Arithmetic of Toric Varieties

Equidistribution of Galois orbits of small points

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XXI Coloquio Latinoamericano de Algebra

Buenos Aires, July 27, 2016

Distribution of polynomial roots

Question: If we choose a polynomial randomly, what can we say about its roots?

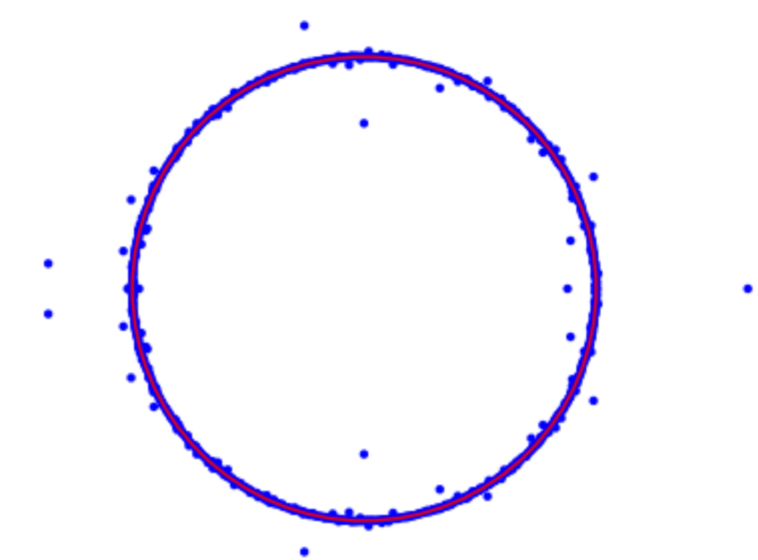
Randomly: Bounded coef with uniform distribution between $-C$ and C. (Not important.)

Naïve answer: Random roots?

→ Experiment.

```
A.set_aspect_ratio(1)
show(A,axes=False,figsize=10)
```

n 1000



```
from sage.rings.polynomial.complex_roots import complex_roots
x = polygen(ZZ)
@interact
def _rotillas(n=(3,100,1)):
    p=(x^n-1)*(x^n-2^n)+5
```

Height of rational numbers

$p = \frac{a}{b} \in \mathbb{Q}^\times$; $a, b \in \mathbb{Z}$, $\mathrm{mcd}(a,b) = 1$

The **Weil height** of p is

$$h(p) = \log(\max(|a|, |b|))$$

height of $p \sim$ "space needed to represent p in a computer".

The Weil height

Let $\xi \in \bar{\mathbb{Q}}^\times$, $P_\xi = \alpha_d X^d + \dots + \alpha_0 \in \mathbb{Z}[X]$ minimal polynomial

$P_\xi = \alpha_d \prod_{\eta \in G\xi} (X - \eta)$ $\quad G\xi$ Galois orbit of ξ

The Weil height of ξ

$$h(\xi) = \frac{1}{d}\left(\sum_{\eta \in G\xi} \log \max(1, |\eta|) + \log \alpha_d \right)$$

- if $\xi = \frac{a}{b} \in \mathbb{Q}^\times$ then $h(x) = \log \max(|a|, |b|)$
- Kronecker $\quad h(\xi) = 0 \iff \xi$ is a root of unity

Bilu's Equidistribution thm

Thm $p_k \in \bar{\mathbb{Q}}^\times$ $k = 1, 2, 3, \ldots$ Sequence of points such that.

* $\#\{k \mid p_k = x\} < \infty \quad \forall x \in \bar{\mathbb{Q}}^\times$
* $h(p_k) \xrightarrow[k\to\infty]{} 0$.

Then: $G.\,p_k \underset{w}{\Longrightarrow} S^1$;

For $f \in C^0(\mathbb{P}^1(\mathbb{C}))$

$$\lim_{k\to\infty} \frac{1}{\# G p_k} \sum_{z \in G p_k} f(z) = \int_{S^1} f \, d\mu_{S^1}$$

(Abelian varieties: Szpiro-Ullmo-Zhang.)

Mahler measure

$$h(\xi) = \frac{1}{d}\Big(\sum_{\eta \in G_S} \log \max(1, |\eta|) + \log a_d\Big)$$

$$= \frac{1}{d}\Big(\int_{S^1} \log |P(z)| \, d\mu_{S^1}\Big)$$

If $d \to \infty$ and the coefficients of P remain bounded then $h(\xi) \to 0$ hence the roots of a random polynomial are equidistributed when the degree growths

Height of Points

- $X^n/\mathbb{Q}$ proper algebraic variety
- D ample Cartier divisor

For each $v \in \mathfrak{M}_{\mathbb{Q}}$ (places of $\mathbb{Q}$.)

- X_v v-analytic space ⟵ $X(\mathbb{C}) + F_\infty$ $v=\infty$; Berkovich space $v \neq \infty$
- $\|\ \|_v$ Semipositive continuous metric on $\mathcal{O}(D)_v$

Almost all defined by a model $/\mathbb{Z}$.

Height of Points

- $\overline{D} = (D, (\|\cdot\|_v)_{v \in M_{\mathbb{Q}}})$ metrized Cartier divisor

The **height** of $p \in X(\mathbb{Q})$ w.r.t. $\overline{D}$ is

$$h_{\overline{D}}(p) = - \sum_{v \in M_{\mathbb{Q}}} \log \|s(p)\|_v$$

for any rational section s regular and $\neq 0$ at p.

Essential minimum

$$\mu^{\mathrm{ess}}_{\bar{D}}(X) = \inf \{\theta \in \mathbb{R} \mid \{p \in X(\bar{\mathbb{Q}}) \mid h(p) \leq \theta\} \text{ Zariski dense}\}$$

Fact: (p_k) a generic sequence in $X(\bar{\mathbb{Q}})$ i.e.
$\forall Y \subsetneq X$ $\quad \#\{k \mid p_k \in Y(\bar{\mathbb{Q}})\} < \infty$. Then

$$\lim_{k \to \infty} h_{\bar{D}}(p_k) \geq \mu^{\mathrm{ess}}_{\bar{D}}(X)$$

Problem: For $(p_k)_{k \geq 0}$ a generic sequence with

$$\lim_{k \to \infty} h_{\bar{D}}(p_k) = \mu^{\mathrm{ess}}_{\bar{D}}(X)$$

study the limit distribution of $G \cdot p_k$ in X_v.

Equidistribution of Galois orbits of small points

THM (Yuan 2008, Szpiro-Ullmo-Zhang, Bilu, Chambert-Loir, Favre-Rivera, Baker-Rumely, Gubler, ...)

$X^n/\mathbb{Q}$ proper, $\overline{D}$ metrized divisor.
With D is ample and $\overline{D}$ is semipositive

Let $(p_k)_{k\geq 0}$ be a generic sequence such that.

$$h_{\overline{D}}(p_k) \xrightarrow[k\to\infty]{} \frac{h_{\overline{D}}(X)}{(n+1)\deg_D(X)}$$

Then, for $v \in \mathfrak{M}_{\mathbb{Q}}$

$$G \cdot p_k \xrightarrow[k\to\infty]{\text{weakly}} \frac{1}{\deg D} \cdot c_1(\mathcal{O}(D), \|\,\|_v)^{\wedge n}$$

probability measure on X_v.

Yuan's equidistribution theorem is very strong, but has a very strong hypothesis.

By Zhang's theorem on successive minima

$$\frac{h_{\bar{D}}(X)}{(n+1)\deg_D(X)} \leq \mu_{\bar{D}}^{\mathrm{ess}}(X)$$

Hence the equidistribution theorem can only be applied when

$$\mu_{\bar{D}}^{\mathrm{ess}}(X) = \frac{h_{\bar{D}}(X)}{(n+1)\deg_D(X)}$$

Toric Varieties

$\mathbb{T} \cong \mathbb{G}_m^n$ a split algebraic torus / $\mathbb{Q}$.

A toric variety (with torus $\mathbb{T}$) is a normal variety X with an open dense immersion $\mathbb{T} \subseteq X$ and action

$\mathbb{T} \times X \to X$ extending $\mathbb{T} \times \mathbb{T} \to \mathbb{T}$.

Combinatorial description

A *fan* is a family of strictly convex rational polyhedral cones $\Sigma = \{\sigma\}$

- $\sigma, \tau \in \Sigma \Rightarrow \sigma \cap \tau$ is a face of σ and τ
- $\sigma \in \Sigma$ all faces of σ are in Σ.

Σ fan on $\mathbb{R}^n$, $\sigma \in \Sigma \rightsquigarrow X_\sigma$ affine toric variety

$\tau, \sigma \in \Sigma \quad \tau \subseteq \sigma \rightsquigarrow \quad X_\tau \hookrightarrow X_\sigma$

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} X_\sigma$$

Toric Cartier Divisors

Assume Σ covers $\mathbb{R}^n$ ($\Leftrightarrow$ X_Σ proper)

A virtual support function is $\psi: \mathbb{R}^n \to \mathbb{R}$. continuous

s.t. $\psi|_\sigma = m_\sigma \in (\mathbb{Z}^n)^\vee \quad \forall \sigma \in \Sigma$

$$\psi \rightsquigarrow D_\psi = (X_\sigma, \chi^{-m_\sigma})_{\sigma\in\Sigma}$$ toric Cartier divisor

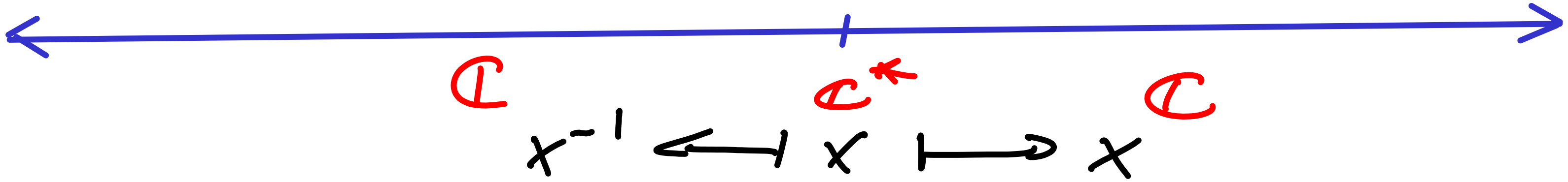
$$\Delta_\psi = \{ x \in (\mathbb{R}^n)^\vee \mid x \geq \psi \}$$ polytope

<u>Prop</u>: D_ψ nef $\Longleftrightarrow$ ψ concave. In this case

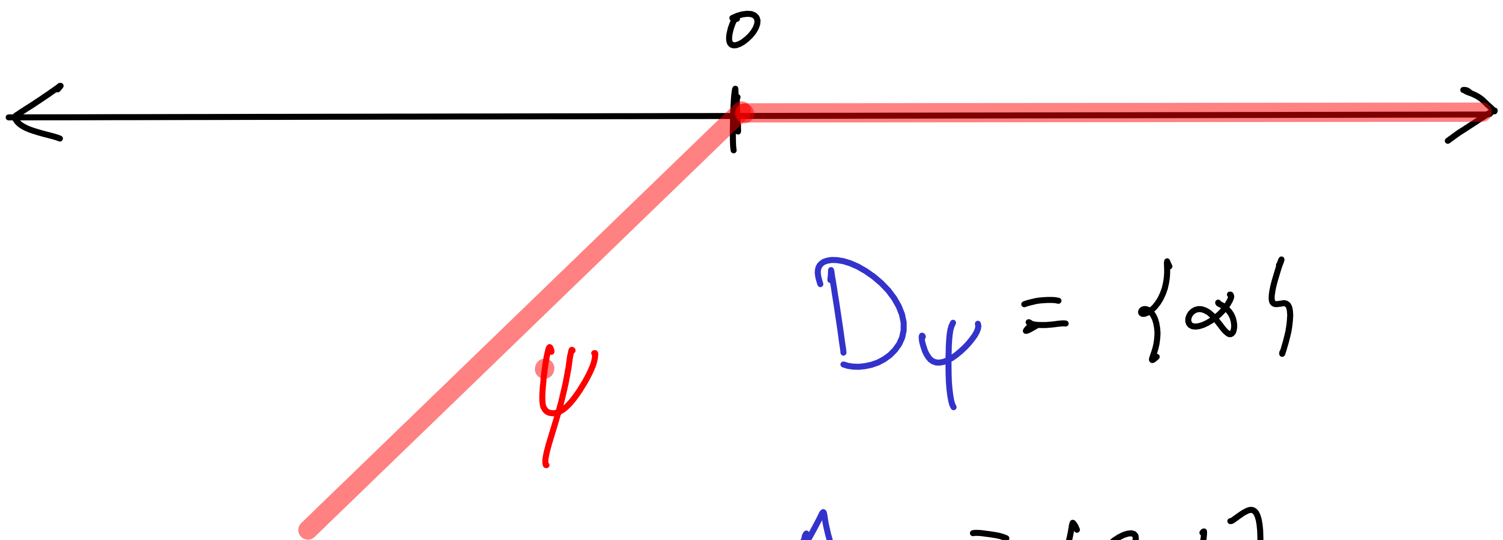
$$\deg_{D_\psi}(X) = n!\ \mathrm{Vol}\,\Delta_\psi .$$

Example: $X_\Sigma = \mathbb{P}^1$, $D = \mathcal{O}(1)$.

Fan:



Support function:



$D_\psi = \{\infty\}$

$\Delta_\psi = [0,1]$

$\deg(D_\psi) = 1 = 1!\,\mathrm{Vol}[0,1].$

Toric Metrics

$v \in M_{\mathbb{Q}}$

$\mathbb{S}_v \subseteq \mathbb{T}_v$ compact torus

Ex $\quad \mathbb{S}_\infty = \{(t_1, \dots, t_n) \in (\mathbb{C}^\times)^n \mid |t_i| = 1\}$

$\simeq (S^1)^n$

toric metric $:=$ $\mathbb{S}_v$-invariant metric.

Description of toric metrics

$$\begin{tikzcd}
\mathbb{T}_v \arrow[d, "\mathrm{val}_v"'] \arrow[drr, "\log \| s_\psi \|_v"] & & \\
\mathbb{T}_v / \mathbb{S}_v = \mathbb{R}^n \arrow[rr, "f_v"'] & & \mathbb{R}
\end{tikzcd}$$

$\mathbb{S}_v$ invariant continuous

$$\mathrm{val}_v(t_1, \dots, t_n) = (-\log |t_1|_v, \dots, -\log |t_n|_v)$$

THEOREM 1 There is a bijection

$$\left\{ \begin{array}{l} f_v : \mathbb{R}^n \to \mathbb{R} \text{ continuous,} \\ \text{concave, } \quad |f_v - \Psi| \text{ bounded} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} v\text{-adic continuous} \\ \text{semipositive metric} \\ \text{on } \mathcal{O}(D_\Psi) \end{array} \right\}$$

Roof function

The local roof function $\vartheta_v : \Delta_\varphi \to \mathbb{R}$.
is the Legendre-Fenchel dual of f_v:

$$\vartheta_v(x) = \inf_{u \in \mathbb{R}^n} \langle x, u \rangle - f_v(u)$$

The global roof function $\vartheta = \sum_v \vartheta_v$

The roof function encodes all the arithmetic information

An abridged toric dictionary

X toric variety	Σ fan
D nef toric divisor	$\Psi : \mathbb{R}^n \to \mathbb{R}$ concave Σ-lin. Δ lattice polytope
$\Vert\ \Vert_v$ Semipositive toric metric on $\mathcal{O}(D)_v$	$f_v : \mathbb{R}^n \to \mathbb{R}$ concave $\vert f_v - \Psi \vert$ bounded $\vartheta_v : \Delta \to \mathbb{R}$. concave
$\overline{D}$ metrized divisor	$\vartheta = \sum_v \vartheta_v$

Arithmetic properties

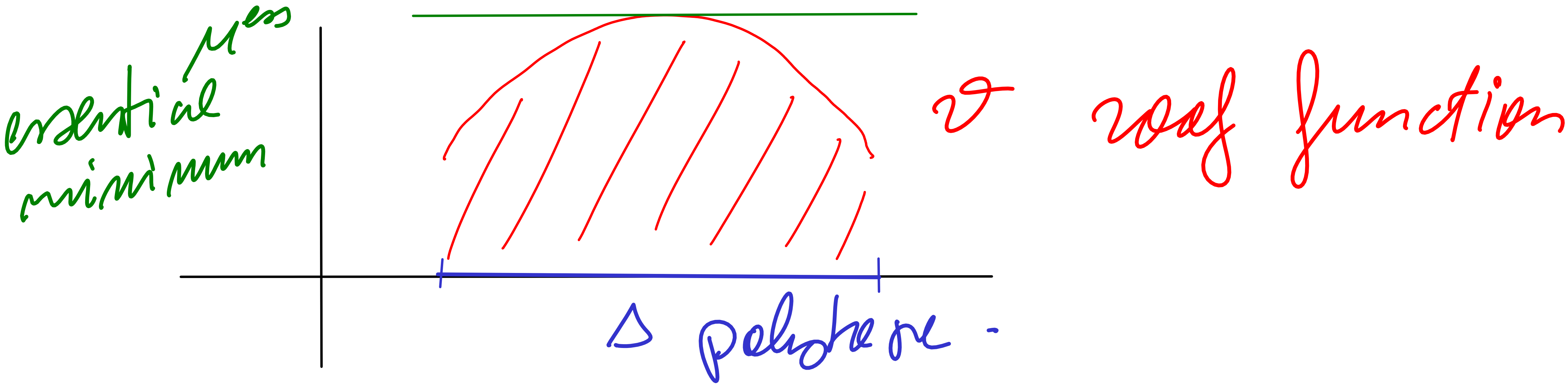
Most Arakelov invariants can be read from the roof function.

THEOREM 2 If $\overline{D}$ is semipositive

$$h_{\overline{D}}(X) = (n+1)! \int_{\Delta} \vartheta(x)\, d\mathrm{vol}$$

THEOREM 3 $$\mu^{\mathrm{ess}}_{\overline{D}}(X) = \max_{x \in \Delta} \vartheta(x)$$

Zhang inequality in the toric case



$$\frac{\mu^{ess}\cdot \deg_D X}{n!} = \mu^{ess}\cdot \mathrm{Vol}\,\Delta \geq \int_\Delta \vartheta \, d\mathrm{vol} = \frac{h_{\bar{D}}(X)}{(n+1)!}$$

$$\mu^{ess} \geq \frac{h_{\bar{D}}(X)}{(n+1)\deg_D X}$$

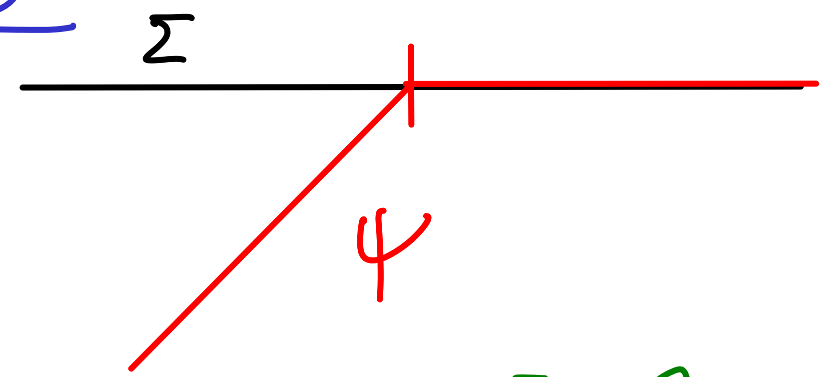
<u>corollary:</u> X toric variety, $\bar{D}$ toric metrized semi-positive, D ample.

$$\mu^{\mathrm{ess}}_{\bar{D}}(X) = \frac{h_{\bar{D}}(X)}{(n+1)\deg_D(X)} \Longleftrightarrow \vartheta \text{ is constant}$$

$\rightsquigarrow$ Yuan's theorem $\Big|_{\substack{\text{toric case}\\ v=\infty}}$ // $\Longleftrightarrow$ "Bilu theorem"

Some examples

$X=\mathbb{P}^1_{\mathbb{Q}}$ $\quad$ $D=(0:1)=\infty$



$\Delta=[0,1]$

1) Weil height. (canonical height)

$$\|s_\ell(p)\| = \frac{|\ell(p)|_v}{\max(|p_0|_v,|p_1|_v)} \rightsquigarrow \overline{D}$$

$p=(p_0:p_1)\in\mathbb{P}^1(\mathbb{C}_v)$; $\quad \ell\in\mathbb{C}_v[x_0,x_1]_1$

$f_v=\psi \quad \forall v\in\mathfrak{M}_{\mathbb{Q}}$ $\qquad \vartheta_v=0$

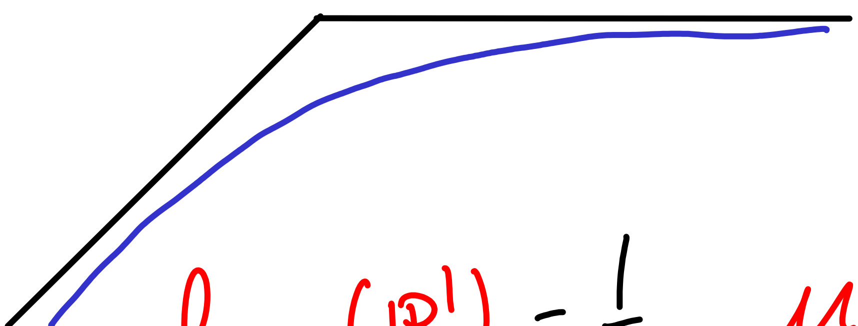
$h_{\overline{D}}(\mathbb{P}^1)=\mu^{ess}_{\overline{D}}(\mathbb{P}^1)=0$ $\qquad \mu^{ess}_{\overline{D}}=\frac{h_{\overline{D}}}{2\cdot\deg_D}$

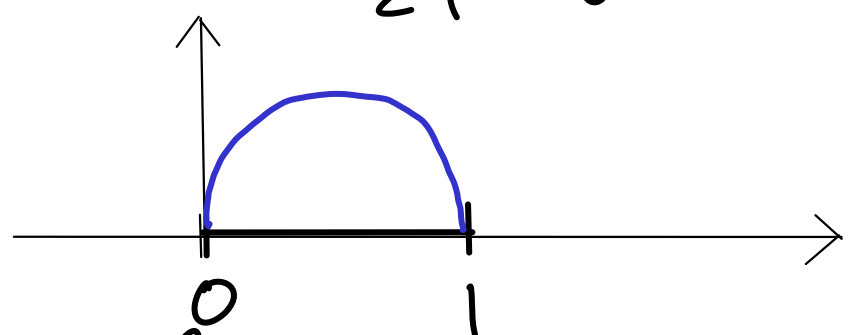
2) Fubini Study

$$\|S_\ell(p)\|_v = \begin{cases} \dfrac{|\ell(p)|_\infty}{\sqrt{|p_0|_\infty^2+|p_1|_\infty^2}} & v=\infty \\ \|S_\ell(p)\|_{v,\mathrm{can}} & v\neq\infty \end{cases}$$

If $\xi = \frac{a}{b} \in \mathbb{Q}^\times$ then $h_{FS}(\xi) = \log\sqrt{a^2+b^2}$

$f_\infty(u) = \frac{1}{2}\log(1+e^{-2u})$ $\quad \vartheta(x) = \vartheta_\infty(x) = \frac{-1}{2}\left(x\log x + (1-x)\log(1-x)\right)$





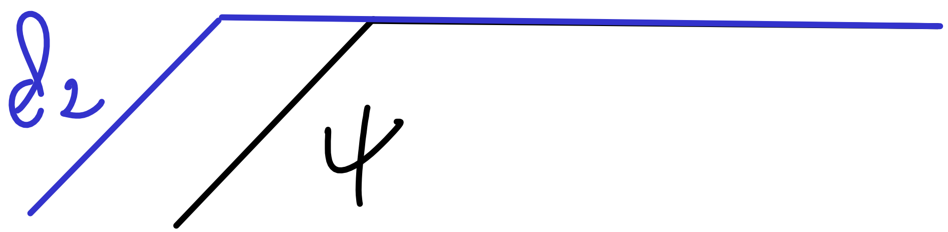
$h_{\bar{D}}(\mathbb{P}^1) = \frac{1}{2}$, $\mu^{ess}_{\bar{D}}(\mathbb{P}^1) = \frac{\log 2}{2}$, $\mu^{ess}_{\bar{D}} > \frac{h_{\bar{D}}}{2}$

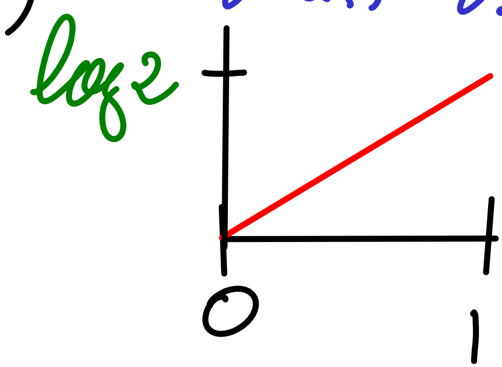
3) Twisted Weil height

$$\|S_\ell(p)\|_v = \begin{cases} \dfrac{|\ell(p)|_2}{\max(|p_0|_2,\ 2\,|p_1|_2)} & v=2 \\[2ex] \|S_\ell(p)\|_{v,\mathrm{can}} & v\neq 2 \end{cases}$$

$\xi = \frac{a}{b}$ $\quad h_{\tilde{w}}(\xi) = \log\max(|a|, 2|b|)$

$f_2(u) = \min(0, u+\log 2)$ $\quad \vartheta(x) = \vartheta_2(x) = x\log 2$





$h_{\bar{D}}(\mathbb{P}^1) = \mu^{ess}_{\bar{D}}(\mathbb{P}^1) = \log 2$ $\quad \mu^{ess}_{\bar{D}} > \dfrac{h_{\bar{D}}}{2\cdot\deg D}$

Equidistribution of G-orbits of small points on toric var.

Theorem 4 $(X, \bar{D})$ toric with D ample & $\bar{D}$ semipos.
Let $x_m \in \Delta$ s.t. $\vartheta(x_m) = \max_{x \in \Delta} \vartheta(x)$.

T.F.A.E.

(1) 0 is a vertex of $\partial_{x_m} \vartheta$ (Sup differential)

(2) $\forall v \in \mathfrak{M}_{\mathbb{Q}}$ $\exists$ μ_v probability measure on X_v
s.t. $\forall (p_k)_{k \geq 1}$ generic, with $\lim_{k \to \infty} h_{\bar{D}}(p_k) = \mu_{\bar{D}}^{\mathrm{ess}}(X)$

$$G p_k \xrightarrow[w]{} \mu_v$$

If so $\exists! \, (u_v)_v$ with $u_v \in \partial_{x_m} \vartheta_v$ and $\sum_v u_v = 0$
s.t. μ_v is "Haar measure" on $\mathrm{val}_v^{-1}(u_v)$

Example again

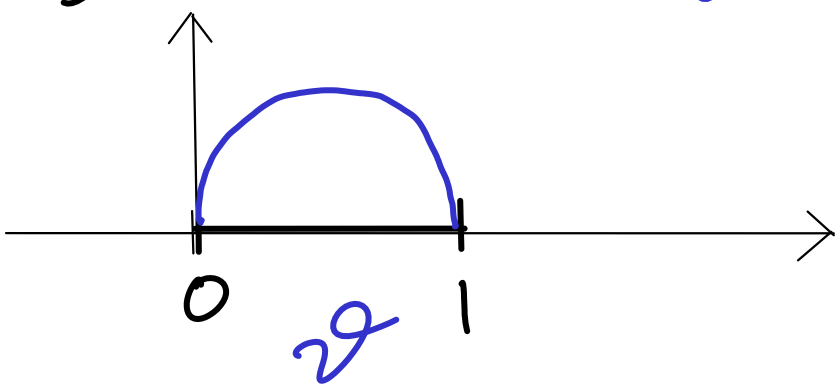
$X = \mathbb{P}^1$, $D = \infty$, $(p_k)_{k \geq 1}$ generic, $h_{\bar{D}}(p_k) \to \mu^{\text{ess}}_{\bar{D}}$

1) Weil height

$\vartheta = 0$ differentiable, $x_m \in (0,1)$ $\quad \partial_{x_m} \vartheta = \{0\}$

$\Rightarrow$ $G p_k \to S^1$ (Bilu's theorem)

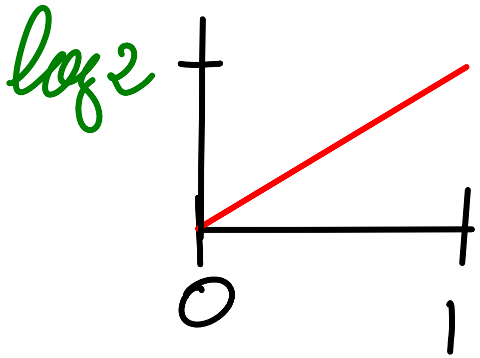
2) Fubini-Study height.



ϑ diff, $x_m = \frac{1}{2}$

$\Rightarrow$ $G p_k \to S^1$

3 Twisted Weil height



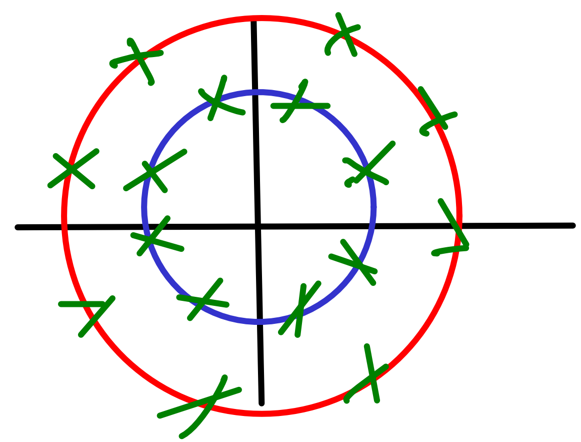
$X_m = \{1\}$ $\qquad \partial_{X_m}\vartheta = [-\infty, \log 2]$

0 not a vertex of $\partial_{X_m}\vartheta$.

$$h_{\bar D}(P) = \log\max(|P_0|_2, 2|P_1|_2) + \sum_{v\neq 2} \log\max(|P_0|_v, |P_1|_v)$$

Take ω_k root of 1 and set.

$$P_{2k} = (1:\omega_k) \qquad P_{2k+1} = (2:\omega_k)$$



$$h(P_k) = \log 2 = \mu^{ess}_{\bar D}(X)$$

No equidistribution.

Modulus concentration

Thm 5 $X, \bar{D}, X_{max}$ as before. Fix $v_0 \in \mathcal{M}_{\mathbb{Q}}$.

put $\vartheta'_{v_0} = \vartheta - \vartheta_{v_0}$.

If ① $\partial_{X_m} \vartheta_{v_0} \cap -\partial_{X_m} \vartheta'_{v_0} = \{u_0\}$ 1 single point

② u_0 is a vertex of $\partial_{X_m} \vartheta_{v_0}$

Then $\forall (p_k)_{k \geq 1}$ generic s.t. $\lim_{k\to\infty} h_{\bar{D}}(p_k) = \mu^{ess}_{\bar{D}}(X)$

and continuous function $g : X_{v_0} \to \mathbb{R}$ $\mathbb{S}_r$-invariant.

$$\mathrm{val}_{v_0 *} G p_k \xrightarrow[w]{} \delta_{u_0}$$

Obs

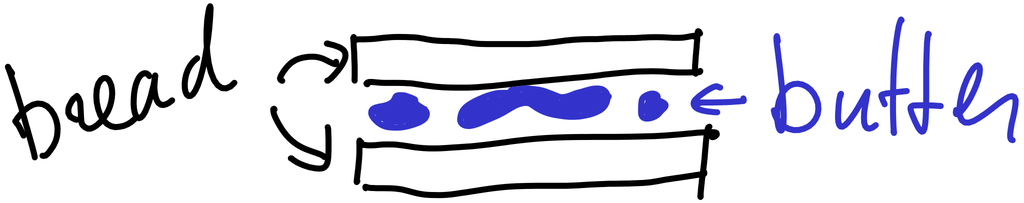
Modulus concentration is local: Can be checked place by place.

Equidistribution is global: Equidistribution at one place may depend on other places.

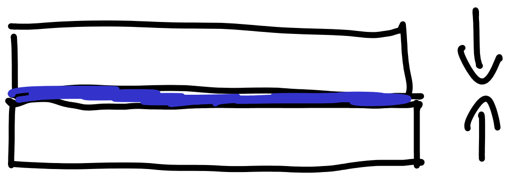
Thm 5 $\Rightarrow$ Thm 4. is a butter and bread principle.

The proof uses Yuan's theorem.

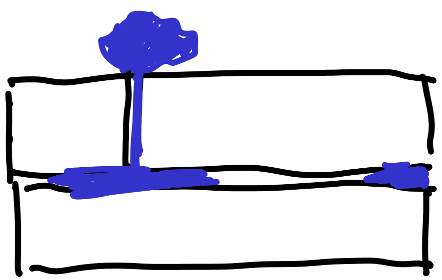
Butter and bread principle



If there is space between the bread slides the butter can have any shape.



If there is no space then the butter is equidistributed.



If there is a hole some butter can scape and equidistribution is lost.

THANK YOU