Probabilidades y estadística (M)
Práctica 1

De la lista de ejercicios de abajo (sacados del libro 'Stochastics' de Georgii), hacer los siguientes:

1.1, 1.6-1.13, 1.15-1.17.

Proof. By (1.29) we have $-\infty < X(u) < \infty$ for all $0 < u < 1$. In fact, X is a left-continuous inverse of F; compare Figure 1.4. Indeed, $X(u) \leq c$ holds if and only if $u \leq F(c)$; this is because, by the right-continuity of F, the infimum in the definition of X is in fact a minimum. In particular, $\{X \leq c\} =]0, F(c)] \cap]0,1[\in \mathscr{B}_{]0,1[}$. Together with Example (1.26) this shows that X is a random variable. Furthermore, the set $\{X \leq c\}$ has Lebesgue measure $F(c)$. Hence F is the distribution function of X. $\diamond$

Since every probability measure P on $(\mathbb{R}, \mathscr{B})$ is uniquely determined by its distribution function, we can rephrase the proposition as follows: Every P on $(\mathbb{R}, \mathscr{B})$ is the distribution of a random variable on the probability space $(]0,1[, \mathscr{B}_{]0,1[}, \mathcal{U}_{]0,1[})$. This fact will repeatedly be useful.

The connection between distribution functions and probability densities is made by the notion of a distribution density.

(1.31) Remark and Definition. *Existence of a distribution density.* Let X be a real random variable on a probability space $(\Omega, \mathscr{F}, P)$. Its distribution $P \circ X^{-1}$ admits a Lebesgue density ϱ if and only if

$$F_X(c) = \int_{-\infty}^{c} \varrho(x)\, dx \quad \text{for all } c \in \mathbb{R}.$$

Such a ϱ is called the *distribution density* of X. In particular, $P \circ X^{-1}$ admits a continuous density ϱ if and only if F_X is continuously differentiable, and then $\varrho = F_X'$. This follows directly from (1.8d) and the uniqueness theorem (1.12).

Problems

1.1. Let $(\Omega, \mathscr{F})$ be an event space, $A_1, A_2, \ldots \in \mathscr{F}$ and

$$A = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\}.$$

Show that (a) $A = \bigcap_{N \geq 1} \bigcup_{n \geq N} A_n$, (b) $1_A = \limsup_{n \to \infty} 1_{A_n}$.

1.2. Let Ω be uncountable and $\mathscr{G} = \{\{\omega\} : \omega \in \Omega\}$ the system of the singleton subsets of Ω. Show that $\sigma(\mathscr{G}) = \{A \subset \Omega : A \text{ or } A^c \text{ is countable}\}$.

1.3. Show that the Borel σ-algebra $\mathscr{B}^n$ on $\mathbb{R}^n$ coincides with $\mathscr{B}^{\otimes n}$, the n-fold product of the Borel σ-algebra $\mathscr{B}$ on $\mathbb{R}$.

1.4. Let $\Omega \subset \mathbb{R}^n$ be at most countable. Show that $\mathscr{B}^n_\Omega = \mathscr{P}(\Omega)$.

1.5. Let E_i, $i \in \mathbb{N}$, be countable sets and $\Omega = \prod_{i \geq 1} E_i$ their Cartesian product. Denote by $X_i : \Omega \to E_i$ the projection onto the ith coordinate. Show that the system

$$\mathscr{G} = \big\{\{X_1 = x_1, \ldots, X_k = x_k\} : k \geq 1,\ x_i \in E_i\big\} \cup \big\{\varnothing\big\}$$

is an intersection-stable generator of the product σ-algebra $\bigotimes_{i \geq 1} \mathscr{P}(E_i)$.

1.6. *Inclusion–exclusion principle*. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $A_i \in \mathscr{F}, i \in I = \{1, \ldots, n\}$. For $J \subset I$ let

$$B_J = \bigcap_{j \in J} A_j \cap \bigcap_{j \in I \setminus J} A_j^c\,;$$

by convention, an intersection over an empty index set is equal to Ω. Show the following:

(a) For all $K \subset I$,

$$P\Big(\bigcap_{k \in K} A_k\Big) = \sum_{K \subset J \subset I} P(B_J).$$

(b) For all $J \subset I$,

$$P(B_J) = \sum_{J \subset K \subset I} (-1)^{|K \setminus J|} P\Big(\bigcap_{k \in K} A_k\Big).$$

What does this imply for $J = \varnothing$?

1.7. *Bonferroni inequality*. Let $A_1, \ldots, A_n$ be any events in a probability space $(\Omega, \mathscr{F}, P)$. Show that

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) \geq \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j)\,.$$

1.8. A certain Chevalier de Méré, who has become famous in the history of probability theory for his gambling problems and their solutions by Pascal, once mentioned to Pascal how surprised he was that when throwing three dice he observed the total sum of 11 more often than the sum of 12, although 11 could be obtained by the combinations 6-4-1, 6-3-2, 5-5-1, 5-4-2, 5-3-3, 4-4-3, and the sum of 12 by as many combinations (which ones?). Can we consider his observation as caused by 'chance' or is there an error in his argument? To solve the problem, introduce a suitable probability space.

1.9. In a pack of six chocolate drinks every carton is supposed to have a straw, but it is missing with probability 1/3, with probability 1/3 it is broken and only with probability 1/3 it is in perfect condition. Let A be the event 'at least one straw is missing and at least one is in perfect condition'. Exhibit a suitable probability space, formulate the event A set-theoretically, and determine its probability.

1.10. Alice and Bob agree to play a fair game over 7 rounds. Each of them pays €5 as an initial stake, and the winner gets the total of €10. At the score of 2:3 they have to stop the game. Alice suggests to split the winnings in this ratio. Should Bob accept the offer? Set up an appropriate model and calculate the probability of winning for Bob.

1.11. *The birthday paradox*. Let p_n be the probability that in a class of n children at least two have their birthday on the same day. For simplicity, we assume here that no birthday is on February 29th, and all other birthdays are equally likely. Show (using the inequality $1-x \leq e^{-x}$) that

$$p_n \geq 1 - \exp\left(-n(n-1)/730\right),$$

and determine the smallest n such that $p_n \geq 1/2$.

1.12. *The rencontre problem*. Alice and Bob agree to play the following game: From two completely new, identical sets of playing cards, one is well shuffled. Both piles are put next to each other face down, and then revealed card by card simultaneously. Bob bets (for a stake of €10) that in this procedure at least two identical cards will be revealed at the same time. Alice, however, is convinced that this is 'completely unlikely' and so bets the opposite way. Who do you think is more likely to win? Set up an appropriate model and calculate the probability of winning for Alice. *Hint:* Use Problem 1.6(b); the sum that appears can be approximated by the corresponding infinite series.

1.13. Let $X, Y, X_1, X_2, \ldots$ be real random variables on an event space $(\Omega, \mathscr{F})$. Prove the following statements.

(a) $(X, Y) : \Omega \to \mathbb{R}^2$ is a random variable.

(b) $X + Y$ and XY are random variables.

(c) $\sup_{n\in\mathbb{N}} X_n$ and $\limsup_{n\to\infty} X_n$ are random variables (taking values in $\bar{\mathbb{R}}$).

(d) $\{X = Y\} \in \mathscr{F}$, $\{\lim_{n\to\infty} X_n \text{ exists}\} \in \mathscr{F}$, $\{X = \lim_{n\to\infty} X_n\} \in \mathscr{F}$.

1.14. Let $(\Omega, \mathscr{F}) = (\mathbb{R}, \mathscr{B})$ and $X : \Omega \to \mathbb{R}$ be an arbitrary real function. Verify the following:

(a) If X is piecewise monotone (i.e., $\mathbb{R}$ may be decomposed into at most countably many intervals, on each of which X is either increasing or decreasing), then X is a random variable.

(b) If X is differentiable with (not necessarily continuous) derivative X', then X' is a random variable.

1.15. *Properties of distribution functions.* Let P be a probability measure on $(\mathbb{R}, \mathscr{B})$ and $F(c) = P(]-\infty, c])$, for $c \in \mathbb{R}$, its distribution function. Show that F is monotone increasing and right-continuous, and (1.29) holds.

1.16. Consider the two cases

(a) $\Omega = [0, \infty[$, $\varrho(\omega) = e^{-\omega}$, $X(\omega) = (\omega/\alpha)^{1/\beta}$ for $\omega \in \Omega$ and $\alpha, \beta > 0$,

(b) $\Omega =]-\pi/2, \pi/2[$, $\varrho(\omega) = 1/\pi$, $X(\omega) = \sin^2 \omega$ for $\omega \in \Omega$.

In each case, show that ϱ is a probability density and X a random variable on $(\Omega, \mathscr{B}_\Omega)$, and calculate the distribution density of X with respect to the probability measure P with density ϱ. (The distribution of X in case (a) is called the *Weibull distribution* with parameters α, β, in case (b) the *arcsine distribution*.)

1.17. *Transformation to uniformity.* Prove the following converse to Proposition (1.30): If X is a real random variable with a *continuous* distribution function $F_X = F$, then the random variable $F(X)$ is uniformly distributed on $[0, 1]$.