$$\Rightarrow \quad \Gamma^1_{11} = \frac{\begin{vmatrix} \frac{1}{2}E_1 & F \\ F_1 - \frac{1}{2}E_2 & G \end{vmatrix}}{EG - F^2} \qquad \Gamma^2_{11} = \frac{\begin{vmatrix} E & \frac{1}{2}E_1 \\ F & F_1 - \frac{1}{2}E_2 \end{vmatrix}}{EG - F^2}$$

Ahora luego $\frac{1}{2}E_2 = f_{12} \cdot f_1 = \Gamma^1_{12} \cdot E + \Gamma^2_{12} \cdot F$

$$\frac{1}{2}G_1 = f_{12} \cdot f_2 = \Gamma^1_{12} \cdot F + \Gamma^2_{12} \cdot G$$

$$\Rightarrow \quad \Gamma^1_{12} = \frac{1}{2}\frac{\begin{vmatrix} E_2 & F \\ G_1 & G \end{vmatrix}}{EG - F^2} \qquad \Gamma^2_{12} = \frac{1}{2}\frac{\begin{vmatrix} E & E_2 \\ F & G_1 \end{vmatrix}}{EG - F^2}$$

Finalmente, $F_2 - \frac{1}{2}G_1 = f_{22} \cdot f_1 = \Gamma^1_{22} \cdot E + \Gamma^2_{22} \cdot F$

$$\frac{1}{2}G_2 = f_{22} \cdot f_2 = \Gamma^1_{22} \cdot F + \Gamma^2_{22} \cdot G$$

$$\left(F_2 = (f_1 \cdot f_2)_2 = f_{12} \cdot f_2 + f_1 \cdot f_{22} = \frac{1}{2}G_1 + f_1 \cdot f_{22} \right)$$

$$\Rightarrow \quad \Gamma^1_{22} = \frac{\begin{vmatrix} F_2 - \frac{1}{2}G_1 & F \\ \frac{1}{2}G_2 & G \end{vmatrix}}{EG - F^2} \qquad \Gamma^2_{22} = \frac{\begin{vmatrix} E & F_2 - \frac{1}{2}G_1 \\ F & \frac{1}{2}G_2 \end{vmatrix}}{EG - F^2}$$

Ej Sup. $F = 0$ (p. ej. sup. de revolución)

$$\Rightarrow \quad \Gamma^1_{11} = \frac{1}{2}\frac{E_1 G}{EG} = \frac{E_1}{2E} \qquad \Gamma^2_{11} = -\frac{1}{2}\frac{E \cdot E_2}{EG} = \frac{-E_2}{2G}$$

$$\Gamma^1_{12} = \frac{E_2}{2G} \qquad \Gamma^2_{12} = \frac{1}{2}\frac{G_1}{G}$$

$$\Gamma^1_{22} = -\frac{G_1}{2E} \qquad \Gamma^2_{22} = \frac{G_2}{2G}$$

a

Teorema (gauss) $S_1, S_2 \subseteq \mathbb{R}^3$ superficies

$f: S_1 \to S_2$ isometría

Entonces $K_{S_1}(x) = K_{S_2}(f(x))$, $\forall x \in S_1$

($K_S : S \to \mathbb{R}$ curvatura de gauss de S)

Dem Sea $\mathcal{U} \xrightarrow{\varphi} \mathbb{R}^3$ parametrización regular

$\mathcal{U} \subseteq \mathbb{R}^2$, $S = \varphi(\mathcal{U})$

$\varphi_1 = \frac{\partial \varphi}{\partial u_1}$, $\varphi_2 = \frac{\partial \varphi}{\partial u_2}$, $N_\varphi := \frac{\varphi_1 \times \varphi_2}{|\varphi_1 \times \varphi_2|} : \mathcal{U} \to \mathbb{S}^2$

Def $N_{\varphi,S} := N_\varphi \circ \varphi^{-1}$

$$\begin{tikzcd} \mathcal{U} \arrow[r, "N_\varphi"] \arrow[d, "\varphi"'] & \mathbb{S}^2 \\ S \arrow[ur, "N_{\varphi,S}"'] & \end{tikzcd}$$

$\odot$

① Sea $\mathcal{V} \xrightarrow{\sigma} \mathcal{U}$ difeo ($\mathcal{U}, \mathcal{V} \subseteq \mathbb{R}^2$ abiertos conexos)

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\sigma"] \arrow[rr, bend right, "\psi"'] & \mathcal{U} \arrow[r, "\varphi"] & \mathbb{R}^3 \end{tikzcd} \qquad \psi = \varphi \circ \sigma$$

Entonces $N_{\varphi,S}$ ~~blah~~ $= \pm N_{\psi,S}$

dem: ejercicio.

Def Para $p \in S$, $\mathcal{G}_\varphi(p) : TS(p) \to T\mathbb{S}^2(N_{\varphi,S}(p)) = TS(p)$

$\mathcal{G}_\varphi(p) := dN_{\varphi,S}(p)$

① $\Rightarrow$ $\mathcal{G}_\varphi(p) = \pm \mathcal{G}_\psi(p)$

$\Rightarrow$ $\det \mathcal{G}_\varphi(p) = \det \mathcal{G}_\psi(p)$

$:= K_S(p)$

$\Rightarrow K_S : S \to \mathbb{R}$ bien definida.

(2) $p = \varphi(u)$, $u \in \mathcal{U}$

$$\begin{tikzcd} T\mathcal{U}(u) = \mathbb{R}^2 \arrow[r, "dN_\varphi(u)"] \arrow[d, "d\varphi(u)"'] & TS(p) \\ TS(p) \arrow[ur, "dg_\varphi(p)"'] & \end{tikzcd}$$

(aplico derivada en ⊙)

vimos: $\mathcal{B} = \{\varphi_1, \varphi_2\}$ $\qquad a_\varphi = [dN_\varphi]_{\mathcal{B}}$ $\qquad$ (en $p = \varphi(u)$)

$$= -[\mathrm{II}_\varphi] \cdot [\mathrm{I}_\varphi]^{-1}$$

donde $[\mathrm{I}_\varphi] = \begin{bmatrix} E_\varphi & F_\varphi \\ F_\varphi & G_\varphi \end{bmatrix}$, $\quad E_\varphi = \varphi_1 \cdot \varphi_1 \quad F_\varphi = \varphi_1 \cdot \varphi_2$

$$G_\varphi = \varphi_2 \cdot \varphi_2$$

$[\mathrm{II}_\varphi] = \begin{bmatrix} e_\varphi & f_\varphi \\ f_\varphi & g_\varphi \end{bmatrix}$, $\quad e_\varphi = \varphi_{11} \cdot N_\varphi \quad f_\varphi = \varphi_{12} \cdot N_\varphi$

$$g_\varphi = \varphi_{22} \cdot N_\varphi$$

$$\Rightarrow \quad K_S \circ \varphi = \det a_\varphi = \frac{e_\varphi \cdot g_\varphi - f_\varphi^2}{E_\varphi \cdot G_\varphi - F_\varphi^2}$$

(3) (mente de Gauss) $K_S \circ \varphi$ se puede expresar como $\mathbb{K}(E_\varphi, F_\varphi, G_\varphi, (E_\varphi)_1, (E_\varphi)_2, \ldots)$

$\mathbb{K}$ expresión explícita, no depende de φ.

(4)

$$\begin{tikzcd} S_1 \arrow[r, "f"] & S_2 \\ \mathcal{U}_1 \arrow[u, "\varphi_1"] \arrow[r, "\bar f"'] & \mathcal{U}_2 \arrow[u, "\varphi_2"'] \end{tikzcd} \quad (*)$$

$\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathbb{R}^2$

φ_1, φ_2 p.r.

$S_i = \varphi_i(\mathcal{U}_i)$

f difeo.

$\bar f = \varphi_2^{-1} f \varphi_1$

Entonces:

i) $[I_{\varphi_1}] = J(\bar f)^t \cdot [I_{\varphi_2}] \circ \bar f \cdot J(\bar f) \iff f$ isometría

Dem: lo vimos

ii) En particular, si S_1, S_2 están parametrizados por el mismo abierto $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{U}$, $\bar f = id_{\mathcal{U}}$

$$\begin{tikzcd} S_1 \arrow[rr, "f"] & & S_2 \\ & \mathcal{U} \arrow[ul, "\varphi_1"] \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

entonces

f isometría $\iff [I_{\varphi_1}] = [I_{\varphi_2}]$

(φ_1, φ_2 tienen mismos E, F, G)

iii) Si S_1, S_2 son isométricos, existen $\mathcal{U} \xrightarrow{\varphi_1} S_1$, $\mathcal{U} \xrightarrow{\varphi_2} S_2$ como en ii).

En efecto, dada f como en (*)

$\bar f$ difeo $\Rightarrow \psi_2 = \varphi_2 \circ \bar f : \mathcal{U}_1 \to S_2$ es p.r.

$$\Rightarrow \begin{tikzcd} S_1 \arrow[r, "f"] & S_2 \\ \mathcal{U}_1 = \mathcal{U} \arrow[u, "\varphi_1"] \arrow[ur, "\psi_2"'] & \end{tikzcd}$$

es como se buscaba.

5 Con la notació de 4,

4 ii) $\Rightarrow$ $[I_{\varphi_1}] = [I_{\psi_2}]$

$\underset{3}{\Rightarrow}$ $K_{S_1} \circ \varphi_1 = K_{S_2} \circ \psi_2$

$\underset{\psi_2 = f \circ \varphi_1}{\Rightarrow}$ $K_{S_1} \circ \varphi_1 = K_{S_2} \circ f \circ \varphi_1$

$\Rightarrow$ $K_{S_1} = K_{S_2} \circ f$ ✓