De los ejercicios de abajo (sacados del libro de *Georgii, Stochastics*) se proponen los siguientes:

6,**2**, **6**,**3**, **6**,**7**, **6**,**8**, **6**,**9**, **6**,**10**, **6**,**14**, **6**,**19**(**a**), **6**,**20**, **6**,**26**.

Finally we use the spatial homogeneity, which is expressed by the fact that X_n coincides with the random walk S_n for $n \le \tau_0$ (by (6.33)). This implies that $\mathbb{E}^y(\tau_{y-1}) = \mathbb{E}^1(\tau_0)$. Altogether, we find that $\mathbb{E}^y(\tau_0) = y\,\mathbb{E}^1(\tau_0)$ for $y \ge 1$. Substituting this into (6.39) we arrive at equation (6.38).

Suppose now that 0 is positive recurrent. Then, (6.33) yields

$$\begin{aligned} 0 &= \mathbb{E}^1(X_{\tau_0}) = \mathbb{E}^1\Big(1 + \sum_{k=1}^{\tau_0} (Z_k - 1)\Big) \\ &= 1 + \sum_{k\ge 1} \mathbb{E}^1\big(Z_k\, 1_{\{\tau_0 \ge k\}}\big) - \mathbb{E}^1(\tau_0) \\ &= 1 + \sum_{k\ge 1} \mathbb{E}(\varrho)\, P^1(\tau_0 \ge k) - \mathbb{E}^1(\tau_0) \\ &= 1 + (\mathbb{E}(\varrho) - 1)\,\mathbb{E}^1(\tau_0)\,. \end{aligned}$$

The third step follows because $\mathbb{E}^1(\tau_0) < \infty$ by (6.38), and because the Z_k are non-negative so that Theorem (4.7c) can be applied. The fourth identity uses the fact that the event $\{\tau_0 \ge k\}$ can be expressed in terms of $Z_1, \dots, Z_{k-1}$, and so, by Theorem (3.24), it is independent of Z_k. The equation just proved shows that $\mathbb{E}(\varrho) < 1$. In particular we get $\mathbb{E}^1(\tau_0) = 1/(1 - \mathbb{E}(\varrho))$ and, by (6.38), $\mathbb{E}^0(\tau_0) = 1/(1 - \mathbb{E}(\varrho))$ as well. That is, the closer the average influx of customers per time approaches the serving rate 1, the longer the average 'busy period'.

Conversely, suppose that $\mathbb{E}(\varrho) < 1$. The same calculation as above, with $\tau_0 \wedge n := \min(\tau_0, n)$ in place of τ_0, then shows that

$$0 \le \mathbb{E}^1(X_{\tau_0 \wedge n}) = 1 + (\mathbb{E}(\varrho) - 1)\,\mathbb{E}^1(\tau_0 \wedge n)\,,$$

thus $\mathbb{E}^1(\tau_0 \wedge n) \le 1/(1 - \mathbb{E}(\varrho))$. For $n \to \infty$ it follows that $\mathbb{E}^1(\tau_0) \le 1/(1-\mathbb{E}(\varrho)) < \infty$ and so, by (6.38), that 0 is positive recurrent.

Problems

6.1. *Iterated random functions.* Let E be a countable set, $(F, \mathscr{F})$ an arbitrary event space, $f : E \times F \to E$ a measurable function, and $(U_i)_{i\ge 1}$ a sequence of i.i.d. random variables taking values in $(F, \mathscr{F})$. Let $(X_n)_{n\ge 0}$ be recursively defined by $X_0 = x \in E$, and $X_{n+1} = f(X_n, U_{n+1})$ for $n \ge 0$. Show that $(X_n)_{n\ge 0}$ is a Markov chain and determine the transition matrix.

6.2. *Functions of Markov chains.* Let $(X_n)_{n\ge 0}$ be a Markov chain with countable state space E and transition matrix Π, and $\varphi : E \to F$ a mapping from E to another countable set F.

(a) Show by example that $(\varphi \circ X_n)_{n\ge 0}$ is not necessarily a Markov chain.

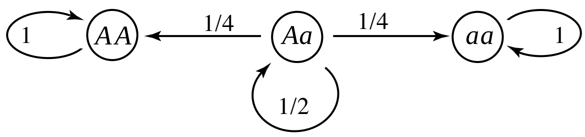
(b) Find a (non-trivial) condition on φ and Π under which $(\varphi \circ X_n)_{n\ge 0}$ is a Markov chain.

6.3. *Embedded jump chain.* Let E be countable and $(X_n)_{n\geq 0}$ a Markov chain on E with transition matrix Π. Let $T_0 = 0$ and $T_k = \inf\{n > T_{k-1} : X_n \neq X_{n-1}\}$ be the time of the kth jump of $(X_n)_{n\geq 0}$. Show that the sequence $X_k^* := X_{T_k}$, $k \geq 0$, is a Markov chain with transition matrix

$$\Pi^*(x, y) = \begin{cases} \Pi(x, y)/(1 - \Pi(x, x)) & \text{if } y \neq x\,, \\ 0 & \text{otherwise.} \end{cases}$$

Show further that, conditional on $(X_k^*)_{k\geq 0}$, the differences $T_{k+1} - T_k - 1$ are independent and geometrically distributed with parameter $1 - \Pi(X_k^*, X_k^*)$.

6.4. *Self-fertilisation.* Suppose the gene of a plant can come in two 'versions', the alleles A and a. A classical procedure to grow pure-bred (i.e. homozygous) plants of genotype AA respectively aa is self-fertilisation. The transition graph



describes the transition from one generation to the next. Let $(X_n)_{n\geq 0}$ be the corresponding Markov chain. Calculate the probability $p_n = P^{Aa}(X_n = Aa)$ for arbitrary n.

6.5. *Hoppe's urn model and Ewens' sampling formula.* Imagine a gene in a population that can reproduce and mutate at discrete time points, and assume that every mutation leads to a new allele (this is the so-called infinite alleles model). If we consider the genealogical tree of n randomly chosen individuals at the times of mutation or birth, we obtain a picture as in Figure 6.7. Here, every bullet marks a mutation, which is the starting point of a new 'clan' of individuals with the new allele. Let us now ignore the family structure of the clans and only record their sizes. The reduced evolution is then described by the following urn model introduced by F. Hoppe (1984).

Let $\vartheta > 0$ be a fixed parameter that describes the mutation rate. Suppose that at time 0 there is a single black ball with weight ϑ in the urn, whereas outside there is an infinite reservoir of balls of different colours and weight 1. At each time step, a ball is drawn from the urn with a probability proportional to its weight. If it is black (which is certainly the case in the first draw), then a ball of a colour that is not yet present in the urn is put in. If the chosen ball is coloured, then it is returned together with another ball of the same colour. The number of balls in the urn thus increases by 1 at each draw, and the coloured balls can be decomposed into clans of the same

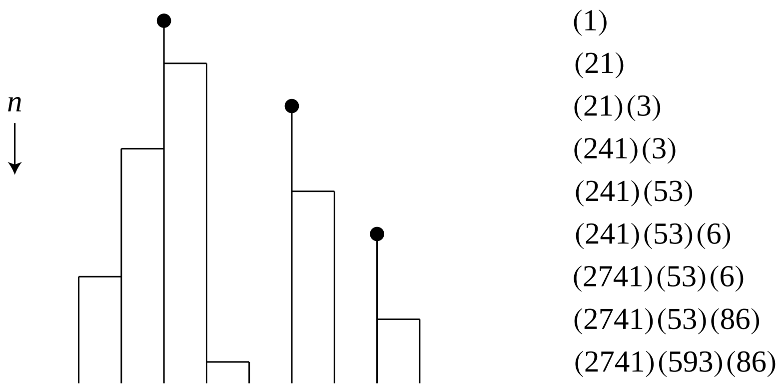


Figure 6.7. A genealogical tree in the infinite alleles model, with corresponding description in terms of cycles as in the Chinese restaurant process from Problem 6.6.

colour. The size distribution of these clans is described by a sequence of the form $x = (x_i)_{i\geq 1}$, where x_i specifies the number of clans of size i. The total number of coloured balls after the nth draw is $N(x) := \sum_{i\geq 1} i\, x_i = n$. Formally, the model is described by the Markov chain $(X_n)_{n\geq 0}$ with state space $E = \{x=(x_i)_{i\geq 1} : x_i \in \mathbb{Z}_+,\ N(x) < \infty\}$ and transition matrix

$$\Pi(x,y) = \begin{cases} \vartheta/(\vartheta + N(x)) & \text{if } y_1 = x_1 + 1\,, \\ j\, x_j/(\vartheta + N(x)) & \text{if } y_j = x_j - 1,\ y_{j+1} = x_{j+1} + 1,\ 1 \leq j \leq N(x)\,, \\ 0 & \text{otherwise.} \end{cases}$$

(The first case corresponds to drawing a black ball, the second to drawing a coloured ball from one of the x_j clans of size j, so that the size of this clan increases to $j+1$.) Let $\mathbf{0} = (0,0,\dots)$ be the initial state, in which the urn does not contain any coloured balls. Show by induction on $n \geq 1$ for arbitrary $x \in E$ with $N(x) = n$:

(a) $P^{\mathbf{0}}(X_n = x) = \varrho_{n,\vartheta}(x)$, where

$$\varrho_{n,\vartheta}(x) := \frac{n!}{\vartheta^{(n)}} \prod_{i\geq 1} \frac{(\vartheta/i)^{x_i}}{x_i!}\,.$$

Here, $\vartheta^{(n)} := \vartheta(\vartheta+1)\dots(\vartheta+n-1)$. Hence, $\varrho_{n,\vartheta}$ is the size distribution of the clans of a random sample of n individuals from a population with mutation rate ϑ. This is the sampling formula by W. J. Ewens (1972).

(b) If $Y = (Y_i)_{i\geq 1}$ is a sequence of independent random variables with Poisson distributions $P \circ Y_i^{-1} = \mathcal{P}_{\vartheta/i}$, then $\varrho_{n,\vartheta}(x) = P(Y = x | N(Y) = n)$.

6.6. *Chinese restaurant process and random permutations.* The Hoppe model from Problem 6.5 can be slightly refined by taking the family structure of the clans into consideration. The balls in Hoppe's urn are labelled in the order in which they arrived in the urn. The state of the urn after the nth draw is written as a permutation in cycle notation as in Figure 6.7: For a new colour at time n we add (n) as a new cycle and otherwise the label of the ball is written to the left of the label of the 'mother ball' in its respective cycle. Let Z_n be the permutation created after n draws. The sequence $(Z_n)_{n\geq 0}$ was introduced by D. J. Aldous (1984) as the 'Chinese restaurant process'; the labels are interpreted as the guests of a Chinese restaurant (in the order of their appearance), and each cycle as the seating order at a (round) table. Show the following:

(a) $(Z_n)_{n\geq 0}$ is a Markov chain. For which E and Π?

(b) For each permutation π of $\{1,\dots,n\}$ with k cycles, $P(Z_n = \pi) = \vartheta^k/\vartheta^{(n)}$, where $\vartheta^{(n)}$ is as in Problem 6.5(a). So, Z_n is uniformly distributed when $\vartheta = 1$.

(c) Deduce that the number of all permutations of $\{1,\dots,n\}$ that have x_i cycles of length i is equal to $n!/\prod_{i=1}^n (i^{x_i} x_i!)$ for all $1 \leq i \leq n$.

6.7. *The Wright–Fisher model in population genetics.* Suppose a gene has the two alleles A and a. So in a population of N individuals with a diploid set of chromosomes, the gene occurs $2N$ times. Assume each generation consists of N individuals and is created from the previous generation by random mating: Each gene of the offspring generation 'selects', independently of all others, a parental gene and adopts its allele. Let X_n be the number of A-genes in the nth generation. Clearly, X_n is a Markov chain on $E = \{0,\dots,2N\}$. Determine the transition matrix and compute the absorption probabilities

$$h_{2N}(x) := P^x(X_n = 2N \text{ for all sufficiently large } n)\,, \quad x \in E\,.$$

6.8. Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix Π on a countable set E, and suppose that $P^x(\tau_y < \infty) = 1$ for all $x, y \in E$. Let $h : E \to [0, \infty[$ be harmonic, in that $\Pi h = h$. Show that h must be constant.

6.9. *The asymmetric ruin problem.* A well-known dexterity game consists of a ball in a 'maze' of N concentric rings (numbered from the centre to the outside) that have an opening to the next ring on alternating sides. The aim of the game is to get the ball to the centre ('ring no. 0') by suitably tilting the board. Suppose that the ball is initially in the mth ring ($0 < m < N$), and that with probability $0 < p < 1$ the player manages to get the ball from the kth to the $(k-1)$st ring, but that with probability $1-p$ the ball rolls back to the $(k+1)$st ring. The player stops if the ball enters either into ring 0 (so that the player succeeds), or into the Nth ring (due to demoralisation). Describe this situation as a Markov chain and find the probability the probability of success.

6.10. Find the extinction probability for a Galton–Watson process with offspring distribution ϱ in the cases

(a) $\varrho(k) = 0$ for all $k > 2$,

(b) $\varrho(k) = ba^{k-1}$ for all $k \geq 1$ and $a, b \in]0, 1[$ with $b \leq 1 - a$. (According to empirical studies by Lotka in the 1930s, for $a = 0.5893$ and $b = 0.2126$ this ϱ describes the distribution of the number of sons of American men quite well, whereas, according to Keyfitz [34], the parameters $a = 0.5533$ and $b = 0.3666$ work best for the number of daughters of Japanese women.)

6.11. *Total size of a non-surviving family tree.* Consider a Galton–Watson process $(X_n)_{n\geq 0}$ with offspring distribution ϱ, and suppose that $\mathbb{E}(\varrho) \leq 1$ and $X_0 = 1$. Let $T = \sum_{n\geq 0} X_n$ be the total number of descendants of the progenitor. (Note that $T < \infty$ almost surely.) Show that the generating function φ_T of T satisfies the functional equation $\varphi_T(s) = s\,\varphi_\varrho \circ \varphi_T(s)$, and determine $\mathbb{E}^1(T)$, the expected total number of descendants.

6.12. *Branching process with migration and annihilation.* Consider the following modification of the Galton–Watson process. Given $N \in \mathbb{N}$, assume that at each site $n \in \{1, \dots, N\}$ there is a certain number of 'particles' that behave independently of each other as follows. During a time unit, a particle at site n first moves to $n-1$ or $n+1$, each with probability 1/2. There it dies and produces k offspring with probability $\varrho(k)$, $k \in \mathbb{Z}_+$. If $n-1 = 0$ or $n+1 = N+1$, the particle is annihilated and does not produce any offspring. Let $\varphi(s) = \sum_{k\geq 0} \varrho(k)s^k$ be the generating function of $\varrho = (\varrho(k))_{k\geq 0}$, and for $1 \leq n \leq N$ let $q(n)$ be the probability that the progeny of a single particle at n becomes eventually extinct. By convention, $q(0) = q(N+1) = 1$.

(a) Describe the evolution of all particles by a Markov chain on $\mathbb{Z}_+^N$ and find the transition matrix.

(b) Justify the equation $q(n) = \frac{1}{2}\varphi(q(n-1)) + \frac{1}{2}\varphi(q(n+1))$, $1 \leq n \leq N$.

(c) In the subcritical case $\varphi'(1) \leq 1$, show that $q(n) = 1$ for all $1 \leq n \leq N$.

(d) On the other hand, suppose that $\varphi(s) = (1 + s^3)/2$. Show that $q(1) = q(2) = 1$ when $N = 2$, whereas $q(n) < 1$ for all $1 \leq n \leq 3$ when $N = 3$.

6.13. Let E be finite and Π a stochastic matrix on E. Show that Π satisfies the assumptions of the ergodic theorem (6.13) if and only if Π is irreducible and *aperiodic* in the sense that for one (and thus all) $x \in E$ the greatest common divisor of the set $\{k \geq 1 : \Pi^k(x, x) > 0\}$ is 1.

6.14. *Random replacement I.* Consider an urn containing at most N balls. Let X_n be the number of balls in the urn after performing the following procedure n times. If the urn is non-empty, one of the balls is removed at random; by flipping a fair coin, it is then decided whether or not the ball is returned to the urn. If the urn is empty, the fair coin is used to decide whether it remains empty or whether it is filled with N balls. Describe this situation as a Markov chain and find the transition matrix. What is the distribution of X_n as $n \to \infty$?

6.15. *Random replacement II.* As in the previous problem, we are given an urn holding at most N balls, but now they come in two colours, either white or red. If the urn is non-empty, a ball is picked at random and is or is not replaced according to the outcome of the flip of a fair coin. If the urn is empty, the coin is flipped to decide whether the urn should be filled again; if so, it is filled with N balls, each of which is white or red depending on the outcomes of further independent coin flips. Let W_n and R_n be the numbers of white and red balls, respectively, after performing this procedure n times. Show that $X_n = (W_n, R_n)$ is a Markov chain, and determine its asymptotic distribution.

6.16. *A variant of Pólya's urn model.* Again, we are given an urn containing no more than $N > 2$ balls in the colours white and red; suppose it contains at least one ball of each colour. If there are less than N balls, one of them is chosen at random and returned together with a further ball of the same colour (taken from an external reserve). If there are already N balls in the urn, then by tossing a coin it is decided whether the urn should be modified. If so, all balls except one of each colour are removed. Let W_n and R_n be the respective numbers of white and red balls after performing this procedure n times. Show the following:

(a) The total number $Y_n := W_n + R_n$ of balls is a Markov chain. Find the transition matrix. Will the chain eventually come to an equilibrium? If so, which one?

(b) $X_n := (W_n, R_n)$ is also a Markov chain. Find the transition matrix and (if it exists) the asymptotic distribution.

6.17. *A cycle condition for reversibility.* Under the assumptions of the ergodic theorem (6.13), show that Π has a reversible distribution if and only if

$$\Pi(x_0, x_1) \dots \Pi(x_{n-1}, x_n) = \Pi(x_n, x_{n-1}) \dots \Pi(x_1, x_0)$$

for all $n \geq 1$ and all cycles $x_0, \dots, x_{n-1} \in E$ with $x_n = x_0$. Check whether this holds for the house-of-cards process in Example (6.29).

6.18. *Time reversal for renewals.* In addition to the age process (X_n) in Example (6.29), consider the process $Y_n = \min\{T_k - n : k \geq 1,\ T_k \geq n\}$ that indicates the remaining life span of the appliance used at time n.

(a) Show that $(Y_n)_{n\geq 0}$ is also a Markov chain, find its transition matrix $\tilde{\Pi}$ and re-derive the renewal theorem.

(b) Find the stationary distribution α of $(X_n)_{n\geq 0}$ and show that α is also a stationary distribution of $(Y_n)_{n\geq 0}$.

(c) Which connection does α create between the transition matrices Π and $\tilde{\Pi}$ of $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$?

6.19. (a) *Random walk on a finite graph.* Let E be a finite set and $\sim$ a symmetric relation on E. Here, E is interpreted as the vertex set of a graph, and the relation $x \sim y$ means that x and y are connected by an edge. Suppose that each vertex is connected by an edge to at least one other vertex or to itself. Let $d(x) = |\{y \in E : x \sim y\}|$ be the degree of the vertex $x \in E$, and set $\Pi(x, y) = 1/d(x)$ if $x \sim y$, and $\Pi(x, y) = 0$ otherwise. The Markov chain with transition matrix Π is called the random walk on the graph $(E, \sim)$. Under which conditions on the graph $(E, \sim)$ is Π irreducible? Find a reversible distribution for Π.

(b) *Random walk of a knight.* Consider a knight on an (otherwise empty) chess board, which chooses each possible move with equal probability. It starts (i) in a corner, (ii) in one of the 16 squares in the middle of the board. How many moves does it need on average to get back to its starting point?

6.20. Let $0 < p < 1$ and consider the stochastic matrix Π on $E = \mathbb{Z}_+$ defined by

$$\Pi(x, y) = \mathcal{B}_{x,p}(\{y\}), \quad x, y \in \mathbb{Z}_+ .$$

Find Π^n for arbitrary $n \geq 1$. Can you imagine a possible application of this model?

6.21. *Irreducible classes.* Let E be countable, Π a stochastic matrix on E, and E_{rec} the set of all recurrent states. Let us say a state y is accessible from x, written as $x \to y$, if there exists some $k \geq 0$ such that $\Pi^k(x, y) > 0$. Show the following:

(a) The relation '$\to$' is an equivalence relation on E_{rec}. The corresponding equivalence classes are called *irreducible classes*.

(b) If x is positive recurrent and $x \to y$, then y is also positive recurrent, and

$$\mathbb{E}^x\Big(\sum_{n=1}^{\tau_x} 1_{\{X_n = y\}}\Big) = \mathbb{E}^x(\tau_x)/\mathbb{E}^y(\tau_y) .$$

In particular, all states within an irreducible class are of the same recurrence type.

6.22. *Extinction or unlimited growth of a population.* Consider a Galton–Watson process $(X_n)_{n \geq 0}$ with supercritical offspring distribution ϱ, i.e., suppose $\mathbb{E}(\varrho) > 1$. Show that all states $k \neq 0$ are transient, and that

$$P^k\big(X_n \to 0 \text{ or } X_n \to \infty \text{ for } n \to \infty\big) = 1 .$$

6.23. *Birth-and-death processes.* Let Π be a stochastic matrix on $E = \mathbb{Z}_+$. Suppose that $\Pi(x, y) > 0$ if and only if either $x \geq 1$ and $|x - y| = 1$, or $x = 0$ and $y \leq 1$. Find a necessary and sufficient condition on Π under which Π has a stationary distribution α. If α exists, express it in terms of the entries of Π.

6.24. *A migration model.* Consider the following simple model of an animal population in an open habitat. Each animal living there leaves the habitat, independently of all others, with probability p, and it stays with probability $1 - p$. At the same time, a Poisson number (with parameter $a > 0$) of animals immigrates from the outside world.

(a) Describe the number X_n of animals living in the habitat by a Markov chain and find the transition matrix Π.

(b) Calculate the distribution of X_n when the initial distribution is $\mathcal{P}_\lambda$, the Poisson distribution with parameter $\lambda > 0$.

(c) Determine a reversible distribution α.

6.25. Generalise Theorem (6.30) as follows. For $x, y \in E$, let $F_1(x, y) = P^x(\tau_y < \infty)$ be the probability that y can eventually be reached from x, $N_y = \sum_{n\geq 1} 1_{\{X_n = y\}}$ the number of visits to y (from time 1 onwards), $F_\infty(x, y) = P^x(N_y = \infty)$ the probability for infinitely many visits, and $G(x, y) = \delta_{xy} + \mathbb{E}^x(N_y)$ the expected number of visits (including at time 0), the so-called *Green function*. Show that

$$P^x(N_y \geq k+1) = F_1(x, y)\, P^y(N_y \geq k) = F_1(x, y)\, F_1(y, y)^k$$

for all $k \geq 0$, and therefore

$$F_\infty(x, y) = F_1(x, y)\, F_\infty(y, y)\,, \quad G(x, y) = \delta_{xy} + F_1(x, y)\, G(y, y)\,.$$

What does this mean when y is recurrent and transient, respectively?

6.26. *Excursions from a recurrent state.* Consider a Markov chain with a countable state space E and transition matrix Π that starts in a recurrent state $x \in E$. Let $T_0 = 0$ and, for $k \geq 1$, let $T_k = \inf\{n > T_{k-1} : X_n = x\}$ be the time of the kth return to x and $L_k = T_k - T_{k-1}$ the length of the kth 'excursion' from x. Show that, under P^x, the random variables L_k are (almost surely well-defined and) i.i.d.

6.27. *Busy period of a queue viewed as a branching process.* Recall Example (6.32) and the random variables X_n and Z_n defined there. Interpret the queue as a population model, by interpreting the customers newly arriving at time n as the children of the customer waiting at the front of the queue; a generation is complete if the last member of the previous generation has been served. Correspondingly, define $Y_0 = X_0$, $Y_1 = \sum_{n=1}^{Y_0} Z_n$, and for general $k \geq 1$ set

$$Y_{k+1} = \sum_{n\geq 1} 1_{\left\{\sum_{i=0}^{k-1} Y_i < n \leq \sum_{i=0}^{k} Y_i\right\}} Z_n\,.$$

Show the following:

(a) (Y_k) is a Galton–Watson process with offspring distribution ϱ.

(b) P^x-almost surely for every $x \geq 1$, one has $Y_{k+1} = X_{T_k}$ for all $k \geq 0$, and therefore

$$\{X_n = 0 \text{ for some } n \geq 1\} = \{Y_k = 0 \text{ for all sufficiently large } k\}\,.$$

Here, the random times T_k are recursively defined by setting $T_0 = X_0$, $T_{k+1} = T_k + X_{T_k}$. (Verify first that these times are not larger than the first time τ_0 at which the queue is empty.)

Deduce (without using the result from Example (6.32)) that the queue is recurrent if and only if $\mathbb{E}(\varrho) \leq 1$. (In this case, Problem 6.11 yields the average number of customers that are served during a busy period, and so gives an alternative proof for the result from Example (6.37).)

6.28. *Markov chains in continuous time.* Let E be countable and $G = (G(x, y))_{x,y\in E}$ a matrix satisfying the properties

(i) $G(x, y) \geq 0$ for $x \neq y$,

(ii) $-a(x) := G(x, x) < 0$, $\sum_{y\in E} G(x, y) = 0$ for all x, and

(iii) $a := \sup_{x\in E} a(x) < \infty$.