projection to a k-plane V have a Hölder inverse? Of course, in order to have an inverse at all, the projection has to be injective. Let $\overline{\dim}_B$ denote upper box dimension. It follows from elementary dimension inequalities that if $\overline{\dim}_B(A) < (d-1)/2$, then for almost all $v \in S^{d-1}$, the orthogonal projection $P_v : \mathbb{R}^d \to \langle v \rangle^\perp$ is indeed injective. In 1999, Hunt and Kaloshin proved that, in this case, for almost all $v \in S^{d-1}$, the set A can be covered by the graph of a Hölder function $f_v : \langle v \rangle^\perp \to \langle v \rangle$.

For any k, we show that if $\overline{\dim}_B(A) < (d-k)/2$, then A can be covered by a graph of a Hölder function $f_V : V^\perp \to V$ for all but a small set of exceptional k-planes V. Further, we give sharp bounds for the dimension of the exceptional set, improving a result of B. Hunt and V. Kaloshin. We also observe that, as a consequence, Hölder graphs can have positive doubling measure, answering a question of T. Ojala and T. Rajala.

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