Embeddings between Grand, Small, and Variable Lebesgue Spaces

Oscar M. Guzman

Universidad Nacional de Colombia

omguzmanf@unal.edu.co

UNIVERSIDAD NACIONAL DE COLOMBIA

Abstract

We give conditions on the exponent function $p(\cdot)$ that imply the existence of embeddings between the grand, small and variable Lebesgue spaces. We construct examples to show that our results are close to optimal. Our work extends recent results by the second author, Rakotoson and Sbordone [8].

Basic Concepts

Grand, Small Lebesgue Spaces

The grand Lebesgue space $L^{p)}(\Omega)$ was introduced by Iwaniec and Sbordone [9]. Given a set $\Omega \subset \mathbb{R}^n$, $|\Omega| = 1$, $1 < p < \infty$, and $\theta > 0$, the generalized grand Lebesgue space $L^{p),\theta}(\Omega)$ consists of all measurable functions f such that

$$\|f\|_{p),\theta} = \sup_{0<\epsilon<p-1} \left(\epsilon^{\theta} \int_{\Omega} |f(x)|^{p-\epsilon}\, dx \right)^{\frac{1}{p-\epsilon}}.$$

The small Lebesgue space $L^{(p,\theta}$ is defined as the associate space of $L^{p'),\theta}$, and so has the norm

$$\|f\|_{(p,\theta} = \sup \left\{ \int_{\Omega} f(x) g(x)\, dx : \|f\|_{p'),\theta} \le 1 \right\}.$$

These expressions were quite complicated, but much simpler expressions were found in [4, 6]:

$$\|f\|_{p),\theta} \approx \sup_{0<t<1} \log\left(\frac{e}{t}\right)^{-\frac{\theta}{p}} \left(\int_t^1 f_*(s)^p\, ds \right)^{\frac{1}{p}} \tag{1}$$

$$\|f\|_{(p,\theta} \approx \int_0^1 \log\left(\frac{e}{t}\right)^{\frac{\theta}{p}-1} \left(\int_0^t f_*(s)^p\, ds \right)^{\frac{1}{p}} \frac{dt}{t}. \tag{2}$$

Variable Lebesgue Spaces

Given a measurable function $p(\cdot) : \Omega \to [1, \infty)$ we define $L^{p(\cdot)}(\Omega)$ to be the collection of all measurable functions such that for some $\lambda > 0$

$$\rho(f/\lambda) = \int_{\Omega} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty, \tag{3}$$

$L^{p(\cdot)}$ becomes a Banach function space with the norm

$$\|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho(f/\lambda) \le 1\}.$$

Given $p(\cdot) \in \mathcal{P}(\Omega)$, let $p_*(\cdot), p^*(\cdot) : [0,1] \to [1,\infty)$ denote the decreasing rearrangement and the increasing rearrangement of $p(\cdot)$ respectively.

$L^{\varphi}(\Omega)$ Spaces

Given a decreasing function $\sigma_* : [0,1] \to \mathbb{R}$, define the function $\varphi : [0,1] \times \mathbb{R} \to [0,\infty)$ by

$$\varphi(a,b) = b^{p_*(a)} \log(e+b)^{\sigma_*(a)}.$$

We define the space $L^{\varphi(\cdot)}([0,1])$ to consist of all measurable functions f_* defined on $[0,1]$ such that for some $\lambda > 0$,

$$\rho_{\varphi}(f/\lambda) = \int_0^1 \varphi\left(t, \frac{|f(t)|}{\lambda}\right) dt < \infty.$$

With a norm defined as above for the variable Lebesgue spaces, $L^{\varphi(\cdot)}([0,1])$ becomes a Banach function space, a particular case of the Musielak-Orlicz spaces, also referred to as generalized Orlicz spaces.

Motivation

For all $1 < p < \infty$ and $\epsilon > 0$ we have

$$L^{p+\epsilon}(\Omega) \subsetneq L^{(p,\theta}(\Omega) \subsetneq L^{p}(\Omega) \subsetneq L^{p),\theta}(\Omega) \subsetneq L^{p-\epsilon}(\Omega) \tag{4}$$

$$L^{p_+}(\Omega) \subset L^{p(\cdot)}(\Omega) \subset L^{p_-}(\Omega). \tag{5}$$

Q1: Are the stronger embeddings

$$L^{p(\cdot)}(\Omega) \subset L^{(p_-,\theta}(\Omega)$$
$$L^{p_+),\theta}(\Omega) \subset L^{p(\cdot)}(\Omega)$$

possible?

Given the inequality (see in [7, 8].)

$$c\|f_*\|_{p^*(\cdot)} \leqslant \|f\|_{p(\cdot)} \leqslant C\|f_*\|_{p_*(\cdot)}, \tag{6}$$

Q2: If $\|f_*\|_{p_*(\cdot)} < \infty$ then in what space the function f would be? .

Results and Examples

Theorem 0.1. *Given an exponent $p(\cdot) \in \mathcal{P}(\Omega)$, $1 < p_- \le p_+ < \infty$, and $\theta > 0$, suppose that there exists $0 < t_0 \le 1$ and $\epsilon > 0$ such that for all $t \in [0, t_0]$,*

$$\frac{1}{p^*(0)} - \frac{1}{p^*(t)} \ge \left(\frac{\theta}{p'_-} + \epsilon \right) \frac{\log\log(\frac{e}{t})}{\log(\frac{e}{t})}. \tag{7}$$

Then

$$L^{p(\cdot)}(\Omega) \hookrightarrow L^{(p_-,\theta}(\Omega). \tag{8}$$

Theorem 0.2. *Given an exponent $p(\cdot) \in \mathcal{P}(\Omega)$, $1 < p_- \le p_+ < \infty$, and $\theta > 0$, suppose that there exists $0 < t_0 \le 1$ and $\epsilon > 0$ such that for all $t \in [0, t_0]$,*

$$\frac{1}{p_*(t)} - \frac{1}{p_*(0)} \ge \left(\frac{\theta}{p_+} + \epsilon \right) \frac{\log\log(\frac{e}{t})}{\log(\frac{e}{t})}. \tag{9}$$

Then

$$L^{p_+),\theta}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega). \tag{10}$$

Q3: Are continuity conditions in Theorem 0.1 and 0.2 are in some sense sharp?

Example 0.3. *Given $\theta > 0$, there exists an increasing function $p(\cdot) \in \mathcal{P}([0,1])$, $1 < p_- \le p_+ < \infty$, such that for $t \in [0, e^{-2}]$,*

$$\frac{1}{p(0)} - \frac{1}{p(t)} \le \frac{\theta}{p'_-} \frac{\log\log(\frac{e}{t})}{\log(\frac{e}{t})}, \tag{11}$$

and there exists $f \in L^{p(\cdot)}([0,1])$ such that $f \notin L^{(p_-,\theta}([0,1]))$.

Q4: For which $p(\cdot)$ do we have that $L^{(p_-,\theta}(\Omega) \subset L^{p(\cdot)}(\Omega)$ or $L^{p(\cdot)}(\Omega) \subset L^{p_+),\theta}(\Omega)$?

Theorem 0.4. *Given $p(\cdot) \in \mathcal{P}(\Omega)$, $1 < p_- \le p_+ < \infty$, and $\theta \ge 1$, suppose there exist $A \in \mathbb{R}$ and $0 < t_0 \leqslant 1$ such that for all $t \in [0, t_0]$,*

$$\frac{1}{p_*(t)} - \frac{1}{p_*(0)} \le \frac{A}{\log(\frac{e}{t})} + \frac{\theta - 1}{p_*(0)} \frac{\log\log(\frac{e}{t})}{\log(\frac{e}{t})}. \tag{12}$$

Then for all $u \in L^{p(\cdot)}(\Omega)$ such that $u_* \in L^{p_*(\cdot)}([0,1])$, $u_* \in L^{p_+),\theta}([0,1])$.

The condition (12) is close to optimal as the following example shows.

Example 0.5. *Given $p(\cdot) \in \mathcal{P}(\Omega)$, $1 < p_- \le p_+ < \infty$, suppose there exists $\theta > 0$, $\epsilon > 0$ and $0 < t_0 \leqslant 1$, such that for all $t \in [0, t_0]$,*

$$\frac{1}{p_*(t)} - \frac{1}{p_*(0)} \ge \left(\frac{\theta + \epsilon}{p_*(0)} \right) \frac{\log\log(\frac{e}{t})}{\log(\frac{e}{t})}. \tag{13}$$

Then there exist a (decreasing) function $f_* \in L^{p_*(\cdot)}([0,1]) \setminus L^{p_+),\theta}([0,1])$.

We can extend Theorem 0.4 to the range $0 < \theta < 1$, and generalize it for $\theta \ge 1$, if we pass to a larger scale of spaces, that is, $L^{\varphi(\cdot)}([0,1])$.

Theorem 0.6. *Given $\theta > 0$, let $\sigma_*(\cdot) : [0,1] \to \mathbb{R}$ be a bounded, decreasing function such that $\sigma_*(0) \ge 1 - \theta$. Suppose further that there exists $B > 0$ and $0 < t_0 \le 1$ such that for $t \in [0, t_0]$,*

$$\sigma_*(0) - \sigma_*(t) \le \frac{B}{\log\log(\frac{e}{t})}. \tag{14}$$

Given $p(\cdot) \in \mathcal{P}(\Omega)$, $1 < p_- \le p_+ < \infty$, suppose there exists $A \in \mathbb{R}$ such that for $t \in [0, t_0]$,

$$\frac{1}{p_*(t)} - \frac{1}{p_*(0)} \le \frac{A}{\log(\frac{e}{t})} + \frac{\theta - 1 + \sigma_*(0)}{p_*(0)} \frac{\log\log(\frac{e}{t})}{\log(\frac{e}{t})}. \tag{15}$$

Let $\varphi(a,b) = b^{p_*(a)} \log(e+b)^{\sigma_*(a)}$. Then, for all $u \in L^{p(\cdot)}(\Omega)$ such that $u_* \in L^{\varphi(\cdot)}([0,1])$, $u_* \in L^{p_+),\theta}([0,1])$.

Remark 0.7. If $\sigma_*(\cdot) \equiv 0$, then Theorem 0.6 reduces to Theorem 0.4. Theorem 0.6 is a more general result: for example, when $\theta > 1$, if $\sigma_*(\cdot) \equiv 1 - \theta$, then $L^{p_*(\cdot)}([0,1]) \subsetneq L^{\varphi(\cdot)}([0,1])$. (See [10, Chapter II.8].)

Open Problems

- We conjecture that some version of Theorem 0.4 is true for $0 < \theta < 1$, but we have not been able to prove it.
- We conjecture that if (12) holds, then a "dual" result holds as well. More precisely, we conjecture that given any decreasing function $u_* \in L^{(q_-,\theta}$, we have $u_* \in L^{q^*(\cdot)}$. However, unlike in the proof of Theorem 0.2, we cannot use associativity to prove this since we are not dealing with a subspace but rather the cone of decreasing functions

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