Fractional Integral Operators acting on Sobolev-BMO spaces

Lucas Oliveira

(joint work with Lucas Chaffee and Jarod Hart)

Instituto de Matemática e Estatística Pura e Aplicada
Departamento de Matemática Pura e Aplicada
UFRGS/RS - Avenida Bento Gonçalves, 9500 - Brasil
`lucas.oliveira@ufrgs.br`

1. Introduction

Suppose that T is a linear operator, continuous from $\mathcal{S}$ (Schwartz class) into $\mathcal{S}'$ (tempered distributions), and let $\nu \in \mathbb{R}$, $M \geq 0$ be an integer, $0 < \gamma \leq 1$ and $\mathcal{D}_M$ the space of smooth compactly supported functions with vanishing moments up to order $M \in \mathbb{N}$. If there is a $P \in \mathbb{N}$ and a kernel $K(x, y)$ such that

$$\langle Tf, g\rangle = \int_{\mathbb{R}^{2n}} K(x,y) f(y) g(x)\, dy\, dx,$$

for all $f, g \in \mathcal{D}_P$ with disjoint support, and if K satisfies

$$|D_x^\alpha D_y^\beta K(x,y)| \lesssim \frac{1}{|x-y|^{n+|\alpha|+\nu+|\alpha|+|\beta|}}$$

for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| \leq M$ and for all $x, y, h \in \mathbb{R}^n$ with $x \neq y$, and

$$\begin{aligned}&|D_x^\alpha D_y^\beta K(x+h,y) - D_x^\alpha D_y^\beta K(x,y)| \\ &+ |D_x^\alpha D_y^\beta K(x,y) - D_x^\alpha D_y^\beta K(x,y+h)| \lesssim \frac{|h|^\gamma}{|x-y|^{n+\nu+M+\gamma}}\end{aligned}$$

for all $x, y, h \in \mathbb{R}^n$ satisfying $|h| < |x-y|/2$ and $\alpha, \beta \in \mathbb{N}_0^n$ satisfying $|\alpha| + |\beta| = M$, We will say that T is in the **class of ν-order Singular Integral Operators**, denoted $T \in SIO_\nu(M+\gamma)$.

Observe that, depending on the values of ν, M and γ, we can (essentially) recover important class of operators from the above definition:

- If $-n < \nu < 0$, $M \geq 0$ and $0 < \gamma < 1$, the above class (without necessarily impose regularity in the y variable) contain the entire class of fractional integral type operators;
- If $\nu = 0$, and we can extend the operator to a bounded linear operator from $L^p \to L^p$, we recover (essentially) the class of Calderón-Zygmund operators;
- Hypersingular integral operators arose from $\nu > 0$ operators in $SIO_\nu(M+\gamma)$ that can be extended to a bounded linear operator from $W^{\nu,p} \to L^p$.

In the study of this large class of operators, we discovered that some Sobolev-BMO spaces play and important role. In what follows, we will discuss shortly some interesting properties of the Sobolev-BMO spaces $I_s(BMO)$ and we will give emphasis to some results obtained for the case $\nu < 0$, that is, the fractional integral type operators. We will conclude with some results concerning the singular and hypersingular integral type operators.

2. Sobolev-BMO spaces

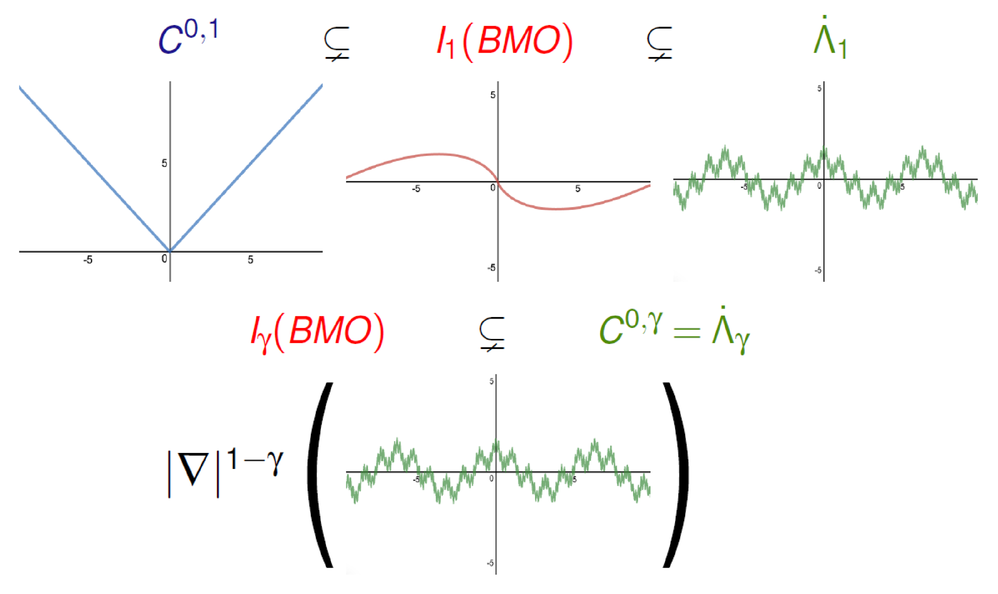
The Sobolev-BMO spaces, denoted by$I_s(BMO)$, were initially introduced by Neri and further developed by Strichartz in the seventies. We simply note that $I_s(BMO)$ is the collection of tempered distributions f such that $|\nabla|^s f \in BMO$ where $|\nabla|^s$ is the Fourier multiplier $\widehat{|\nabla|^s f}(\xi) = |\xi|^s \widehat{f}(\xi)$. Moreover, we define a norm in this space simply as

$$\|f\|_{I_s(BMO)} := \||\nabla|^s f\|_{BMO}.$$

These spaces have some interesting properties:

1. **Theorem (Strichartz 1980)** For $s = k+1$ is an integer, we have that
$$C^{k,1} \subsetneq I_s(BMO) \subsetneq \dot{\Lambda}_s;$$
2. **Theorem (Zygmund 1945, Strichartz 1980 - Strichartz)** When $s = k + \gamma > 0$ is a non-integer, we have that
$$I_s(BMO) \subsetneq C^{k,\gamma} = \dot{\Lambda}_s;$$
3. **Theorem (Muckenhoupt and Wheeden 1975 for $s = 0$, Hart-O. 2017 for $s > 0$)** If $w \in A^\infty$, we have that
$$I_s(BM0)_w = I_s(BMO);$$
4. **Theorem (Strichartz 1980, Chaffee-Hart-O. 2017)** For $0 < s < 1$ and $f \in I_s(BMO)$, we have that
$$|(f)(x) - f(y)| \leq C|x-y|^s \|f\|_{I_s(BMO)}.$$

Below, we include a picture describing the different *typical* kinds of behavior of functions and how they relate to the Sobolev-BMO spaces:



*Graphs created at Desmos.com

3. Linear fractional integral type operators

The Riesz potential I_ν is given by

$$I_\nu f(x) = \int_{\mathbb{R}^n} f(y)|x-y|^{-(n-\nu)}\, dy$$

is the prototypical ν-order linear fractional integral operator and for it we can explain easily in what sense this operators are *smoothing operators*:

- (Smoothing in Lebesgue spaces) $I_\nu : L^p \to L^q$ continuously for any $1 < p < \frac{n}{\nu}$ with $\frac{1}{p} = \frac{1}{q} + \frac{\nu}{n}$ when $0 < \nu < n$;
- (Smoothing in Sobolev spaces) $I_\nu : L^p \to W^{\nu,q}$ continuously for any $1 < p < \infty$;
- (Smoothing in Sobolev-BMO spaces) $I_\nu : L^p \to I_s(BMO)$ continuously for any $\frac{n}{\nu} < p < \infty$ with $\frac{1}{p} = \frac{\nu - s}{n}$ and also $I_\nu : L^{n/\nu} \to (BMO)$.

Let us see what happens for a general $T \in SIO_{-\nu}(M+\delta)$:

- We have the bound
$$|Tf(x)| \leq CI_\nu(|f|)(x)\,,$$
for these operators; so, it trivially follows that T satisfies the same L^p-smoothing properties as I_ν;
- For the critical index $p = \frac{n}{\nu}$, any such T can be extended to a bounded linear operator from $L^{n/\nu}$ into BMO (as long as $0 < \nu < n$);
- Harboure, Salinas, and Viviani extended the definition of the Riesz potential I_ν to L^p for a restricted range of values for $p \geq n/\nu$, and had proved some estimates for I_ν mapping into Campanato type spaces;
- For non-convolution type operators, less is known about the behavior of fractional integral operators when $n/\nu < p < \infty$.

Theorem 1. If $\nu > 0$, and $T \in SIO_{-\nu}(M+\delta)$, then T is bounded from L^p into $I_s(BMO)$, for all $0 \leq s < \min(\nu, M+\delta)$, with $1/p = (\nu - s)/n < 1$.

Remarks:

- The $L^{n/\nu} \to BMO$ was already know (probably due to Stein-Zygmund or to Muckenhoupt-Wheden) when $0 < \nu < n$;
- There is no restriction on $\nu > 0$;
- The exponent restriction is $n/\nu \leq p < \infty$.

4. Bilinear fractional integral type operators

We can obtain also a bilinear version of the above result. Precisely, if we Consider the bilinear operator T defined via

$$T(f,g)(x) = \int_{\mathbb{R}^{2n}} K(x,y,z) f(y) g(z) dy\, dz$$

for appropriate functions $f, g : \mathbb{R}^n \to \mathbb{C}$, we will say that T **is a ν-order bilinear fractional integral operator** with kernel regularity $M + \gamma$ for some integer $M \geq 0$ and $0 < \gamma \leq 1$, denoted $T \in BSIO_{-\nu}(M+\gamma)$, if the kernel K satisfies

$$|D_x^\alpha K(x,y,z)| \lesssim \frac{1}{(|x-y|+|x-z|)^{2n+|\alpha|-\nu}}$$

for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq M$ and

$$\begin{aligned}&|D_x^\alpha K(x,y,z) - D_x^\alpha K(x+h,y,z)| \\ &\lesssim \frac{|h|^\gamma}{(|x-y|+|x-z|)^{2n+|\alpha|+\gamma-\nu}}\end{aligned}$$

for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = M$ for all $x, y, z, h \in \mathbb{R}^n$ with $|x-y| + |x-z| \neq 0$ and $|h| < (|x-y| + |x-z|)/4$. For this class of operators, we can prove the following result:

Theorem 2. Let $T \in BSIO_{-\nu}(M+\gamma)$ for some integer $M \geq 0$, $0 < \gamma \leq 1$, $\nu > 0$, and $0 \leq s < \min(M+\gamma, \nu)$.

- If $0 < \nu < n$ and $1 < p_1, p_2 \leq \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{\nu - s}{n}$, then
$$||T(f,g)||_{I_s(BMO)} \lesssim ||f||_{L^{p_1}} ||g||_{L^{p_2}}$$
for all $f \in L^{p_1}$ and $g \in L^{p_2}$.
- If $\nu = n$, $s \neq 0$, and $1 < p_1, p_2 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{\nu - s}{n}$, then
$$||T(f,g)||_{I_s(BMO)} \lesssim ||f||_{L^{p_1}} ||g||_{L^{p_2}}$$
for all $f \in L^{p_1}$ and $g \in L^{p_2}$.
- If $n < \nu < 2n$, $s > \nu - n$, and $1 < p_1, p_2 < \infty$ such that $\min(p_1, p_2) < \frac{n}{\nu - n}$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{\nu - s}{n}$, then
$$||T(f,g)||_{I_s(BMO)} \lesssim ||f||_{L^{p_1}} ||g||_{L^{p_2}}$$
for all $f \in L^{p_1}$ and $g \in L^{p_2}$.

Remarks:

- The theorem above does in fact exhibit the appropriate behavior in terms of smoothing action according to the size of ν.
- $\nu = n$ is a threshold value, where the operators in $BSIO_{-\nu}(M+\gamma)$ behave significantly different on either side of the $\nu = n$ mark.
- For $0 < \nu < n$, the exponents p_1 and p_2 must belong to an interval of the form $[\frac{n}{\nu}, \infty)$, whereas for larger values $n < \nu < 2n$ the exponents are allowed to range all the way down to 1 and at least one of p_1 or p_2 must belong to $(1, \frac{n}{\nu - n})$.
- More can be said situation where ν is large, but we will not include this result here.

5. Final Remarks

To finish we would like just to indicate that the results above can be used to study problems about singular and hypersingular integral operators:

- In (Hart-O 2017), we use the weight invariance of the Sobolev-BMO Spaces to obtain weighted Hardy space estimates for singular integrals. To be precise, if T is a CZO bounded on L^2 and if $T^*(x^\alpha) = 0$ for all $\alpha \in \mathbb{N}_0$ verifying $|\alpha| \leq M$, then $T : H_w^p \to H_w^p$ for all $n/(n+M+1) < p < \infty$ and $w \in A_{p(n+M+1)/n}$.
- Currently, in a joint work with Lucas Chaffee and Jarod Hart, we are extending the **Theorem 1** and **Theorem 2** to the case of singular integral ($\nu = 0$) and hypersingular integral ($\nu > 0$) cases.

References

[1] E. Harboure, O. Salinas, and B. Viviani, *Boundedness of the fractional integral on weighted Lebesgue and Lipschitz spaces*, Trans. Amer. Math. Soc. **349**, (1997), no. 1, 235-255.

[2] B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **192**, (1974), 261-274.

[3] R. Strichartz, *Bounded mean oscillation and Sobolev spaces*, Indiana Univ. Math. J. **29**, (1980), no. 4, 539-558.

[4] L. Chaffee, J. Hart and L. Oliveira, *Sobolev-BMO and fractional integrals on super-critical ranges of Lebesgue spaces*, J. Funct. Anal., **272** (2017), 631–660.

[5] J. Hart and L. Oliveira, *Hardy space estimates for limited ranges of Muckenhoupt weights*, Adv. Math., **313** (2017) 803–838.

[6] L. Chaffee, J. Hart and L. Oliveira, *Operator Calculus for Singular Integrals, work in progress.*