Hausdorff dimension of unions of subsets of lines or k-planes

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There are several problems gathering around the general principle that an s-dimensional collection of d-dimensional sets in $\mathbf{R}^n$ must have positive measure if $s+d>n$ and Hausdorff dimension $s+d$ if $s+d \le n$, unless the sets have large intersections; the most famous example is the Kakeya conjecture. The principle was proved in the plane for circles about 20 years ago by Tom Wolff and for lines in the plane (and more generally for positive measure subsets of $n-1$-dimensional hyperplanes) recently by Kenneth Falconer and Pertti Mattila. We study the cases when in $\mathbf{R}^n$ we have lines (or more generally k-planes) or subsets of lines (or k-planes) of given Hausdorff dimension.

We prove that for any $1 \le k < n$ and $s \le 1$, the union of any nonempty s-Hausdorff dimensional family of k-planes of $\mathbf{R}^n$ has Hausdorff dimension $k+s$. More generally, we show that for any $0 < \alpha \le k$, if $B \subset \mathbf{R}^n$ and E is a nonempty collection of k-planes of $\mathbf{R}^n$ such that every $P \in E$ intersects B in a set of Hausdorff dimension at least α, then $\dim B \ge 2\alpha - k + \min(\dim E, 1)$, where dim denotes the Hausdorff dimension. This strengthens some of the mentioned results of Falconer and Mattila and extends the well known Furstenberg-type estimate that every α-Furstenberg set has Hausdorff dimension at least 2α and its generalization by Ursula Molter and Ezequiel Rela.

As an application we show that for any $0 \le k < n$, if a set $A \subset \mathbf{R}^n$ contains the k-skeleton of a rotated unit cube around every point of $\mathbf{R}^n$, or if A contains a k-plane at a fixed positive distance from every point of $\mathbf{R}^n$, then the Hausdorff dimension of A is at least $k+1$. These results are sharp by our recent results with Marianna Csörnyei, Alang Chang and Kornélia Héra.

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