Normal Multivariada

Graciela Boente

Cuál es la conjunta?

- $Z_1, \ldots, Z_p$ i.i.d $\quad Z_j \sim N(0,1) \quad \mathbf{Z} = (Z_1, \ldots, Z_p)^{\mathrm{T}}$

Cuál es su distribución conjunta?

$$f(\mathbf{z}) \;=\; f(z_1, z_2, \cdots, z_p) = f_1(z_1)\, f_2(z_2) \cdots f_p(z_p)$$

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&= \prod_{j=1}^{p} f(z_j) = \prod_{j=1}^{p} \frac{1}{(2\,\pi)^{\frac{1}{2}}} e^{-\frac{1}{2} z_j^2}
\end{aligned}
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&= \frac{1}{(2\,\pi)^{\frac{p}{2}}} e^{-\frac{1}{2}\mathbf{z}^{\mathrm{T}}\mathbf{z}}
\end{aligned}
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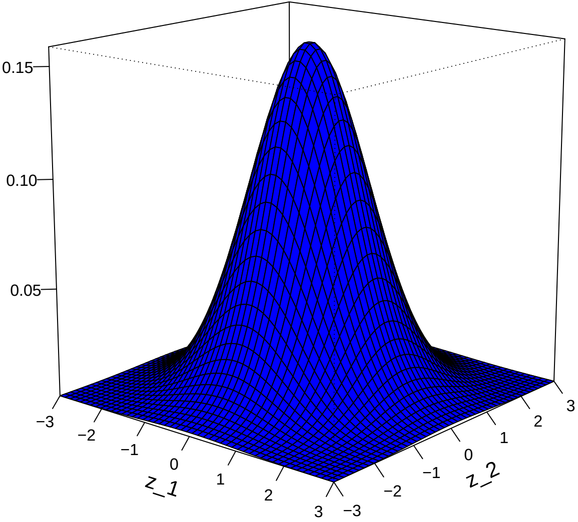
Diremos que $\mathbf{Z} \sim N(\mathbf{0}_p, \mathbf{I}_p)$

Normal Multivariada $p = 2$

Densidad de $\mathbf{Z} \sim N(\mathbf{0}_2, \mathbf{I}_2)$



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- $X_1, \ldots, X_p$ i.i.d $\quad X_j \sim N(0, \sigma_j^2) \quad \mathbf{X} = (X_1, \ldots, X_p)^{\mathrm{T}}$

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Sea $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1^2, \cdots, \sigma_p^2)$

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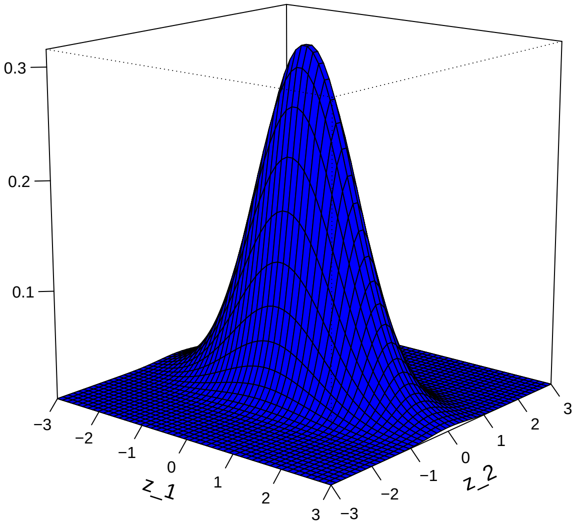
Diremos que $\mathbf{X} \sim N(\mathbf{0}_p, \boldsymbol{\Sigma})$

Normal Multivariada $p = 2$

Densidad de $\mathbf{X} \sim N(\mathbf{0}_2, \mathbf{\Sigma})$, $\mathbf{\Sigma} = \mathrm{diag}(1, 0.25)$



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- $X_1, \ldots, X_p$ i.i.d $\quad X_j \sim N(\mu_j, \sigma_j^2) \quad \mathbf{X} = (X_1, \ldots, X_p)^{\mathrm{T}}$

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Diremos que $\mathbf{X} \sim N(\mu, \boldsymbol{\Sigma})$

Definición 1

- Sea $\boldsymbol{\mu} \in \mathbb{R}^p$ y $\boldsymbol{\Sigma} \in \mathbb{R}^{p \times p}$ simétrica y definida positiva
  Se dice que $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ si su densidad está dada por

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{p}{2}}} \frac{1}{|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \mu) \right\}$$

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- Si $\mathbf{x} \sim N(\mathbf{0}, \mathrm{diag}(\lambda_1, \ldots, \lambda_p))$

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{p}{2}}} \frac{1}{\prod_{j=1}^{p} \lambda_j^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2} \sum_{j=1}^{p} \frac{x_j^2}{\lambda_j} \right\}$$

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  Por lo tanto, $x_1, \ldots, x_p$ son independientes $x_j \sim N(0, \lambda_j)$.
- En particular, si $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I}_p)$, $x_1, \ldots, x_p$ son i.i.d. $N(0,1)$.
- Si $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ y $\mathbf{A} \in \mathbb{R}^{p \times p}$ es no singular $\Longrightarrow$
  $\mathbf{A}\mathbf{x} + \mathbf{b} \sim N(\mathbf{A}\mu + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathrm{T}})$

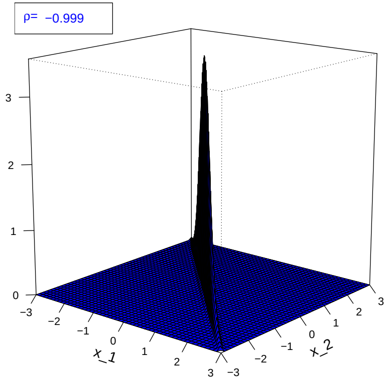
Caso $p = 2$

- Sea $\boldsymbol{\mu} \in \mathbb{R}^2$ y $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$ definida positiva $(|\rho| \neq 1)$

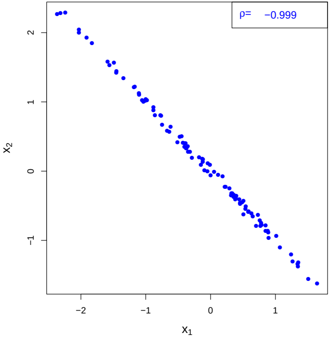
$$f(\mathbf{x}) = \frac{1}{2\pi}\frac{1}{\sigma_1\sigma_2(1-\rho^2)^{\frac{1}{2}}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]\right\}$$

Caso $p = 2$

Densidad



Datos generados



Propiedades

- Si $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I}_p) \Longrightarrow \varphi_{\mathbf{x}}(\mathbf{t}) = \exp\{-\frac{1}{2}\|\mathbf{t}\|^2\}$

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- $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longleftrightarrow \mathbf{a}^{\mathrm{T}}\mathbf{x} \sim N(\mathbf{a}^{\mathrm{T}}\boldsymbol{\mu}, \mathbf{a}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{a})$, $\forall \mathbf{a} \in \mathbb{R}^p$

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- $\mathbf{x} = (x_1, \ldots, x_p)^{\mathrm{T}} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \mathbb{E}(\mathbf{x}) = \boldsymbol{\mu}$ y $\mathrm{Cov}(\mathbf{x}) = (\mathrm{Cov}(x_i, x_j))_{1 \le i,j \le p} = \boldsymbol{\Sigma}$

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- Sea $\mathbf{x} = (x_1, \ldots, x_p)^{\mathrm{T}} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, entonces, $x_1, \ldots, x_p$ son independientes $\Longleftrightarrow$ $\boldsymbol{\Sigma}$ es diagonal.

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Definición 2

Se dice que $\mathbf{x}$ es normal multivariada si y sólo si $\forall\,\mathbf{t} \in \mathbb{R}^p$ se tiene que $\mathbf{t}^{\mathrm{T}}\mathbf{x}$ es normal univariada.

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Definición 3

Se dice que $\mathbf{x}$ es normal multivariada si y sólo si $\exists \boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma} \in \mathbb{R}^{p\times p}$ simétrica $\boldsymbol{\Sigma} > 0$, tal que

$$\varphi_{\mathbf{x}}(\mathbf{t}) = \exp\{i\mathbf{t}^{\mathrm{T}}\boldsymbol{\mu}\}\exp\{-\frac{1}{2}\mathbf{t}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{t}\}$$

Propiedades

Vimos que si $\mathbf{x} \sim N(\mathbf{0}, \mathbf{\Sigma})$, entonces

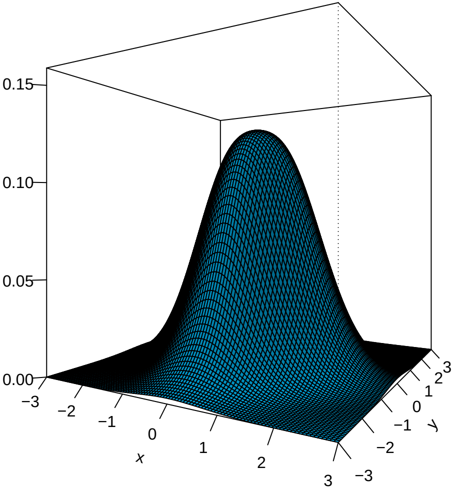
- $x_1, \ldots, x_p$ son independientes $\iff$ $\mathbf{\Sigma}$ es diagonal.
- $x_j \sim N(0, \sigma_{jj})$ donde $\mathbf{\Sigma} = (\sigma_{ij})_{1 \leq i,j \leq p}$

El siguiente ejemplo muestra que existen vectores aleatorios $\mathbf{x} = (x_1, x_2)$ tales que

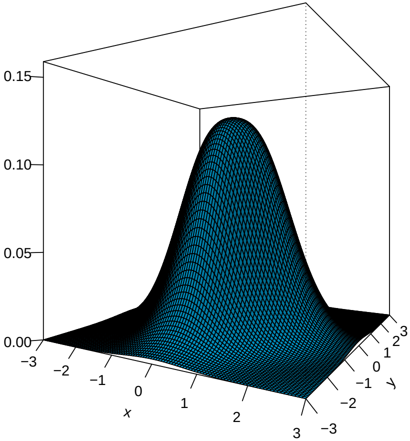
- $\mathrm{Cov}(\mathbf{x}) = \mathbf{I}_p$.
- $x_j \sim N(0, 1)$
- x_1 y x_2 no son independientes.

Sea **x** con densidad

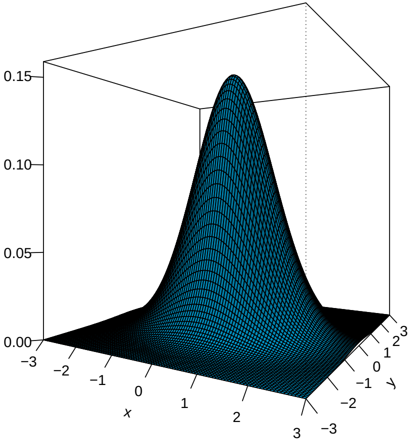
$$h(x,y) = \frac{1}{2\pi}\left\{\left(\sqrt{2}\,e^{-\frac{x^2}{2}} - e^{-x^2}\right)e^{-y^2} + \left(\sqrt{2}\,e^{-\frac{y^2}{2}} - e^{-y^2}\right)e^{-x^2}\right\}$$



Densidad h



Densidad $N(\mathbf{0}, \mathbf{I}_2)$



Propiedades

Sea $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ con $\boldsymbol{\Sigma} > 0$. Definamos

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{pmatrix} \qquad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$$

con $\mathbf{x}^{(i)}, \boldsymbol{\mu}^{(i)} \in \mathbb{R}^{p_i}$, $\boldsymbol{\Sigma}_{ii} \in \mathbb{R}^{p_i \times p_i}$, $p_1 + p_2 = p$.

Entonces,

a) $\mathbf{x}^{(1)} \sim N(\boldsymbol{\mu}^{(1)}, \boldsymbol{\Sigma}_{11})$ y $\mathbf{x}^{(2)} \sim N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22})$.

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Entonces,

a) $\mathbf{x}^{(1)} \sim N(\boldsymbol{\mu}^{(1)}, \boldsymbol{\Sigma}_{11})$ y $\mathbf{x}^{(2)} \sim N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22})$.

b) Más aún, $\mathbf{x}^{(1)}$ y $\mathbf{x}^{(2)}$ son independientes $\iff \boldsymbol{\Sigma}_{21} = 0$.

Propiedades

Sea $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ con $\boldsymbol{\Sigma} > 0$. Definamos

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{pmatrix} \qquad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$$

con $\mathbf{x}^{(i)}, \boldsymbol{\mu}^{(i)} \in \mathbb{R}^{p_i}$, $\boldsymbol{\Sigma}_{ii} \in \mathbb{R}^{p_i \times p_i}$, $p_1 + p_2 = p$.

Entonces,

a) $\mathbf{x}^{(1)} \sim N(\boldsymbol{\mu}^{(1)}, \boldsymbol{\Sigma}_{11})$ y $\mathbf{x}^{(2)} \sim N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22})$.

b) Más aún, $\mathbf{x}^{(1)}$ y $\mathbf{x}^{(2)}$ son independientes $\Longleftrightarrow \boldsymbol{\Sigma}_{21} = 0$.

c) Dada $\mathbf{A} \in \mathbb{R}^{q \times p}$, $rg(\mathbf{A}) = q \Longrightarrow \mathbf{Ax} \sim N(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathrm{T}})$.

Propiedades

Sea $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ con $\boldsymbol{\Sigma} > 0$. Definamos

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{pmatrix} \qquad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$$

con $\mathbf{x}^{(i)}, \boldsymbol{\mu}^{(i)} \in \mathbb{R}^{p_i}$, $\boldsymbol{\Sigma}_{ii} \in \mathbb{R}^{p_i \times p_i}$, $p_1 + p_2 = p$.

Entonces,

a) $\mathbf{x}^{(1)} \sim N(\boldsymbol{\mu}^{(1)}, \boldsymbol{\Sigma}_{11})$ y $\mathbf{x}^{(2)} \sim N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22})$.

b) Más aún, $\mathbf{x}^{(1)}$ y $\mathbf{x}^{(2)}$ son independientes $\iff \boldsymbol{\Sigma}_{21} = 0$.

c) Dada $\mathbf{A} \in \mathbb{R}^{q \times p}$, $rg(\mathbf{A}) = q \Longrightarrow \mathbf{A}\mathbf{x} \sim N(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathrm{T}})$.

En particular, si $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I}_p)$ y $\mathbf{H} = (\mathbf{h}_1, \ldots, \mathbf{h}_q) \in \mathbb{R}^{p \times q}$, es ortogonal incompleta, o sea, $\mathbf{H}^{\mathrm{T}}\mathbf{H} = \mathbf{I}_q$, entonces $\mathbf{y} = \mathbf{H}^{\mathrm{T}}\mathbf{x} \sim N(\mathbf{0}, \mathbf{I}_q)$.

Propiedades

d) Sea $\boldsymbol{\Sigma} = \mathbf{H}\boldsymbol{\Lambda}\mathbf{H}^{\mathrm{T}}$, con $\mathbf{H}$ ortogonal y $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \ldots, \lambda_p)$, $\lambda_1 \geq \ldots, \lambda_p$.
Si $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \mathbf{H}^{\mathrm{T}}(\mathbf{x} - \boldsymbol{\mu}) \sim N(\mathbf{0}, \boldsymbol{\Lambda})$.

Propiedades

d) Sea $\mathbf{\Sigma} = \mathbf{H\Lambda H}^{\mathrm{T}}$, con $\mathbf{H}$ ortogonal y $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$, $\lambda_1 \geq \ldots, \lambda_p$.
Si $\mathbf{x} \sim N(\boldsymbol{\mu}, \mathbf{\Sigma}) \Longrightarrow \mathbf{H}^{\mathrm{T}}(\mathbf{x} - \boldsymbol{\mu}) \sim N(\mathbf{0}, \mathbf{\Lambda})$.

e) Si $\mathbf{x} \sim N(\boldsymbol{\mu}, \mathbf{\Sigma}) \Longleftrightarrow \mathbf{x} = \mathbf{Az} + \boldsymbol{\mu}$ con $\mathbf{z} \sim N(\mathbf{0}, \mathbf{I}_p)$ y $\mathbf{AA}^{\mathrm{T}} = \mathbf{\Sigma}$.

Propiedades

d) Sea $\boldsymbol{\Sigma} = \mathbf{H}\boldsymbol{\Lambda}\mathbf{H}^{\mathrm{T}}$, con $\mathbf{H}$ ortogonal y $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \ldots, \lambda_p)$, $\lambda_1 \geq \ldots, \lambda_p$.
Si $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \mathbf{H}^{\mathrm{T}}(\mathbf{x} - \boldsymbol{\mu}) \sim N(\mathbf{0}, \boldsymbol{\Lambda})$.

e) Si $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longleftrightarrow \mathbf{x} = \mathbf{A}\mathbf{z} + \boldsymbol{\mu}$ con $\mathbf{z} \sim N(\mathbf{0}, \mathbf{I}_p)$ y $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \boldsymbol{\Sigma}$.

f) Si $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow (\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \mu) \sim \chi^2_p$

g) Si $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \mathbf{x}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{x} \sim \chi^2_p(\delta^2)$ con $\delta^2 = \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$

Propiedades

Lema 1. Sea $\mathbf{x} \sim N(\mathbf{0}, \sigma^{\mathbf{2}}\mathbf{I_p})$ y sea $\mathbf{H}_1 = (\mathbf{h}_1, \ldots, \mathbf{h}_q) \in \mathbb{R}^{p \times q}$ ortogonal incompleta, o sea,

$$\mathbf{H}_1^{\mathrm{T}}\mathbf{H}_1 = \mathbf{I}_q$$

Sea $\mathbf{H} = (\mathbf{H}_1, \mathbf{H}_2)$ ortogonal, o sea, $\mathbf{H}^{\mathrm{T}}\mathbf{H} = \mathbf{H}\mathbf{H}^{\mathrm{T}} = \mathbf{I}_p$ Entonces

a) $\mathbf{z} = \mathbf{H}_1^{\mathrm{T}}\mathbf{x} \sim N(\mathbf{0}, \sigma^{\mathbf{2}}\mathbf{I_q})$.

b) $\mathbf{z}$ es independiente de $\mathbf{x}^{\mathrm{T}}\mathbf{x} - \mathbf{z}^{\mathrm{T}}\mathbf{z}$.

c) $\dfrac{\mathbf{x}^{\mathrm{T}}\mathbf{x} - \mathbf{z}^{\mathrm{T}}\mathbf{z}}{\sigma^2} \sim \chi^2_{p-q}$.