Spherical analysis on homogeneous vector bundles of the 3-dimensional euclidean motion group

Rocío Díaz Martín and Fernando Levstein

Facultad de Matemática, Astronomía, Física y Computación - Universidad Nacional de Córdoba

Abstract

The goal of this work is to describe translation and rotation invariant bounded linear operators over the sections of each homogeneous vector bundle of the euclidean motion group. From the Schwartz kernel theorem, each such operator can be represented in a unique way as a convolution operator and the composition coincides with the convolution of their kernels. In order to change these operators simultaneously into multiplicative ones we need a condition about commutation of their kernels with respect to the convolution product. This motivates a generalization of the notion of Gelfand pair. The linear functionals which "diagonalize" these operators are called *spherical functions*. We present their computation in three different ways.

Notation: $G = K \ltimes N$, $N = \mathbb{R}^3$, $K = SO(3)$, where $(k,x)(h,y) = (kh, x + k\cdot y)$, $(\tau, V_\tau) \in \widehat{SO(3)}$ of dimension $2m+1$ with $m \in \mathbb{Z}_{\geq 0}$.

Scalar case: τ is the trivial representation $V_\tau = \mathbb{C}$

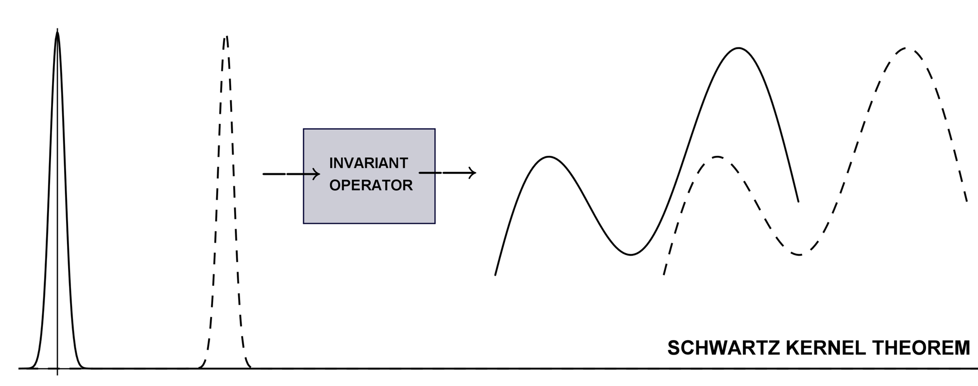
- Homogeneous space: $G/K \simeq \mathbb{R}^3$, $g \sim gk \Rightarrow (I,x) \sim (I,k)(k,0) = (k,x)$.
  Action of $SO(3) \ltimes \mathbb{R}^3$ on $\mathbb{R}^3$: translation & rotation.
- Sections: $\tilde{u} : SO(3) \ltimes \mathbb{R}^3 \to \mathbb{C}$ / $\tilde{u}(gk) = \tilde{u}(g)$ or $u : \mathbb{R}^3 \to \mathbb{C}$.

$$SO(3) \ltimes \mathbb{R}^3 \text{ acts on } \tilde{u} \equiv \begin{cases} \mathbb{R}^3 \text{ acts on } u \text{ by translation} \\ SO(3) \text{ acts on } u \text{ by } (k \cdot u)(x) := u(k^{-1}\cdot x) \end{cases}$$

Matrix case: τ non-trivial

- Homogeneous fiber bundle: $G \times_\tau V_\tau$, $(g,v) \sim (gk, \tau(k)^{-1}v)$.
  Action of G on the fiber bundle: $g' \cdot (g,v) = (g'g, v)$ (left action)
- Sections: $\tilde{u} : SO(3) \ltimes \mathbb{R}^3 \to V_\tau$ / $\tilde{u}(gk) = \tau(k)^{-1}\tilde{u}(g)$ or
  $u : \mathbb{R}^3 \to V_\tau$ via $u(x) \longmapsto u(k,x) = \tau(k)^{-1}u_0(x)$.

$$SO(3) \ltimes \mathbb{R}^3 \text{ acts on } \tilde{u} \equiv \begin{cases} \mathbb{R}^3 \text{ acts on } u \text{ by translation} \\ SO(3) \text{ acts on } u \text{ by } (k \cdot u)(x) := \tau(k)u(k^{-1}\cdot x) \end{cases}$$



Scalar case

- Kernel: $f : \mathbb{R}^3 \to \mathbb{C}$ such that $f(k\cdot x) = f(x)$ i.e. radial function.
- $(K \ltimes N, K)$ is a *Gelfand pair* if the algebra of K-invariant integrable functions on N is commutative with respect to the convolution product. ✓
- rotation-invariant (radial) bounded function ϕ is *spherical* if the map

$$f \longmapsto (\mathcal{F}(f))(\phi) := \int_{\mathbb{R}^3} f(x)\phi(-x)\,dx \quad (\textit{spherical transform})$$

  is a homomorphism from integrable rotation-invariant kernels to $\mathbb{C}$ or equivalently if $\phi(0) = 1$ and ϕ is a joint eigenfunction for all translation & rotation invariant differential operators.
- translation & rotation invariant differential operators: Laplacian Δ.
- Eigenvalues: $-s^2$ with $s \in \mathbb{R}_{\geq 0}$.

Matrix case

- Kernel: $F : \mathbb{R}^3 \to End(V_\tau)$ *bi-τ-equivariant* i.e. $F(k\cdot x) = \tau(k)F(x)\tau(k)^{-1}$
- $(K \ltimes N, K, \tau)$ is a *commutative triple* if the algebra of such integrable kernels is commutative with respect to the convolution product. ✓
- a non-trivial bi-τ-equivariant bounded function Φ is *τ-spherical* if the map

$$F \longmapsto (\mathcal{F}(F))(\Phi) := \frac{1}{2m+1}\int_N Tr[F(x)\Phi(x^{-1})]\,dx \quad (\tau\textit{-spherical transform})$$

  is a homomorphism from integrable bi-τ-equivariant kernels to $\mathbb{C}$ or equivalently if $\Phi(0) = I$ and Φ is a joint eigenfunction for all N-invariant and bi-τ-equivariant differential operators.
- translation-invariant & bi-τ-equivariant differential operators: Δ & $d\tau(\partial_x)$.
- Eigenvalues: $(-s^2, sj)$ with $s \in \mathbb{R}_{\geq 0}$ and $j \in \mathbb{Z}$ $-m \leq j \leq m$.

$\tau-$spherical functions (matrix case)

as linear combination of scalar $SO(3)$-invariant functions times $End(V_\tau)$-valued bi-τ-equivariant polynomials	as Fourier transforms of proyection-valued measures on $SO(3)$-orbits	as bi-τ-equivariant differential operators applied to scalar-valued spherical functions
$\Phi_{s,j}(x) = \phi_s(\lvert x\rvert)I + v_1^{(s,j)}\phi_s^1(\lvert x\rvert)Q_1(x) + \ldots + v_{2m}^{(s,j)}\phi_s^{2m}(\lvert x\rvert)Q_{2m}(x)$	$\Phi_{s,j} = (2m+1)\widehat{P_j(\frac{\cdot}{s})\sigma_s}$ where:	$\Phi_{s,j} = (2m+1)D_{s,j}(\phi_s I)$
where: $\phi_s^k(r)$ multiple of $J_{j+\frac{1}{2}}(sr)/(\frac{sr}{2})^{j+\frac{1}{2}}$ (J Bessel function)	σ_s: $O(3)$-inv measure of s-sphere in $\mathbb{R}^3$	where: ϕ_s is scalar-spherical function
Q_k: bi-τ-equivariant matrix - entries: harmonic homogeneous deg. k polynomials	$P_j(\xi) \sim \Pi\,\sqrt{-1}d\tau_m(\xi) + lI$, $\xi \in S^2$,	$D_{s,j} \sim \Pi\,\frac{1}{s}d\tau(\partial_x) - lI$
$\{Q_k\}_{k=0}^{2m}$ generates (as $\mathbb{C}[\lvert x\rvert^2]$-module) the bi-$\tau$-equivariant polynomials	product over $-m \leq l \neq j \leq m$, which arises from	(product over $l \neq j$; $-m \leq l \leq m$)
$(v_k^{(s,j)})$ eigenvector of $d\tau(\partial_x)$ on $< \{\phi_s^k Q_k\}_{k=0}^{2m} >$	decomposing the action of $\{k \in K / k\cdot\xi = \xi\}$ on V_τ	proof: uses the characteristic polynomial of $d\tau(x)$

Scalar case

- *Inversion formula*: for f radial integrable function such that its Fourier transform is integrable

$$f(x) = \int_0^\infty \mathcal{F}(f)(\phi_s)\,\phi_s(x)\,s^2\,ds.$$

- *Plancherel measure*: $s^2\,ds$,
  the dual space is identified with $[0,\infty)$ via the correspondence $s \longmapsto \phi_s$.

Matrix case

- *Inversion formula*: for F bi-τ-equivariant integrable function such that its classical Fourier transform is integrable

$$F(x) = \sum_{j=-m}^{m}\int_0^\infty \mathcal{F}(F)(\Phi_{s,j})\,\Phi_{s,j}(x)\,s^2\,ds.$$

- *Plancherel measure*: is the product measure of the Plancherel measure associated to the Gelfand pair and a finite sum of deltas.

Main References

[1] R. Camporesi, *The spherical transform for homogeneous vector bundles over Riemannian symmetric spaces*, Journal of Lie Theory, **7**, 29-67 (1997).

[2] F. Ricci, and A. Samanta, *Spherical analysis on homogeneous vector bundles* (2016) http://arxiv.org/abs/1604.07301

[3] N. Wallach, *Harmonic Analysis on Homogeneous Spaces*. Marcel Dekker, NY (1973).

Acknowledgements

We are immensely grateful to Linda Saal and Fulvio Ricci for their guidance and hospitality.

Contact Information

Email: rocio.dm13@gmail.com

CIMPA Buenos Aires 2017 Research School: Harmonic Analysis, Geometric Measure Theory and Applications