A sampling theorem for functions in Besov spaces on the sphere

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Introduction

We prove a sampling theorem for d-dimensional sphere

$$\mathbb{S}^d := \{(x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} : x_1^2 + \cdots + x_{d+1}^2 = 1\}$$

in the spirit of that proved by the first author and E. Malinnikova on Euclidean spaces:

Theorem (Jaming-Malinnikova, 2016)**.** *Let $1 \leqslant p \leqslant \infty$. Then, there exists constants $c_0 = c_0(p)$, $c_1 = c_1(p)$ and $c_2 = c_2(p)$ such that for any $r > 0$, every sequence $\{a_n\}_{n\in\mathbb{Z}}$ with $r/2 \leqslant a_{n+1} - a_n \leqslant r$, and every $f \in B^{1/p}_{p,1}(\mathbb{R})$ with $\|f\|_{B^{1/p}_{p,1}} \leqslant (c_0 r^{1/p})^{-1}\|f\|_{L^p}$*

$$c_1 r^{-1/p}\|f\|_{L^p} \leqslant \left(\sum_{n\in\mathbb{Z}} |f(a_n)|^p\right)^{1/p} \leqslant c_2 r^{-1/p}\|f\|_{L^p}.$$

Our aim is to show that a similar result holds on the sphere:

Main Theorem. *Let $1 \leqslant p \leqslant \infty$. Then, there exists constants $c_0 = c_0(d,p)$, $c_1 = c_1(d,p)$ and $c_2 = c_2(d,p)$ such that for any $0 < r \leqslant 1$, every set $\{\zeta_1, \ldots, \zeta_{N_r}\}$ of almost uniformly r-distributed points on $\mathbb{S}^d$, and every $f \in B^{d/p}_{p,1}(\mathbb{S}^d)$ with $\|f\|_{B^{d/p}_{p,1}} \leqslant (c_0)^{-1}\|f\|_{L^p}$*

$$c_1 r^{-d/p}\|f\|_{L^p} \leqslant \left(\sum_{n=1}^{N_r} |f(\zeta_n)|^p\right)^{1/p} \leqslant c_2 r^{-d/p}\|f\|_{L^p}.$$

To that end we use a suitable representation system whose coefficients allow us to characterize the Besov space on the sphere as wavelets coefficients do in the real case.

Needlets

In order to define the representation system we need to partition the sphere in the same way we do with the dyadic cubes in Euclidean case.

Lemma. *For any $0 < r \leqslant 1$ there exists a partition $\{R_1, \ldots, R_{N_r}\}$ of $\mathbb{S}^d$, together with a set of points $\{\eta_1, \ldots, \eta_{N_r}\} \subset \mathbb{S}^d$ with the proprieties:*

1. $\mathbb{S}^d = R_1 \cup \cdots \cup R_{N_r}$ and $\mathring{R}_k \cap \mathring{R}_l = \emptyset$ if $k \neq l$,

2. there exist a constant $0 < c^* = c^*(d) < 1$ such that $B_{\eta_k}(c^* r) \subset R_k \subset B_{\eta_k}(r)$ holds for every $k = 1, \ldots, N_r$,

3. there exist a constant $c^{**} = c^{**}(d) > 0$ such that $N_r \leqslant c^{**} r^{-d}$.

A set $\{\eta_1, \ldots, \eta_{N_r}\} \subset \mathbb{S}^d$ which along with the associated partition $\{R_1, \ldots, R_{N_r}\}$ of $\mathbb{S}^d$ has the proprieties given in the Lemma it is called a *set of almost uniformly r-distributed points* on $\mathbb{S}^d$.

Theorem (Narcowich-Petrushev-Ward, 2006)**.** *For each $j \geqslant 0$ let $\{\eta_{j,k}\}_{k\in I_j}$ be a set of almost uniformly 2^{-j}-distributed points on $\mathbb{S}^d$. Then there exist a family of functions $\{\varphi_{j,k}\}_{j,k}$ with the proprieties:*

1. (size condtion) for every $n \geqslant 0$ there exist a constant $c_n = c(n,d)$ such that

$$|\varphi_{j,k}(\xi)| \leqslant \frac{c_n 2^{jd/2}}{(1 + 2^j \,\mathrm{d}(\xi, \eta_{j,k}))^n} \quad \forall \xi \in \mathbb{S}^d,$$

2. (smoothness condition) there exists another constant $\kappa = \kappa(d)$ such that for every $n \geqslant 0$

$$|\varphi_{j,k}(\xi) - \varphi_{j,k}(\theta)| \leqslant \frac{c_n 2^{jd/2} 2^j \,\mathrm{d}(\xi,\theta)}{(1 + 2^j \,\mathrm{d}(\xi, \eta_{j,k}))^n} \quad \text{if } \mathrm{d}(\xi,\theta) \leqslant \kappa 2^{-j},$$

3. (decomposition) for all $f \in L^2(\mathbb{S}^d)$

$$f = \sum_{j\geqslant 0}\sum_{k\in I_j} \langle f, \varphi_{j,k}\rangle \varphi_{j,k} \quad \text{in } L^2(\mathbb{S}^d).$$

The functions $\varphi_{j,k}$ are called *needlets*.

Besov spaces on the sphere

Definition. Let $\{\varphi_{j,k}\}_{j,k}$ be a needlet system. Given $0 < p, q \leqslant \infty$ and $s \in \mathbb{R}$, the Besov space $B^s_{p,q}(\mathbb{S}^d)$ is defined as the set of all $f \in \mathcal{S}'$ such that the norm

$$\|f\|_{B^s_{p,q}} := \left(\sum_{j=0}^{\infty}\left[2^{j(s+d/2-d/p)}\left(\sum_{k\in I_j} |\langle f, \varphi_{j,k}\rangle|^p\right)^{1/p}\right]^q\right)^{1/q}$$

is finite. Where the L^p, ℓ^q norms are replaced by the sup-norms when $p = \infty$ or $q = \infty$.

Sampling result

Let us reformulate our theorem as follows:

Theorem. *Let $1 \leqslant p \leqslant \infty$, $s \geq d/p$ and set $\alpha := \frac{s-d/p}{1+s-d/p}$. Then, there exist a constant $c_0 = c_0(d,p)$ such that for any $0 < r \leqslant 1$, for every set of almost uniformly r-distributed points $\{\zeta_1, \ldots, \zeta_{N_r}\}$ with associated partition $\{R_1, \ldots, R_{N_r}\}$ and every $f \in B^s_{p,1}(\mathbb{S}^d)$*

$$\left(\int_{\mathbb{S}^d}\left|f(\xi) - \sum_{n=1}^{N_r} f(\zeta_n)\mathbf{1}_{R_n}(\xi)\right|^p \mathrm{d}\sigma(\xi)\right)^{1/p} \leqslant c_0 r^{\alpha}\|f\|_{B^s_{p,1}}.$$

Sketch of proof. First we note that

$$\left(\int_{\mathbb{S}^d}\left|f(\xi) - \sum_{n=1}^{N_r} f(\zeta_n)\mathbf{1}_{R_n}(\xi)\right|^p \mathrm{d}\sigma(\xi)\right)^{1/p} = \left\|\left(\int_{R_n} |f(\xi) - f(\zeta_n)|^p \,\mathrm{d}\sigma(\xi)\right)^{1/p}\right\|_{\ell^p_n}.$$

Then we use the needlet decomposition and the Hölder inequality to obtain

$$|f(\xi) - f(\zeta_n)| \leqslant c_0 \sum_{j=0}^{j_0} 2^{jd/2} 2^j \,\mathrm{d}(\xi, \zeta_n) E_j(f) + c_0 \sum_{j=j_0+1}^{\infty} 2^{jd/2} E_j(f).$$

where for each $j \geqslant 0$, $E_j(f) := \left(\sum_{k\in I_j} |\langle f, \varphi_{j,k}\rangle|^p\right)^{1/p}$ and j_0 is chosen such that $2^{-j_0} \approx r^{\alpha-1}$. Finally we use the proprieties from the partition Lemma when we integrate over R_n to get the right bound. □

Conclusion and future work

In this work, we have used the needlet decomposition of Besov spaces to establish a sampling theorem in $B^s_{p,1}$ when the smoothness index is large enough. This opens the path to future work in two directions:

– establish efficient algorithms for reconstructions of function on a sphere from their samples;

– extend this work to more general compact Riemannian manifolds and then to more general homogeneous spaces for which appropriate decompositions systems have been established.

References

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