Oblique Duality for Fusion Frames

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Abstract

We introduce and develop the concept of oblique duality for fusion frames. This concept provides a mathematical framework to deal with problems in distributed signal processing where the signals considered as elements in a Hilbert space are, under certain requirements, analyzed in one subspace and reconstructed in another subspace. The requirements are, on one side, the uniqueness of the reconstructed signal, and on the other what we call consistency of the sampling for fusion frames. Both conditions are naturally related to oblique projections. We study the main properties of oblique dual fusion frames and oblique dual fusion frame systems and present several results that provide alternative methods for their construction.

Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and $\mathcal{W}$ a closed subspace of $\mathcal{H}$.

Definition. *We say that $\{(W_i, w_i)\}_{i\in I}$ is a fusion frame (FF) for $\mathcal{W}$, if there exist constants $0 < \alpha \leq \beta < \infty$ such that*

$$\alpha\|f\|^2 \leq \sum_{i\in I} w_i^2 \|\pi_{W_i}(f)\|^2 \leq \beta\|f\|^2 \text{ for all } f \in \mathcal{W}. \quad (1)$$

If the right inequality is satisfied, then $\{(W_i, w_i)\}_{i\in I}$ is a *Bessel fusion sequence* of $\mathcal{W}$.

Definition. *Let $\mathcal{W}$ be a closed subspace of $\mathcal{H}$, let $\{(W_i, w_i)\}_{i\in I}$ be a FF (Bessel fusion sequence) for $\mathcal{W}$, and let $\{f_{i,l}\}_{l\in L_i}$ be a frame for W_i for $i \in I$. Then $\{(W_i, w_i, \{f_{i,l}\}_{l\in L_i})\}_{i\in I}$ is called a FF system (Bessel fusion system) for $\mathcal{W}$.*

Notation:

- $\{w_i\}_{i\in I} := \mathbf{w}$.
- $\{(W_i, w_i)\}_{i\in I} = (\mathbf{W}, \mathbf{w})$.
- $\mathcal{K}_\mathcal{W} = \{(f_i)_{i\in I} : f_i \in \mathcal{W}_i$ and $\{\|f_i\|\}_{i\in I} \in \ell^2(I)\}$ with inner product $\langle (f_i)_{i\in I}, (g_i)_{i\in I}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle$.
- $\mathcal{F}_i = \{f_{i,l}\}_{l\in L_i}$, $\mathcal{F} = \{\mathcal{F}_i\}_{i\in I}$, $\mathbf{w}\mathcal{F} = \{w_i\mathcal{F}_i\}_{i\in I}$, and $(\mathbf{W}, \mathbf{w}, \mathcal{F}) = \{(W_i, w_i, \{f_{i,l}\}_{l\in L_i})\}_{i\in I}$.
- For $T \in L(\mathcal{H}, \mathcal{K})$: $\{(TW_i, v_i)\}_{i\in I} := (T\mathbf{W}, \mathbf{v})$, $T\mathcal{F} = \{\{Tf_{i,l}\}_{l\in L_i}\}_{i\in I}$ and $T\mathcal{F}_i = \{Tf_{i,l}\}_{l\in L_i}$.

Associated operators:

Let $(\mathbf{W}, \mathbf{w})$ be a Bessel fusion sequence of $\mathcal{W}$

- *Synthesis operator :* $T_{\mathbf{W},\mathbf{w}} : \mathcal{K}_\mathcal{W} \to \mathcal{H}$, $T_{\mathbf{W},\mathbf{w}}(f_i)_{i\in I} = \sum_{i\in I} w_i f_i$.
- *Analysis operator:* $T^*_{\mathbf{W},\mathbf{w}} : \mathcal{H} \to \mathcal{K}_\mathcal{W}$, $T^*_{\mathbf{W},\mathbf{w}} f = \big(w_i\pi_{W_i}(f)\big)_{i\in I}$
- *Fusion frame operator:* $S_{\mathbf{W},\mathbf{w}} = T_{\mathbf{W},\mathbf{w}}T^*_{\mathbf{W},\mathbf{w}}$.

$(\mathbf{W}, \mathbf{w})$ is a *Riesz fusion basis* (RFB) for $\mathcal{W}$ if $\mathcal{W}$ is the direct sum of the W_i. A FF which is not a Riesz basis is called an *overcomplete* FF.

Oblique projections:

Definition. *Let $\mathcal{W}$ y $\mathcal{V}$ be two closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^\perp$ (or equivalently $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$)*
The oblique projection onto $\mathcal{V}$ along $\mathcal{W}^\perp$ is the unique operator that satisfies

$$\pi_{\mathcal{V},\mathcal{W}^\perp} f = f \text{ para todo } f \in \mathcal{V},$$
$$\pi_{\mathcal{V},\mathcal{W}^\perp} f = 0 \text{ para todo } f \in \mathcal{W}^\perp.$$

We will note $\mathfrak{L}_T^{\mathcal{V},\mathcal{W}^\perp} = \{U \in L(\mathcal{K}, \mathcal{H}) : UT = \pi_{\mathcal{V},\mathcal{W}^\perp}$ and $\mathrm{Im}(U) = \mathcal{V}\}$.

Oblique duality for fusion frames and fusion frame systems

Problem: Let $(\mathbf{W}, \mathbf{w})$ be a FF for $\mathcal{W}$. Let $f \in \mathcal{H}$ be un unknown signal that we want to reconstruct from its samples $T^*_{\mathbf{W},\mathbf{w}} f = (w_i\pi_{W_i}f)_{i\in I}$ using a FF $(\mathbf{V}, \mathbf{v})$ of $\mathcal{V}$, such that the reconstruction $\widetilde{f}$ is a good approximation of f i.e. we want:

(i) *Uniqueness of the reconstructed signal*: If $f, g \in \mathcal{V}$ and $T^*_{\mathbf{W},\mathbf{w}} f = T^*_{\mathbf{W},\mathbf{w}} g$, then $f = g$.

(ii) *Consistent sampling*: $T^*_{\mathbf{W},\mathbf{w}}\widetilde{f} = T^*_{\mathbf{W},\mathbf{w}} f$ for all $f \in \mathcal{H}$.

Requirement (i) is equivalent to $\mathcal{V} \cap \mathcal{W}^\perp = \{0\}$.
In case that (ii) is satisfied we say that $\widetilde{f} \in \mathcal{V}$ is a *consistent reconstruction* of $f \in \mathcal{H}$.
From (i) and (ii), we deduce that if $f \in \mathcal{V}$ then $\widetilde{f} = f$. So in this case, f can be perfectly reconstructed.
The next result shows how consistent reconstruction is linked to oblique projections.

Theorem. *Let $\mathcal{W}$ and $\mathcal{V}$ be two closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^\perp$. Let $(\mathbf{W}, \mathbf{w})$ be a FF for $\mathcal{W}$. Then $\widetilde{f} \in \mathcal{V}$ is a consistent reconstruction of $f \in \mathcal{H}$ if and only if $\widetilde{f} = \pi_{\mathcal{V},\mathcal{W}^\perp} f$.*

In order to have an adequate instrument to solve the problem described before we introduce the definition of oblique dual fusion frames.

Definition. *Let $\mathcal{W}$ and $\mathcal{V}$ be two closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^\perp$. Let $(\mathbf{W}, \mathbf{w})$ be a FF for $\mathcal{W}$ and $(\mathbf{V}, \mathbf{v})$ be a FF for $\mathcal{V}$. We say that $(\mathbf{V}, \mathbf{v})$ is an oblique dual fusion frame (ODFF) of $(\mathbf{W}, \mathbf{w})$ on $\mathcal{V}$ if there exists $Q \in L(K_\mathcal{W}, K_\mathcal{V})$ such that*

$$T_{\mathbf{V},\mathbf{v}} Q T^*_{\mathbf{W},\mathbf{w}} = \pi_{\mathcal{V},\mathcal{W}^\perp}. \quad (2)$$

The operator Q is actually important in the definition. If we need to do an explicit reference to it we say that $(\mathbf{V}, \mathbf{v})$ is a Q-ODFF of $(\mathbf{W}, \mathbf{w})$.
If in the previous definition $\mathcal{W} = \mathcal{V} = \mathcal{H}$ we say that $(\mathbf{V}, \mathbf{v})$ is a Q-dual FF of $(\mathbf{W}, \mathbf{w})$ (see [2, 3]).

Corollary. *Let $\mathcal{W}$ and $\mathcal{V}$ be two closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^\perp$. Let $(\mathbf{W}, \mathbf{w})$ be a FF for $\mathcal{W}$, $(\mathbf{V}, \mathbf{v})$ be a FF for $\mathcal{V}$ and $Q \in L(K_\mathcal{W}, K_\mathcal{V})$. Then $\widetilde{f} := T_{\mathbf{V},\mathbf{v}} Q T^*_{\mathbf{W},\mathbf{w}} f$ is a consistent reconstruction of f for all $f \in \mathcal{H}$ if and only if $(\mathbf{V}, \mathbf{v})$ is a Q-ODFF of $(\mathbf{W}, \mathbf{w})$ on $\mathcal{V}$.*

We now present two special types of linear transformations Q that make the reconstruction formula that follows from (2) simpler. We need the selfadjoint operator $M_{J,\mathbf{W}} : K_\mathcal{W} \to K_\mathcal{W}$, $M_{J,\mathbf{W}}(f_i)_{i\in I} = (\chi_J(i) f_i)_{i\in I}$. $M_{\{j\}} = M_j$.

Definition. *Let $Q \in L(K_\mathcal{W}, K_\mathcal{V})$.*

1. *If $QM_{j,\mathbf{W}}K_\mathcal{W} \subseteq M_{j,\mathbf{V}}K_\mathcal{V}$ for each $j \in I$, Q is called block-diagonal.*
2. *If $QM_{j,\mathbf{W}}K_\mathcal{W} = M_{j,\mathbf{V}}K_\mathcal{V}$ for each $j \in I$, Q is called component preserving.*

Oblique dual fusion frame systems:

Let $(\mathbf{W}, \mathbf{w})$ be a Bessel fusion sequence for $\mathcal{W}$ and $\mathcal{F}_i$ be a frame for W_i with frame bounds α_i, β_i such that $\sup_{i\in I}\beta_i = \beta < \infty$. Let

$$C_\mathcal{F} : \oplus_{i\in I}\ell^2(L_i) \to K_\mathcal{W},\ C_\mathcal{F}((x_{i,l})_{l\in L_i})_{i\in I} = (T_{\mathcal{F}_i}(x_{i,l})_{l\in L_i})_{i\in I}.$$

Definition. *Let $\mathcal{W}$ and $\mathcal{V}$ be two closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^\perp$. Let $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ be a FF system for $\mathcal{W}$ with upper local frame bounds β_i such that $\sup_{i\in I}\beta_i < \infty$, $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ be a FF system for $\mathcal{V}$ with local upper frame bounds $\widetilde{\beta}_i$ such that $\sup_{i\in I}\widetilde{\beta}_i < \infty$ and $|\mathcal{F}_i| = |\mathcal{G}_i|$ for each $i \in I$. Then $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ is an oblique dual FF system of $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ on $\mathcal{V}$ if $(\mathbf{V}, \mathbf{v})$ is a $C_\mathcal{G}C^*_\mathcal{F}$-ODFF of $(\mathbf{W}, \mathbf{w})$ on $\mathcal{V}$.*

If in the previous definition $\mathcal{W} = \mathcal{V} = \mathcal{H}$ we say that $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ is a dual FF system of $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ on $\mathcal{V}$.

Relation between block-diagonal oblique dual frames, oblique dual fusion frame systems and oblique dual frames:

The next theorem asserts that a block-diagonal oblique dual fusion frame pair can always be viewed as an oblique dual fusion frame system pair.

Theorem. *Let $\mathcal{W}$ and $\mathcal{V}$ be two closed subspaces of $\mathcal{H}$. Let $(\mathbf{W}, \mathbf{w})$ be a FF for $\mathcal{W}$ and let $(\mathbf{V}, \mathbf{v})$ be a block diagonal Q-ODFF of $(\mathbf{W}, \mathbf{w})$ on $\mathcal{V}$. Then there exists a frame $\mathcal{F}_i$ for W_i with frame bounds α_i, β_i such that $0 < \inf_{i\in I}\alpha_i \leq \sup_{i\in I}\beta_i < \infty$ and a frame $\mathcal{G}_i$ for V_i with frame bounds $\widetilde{\alpha}_i, \widetilde{\beta}_i$ such that $0 < \inf_{i\in I}\widetilde{\alpha}_i \leq \sup_{i\in I}\widetilde{\beta}_i < \infty$, such that $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ is an ODFF system of $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ on $\mathcal{V}$ and $Q = C_\mathcal{G}C^*_\mathcal{F}$.*

The following results establishes the connection between the notions of oblique dual fusion frame system and oblique dual frame.

Theorem. *Let $\mathcal{W}$ and $\mathcal{V}$ be two closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^\perp$. Let $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ be a Bessel fusion system for $\mathcal{W}$ such that $\mathcal{F}_i$ has upper frame bound β_i with $\sup_{i\in I}\beta_i < \infty$, and let $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ be a Bessel fusion system for $\mathcal{V}$ such that $\mathcal{G}_i$ has upper frame bound $\widetilde{\beta}_i$ with $\sup_{i\in I}\widetilde{\beta}_i < \infty$. If $|\mathcal{F}_i| = |\mathcal{G}_i|$ for each $i \in I$ then the following conditions are equivalent:*

1. *$\mathbf{v}\mathcal{G}$ is an oblique dual frame of $\mathbf{w}\mathcal{F}$ on $\mathcal{V}$.*
2. *$(\mathbf{V}, \mathbf{v}, \mathcal{G})$ is an ODFF system of $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ on $\mathcal{V}$.*

Duals and Oblique Duals:

We can obtain dual FF systems from oblique dual FF systems and vice versa:

Proposition. *Let $\mathcal{W}$ and $\mathcal{V}$ be two closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^\perp$. Let $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ be a FF system for $\mathcal{W}$ with local upper frame bounds β_i such that $\sup_{i\in I}\beta_i < \infty$, $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ be a FF system for $\mathcal{V}$ with local upper frame bounds $\widetilde{\beta}_i$ such that $\sup_{i\in I}\widetilde{\beta}_i < \infty$ and $|\mathcal{F}_i| = |\mathcal{G}_i|$ for each $i \in I$. If $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ is an ODFF system of $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ on $\mathcal{V}$, then $(\pi_\mathcal{W}(\mathbf{V}), \mathbf{v}, \pi_\mathcal{W}(\mathcal{G}))$ is a dual FF system of $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ for $\mathcal{W}$ and $(\pi_\mathcal{V}(\mathbf{W}), \mathbf{w}, \pi_\mathcal{V}(\mathcal{F}))$ is a dual FF system of $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ for $\mathcal{V}$.*

Proposition. *Let $\mathcal{W}$ and $\mathcal{V}$ be two closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^\perp$. Let $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ be a FF system for $\mathcal{W}$ with local upper frame bounds β_i such that $\sup_{i\in I}\beta_i < \infty$, $(\widetilde{\mathbf{W}}, \widetilde{\mathbf{w}}, \widetilde{\mathcal{F}})$ be a FF system for $\mathcal{W}$ with local upper frame bounds $\widetilde{\beta}_i$ such that $\sup_{i\in I}\widetilde{\beta}_i < \infty$ and $|\mathcal{F}_i| = |\widetilde{\mathcal{F}}_i|$ for each $i \in I$. If $(\widetilde{\mathbf{W}}, \widetilde{\mathbf{w}}, \widetilde{\mathcal{F}})$ is a dual FF system of $(\mathbf{W}, \mathbf{w}, \mathcal{F})$, then $(\pi_{\mathcal{V},\mathcal{W}^\perp}\widetilde{\mathbf{W}}, \widetilde{\mathbf{w}}, \pi_{\mathcal{V},\mathcal{W}^\perp}\widetilde{\mathcal{F}})$ is an ODFF system of $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ on $\mathcal{V}$.*

As a consequence, analogous results are also true for block-diagonal fusion frames.

We can construct ODFF systems from a given FF for a closed subspace of $\mathcal{H}$ via local dual frames and an oblique left inverse of its analysis operator:

Proposition. *Let $\mathcal{V}$ and $\mathcal{W}$ be two closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^\perp$. Let $(\mathbf{W}, \mathbf{w})$ be a FF for $\mathcal{W}$, $A \in \mathfrak{L}^{\mathcal{V},\mathcal{W}^\perp}_{T^*_{\mathbf{W},\mathbf{w}}}$ and $\mathbf{v}$ be a collection of weights such that $\inf_{i\in I} v_i > 0$. For each $i \in I$ let $\{f_{i,l}\}_{l\in L_i}$ and $\{\tilde{f}_{i,l}\}_{l\in L_i}$ be dual frames for W_i, β_i upper frame bound of $\{f_{i,l}\}_{l\in L_i}$ such that $\sup_{i\in I}\beta_i < \infty$, $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ frame bounds of $\{\tilde{f}_{i,l}\}_{l\in L_i}$ such that $\sup_{i\in I}\widetilde{\beta}_i < \infty$, $\mathcal{G}_i = \{\frac{1}{v_i}A(\chi_i(j)\tilde{f}_{i,l})_{j\in I}\}_{l\in L_i}$ and $V_i = \overline{\mathrm{span}}\mathcal{G}_i$. Then*

1. *$\mathcal{G}_i$ is a frame for V_i with frame bounds $\|A^\dagger\|^{-2}\frac{\widetilde{\alpha}_i}{v_i^2}$ and $\|A\|^2\frac{\widetilde{\beta}_i}{v_i^2}$.*
2. *$(\mathbf{V}, \mathbf{v}, \mathcal{G})$ is a component preserving $Q_{A,\mathbf{v}}$-ODFF system of $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ on $\mathcal{V}$.*

The following proposition presents a way to construct component preserving ODFF systems from a given frame for a subspace, using an oblique left inverse of its analysis operator.

Proposition. *Let $\mathcal{V}$ and $\mathcal{W}$ be two closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^\perp$. Let $\mathbf{w}$ and $\mathbf{v}$ be two collections of weights such that $\inf_{i\in I} v_i > 0$. Let $\mathbf{w}\mathcal{F}$ be a frame for $\mathcal{W}$ with local upper frame bounds β_i such that $\sup_{i\in I}\beta_i < \infty$, $A \in \mathfrak{L}^{\mathcal{V},\mathcal{W}^\perp}_{T^*_{\mathbf{w}\mathcal{F}}}$ and $\{\{e_{i,l}\}_{l\in L_i}\}_{i\in I}$ be the standard basis for $\oplus_{i\in I}\ell^2(L_i)$. For each $i \in I$, set $W_i = \overline{span}\{f_{i,l}\}_{l\in L_i}$ and $V_i = \overline{span}\{\frac{1}{v_i}Ae_{i,l}\}_{l\in L_i}$. Let $\mathcal{G} = \{\{\frac{1}{v_i}Ae_{i,l}\}_{l\in L_i}\}_{i\in I}$. Then*

1. *$\{\frac{1}{v_i}Ae_{i,l}\}_{l\in L_i}$ is a frame for V_i with frame bounds $\frac{\|A^\dagger\|^{-2}}{v_i^2}$ and $\frac{\|A\|^2}{v_i^2}$.*
2. *$(\mathbf{V}, \mathbf{v}, \mathcal{G})$ is an ODFF system of $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ on $\mathcal{V}$.*

The canonical oblique dual fusion frame

Let $(\mathbf{W}, \mathbf{w})$ be a fusion FF for $\mathcal{W}$. Let $A = \pi_{\mathcal{V},\mathcal{W}^\perp}S^\dagger_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}} \in \mathfrak{L}^{\mathcal{V},\mathcal{W}^\perp}_{T^*_{\mathbf{W},\mathbf{w}}}$ $\mathbf{v}$ such that $(\pi_{\mathcal{V},\mathcal{W}^\perp}S^\dagger_{\mathbf{W},\mathbf{w}}\mathbf{W}, \mathbf{v})$ is a Bessel fusion sequence for $\mathcal{V}$. Assume that $Q_{A,\mathbf{v}} : K_\mathcal{W} \to \oplus_{i\in I}\pi_{\mathcal{V},\mathcal{W}^\perp}S^\dagger_{\mathbf{W},\mathbf{w}}W_i$ given by $Q_{A,\mathbf{v}}(f_i)_{i\in I} = (\frac{w_i}{v_i}\pi_{\mathcal{V},\mathcal{W}^\perp}S^\dagger_{\mathbf{W},\mathbf{w}}f_i)_{i\in I}$ is a well defined bounded operator. Then $(\pi_{\mathcal{V},\mathcal{W}^\perp}S^\dagger_{\mathbf{W},\mathbf{w}}\mathbf{W}, \mathbf{v})$ is a component preserving $Q_{A,\mathbf{v}}$- ODFF of $(\mathbf{W}, \mathbf{w})$ on $\mathcal{V}$. Given $\mathbf{v}$ we will refer to this oblique dual as the *canonical oblique dual with weights* $\mathbf{v}$ and to

$$Q^*_{\pi_{\mathcal{V},\mathcal{W}^\perp}S^\dagger_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}},\mathbf{v}}\, T^*_{\pi_{\mathcal{V},\mathcal{W}^\perp}S^\dagger_{\mathbf{W},\mathbf{w}}\mathbf{W},\mathbf{v}} f = T^*_{\mathbf{W},\mathbf{w}}S^\dagger_{\mathbf{W},\mathbf{w}}\pi_{\mathcal{W},\mathcal{V}^\perp} f$$

as the *oblique fusion frame coefficients* of $f \in \mathcal{H}$ with respect to $(\mathbf{W}, \mathbf{w})$ on $\mathcal{V}$.
The following lemma implies that oblique fusion frame coefficients are those which have minimal norm among all other coefficients.

Lemma. *Let $\mathcal{W}$ and $\mathcal{V}$ be two closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^\perp$. Let $(\mathbf{W}, \mathbf{w})$ be a fusion frame for $\mathcal{W}$ and $f \in \mathcal{H}$. For all $(f_i)_{i\in I} \in K_\mathcal{W}$ satisfying $T_{\mathbf{W},\mathbf{w}}(f_i)_{i\in I} = \pi_{\mathcal{W},\mathcal{V}^\perp} f$, we have*

$$||(f_i)_{i\in I}||^2 = ||T^*_{\mathbf{W},\mathbf{w}}S^\dagger_{\mathbf{W},\mathbf{w}}\pi_{\mathcal{W},\mathcal{V}^\perp} f||^2 + ||(f_i)_{i\in I} - T^*_{\mathbf{W},\mathbf{w}}S^\dagger_{\mathbf{W},\mathbf{w}}\pi_{\mathcal{W},\mathcal{V}^\perp} f||^2.$$

Proposition. *Let $(\mathbf{W}, \mathbf{w})$ be a fusion frame for a closed subspace $\mathcal{W} \subseteq \mathcal{H}$ and let $\mathcal{V}$ be a closed subspace such that $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^\perp$. Let $(\mathbf{W}, \mathbf{w})$ be an overcomplete fusion frame for $\mathcal{W}$ such that $W_i \neq \{0\}$ for every $i \in I$. Then there exist component preserving oblique dual fusion frames $(\mathbf{V}, \mathbf{v})$ of $(\mathbf{W}, \mathbf{w})$ different from $(\pi_{\mathcal{V},\mathcal{W}^\perp}S^\dagger_{\mathbf{W},\mathbf{w}}\mathbf{W}, \mathbf{w})$.*

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