Probabilidades y estadística (M)
Segundo cuatrimestre 2018
Práctica 6

De la lista de ejercicios de abajo (sacados del libro 'Stochastics' de Georgii), hacer los siguientes:

5.1, 5.2, 5.6, 5.7, 5.10, 5.12, 5.13, 5.15, 5.17-5.19, 5.22-5.24.

In the limit as $n \to \infty$ and $p = p_n \to 0$ such that $np_n \to \lambda > 0$, we get back our earlier Theorem (2.17). In fact we obtain the error bound $\|\mathcal{B}_{n,p_n} - \mathcal{P}_\lambda\| \le 2(np_n^2 + |np_n - \lambda|)$; see the subsequent Remark (5.34). By the way, in the discrete case of probability measures on $\mathbb{Z}_+$, the convergence with respect to the so-called *total variation distance* $\|\cdot\|$ used here is equivalent to convergence in distribution, see Problem 5.24. We also note that the preceding proof can be directly generalised to the case of Bernoulli variables X_i with *distinct* success probabilities p_i. The distribution of the sum $S = \sum_{i=1}^n X_i$ is then no longer binomial, but can still be approximated by the Poisson distribution $\mathcal{P}_{\sum_i p_i}$ up to an error of at most $2\sum_i p_i^2$.

(5.34) Remark. *Variation of the Poisson parameter.* If $\lambda, \delta > 0$ then

$$\|\mathcal{P}_{\lambda+\delta} - \mathcal{P}_\lambda\| \le 2\delta\,.$$

For, if X and Y are independent with distribution $\mathcal{P}_\lambda$ and $\mathcal{P}_\delta$ respectively, $X+Y$ has the distribution $\mathcal{P}_{\lambda+\delta}$, and arguing as in the above proof one finds that $\|\mathcal{P}_{\lambda+\delta} - \mathcal{P}_\lambda\|$ is bounded by $2\,P(Y \ge 1) = 2(1-e^{-\delta}) \le 2\delta$.

When is the Poisson distribution appropriate to approximate the binomial distribution, and when the normal distribution? By (5.33) the Poisson approximation works well when np^2 is small. On the other hand, the Berry–Esséen theorem discussed in Remark (5.30c) shows that the normal approximation performs well when

$$\frac{p(1-p)^3 + (1-p)p^3}{(p(1-p))^{3/2}}\,\frac{1}{\sqrt{n}} = \frac{p^2 + (1-p)^2}{\sqrt{np(1-p)}}$$

is small, and since $1/2 \le p^2+(1-p)^2 \le 1$, this is precisely the case when $np(1-p)$ is large. If p is very close to 1, then neither approximation is appropriate. But then we can replace p by $1-p$ and k by $n-k$ to obtain $\mathcal{B}_{n,p}(n-k) = \mathcal{P}_{n(1-p)}(\{k\}) + O\big(n(1-p)^2\big)$.

Problems

5.1. *The Ky Fan metric for convergence in probability.* For two real-valued random variables X, Y on an arbitrary probability space let

$$d(X,Y) = \min\{\varepsilon \ge 0 : P(|X-Y| > \varepsilon) \le \varepsilon\}\,.$$

Show the following:

(a) The minimum is really attained, and d is a metric on the space of all real-valued random variables.

(b) For every sequence of real-valued random variables on Ω we have $Y_n \xrightarrow{P} Y$ if and only if $d(Y_n, Y) \to 0$.

5.2. *Collectibles.* Consider the problem of collecting a complete series of stickers, as described in Problem 4.20. What is the minimal number of items you have to buy, so that with probability at least 0.95 you have collected all $N = 20$ stickers? Use Chebyshev's inequality to give a best possible lower bound.

5.3. (a) A particle moves randomly on a plane according to the following rules. It moves one unit along a randomly chosen direction Ψ_1, then it chooses a new direction Ψ_2 and moves one unit along that new direction, and so on. We suppose that the angles Ψ_i are independent and uniformly distributed on $[0, 2\pi]$. Let D_n be the distance between the starting point and the location after the nth step. Calculate the mean square displacement $\mathbb{E}(D_n^2)$.

(b) At the centre of a large plane there are exactly 30 particles at time $t = 0$, which move according to the rules set out in (a). For every step they take, the particles need one time unit. Determine, for every $n \geq 1$, the smallest number $r_n > 0$ with the following property: With probability ≥ 0.9 there are, at time $t = n$, more than 15 particles in a circle of radius r_n around the centre of the plane. *Hint:* Determine first some $\delta > 0$ such that $P\big(\sum_{i=1}^{30} Z_i > 15\big) \geq 0.9$ for any Bernoulli sequence $Z_1, \dots, Z_{30}$ with parameter $p \geq \frac{1}{2} + \delta$.

5.4. *Large deviations of empirical averages from the mean.* Let $(X_i)_{i\geq 1}$ be a Bernoulli sequence with $0 < p < 1$. Show that for all $p < a < 1$ we have:

$$P\Big(\frac{1}{n}\sum_{i=1}^{n} X_i \geq a\Big) \leq e^{-nh(a;p)},$$

where $h(a; p) = a \log \frac{a}{p} + (1-a) \log \frac{1-a}{1-p}$. Show first that for all $s \geq 0$

$$P\Big(\frac{1}{n}\sum_{i=1}^{n} X_i \geq a\Big) \leq e^{-nas}\, \mathbb{E}(e^{sX_1})^n .$$

5.5. *Convexity of the Bernstein polynomials.* Let $f : [0, 1] \to \mathbb{R}$ be continuous and convex. Show that, for every $n \geq 1$, the corresponding Bernstein polynomial f_n is also convex. *Hint:* Let $p_1 < p_2 < p_3$, consider the frequencies $Z_k = \sum_{i=1}^{n} 1_{[0,p_k]} \circ U_i$, $k = 1, 2, 3$, and represent Z_2 as a convex combination of Z_1 and Z_3. Use that the vector $(Z_1, Z_2 - Z_1, Z_3 - Z_2, n - Z_3)$ has a multinomial distribution!

5.6. *Law of large numbers for random variables without expectation.* Let $(X_i)_{i\geq 1}$ be i.i.d. real-valued random variables having no expectation, i.e., $X_i \notin \mathscr{L}^1$. Let $a \in \mathbb{N}$ be arbitrary. Show the following:

(a) $P(|X_n| > an$ infinitely often$) = 1$. *Hint:* Use Problem 4.5.

(b) For the sums $S_n = \sum_{i=1}^{n} X_i$ we have $P(|S_n| > an$ infinitely often$) = 1$ and thus $\limsup_{n\to\infty} |S_n|/n = \infty$ almost surely.

(c) If all $X_i \geq 0$, we even obtain $S_n/n \to \infty$ almost surely.

5.7. *Renewals of, say, light bulbs.* Let $(L_i)_{i\geq 1}$ be i.i.d. non-negative random variables with finite or infinite expectation. One can interpret L_i as the life time of the ith light bulb (which is immediately replaced when it burns out); see also Figure 3.7. For $t > 0$ let

$$N_t = \sup\Big\{N \geq 1 : \sum_{i=1}^{N} L_i \leq t\Big\}$$

be the number of bulbs used up to time t. Show that $\lim_{t\to\infty} N_t/t = 1/\mathbb{E}(L_1)$ almost surely; here we set $1/\infty = 0$ and $1/0 = \infty$. (In the case $\mathbb{E}(L_1) = \infty$ use Problem 5.6(c); the case $\mathbb{E}(L_1) = 0$ is trivial.) What does the result mean in the case of a Poisson process?

5.8. *Inspection, or waiting time, paradox.* As in the previous problem, let $(L_i)_{i\geq 1}$ be i.i.d. non-negative random variables representing the life times of machines, or light bulbs, which are immediately replaced when defect. (In a waiting time interpretation, one can think of the L_i as the time spans between the arrivals of two consecutive buses at a bus stop.) We assume $0 < \mathbb{E}(L_i) < \infty$. For $s > 0$, let $L_{(s)}$ be the life time of the machine working at instant s; so $L_{(s)} = L_i$ for $s \in [T_{i-1}, T_i[$, where the T_i are defined as in Figure 3.7 on p. 74. Use Problem 5.7 and the strong law of large numbers to show that, for every random variable $f : [0, \infty[\to [0, \infty[$,

$$\frac{1}{t}\int_0^t f(L_{(s)})\,ds \underset{t\to\infty}{\longrightarrow} \frac{\mathbb{E}\big(L_1 f(L_1)\big)}{\mathbb{E}(L_1)} \quad \text{almost surely.}$$

For $f = \mathrm{Id}$ this means that, for large t, the life time of the machine inspected at a random instant in $[0, t]$ is approximately equal to $\mathbb{E}(L_1^2)/\mathbb{E}(L_1)$, which is larger than $\mathbb{E}(L_1)$ unless L_1 is almost surely constant. Compare this result with Example (4.16), where the L_i are exponentially distributed. The probability measure $Q(A) := \mathbb{E}(L_1 1_{\{L_1\in A\}})/\mathbb{E}(L_1)$ on $([0, \infty[, \mathscr{B}_{[0,\infty[})$, which shows up in the above limit when $f = 1_A$, is called the *size-biased distribution* of L_i, and with slightly more effort one can even show that $\frac{1}{t}\int_0^t \delta_{L_{(s)}}\,ds \xrightarrow{d} Q$ almost surely for $t \to \infty$.

5.9. Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables that are exponentially distributed with parameter $\alpha > 0$. Show that

$$\limsup_{n\to\infty} X_n/\log n = 1/\alpha \quad \text{and} \quad \liminf_{n\to\infty} X_n/\log n = 0 \quad \text{almost surely.}$$

5.10. *Expectation versus probability.* Bob suggests the following game to Alice: ‘Here is a biased coin, which shows heads with probability $p \in\,]1/3, 1/2[$. Your initial stake is €100; each time the coin shows heads, I double your capital, otherwise you pay me half of your capital. Let X_n denote your capital after the nth coin flip. As you can easily see, $\lim_{n\to\infty} \mathbb{E}(X_n) = \infty$, so your expected capital will grow beyond all limits.’ Is it advisable for Alice to play this game? Verify Bob’s claim and show that $\lim_{n\to\infty} X_n = 0$ almost surely.

5.11. *Asymptotics of the Pólya model.* Consider Pólya’s urn model with parameters $a = r/c$ and $b = w/c$, as introduced in Example (3.14). Let R_n be the number of red balls drawn after n iterations.

(a) Use Problem 3.4 and the law of large numbers to show that R_n/n converges in distribution to the beta distribution $\boldsymbol{\beta}_{a,b}$.

(b) What does this mean for the long-term behaviour of the competing populations? Consider the cases (i) $a, b < 1$, (ii) $a, b > 1$, (iii) $b < 1 < a$, (iv) $a = b = 1$.

5.12. Give a sequence of random variables in $\mathscr{L}^2$ for which neither the (strong or weak) law of large numbers nor the central limit theorem holds.

5.13. *Decisive power of determined minorities.* In an election between two candidates A and B one million voters cast their votes. Among these, 2 000 know candidate A from his election campaign and vote unanimously for him. The remaining 998 000 voters are more or less undecided and make their decision independently of each other by tossing a fair coin. What is the probability p_A of a victory of candidate A?

5.14. *Local normal approximation of Poisson distributions.* Let $\lambda > 0$ and $x_n(k) = (k - \lambda n)/\sqrt{\lambda n}$. Show that, for any $c > 0$,

$$\lim_{n\to\infty} \max_{k\in\mathbb{Z}_+:|x_n(k)|\le c} \left| \frac{\sqrt{\lambda n}\,\mathcal{P}_{\lambda n}(\{k\})}{\phi(x_n(k))} - 1 \right| = 0\,.$$

5.15. *Asymptotics of* Φ. Establish the sandwich estimate

$$\phi(x)\left(\frac{1}{x} - \frac{1}{x^3}\right) \le 1 - \Phi(x) \le \phi(x)\,\frac{1}{x} \quad \text{for all } x > 0,$$

and hence the asymptotics $1 - \Phi(x) \sim \phi(x)/x$ for $x \to \infty$. *Hint:* Compare the derivatives of the functions on the left- and right-hand sides with ϕ.

5.16. *Effect of the discreteness corrections.* Determine a lower bound for the error term in (5.23) when the discreteness corrections $\pm 1/2$ are omitted, by considering the case $k = l = np \in \mathbb{N}$. Compare the result with Figure 5.6.

5.17. *No-Shows.* Frequently, the number of passengers turning up for their flight is smaller than the number of bookings made. This is the reason why airlines overbook their flights (i.e., they sell more tickets than seats are available), at the risk of owing compensation to an eventual surplus of people. Suppose the airline has an income of $a = 300$€ per person flying, and for every person that cannot fly it incurs a loss of $b = 500$€; furthermore, suppose that every person that has booked shows up for the flight independently with probability $p = 0.95$. How many places would you sell for an

(a) Airbus A319 with $S = 124$ seats,

(b) Airbus A380 with $S = 549$ seats

to maximise the expected profit?

Hint: Let $(X_n)_{n\ge1}$ be a Bernoulli sequence with parameter p, and $S_N = \sum_{k=1}^N X_k$. The profit G_N by selling N places then satisfies the recursion

$$G_{N+1} - G_N = a\, 1_{\{S_N < S\}} X_{N+1} - b\, 1_{\{S_N \ge S\}} X_{N+1}\,.$$

Deduce that $\mathbb{E}(G_{N+1}) \ge \mathbb{E}(G_N)$ if and only if $P(S_N < S) \ge b/(a+b)$, and then use the normal approximation.

5.18. Estimate the error of a sum of rounded numbers as follows. The numbers $R_1, \dots, R_n \in \mathbb{R}$ are rounded to the next integer, i.e., they can be represented as $R_i = Z_i + U_i$ with $Z_i \in \mathbb{Z}$ and $U_i \in [-1/2, 1/2[$. The deviation of the sum of rounded numbers $\sum_{i=1}^n Z_i$ from the true sum $\sum_{i=1}^n R_i$ is $S_n = \sum_{i=1}^n U_i$. Suppose the $(U_i)_{1\le i\le n}$ are independent random variables having uniform distribution on $[-1/2, 1/2[$. Using the central limit theorem, determine a bound $k > 0$ with the property $P(|S_n| < k) \approx 0.95$ for $n = 100$.

5.19. In a sales campaign, a mail order company offers their first 1000 customers a complimentary ladies' respectively men's watch with their order. Suppose that both sexes are equally attracted by the offer. How many ladies' and how many men's watches should the company keep in stock to ensure that, with a probability of at least 98%, all 1000 customers receive a matching watch? Use (a) Chebyshev's inequality, (b) the normal approximation.

5.20. A company has issued a total of $n = 1000$ shares. At a fixed time, every share is sold with probability $0 < p < 1$, independently of all other shares. Every shareholder decides for each share with probability $0 < p < 1$ to sell it. This decision is independent for every share. The market can take in $s = 50$ shares without the price falling. What is the largest value of p such that the price remains stable with 90% probability?

5.21. *Error propagation for transformed observations.* Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables taking values in a (possibly unbounded) interval $I \subset \mathbb{R}$, and suppose the variance $v = \mathbb{V}(X_i) > 0$ exists. Let $m = \mathbb{E}(X_i)$ and $f : I \to \mathbb{R}$ be twice continuously differentiable with $f'(m) \neq 0$ and bounded f''. Show that

$$\frac{\sqrt{n/v}}{f'(m)} \Big(f\big(\tfrac{1}{n} \sum_{i=1}^{n} X_i\big) - f(m) \Big) \xrightarrow{d} \mathcal{N}_{0,1} \quad \text{for } n \to \infty.$$

Hint: Use a Taylor's expansion of f at the point m and control the remainder term by means of Chebyshev's inequality.

5.22. *Brownian motion.* A heavy particle is randomly hit by light particles, so that its velocity is randomly reversed at equidistant time points. That is, its spatial coordinate (in a given direction) at time t satisfies $X_t = \sum_{i=1}^{\lfloor t \rfloor} V_i$ with independent velocities V_i, where $P(V_i = \pm v) = 1/2$ for some $v > 0$. If we pass to a macroscopic scale, the particle at time t can be described by the random variable $B_t^{(\varepsilon)} = \sqrt{\varepsilon} X_{t/\varepsilon}$, where $\varepsilon > 0$. Determine the distributional limit B_t of $B_t^{(\varepsilon)}$ for $\varepsilon \to 0$ and also the limiting Lebesgue density ϱ_t. Verify that this family of densities satisfies the heat equation

$$\frac{\partial \varrho_t(x)}{\partial t} = -\frac{D}{2} \frac{\partial^2 \varrho_t(x)}{\partial x^2}$$

for some appropriately chosen diffusion coefficient $D > 0$.

5.23. *Convergence in probability versus convergence in distribution.* Let X and X_n, $n \geq 1$, be real-valued random variables on the same probability space. Show the following:

(a) $X_n \xrightarrow{P} X$ implies $X_n \xrightarrow{d} X$.

(b) The converse of (a) does not hold in general, but it does when X is almost surely constant.

5.24. *Convergence in distribution of discrete random variables.* Let X and X_n, $n \geq 1$, be random variables on the same probability space that take values in $\mathbb{Z}$. Prove that the following statements are equivalent:

(a) $X_n \xrightarrow{d} X$ for $n \to \infty$.

(b) $P(X_n = k) \to P(X = k)$ for $n \to \infty$ and every $k \in \mathbb{Z}$.

(c) $\sum_{k\in\mathbb{Z}} |P(X_n = k) - P(X = k)| \to 0$ for $n \to \infty$.

5.25. *The arcsine law.* Consider the simple symmetric random walk $(S_j)_{j\leq 2N}$ introduced in Problem 2.7, for fixed $N \in \mathbb{N}$. Let $L_{2N} = \max\{2n \leq 2N : S_{2n} = 0\}$ be the last time both candidates have the same number of votes before the end of the count. (In a more general context, one would speak of the last visit of the random walk to 0 before time $2N$.) Show the following.

(a) For all $0 \leq n \leq N$, $P(L_{2N} = 2n) = u_n u_{N-n}$, where $u_k = 2^{-2k} \binom{2k}{k}$.