Tame topology and Complex Analytic Geometry

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O-minimal structures, introduced by L. Van den Dries, C. Steinhorn and A. Pillay around 1982, may be viewed as a realization of the idea of tame topologies proposed by A. Grothendieck in “Esquisse d’un Programme” (1984).

Over the past decade there have been very exciting developments in number theory related to André–Oort conjecture based on so called Pila-Zannier strategy (mainly due to Habbeger, Klingler, Masser, Pila, Tsimerman, Yafaev, Zannier, ...) and o-minimality plays an essential part in this approach.

Unfortunately o-minimality is not well-known to general mathematicians, and in this talk I present some basics of o-minimality.

"André-Oort-Mordell-Lang-Manin-Mumford" type of statements.

A typical statement

Let A be an algebraic variety of a certain type, and $V \subseteq A$ a subvariety. If V contains infinitely many "special" points then V contains a non-trivial "special" subvariety.

Theorem (M. Laurent (1984))

Let $V \subseteq (\mathbb{C}^*, \cdot)^n$ be an algebraic subvariety (i.e. V is the zero locus of finitely many polynomials in $z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}$.)
If V contains infinitely many torsion points then V contains a coset of a non-trivial algebraic subgroup of $(\mathbb{C}^*, \cdot)^n$.

Remark. $(\xi_1, \ldots, \xi_n) \in (\mathbb{C}^*, \cdot)^n$ is a torsion point if and only if each ξ_i is a root of unity.

Pila–Zannier Approach.

Given: An algebraic subset $V \subseteq (\mathbb{C}^*, \cdot)^n$ containing infinitely many torsion points.
Goal: Show that V contains a coset of an infinite subgroup of $(\mathbb{C}^*, \cdot)^n$.
Idea: *Switch to the analytic side!*

Analytic Side

Let $E\colon \mathbb{C} \to \mathbb{C}^*$ be the map $E(z) = e^{2\pi i z}$. It is a $\mathbb{C}$-analytic $\mathbb{Z}$-periodic map.
Let $\mathrm{Exp}(\bar{z})\colon (\mathbb{C}, +)^n \to (\mathbb{C}^*, \cdot)^n$ be the covering map

$$\mathrm{Exp}(z_1, \dots, z_n) = (E(z_1), \dots, E(z_n)).$$

It is a $\mathbb{C}$-analytic surjective group homomorphism whose kernel is $\mathbb{Z}^n$.
Let $X = \mathrm{Exp}^{-1}(V)$. It is a complex analytic subset of $\mathbb{C}^n$.

Observation

$\mathrm{Exp}(\bar{z})$ is a torsion point of $(\mathbb{C}^*, \cdot)^n$ if and only if $\bar{z} \in \mathbb{Q}^n$.

On the analytic side

We have:

1. A complex analytic subset $X \subseteq \mathbb{C}^n$, i.e. locally at every point $\bar{z} \in \mathbb{C}^n$ it is the zero set of finitely many complex analytic functions.
2. X is $\mathbb{Z}^n$-invariant, i.e. $\bar{z} \in X$ and $\bar{n} \in \mathbb{Z}^n$ implies $\bar{z} + \bar{n} \in X$.
3. X contains infinitely many rational points.

Want to show:

4. X contains a translate of a linear subspace of $\mathbb{C}^n$.

A problem. In general $1 - 3$ above do not imply 4.
Complex analytic sets may be very complicated.

A remedy. The map $E(z) = e^{2\pi i z}$ is $\mathbb{Z}$-periodic! It maps the strip $\mathcal{F} = \{z \in \mathbb{C} \colon 0 \leqslant Re(z) \leqslant 1\}$ onto $\mathbb{C}^*$.

Let $\widetilde{X} = X \cap \mathcal{F}^n = \{(z_1, \ldots, z_n) \in X \colon 0 \leqslant Re(z_i) \leqslant 1\}$.

We have: $X = \widetilde{X} + \mathbb{Z}^n$, $\text{Exp}(\widetilde{X}) = V$, and $\widetilde{X}$ contains infinitely many rational points.

Analytic side: o-minimality

For $z = x + iy$ we have $E(z) = e^{2\pi iz} = e^{-2\pi y}(\cos(2\pi x) + i\sin(2\pi x))$.

A key point. Real and imaginary parts of the restriction $\widetilde{E} = E\restriction\mathcal{F}$ use only real functions $\exp(x)$, $\cos(2\pi x)\restriction[0,1]$, and $\sin(2\pi x)\restriction[0,1]$.
All these functions are definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$.
Also identifying $\mathbb{C}$ with $\mathbb{R}^2$, we obtain that $\widetilde{X} = \widetilde{E}^{-1}(V)$, as a subset of $\mathbb{R}^{2n}$, is also definable in $\mathbb{R}_{\mathrm{an,exp}}$.

We have:

1. A $\mathbb{Z}$-invariant $\mathbb{C}$-analytic subset $X \subseteq \mathbb{C}^n$ such that $\widetilde{X} = X \cap \mathcal{F}^n$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$.
2. $\widetilde{X}$ contains infinitely many rational points.

We need to show:

3. There is a non-trivial linear subspace $L \subseteq \mathbb{C}^n$ with $L \cap \mathcal{F}^n \subseteq \widetilde{X}$.

Remark

The statement "$1 - 2$ implies 3" is equivalent to Laurent's theorem

A. Grothendieck, "Esquisse d'un Programme" (1984)

I would like to say a few words now about some topological considerations which have made me understand the necessity of new foundations for geometric topology.

"General topology" was developed by analysts and in order to meet the needs of analysis.
When one tries to do topological geometry in the technical context of topological spaces, one is confronted at each step with spurious difficulties related to wild phenomena.

Grothendieck's Program :

Develop a "tame topology" where no "wild" phenomena may take place (such as curves filling squares, continuous nowhere differentiable functions etc.).

A. Grothendieck, "Esquisse d'un Programme" (1984)

My approach has been an axiomatic one. I preferred to work on extracting which exactly, among the geometrical properties of the semianalytic sets in a space $\mathbb{R}^n$, make it possible to use these as local "models" for a notion of "tame space".

Grothendieck's Plan :

Consider the sequence $(\mathfrak{D}_k)_{k\in\mathbb{N}}$ of semianalytic subsets of $\mathbb{R}^k$, and understand what makes them to be "tame".

A Model Theoretic Solution.

O-minimal structures.

L. Van den Dries; C. Steinhorn and A. Pillay (1982).

Tarski Systems on the field $\mathbb{R}$

By a Tarski system on $\mathbb{R}$ we mean a a sequence $\mathfrak{D} = \{\mathcal{D}_k : n \in \mathbb{N}\}$, where each $\mathcal{D}_k$ is a family of subsets of $\mathbb{R}^k$, satisfying the following properties:

1. Each $\mathcal{D}_k$ contains all algebraic subsets of $\mathbb{R}^k$, i.e. zero sets of real polynomials in k variables.
2. $\mathcal{D}_k$ is a Boolean subalgebra of $\mathcal{P}(\mathbb{R}^k)$.
3. If $A \in \mathcal{D}_k$ and $B \in \mathcal{D}_l$ then $A \times B \in \mathcal{D}_{k+l}$.
4. If $A \in \mathcal{D}_k$ and $\pi : \mathbb{R}^k \to \mathbb{R}^l$ is a projection then $\pi(A) \in \mathcal{D}_l$.

Example

If we take $\mathcal{D}_k = \mathcal{P}(\mathbb{R}^k)$ then $\{\mathcal{D}_k : k \in \mathbb{N}\}$ is a Tarski system on $\mathbb{R}$.

Definition

Let $\mathfrak{D}$ be a Tarski system. If $X \in \mathcal{D}_k$ then we say that X is $\mathfrak{D}$-definable, or definable in $\mathfrak{D}$.

Definability

Let $\mathfrak{D} = (\mathcal{D}_k)$ be a Tarski system.

We already know that $\mathfrak{D}$-definable sets are closed under taking finite Boolean combinations.

Let A, B and $C \subseteq A \times B$ be $\mathfrak{D}$-definable sets.

1. The set $\{a \in A \colon \exists b \in B$ such that $(a, b) \in C\}$ is $\mathfrak{D}$-definable as well: it is the projection of C onto the first coordinates.
2. The set $\{a \in A \colon \forall b \in B \quad (a, b) \in C\}$ is also $\mathfrak{D}$-definable.
3. If $A \subseteq \mathbb{R}$ is $\mathfrak{D}$-definable then the sets $A^{\geqslant 0}$, $A^{>0}$, $A^{\leqslant 0}$, $A^{<0}$ are also $\mathfrak{D}$-definable.

$$A^{\geqslant 0} = \{a \in A \colon \exists x \; a - x^2 = 0\}.$$

Remark

If $A \subseteq \mathbb{R}^k$ can be obtained from $\mathfrak{D}$-definable sets using finitely many Boolean operations, equalities, inequalities and finitely many quantifiers *"there is a real number x ..."*, *"for all real numbers x ..."*, then A is $\mathfrak{D}$-definable.

More on definability

Let $\mathfrak{D}$ be a Tarski system on $\mathbb{R}$.

Example

If $A \subseteq \mathbb{R}^k$ is $\mathfrak{D}$-definable set then its topological closure is $\mathfrak{D}$-definable as well.
Indeed, $\mathrm{cl}(A) = \{\bar{y} \in \mathbb{R}^k : \forall \epsilon(\epsilon > 0 \to \exists \bar{x}[\bar{x} \in A \,\&\, \sum(x_i - y_i)^2 < \epsilon.])\}$

Definition

A function $f \colon A \to B$ is $\mathfrak{D}$-definable if its graph is $\mathfrak{D}$-definable.

Claim

1. *A composition of $\mathfrak{D}$-definable functions is $\mathfrak{D}$-definable.*
2. *If $f \colon \mathbb{R}^m \to \mathbb{R}^n$ is $\mathfrak{D}$-definable then:*
    - *The set $\{x \in \mathbb{R}^m : f \text{ is continuous at } x\}$ is $\mathfrak{D}$-definable.*
    - *The set $\{x \in \mathbb{R}^m : f \text{ is differentiable at } x\}$ is $\mathfrak{D}$-definable.*

The smallest Tarski system: Semialgebraic Sets

If $\mathfrak{D} = (\mathcal{D}_k)_{k\in\mathbb{N}}$ is a system of subsets of $(\mathbb{R}^k)_{k\in\mathbb{N}}$ then there is the list Tarski system $\widetilde{\mathfrak{D}}$ containing $\mathfrak{D}$.
In particular there is the smallest Tarski system on $\mathbb{R}$.

Definition

We say that a subset $A \subseteq \mathbb{R}^k$ is semialgebraic if it is a finite Boolean combination of sets of the form $\{\bar{x} \in \mathbb{R}^k \colon f(\bar{x}) = 0\}$ and $\{\bar{x} \in \mathbb{R}^k \colon g(\bar{x}) > 0\}$, where f and g are polynomials.

Lest $\mathfrak{D}_{sa}$ be the system of semialgebraic sets. Obviously every Tarski system contains $\mathfrak{D}_{sa}$.

Theorem (Tarski-Seidenberg)

A projection of a semialgebraic set is semialgebraic.

Corollary

Semialgebraic sets form the smallest Tarski system on $\mathbb{R}$*.*

Tameness of Semialgebraic Sets

Theorem

1. *(O-minimality) If $A \subseteq \mathbb{R}$ is semialgebraic then A is a finite union of points and intervals.*
2. *Every semialgebraic set has finitely many connected components.*
3. *If A is a semialgebraic set then $\dim(cl(A) \setminus A) < \dim(A)$.*
4. *If ,$f : \mathbb{R} \to \mathbb{R}$ is a semialgebraic function then $\mathbb{R}$ can be partition into finitely many intervals so that f is continuous, monotone and differentiable on each interval.*
5. *If $A \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is a semialgebraic set then there is $K \in \mathbb{N}$ such for any $\bar{a} \in \mathbb{R}^m$ for the set $A_{\bar{a}} = \{\bar{x} \in \mathbb{R}^n : (\bar{a}, \bar{x}) \in A\}$, if $|A_{\bar{a}}| > K$ then it is infinite.*

Theorem (van den Dries)

In any Tarski system 1 implies 2–5.

Tame Tarski systems

Definition

A Tasrki system $\mathfrak{D} = (\mathcal{D}_k)_{k \in \mathbb{N}}$ is o-minimal if every $A \in \mathcal{D}_1$ is a finite union of points and intervals.

Theorem (van den Dries)

Let $\mathfrak{D}$ be an o-minimal Tarski system.

1. *If A is $\mathfrak{D}$-definable then $\dim(cl(A) \setminus A) < \dim(A)$.*
2. *Every $\mathfrak{D}$-definable set has finitely many connected components.*
3. *If $f\colon \mathbb{R}^n \to \mathbb{R}$ is $\mathfrak{D}$-definable then, for any $r \in \mathbb{N}$, $\mathbb{R}^n$ can be partition into finitely many $\mathfrak{D}$-definable sets so that f is C^r on each of them.*
4. *Any $\mathfrak{D}$-definable set admits a $\mathfrak{D}$-definable triangulation.*
5. *If $A \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is a $\mathfrak{D}$-definable set then there is $K \in \mathbb{N}$ such for any $\bar{a} \in \mathbb{R}^m$ for the set $A_{\bar{a}} = \{\bar{x} \in \mathbb{R}^n\colon (\bar{a}, \bar{x}) \in A\}$, if $|A_{\bar{a}}| > K$ then it is infinite.*

Structures and definable sets

In this talk by a structure on $\mathbb{R}$ we mean an expansion of the real filed by functions, i.e. a structure is a collection $\mathcal{F}$ of functions $f \colon \mathbb{R}^{m_f} \to \mathbb{R}^{n_f}$.

Definition

Let $\mathbb{R}_{\mathcal{F}} = (\mathbb{R}, +, \cdot, \mathcal{F})$ be a structure, and $\mathfrak{D}_{\mathcal{F}}$ be the smallest Tarski system containing all $f \in \mathcal{F}$.

We say that a subset $A \subseteq \mathbb{R}^n$ is $\mathbb{R}_{\mathcal{F}}$-definable if it is $\mathfrak{D}_{\mathcal{F}}$-definable.

Also we say that the structure $\mathbb{R}_{\mathcal{F}}$ is o-minimal if $\mathfrak{D}_{\mathcal{F}}$ is o-minimal.

Example

If $\mathcal{F}$ contains the function $\sin(x)$ then the strucrure $\mathcal{R}_{\mathcal{F}}$ it is not o-minimal:
the set $\{x \in \mathbb{R} \colon sin(x) = 0\}$ is $\mathbb{R}_{\mathcal{F}}$-definable, but it is not a union of finitely many points and intervals.

Semianalytic sets

A restricted analytic function is a function $f\colon \mathbb{R}^n \to \mathbb{R}$ with the property that there exists a real analytic function $\tilde{f}\colon U \to \mathbb{R}$, where U is an open neighborhood of the unit cube $[0,1]^n$, such that

$$f(x) = \begin{cases} \tilde{f}(x) & x \in [0,1]^n \\ 0 & \text{otherwise.} \end{cases}$$

Ler $\mathcal{F}_{\mathrm{an}}$ be the set of all restricted analytic functions and $\mathbb{R}_{\mathrm{an}} = \mathbb{R}_{\mathcal{F}_{\mathrm{an}}}$. We are interested in the Tarski system $\mathfrak{D}_{\mathrm{an}}$ of $\mathbb{R}_{\mathrm{an}}$-definable sets.

Definition

A subset $A \subseteq \mathbb{R}^n$ is called (globally) semianalytic if it is a finite Boolean combination of sets of the from $f(\bar{x}) = 0$ and $g(\bar{x}) > 0$, wehre f, g are compositions of polynomials and restricted analytic functions.

Obviously every semianalytic set is $\mathbb{R}_{\mathrm{an}}$-definable.

Subanalytic sets

Remark

A projection of a semianalytic set does not have to be semianalytic. Hence semianalytic sets do not form a Tarski system.

Definition

A projection of a semianalytic set is called (globally) sabanalytic.

Obviously every subanalytic set is $\mathbb{R}_{\mathrm{an}}$-definable.

Theorem (Gabrielov; Denef, van den Dries)

Subanalytic sets from a Tarski system, i.e. $\mathfrak{D}_{\mathrm{an}}$ consists of subanalytics sets.

This Tarski system is o-minimal.

The structure $\mathbb{R}_{\mathrm{an,exp}}$

Let $\mathbb{R}_{\mathrm{an,exp}}$ be the structure $\mathbb{R}_{\mathrm{an}}$ together with the function $\exp(x)$.

Theorem (van den Dries, Macintyre, Marker)

The structure $\mathbb{R}_{\mathrm{an,exp}}$ is o-minimal.

On Density of integer points

A key part in Pilla–Zannir strategy is Pila–Wilkie Theorem.

Idea. Transcendental sets should not contain "many" rational points.

Example

Let Γ_f be the graph of the function $f(x) = x^2$, and Γ_h be the graph of the function $h(x) = 2^x$. Both graphs contain infinitely many integer points.

But, for $T \in \mathbb{N}$ we have:
$\left|\Gamma_f \cap \mathbb{N}^2 \cap [0, T]^2\right| \approx \sqrt{T}$ and $\left|\Gamma_h \cap \mathbb{N}^2 \cap [0, T]^2\right| \approx \log T$

Example

For the graph Γ of the function $\sin(\pi x)$ we have $\left|\Gamma \cap \mathbb{N}^2 \cap [0, T]^2\right| \approx T$.

Remark

The function $\sin(\pi x)$ is not definable in any o-minimal structure.

Density of Rational Points: Pila–Wilkie Theorem

For $q = \frac{a}{b} \in \mathbb{Q}$ with $(a, b) = 1$, the height of q, denoted by $ht(q)$, is $\max\{|a|, |b|\}$.

For $(q_1, \ldots, q_n) \in \mathbb{Q}^n$ let $ht(q_1, \ldots, q_n) = \max\{ht(q_1), \ldots, ht(q_n)\}$.

For $S \subseteq \mathbb{R}^n$ and a real number $T \geqslant 1$ let

$$S_{\mathbb{Q}}(T) = \{p \in S \cap \mathbb{Q}^n \colon ht(p) \leqslant T\}.$$

Theorem (Pila–Wilkie)

Let $S \subseteq \mathbb{R}^n$ be definable in an o-minimal structure. Assume there is $\varepsilon > 0$ such that $|S_{\mathbb{Q}}(T)| > T^{\varepsilon}$ for all sufficiently large T. Then S contains a piece of an algebraic curve.

Complex Analytic Subsets

Recall that a subset S of a complex manifold M is called a complex analytic subset of M if for every $p \in M$ there is an open neighborhood U of p such that $S \cap U$ is the zero locus of finitely many holomorphic on U functions.

Example

Every algebraic subvariety of $\mathbb{C}^n$ (or $\mathbb{P}^n(\mathbb{C})$) is complex analytic.

Over the field of complex numbers Chow's theorem (or more general Serre's GAGA) allows quite a free use of complex analytic methods within projective algebraic geometry.

Theorem (Chow)

A complex analytic subset S of a projective space $\mathbb{P}^n(\mathbb{C})$ is an algebraic variety.

O-minimality provides a way to use complex analytic methods for arbitrary algebraic varieties over $\mathbb{C}$.

Definable complex analytic sets

Let $\mathfrak{D}$ be a Tarski system on $\mathbb{R}$.

Identifying as usual $\mathbb{C}$ with $\mathbb{R}^2$ we say that a subset $A \subseteq \mathbb{C}^n$ is $\mathfrak{D}$-definable if A is $\mathfrak{D}$-definable as a subset of $\mathbb{R}^{2n}$.

Remark

- Every complex analytics subset of $\mathbb{C}^n$ is locally $\mathbb{R}_{\mathrm{an}}$-definable.
- Every complex analytics subset of $\mathbb{P}^n(\mathbb{C})$ is $\mathbb{R}_{an}$-definable.

Theorem (Peterzil–S.)

Let S be a complex analytic subset of $\mathbb{C}^n$ (or $(\mathbb{C}^*)^n$ or any algebraic variety over $\mathbb{C}$).
If S is definable in some o-minimal structure over $\mathbb{R}$ then S is an algebraic variety.

Remark

The above theorem implies Chow's theorem.