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Álgebras Jacobianas asociadas a superficies a través del Lema del Diamante

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Introducción

Un problema recurrente en diversas áreas del álgebra consiste en, dados un objeto algebraico A y un subobjeto B, entender la estructura del cociente A/B. En su influyente trabajo [Ber78], Bergman presenta el *lema del diamante*, una técnica para poder atacar este problema en el contexto de la teoría de k-álgebras. Esencialmente, uno puede entender a las relaciones dadas por los elementos del ideal B como reglas de reescritura para elementos del cociente A/B. Si el sistema de reglas de reescritura verifica una cierta condición de confluencia, entonces somos capaces de realizar cómputos explícitos en el cociente de una manera sumamente sencilla.

El lema del diamante de Bergman es un resultado muy versátil; en la misma publicación donde fue enunciado aparecen diversas versiones del mismo para múltiples estructuras algebraicas. Más recientemente, en [SAV15] se describe un resultado análogo para completaciones de k-álgebras. En este trabajo empleamos esta variación para estudiar una familia particular de cocientes de este tipo de objetos.

En [DWZ08] se describe un procedimiento que permite asignarle un *quiver con potencial*, o *QP*, a una triangulación de una superficie. Un quiver es un multigrafo dirigido, y un potencial es una combinación lineal de ciclos en la completación del álgebra de caminos asociada al quiver. El *álgebra Jacobiana* asociada al QP es un cociente particular de dicha completación; explícitamente, es aquel que se obtiene al cocientar por el ideal cerrado generado por las derivadas cíclicas del potencial.

En trabajos como [Lad12] y [TVD12] fue determinada una propiedad clave de estos objetos: el álgebra Jacobiana inducida por un QP asociado a una triangulación de una superficie Σ es de dimensión finita si Σ no es una esfera con cuatro punciones. En particular, el hecho de que el álgebra sea o no de dimensión finita es independiente de la elección de escalares del potencial.

En esta tesis estudiamos este tipo de problemas, pero empleando como herramienta principal el lema del diamante. Además, presentamos un procedimiento análogo para generar un QP a partir de una subdivisión poligonal arbitraria de una superficie y abordamos este mismo tipo de problemas en esta nueva situación.

Nuestro trabajo se organiza de la siguiente manera. En el primer capítulo desarrollamos los preliminares necesarios para definir con precisión los QPs asociados a una triangulación de una superficie, y sus correspondientes álgebras Jacobianas. En el segundo capítulo presentamos el lema del diamante de Bergman en su versión para k-álgebras y luego en una variante para completaciones de álgebras de caminos, que será la que usaremos principalmente. Incluimos además varios ejemplos para ilustrar cómo se utiliza el lema del diamante en situaciones concretas.

El tercer capítulo contiene los resultados principales de la tesis. Comenzamos estudiando el caso de la esfera con cuatro punciones, que era el único caso sin cubrir en [Lad12], y determinamos que el álgebra Jacobiana asociada es de dimensión infinita (más aún, calculamos su serie de Hilbert). Luego, estudiamos la dimensión de álgebras Jacobianas procedentes de subdivisiones arbitrarias. Construimos familias infinitas de dichas álgebras tanto de dimensión finita como infinita, mostrando que la situación en el caso poligonal es notablemente diferente al caso triangular. Finalmente, producimos sistemas de reescritura confluentes para tres familias de subdivisiones poligonales de la esfera: las pirámides, los prismas y los antiprismas. Las pirámides proveen además de una familia de contraejemplos a la generalización de un teorema de Ladkani (acerca de la relación entre la dimensión del álgebra y la elección de escalares en el potencial) al caso poligonal.

En el último capítulo, utilizamos estos sistemas de reescritura para calcular invariantes de tipo cohomológico para sus álgebras Jacobianas asociadas. En particular, computamos el centro de dichas álgebras y probamos que admiten derivaciones no triviales.

Finalmente, incluimos un apéndice en el que anexamos y explicamos el funcionamiento de un programa escrito en `SageMath`, el cual desarrollamos para facilitar el cálculo de sistemas de reescritura.

Introduction

A recurring problem in various branches of algebra consists in, given an algebraic object A and a subobject B, understanding the structure of the quotient A/B. In his influential work [Ber78], Bergman introduces the *diamond lemma*, a tool used to tackle this problem in the context of k-algebras. Essentially, one may think of the relations given by elements in the ideal B as rewriting rules for the elements in the quotient A/B. If the set of rewriting rules verifies a certain confluence condition, then we are able to carry out explicit computations in the quotient in a simple fashion.

Bergman's diamond lemma is a very versatile result; there are several versions of it for various algebraic structures on the same paper where it was first stated. More recently, in [SAV15], an analogous result for completions of k-algebras is described. In this work, we make use of this variation to study a particular family of quotients of this kind of objects.

In [DWZ08], the authors describe a procedure that assigns a *quiver with potential*, or *QP* for short, to a triangulation of a surface. A quiver is a directed multigraph, and a potential is a linear combination of cycles in the completion of the path algebra associated to said quiver. The *Jacobian algebra* associated to the QP is a particular quotient of said completion; explicitly, it is the one obtained after modding out the closed ideal spanned by the set of cyclic derivatives of the potential.

In works such as [Lad12] and [TVD12], the authors determine a key property of these objects: the Jacobian algebra induced by a QP associated to a triangulation of a surface Σ is finite-dimensional if Σ is not a sphere with four punctures. In particular, the finite-dimensionality of the algebra is independent of the choice of scalars in the potential.

In this thesis we study this kind of problems, but using the diamond lemma as our main tool. Moreover, we introduce a procedure that assigns a QP to an arbitrary polygonal subdivision of a surface, and we tackle the same kind of problems in this new setting.

Our work is organized as follows. In the first chapter we introduce the necessary preliminaries in order to define the QP associated to a triangulation of a surface, and its corresponding Jacobian algebra. In the second chapter we introduce Bergman's diamond lemma, first in the classical k-algebra setting and then we present a variation of it that applies over completions of path algebras, which we will use the most. We also include several examples, with the purpose of showing how the diamond lemma is used in concrete situations.

The third chapter contains our main results. We start off by studying the case of the sphere with four punctures, which is the only case not discussed in [Lad12], and we prove that the associated Jacobian algebra is infinite-dimensional (moreover, we compute its Hilbert series). We then study the finite-dimensionality problem in the case of Jacobian algebras arising from arbitrary polygonal subdvisions. We construct infinite families of said algebras, of both finite and infinite dimension, showing that the situation in the polygonal case is remarkably different to the triangular case. Finally, we produce confluent rewriting systems for three families of polygonal subdivisions of the sphere: pyramids, prisms and antiprisms. Pyramids also provide a family of counterexamples to the generalization of a theorem of Ladkani (concerning the relation between the dimension of the Jacobian algebra and the choice of scalars appearing in the potential) to the polygonal case.

In the last chapter, we use these rewriting systems to compute cohomological invariants of the associated Jacobian algebras. In particular, we compute the center of said algebras and we prove that they admit non-trivial derivations.

Finally, we include an appendix on which we present a program written in `SageMath`, which we developed to ease computations related to rewriting systems.

Chapter 1

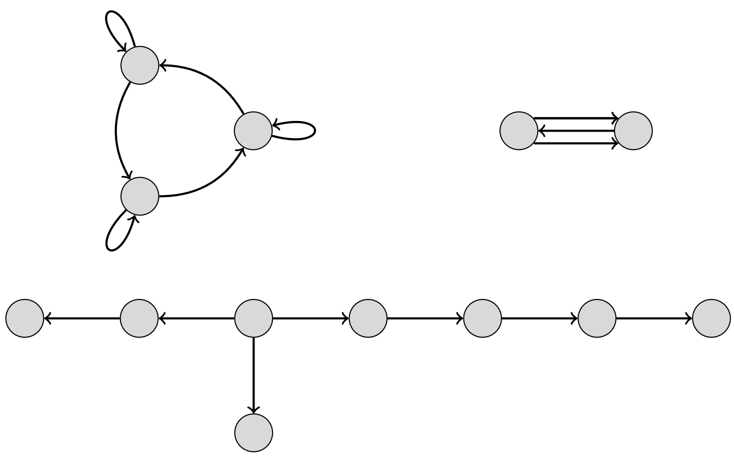
Preliminaries

Throughout the text, k will denote a field of characteristic zero.

1.1 The path algebra of a quiver

A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 and Q_1 are finite sets whose elements are called *vertices* and *arrows* respectively, and $s, t : Q_1 \to Q_0$ are functions that associate to each arrow $\alpha \in Q_1$ its *source* $s(\alpha) \in Q_0$ and its *target* $t(\alpha) \in Q_0$. We will usually abbreviate the fact that an arrow $\alpha \in Q_1$ has source a and target b using the notation $\alpha : a \to b$. We will also omit mentioning s and t explicitly when they are clear from context.

One can represent a quiver graphically as an oriented graph allowing loops and multiple arrows between the same pair of vertices. The following are some examples of quivers:



Let $Q = (Q_0, Q_1, s, t)$ be a quiver and consider the k-vector spaces $R = k^{Q_0}$ and $A = k^{Q_1}$, which we will call the *vertex span* and *arrow span* of Q, respectively. The space R is a commutative k-algebra with the product given by pointwise multiplication. We can consider an R-bimodule structure on A given as follows: if $r \in Q_0$, $\alpha \in Q_1$ then we define $r\alpha = \delta_{r,t(\alpha)}\alpha$ and analogously $\alpha r = \delta_{r,s(\alpha)}\alpha$, and extend the action linearly. A vertex r acts as the identity on the left (right) of an arrow α if the target (source) of α is r, and otherwise it acts as zero. The *path algebra* $k\langle Q\rangle$ associated to the quiver Q is the graded tensor algebra

$$k\langle Q\rangle = \bigoplus_{n=0}^{\infty} A^{\otimes_R n},$$

where we set $A^{\otimes_R 0} = R$. For the sake of simplicity we will usually notate $A^{\otimes_R n}$ as A^n and an elementary tensor $\alpha_n \otimes \cdots \otimes \alpha_1$ as $\alpha_n \dots \alpha_1$. Notice that a non-zero element of the form $\alpha_n \dots \alpha_1$ consists of a sequence of *concatenable* arrows α_i, that is, arrows such that $s(\alpha_{i+1}) = t(\alpha_i)$. We will call such an element a *path of length n*. It is worth observing that the collection of all paths of length n form a basis of A^n as a k-vector space. Since Q_0 is a basis of $A^0 = R$, we will refer to elements of Q_0 as *paths of length 0*, which we will usually call *trivial* or *stationary* paths. Considering the fact that Q_0 and Q_1 are in bijection with paths of length 0 and 1 respectively, we will denote the set of paths of length n as Q_n and the set of all paths as Q_*. We can now define source and target functions $s, t : Q_* \to Q_0$ as follows: if $u = \alpha_n \dots \alpha_0 \in Q_n$ with $n > 0$, then $s(u) = s(\alpha_0)$ and $t(u) = t(\alpha_n)$. Otherwise, if $u = r \in Q_0$ then $s(u) = t(u) = r$. One easily checks that if $a, b \in Q_0$, the space spanned by paths with source a and target b is exactly $bk\langle Q\rangle a$.

The path algebra satisfies the following useful universal property:

Proposition 1.1. *Let Q be a quiver and Λ be an associative k-algebra with unit. Suppose $f_0 : Q_0 \to \Lambda$, $f_1 : Q_1 \to \Lambda$ are maps satisfying:*

1. $\sum_{a \in Q_0} f_0(a) = 1$.
2. *If $a \in Q_0$, then $f_0(a)^2 = f_0(a)$.*
3. *If $a \neq b$, with $a, b \in Q_0$, then $f_0(a)f_0(b) = 0$.*
4. *If $\alpha : a \to b$ is an arrow in Q_1, then $f_1(\alpha) = f_0(b)f_1(\alpha)f_0(a)$.*

Then, there exists a unique k-algebra morphism $f : k\langle Q\rangle \to \Lambda$ extending f_0 and f_1.

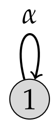
Proof. For $n \geq 1$, define $f(\alpha_n \dots \alpha_1) = f_1(\alpha_n) \dots f_1(\alpha_0)$ and let $f(r) = f_0(r)$ for $r \in Q_0$. As the set of all paths forms a basis of $k\langle Q\rangle$ as a vector space, this defines a k-linear map $f : k\langle Q\rangle \to \Lambda$. Condition 1 guarantees that such a map preserves the unit and conditions 2, 3 and 4 guarantee that it preserves products involving stationary paths. Since by definition f preserves products of paths of positive length, we conclude that f is a k-algebra morphism extending f_0 and f_1. As for uniqueness, consider another extension $g : k\langle Q\rangle \to \Lambda$. Since g is a k-algebra morphism, we have that $g(\alpha_n \dots \alpha_1) = g(\alpha_n) \dots g(\alpha_1) = f_1(\alpha_n) \dots f_1(\alpha_1) = f(\alpha_n \dots \alpha_1)$ for $n \geq 1$ and $g(r) = f_0(r) = f(r)$ for all $r \in Q_0$. Thus g agrees with f in a basis, and so $g = f$ as wanted. $\square$

Let us now consider some examples:

Example 1.2. Let Q be the quiver with vertices $\{1, \dots, n\}$ and no arrows:

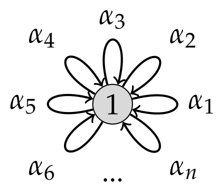
The path algebra $k\langle Q\rangle$ is then isomorphic to $k^n = ke_1 \oplus \cdots \oplus ke_n$, where multiplication is given by $e_i e_j = \delta_{i,j} e_i$ and extended linearly.

Example 1.3. Let Q be the quiver with a single vertex 1 and a single arrow $\alpha : 1 \to 1$.



The path algebra $k\langle Q\rangle$ has a basis given by $\{e_1, \alpha, \alpha^2, \dots\}$. It is easily seen that $k\langle Q\rangle$ is isomorphic to the polynomial algebra $k[x]$, via the map sending $\alpha \mapsto x$ and $1 \mapsto 1$.

Example 1.4. More generally, let Q be the quiver with a single vertex 1 and n arrows $\alpha_n : 1 \to 1$.



The path algebra $k\langle Q\rangle$ is isomorphic to the free algebra on n generators $k\langle x_1, \dots, x_n\rangle$, via the map sending $\alpha_i \mapsto x_i$ and $1 \mapsto 1$.

A path $u \in Q_n$ with $n > 1$ such that $s(u) = t(u)$ is called a *cycle*, and a quiver containing no cycles is said to be *acyclic*. As we can infer from the previous examples, the existence of cycles in the quiver is closely related to the dimension of the path algebra. More precisely, we have:

Proposition 1.5. *Let Q be a quiver and $k\langle Q\rangle$ its associated path algebra. Then $k\langle Q\rangle$ is finite-dimensional iff Q is acyclic.*

Proof. Suppose $k\langle Q\rangle$ is infinite-dimensional. Then the set of all paths Q_*, which is a basis for $k\langle Q\rangle$, must be infinite. Since the quiver has only a finite number of arrows, there is only a finite number of paths of less than a fixed length. Therefore, if the set of paths Q_* is infinite, then there exist arbitrarily long paths. Let n be the number of vertices in Q_0 and pick a path $\alpha_m \dots \alpha_1$ with $m > n$. Then $s(\alpha_i) = t(\alpha_j)$ for some $1 \leq i < j \leq m$ and so $\alpha_j \dots \alpha_i$ is a cycle in Q.

Conversely, if Q contains a cycle u, then $\{u, u^2, u^3, \dots\}$ is an infinite linearly independent set, and so $k\langle Q\rangle$ is infinite-dimensional. $\square$

1.2 Completions

In this section we will outline the construction of the completion of a path algebra. For a more detailed treatment of this subject, the reader may consult [SAV15].

Given a quiver Q, the path algebra $k\langle Q\rangle$ admits a k-algebra norm $|\cdot|$ such that for each non-zero $x = \sum_{u\in Q_*} \lambda_u u$ we have $|x| = e^{-\nu(x)}$, where $\nu(x) = \min\{i \in \mathbb{N}_0 : |u| = i \text{ and } \lambda_u \neq 0\}$. In fact, this is not just a norm but an *ultranorm*, which means we have a stronger triangle inequality, namely

$$|x+y| \leq \max\{|x|, |y|\}.$$

This ultranorm is compatible with multiplication as well, since

$$|xy| = e^{-\nu(xy)} = e^{-\nu(x)-\nu(y)} = e^{-\nu(x)}e^{-\nu(y)} = |x|\cdot|y|.$$

It is thus straightforward to check that the k-algebra operations of $k\langle Q\rangle$ are continuous in the topology induced by this ultranorm. These operations extend to the completion of $k\langle Q\rangle$ as a metric space, making it a k-algebra itself, which we will call the *complete path algebra* $k\langle\langle Q\rangle\rangle$.

Suppose (x_n) is a Cauchy sequence in $k\langle Q\rangle$. If $x_n = \sum_{u\in Q_*} \lambda_u^n u$, the sequence (λ_u^n) is eventually constant, since $|x_n - x_m| < e^{-|u|}$ for big enough n, m. Therefore, we may identify the Cauchy sequence (x_n) with its termwise limit $x = \sum_{u\in Q_*} \lambda_u u$, where λ_u denotes the constant value the sequence (λ_u^n) eventually takes. Notice that the termwise limit x is not necessarily finitely supported. One can now check that $k\langle\langle Q\rangle\rangle$ may be identified with the k-algebra $\prod_{n=0}^{\infty} A^n$, where, as before, A^n stands for the k-vector spanned by paths of length n. The norm $|\cdot|$ extends naturally to this algebra, and it is easy to see that the usual path algebra $k\langle Q\rangle$ is a dense k-subalgebra and R-subbimodule of $k\langle\langle Q\rangle\rangle$.

We now consider the completions of the examples of path algebras mentioned previously:

Example 1.6. Let $k\langle Q\rangle$ be a finite-dimensional path algebra. Since we must have $A^n = 0$ for big enough n, we conclude that $\bigoplus A^n = \prod A^n$ and thus the path algebra coincides with its completion. As we proved in Proposition 1.5, this can only happen if the quiver Q is acyclic.

Example 1.7. Consider the quiver Q from Example 1.4. The isomorphism $k\langle Q\rangle \simeq k\langle x_1,\ldots,x_n\rangle$ extends to an isomorphism between the respective completions $k\langle\langle Q\rangle\rangle \simeq k\langle\langle x_1,\ldots,x_n\rangle\rangle$, where the right-hand side term denotes the algebra of non-commutative formal power series in n variables.

We end this section with a trivial remark that will be used extensively later:

Remark 1.8. *Let I be a closed ideal in $k\langle\langle Q\rangle\rangle$. Suppose x is a path such that, for all $n \in \mathbb{N}$, there exists a path x_n of length at least n such that $x = x_n$ in $k\langle\langle Q\rangle\rangle/I$. Then, $x = 0$ in $k\langle\langle Q\rangle\rangle/I$.*

Proof. Let (x_n) be a sequence of paths such that $x = x_n$ in $k\langle\langle Q\rangle\rangle/I$ and x_n is of length at least n. Then $x - x_n \in I$ and since $|x_n| \leq e^{-n} \to 0$ we have that $x - x_n \to x$. Since the ideal I is closed and the sequence $(x - x_n)$ is contained in I, then $x \in I$ and therefore $x = 0$ in $k\langle\langle Q\rangle\rangle/I$. □

1.3 The Jacobian algebra of a quiver with potential

Let $k\langle\langle Q\rangle\rangle = \prod A^n$ be the complete path algebra associated to a quiver Q. For $n \geq 1$, we define the *cyclic part* of A^n as

$$A^n_{\text{cyc}} = \bigoplus_{r \in Q_0} rA^n r,$$

that is, the R-subbimodule spanned by cycles of length n. A *potential* P is an element of the closed R-subbimodule $k\langle\langle Q\rangle\rangle_{\text{cyc}} \subseteq k\langle\langle Q\rangle\rangle$, which we define as

$$k\langle\langle Q\rangle\rangle_{\text{cyc}} = \prod_{n=1}^{\infty} A^n_{\text{cyc}}.$$

In other words, a potential is a possibly infinitely supported linear combination of cycles. We will call a pair (Q, P) a *quiver with potential*, or *QP* for short.

In their work [RSS80], Rota, Sagan and Stein introduced a notion of derivative for non-commutative algebras, called the *cyclic derivative*. Here we will work with this concept within the context of the complete path algebra of a quiver. Given an arrow $\alpha \in Q_1$, the cyclic derivative with respect to α is the morphism $\partial_\alpha : k\langle\langle Q\rangle\rangle_{\text{cyc}} \to k\langle\langle Q\rangle\rangle$ defined for a cycle $u = \alpha_n \ldots \alpha_1$ as

$$\partial_\alpha(u) = \sum_{k=1}^{n} \delta_{\alpha,\alpha_k} \alpha_{k-1} \ldots \alpha_1 \alpha_n \ldots \alpha_{k+1}$$

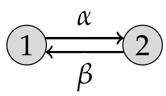
and extended linearly and continuously.

We are now able to introduce the *Jacobian algebra* associated to a QP, which is the main algebraic object of study in this work. Given a QP (Q, P), the *Jacobian ideal* $J(P)$ is the *closed* ideal in $k\langle\langle Q\rangle\rangle$ generated by the set of all cyclic derivatives of the potential, that is, $\{\partial_\alpha(P) : \alpha \in Q_1\}$. The Jacobian algebra $J(Q, P)$ is then the quotient $k\langle\langle Q\rangle\rangle/J(P)$.

Example 1.9. As in Example 1.3, consider the quiver Q with a single vertex and a unique arrow from that vertex to itself. As we have seen, the complete path algebra $k\langle\langle Q\rangle\rangle$ is isomorphic to the algebra of formal power series $k[\![x]\!]$. Identifying x with the only arrow in the quiver, we see that any potential for Q is of the form $\sum_{n=1}^{\infty} \lambda_n x^n$ for some choice of scalars $\lambda_n \in k$. For example, let us consider the potential $P = x^n$. Since x is the only arrow in the quiver, we just have to compute $\partial_x(x^n)$, which turns out to be nx^{n-1} (happily coinciding with the usual, commutative notion of derivation). Therefore, the Jacobian algebra is $k[\![x]\!]/(nx^{n-1})$, which is isomorphic to the truncated polynomial algebra $k[x]/(x^{n-1})$.

A direct computation shows that cyclic derivatives vanish on the subspace V of $k\langle\langle Q\rangle\rangle_{\mathrm{cyc}}$ generated by all commutators. Suppose now that P and P' are two potentials. If every term of P is a cyclic permutation of the factors of a term of P' then the difference $P - P'$ lies in V and so $\partial_\alpha(P - P')$ vanishes for any arrow α. This in turn implies that P and P' induce both the same Jacobian ideal and the same Jacobian algebra. In this case, we will say that P and P' are *cyclically equivalent*.

Example 1.10. Let Q be the quiver with vertices 1 and 2, and arrows $\alpha : 1 \to 2$, $\beta : 2 \to 1$.



Any potential for Q is of the form $\sum_{n=1}^{\infty} \lambda_n(\alpha\beta)^n + \mu_n(\beta\alpha)^n$ for some scalars $\lambda_n, \mu_n \in k$. For instance, the cyclically equivalent potentials $P = (\alpha\beta)^n$ and $P' = (\beta\alpha)^n$ give rise to the same Jacobian ideal, which is $(\partial_\alpha(P), \partial_\beta(P)) = (n(\beta\alpha)^{n-1}\beta, n(\alpha\beta)^{n-1}\alpha)$. As any path of length at least $2n$ has either $(\beta\alpha)^{n-1}\beta$ or $(\alpha\beta)^{n-1}\alpha$ as factors, the Jacobian ideal contains A^d for $d \geq 2n$. Therefore, the Jacobian algebra $J(Q, P)$ turns out to be finite-dimensional.

In the context of Jacobian algebras, having finite dimension is a highly desirable attribute. A usual method for proving this, as seen in the works of [LF09, Lad12, TVD12], consists in showing that enough cycles are contained in the Jacobian ideal, just as we did in the toy example above. We will make use of the diamond lemma in order to produce this kind of arguments in a streamlined fashion.

1.4 Ideal triangulations of surfaces

In this thesis we will focus primarily on a family of QPs (and their associated Jacobian algebras) arising from a particular procedure carried out on a triangulation of a Riemann surface, as described in [LF09]. In order to do this, we start with some basic definitions regarding our geometric objects.

Throughout the text, we will refer to compact, connected, oriented Riemann surface with boundary simply as *surfaces*. We now state a classical result in combinatorial topology that completely classifies these objects:

Proposition 1.11. *Let Σ be a surface with a number b of boundary components. Then Σ is homeomorphic to either a sphere with b open disks removed or to a g-holed torus with b open disks removed.*

Proof. See [Mas77, Section 10]. □

The number g appearing in the statement of the previous theorem is called the *genus* of the surface, which we will set as $g = 0$ in the spherical case.

A *surface with marked points* (Σ, M) is an ordered pair consisting of a surface Σ and a finite, non-empty collection M of points (called *marked points*) of Σ containing at least one point from each of its boundary components. Points in M that belong to the interior of Σ are called *punctures*.

An *arc* on a surface with marked points (Σ, M) is a curve $\gamma : [0,1] \to \Sigma$ such that:

1. Its endpoints lie in M.

2. Its restriction to $(0,1)$ is injective and does not intersect M or the boundary of Σ.

3. Its image is not contractible into M or onto the boundary of Σ.

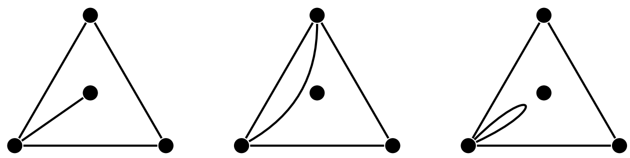


Figure 1.1: While the first is an arc of the punctured triangle, the last two are not, since they are contractible to the boundary or to the set of marked points, respectively.

We will consider arcs on a surface only up to isotopy relative to M. Two arcs will be said to be *compatible* if there exist representatives in their corresponding relative isotopy classes such that they do not intersect in the interior of Σ. An *ideal triangulation* for (Σ, M) is then a maximal family of compatible arcs in Σ.

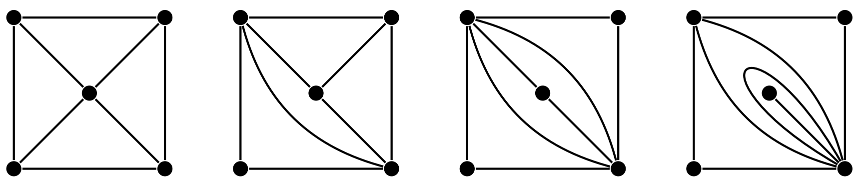


Figure 1.2: Some ideal triangulations of the square with one puncture.

An ideal triangulation divides the surface Σ into *ideal triangles*. As we can see in Figure 1.2, such triangulations are more general than the usual ones, since they admit configurations in which two triangles share more than one side. Even more, the three sides of an ideal triangle may not be distinct. In that case we say that the triangle is *self-folded*.

Most surfaces may be triangulated without using self-folded triangles. In fact, we have the following result:

Proposition 1.12 ([FST08, Proposition 2.13]). *Let (Σ, M) be any surface with marked points different from:*

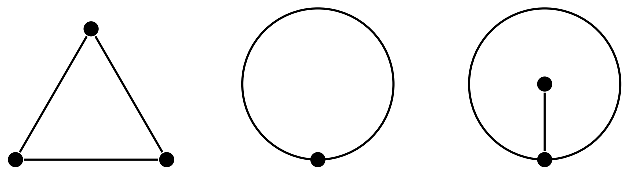


Figure 1.3: The three possible kinds of ideal triangles: the usual *triangle* and the self-folded ones, which are the *monogon* and the *digon*.

1. *A sphere with one, two or three punctures.*
2. *An unpunctured or once-punctured monogon.*
3. *An unpunctured digon.*
4. *An unpunctured triangle.*

Then (Σ, M) *admits an ideal triangulation involving no self-folded triangles.* □

As one may observe, every ideal triangulation of the square pictured in Figure 1.2 consists of exactly 4 arcs. Indeed, the number of arcs in a triangulation is an invariant of the marked surface. More precisely, we have:

Proposition 1.13 ([FST08, Proposition 2.10])**.** *Any ideal triangulation of a surface with marked points* (Σ, M) *involving no self-folded triangles consists of exactly*

$$n = 6g + 3b + 3p + c - 6$$

arcs, where g *is the genus of the surface* Σ*,* b *is its number of boundary components,* p *is the number of punctures and* c *is the number of marked points lying on the boundary.*

Proof. One can show using the classification theorem 1.11 that the Euler characteristic of a surface Σ of genus g with b boundary components is

$$\chi(\Sigma) = 2 - 2g - b \tag{1}$$

On the other hand, we can compute the Euler characteristic of the surface regarding an ideal triangulation as a cellular decomposition. Since an ideal triangulation is a maximal collection of compatible arcs, any marked point must be a vertex of the decomposition and so we have $v = c + p$ vertices. As for edges, we have n of them in the interior of the surface and c of them in the boundary, which amounts to $e = n + c$. Finally, we know that each face is triangular, that any edge in the interior of the surface belongs to two different faces, and that edges in the boundary belong to only one face.

Therefore, by counting faces we arrive at $3f = 2n + c$. Putting it all together, we get that

$$\begin{aligned}\chi(\Sigma) &= v - e + f \\ &= (c+p) - (n+c) + \left(\frac{2}{3}n + \frac{1}{3}c\right) \\ &= p + \frac{1}{3}(c-n)\end{aligned} \tag{2}$$

The result now follows by equating (1) and (2) and solving for n. □

1.5 The QP associated to an ideal triangulation

We will now introduce a way to produce a QP out of an ideal triangulation, so one may speak of the Jacobian algebra associated to the triangulation.

Taking Proposition 1.12 into account, from now on we will restrict ourselves to the case where no self-folded triangles appear. Although there is a general mechanism to produce QPs from arbitrary triangulations, as described in Section 3 of [LF09], technical difficulties arise when dealing with self-folded triangles.

Let T be an ideal triangulation of a surface with marked points (Σ, M). We construct a quiver Q following these steps:

1. Draw a vertex v_i for each edge e_i in the triangulation T that does not belong to the boundary of Σ.

2. Draw an arrow between two vertices v_i, v_j if e_i and e_j are edges of the same triangle.

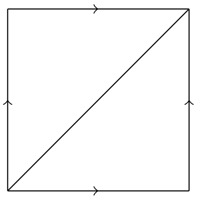
3. Orient all arrows according to the already existing orientation on the surface.

Note that every triangle without edges on the boundary of the surface adds a 3-cycle to the quiver. Consider the set X formed by all of these 3-cyles and all other cycles that circle a puncture, and pick a scalar λ_x in the field k for each cycle $x \in X$. We thus define the potential P arising from this choice of scalars as

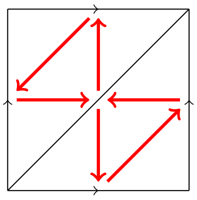
$$P = \sum_{x \in X} \lambda_x x.$$

Notice that this construction is well defined up to cyclic equivalence. Thus, it makes sense to speak of the Jacobian algebra associated from the QP. From now on, we will only consider the potential formed by setting all scalars $\lambda_x = 1$, which we will call *the* potential associated to the triangulation T, unless we specify otherwise. Moreover, we will refer to the Jacobian algebra induced by the triangulation as the one that is construced from this specific QP.

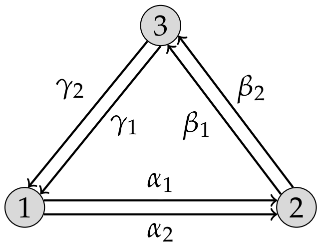
Example 1.14. Consider the following triangulation of the torus with a single puncture:



The puncture is placed at the corners of the square, which are all identified. We follow the steps outlined above to obtain the following configuration:



Keep in mind that, since opposing sides of the square are identified, the two triangles of the triangulation share all of their sides, and so this is actually a quiver with three vertices, which we will draw as:

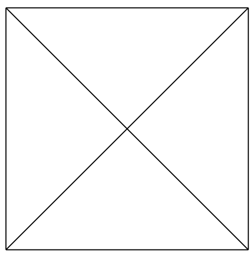


Here we have two 3-cycles arising from the two triangles, which are $\gamma_1\beta_1\alpha_1$ and $\gamma_2\beta_2\alpha_2$, and a single 6-cycle that goes around the puncture, which is $\gamma_2\beta_2\alpha_2\gamma_1\beta_1\alpha_1$. Therefore, the associated potential is

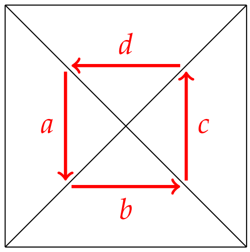
$$P = \gamma_1\beta_1\alpha_1 + \gamma_2\beta_2\alpha_2 + \gamma_2\beta_2\alpha_2\gamma_1\beta_1\alpha_1$$

up to cyclic equivalence.

Example 1.15. Let us now consider a triangulation of a surface with non-empty boundary. The following is a triangulation of the square with a single puncture, placed on its center:



Every triangle shares an edge with the boundary of the square, so each triangle accounts for only one arrow:



Since there are no 3-cycles coming from any triangle, the potential P consists simply of the 4-cycle that circles the puncture, which is

$$P = dcba.$$

Notice that we could have chosen $cbad$, $badc$ or $adcb$ as well, since they all define cyclically equivalent potentials.

Chapter 2

The diamond lemma

Suppose A is a k-algebra given by a finite number of generators and relations, that is $A = k\langle x_1, \dots, x_n\rangle / \langle R\rangle$. A question that arises immediately is that of how to compute in A. Ideally, one should be able to provide a family of normal or canonical forms for monomials, such that every monomial is equal to a unique canonical form after passing to the quotient. In this way, if one knew how to multiply two elements in normal form, one would be able to compute the product of two arbitrary monomials in A by reducing them to their respective canonical forms, taking their product and reducing once again. This procedure would obviously apply to arbitrary elements in A by linearity.

In order to be able to reduce elements into canonical forms, one should be able to test for equality in A; in particular, one should be able to distinguish if a certain element is zero or not. Experience tells us that this problem may very well be untractable: it is in fact the word problem for algebras, which is known to be undecidable in its full generality (see [Sti82]).

Nonetheless, under some relatively mild hypotheses, this kind of argument may be succesfully carried out. In that case, Bergman's diamond lemma, presented on his seminal paper [Ber78], states that not only there exists a set of normal forms, but that they actually form a basis for A as a k-algebra.

2.1 Bergman's diamond lemma

Let X be a set and let $\langle X\rangle$ denote the free monoid on X. A *monomial order* on $\langle X\rangle$ is a partial order $\preceq$ on $\langle X\rangle$ such that:

- $1 \preceq v$ for all $v \in \langle X\rangle$, and
- for all $u, v, v', w \in \langle X\rangle$, if $v \preceq v'$, then $uvw \preceq uv'w$.

We will use the notation $u \prec v$ for strict inequalities and refer to the length of a monomial u as $|u|$.

Example 2.1. Suppose $\leq$ is a total order on X. The *graded lexicographical order*, or *grlex* for short, is the monomial order $\preceq$ defined on $\langle X\rangle$ as follows: given $u, v \in \langle X\rangle$, we have $u \preceq v$ if

- $|u| < |v|$ or
- $|u| = |v|$ and $u = wau'$, $v = wbv'$, with $w, u', v' \in \langle X\rangle$, $a, b \in X$ and $a \leq b$.

In other words, monomials are sorted first by length and then lexicographically according to the total order $\leq$.

A poset $(P, \leq)$ is said to satisfy the *descending chain condition* if there is no sequence $(p_n) \subseteq P$ such that $p_{n+1} < p_n$ for all $n \in \mathbb{N}$. Equivalently, $(P, \leq)$ satisfies the descending chain condition if any sequence (p_n) such that $p_{n+1} \leq p_n$ for all $n \in \mathbb{N}$ eventually stabilizes, that is, there exists some k such that $p_j = p_k$ for all $j > k$.

Lemma 2.2. *Let $(X, \leq)$ be a finite, totally ordered set. Then, the graded lexicographical order on $\langle X\rangle$ satisfies the descending chain condition.*

Proof. Let (x_n) be a decreasing sequence in $\langle X\rangle$, l be the length of x_1 and k be the cardinality of X. There are exactly k^d words in $\langle X\rangle$ of length d, and since a word smaller than x_1 must be of length at most d, there are at most $j = \sum_{l=0}^{d} k^l$ words smaller than x_1. Since j is a finite number, it follows that the sequence (x_n) must eventually stabilize. $\square$

A *rewriting system* on X is a subset $S \subseteq \langle X\rangle \times k\langle X\rangle$ such that for each $\sigma = (w_\sigma, f_\sigma) \in S$ we have $w_\sigma \neq f_\sigma$. Every $\sigma \in S$ is called a *rewriting rule*, which we will sometimes denote $w_\sigma \rightsquigarrow f_\sigma$. If $u, v \in \langle X\rangle$ and $\sigma \in S$ we call the triple $r = (u, \sigma, v)$ a *basic reduction*. We denote the set of all basic reductions associated to a rewriting system S as B_S and call *reductions* the elements of the free monoid $\langle B_S\rangle$.

If $u \in \langle X\rangle$, there exists a unique k-linear function $\mathrm{cf}_u : k\langle X\rangle \to k$ such that $\mathrm{cf}_u(u) = 1$ and $\mathrm{cf}_u(v) = 0$ for all $v \in \langle X\rangle$ such that $v \neq u$. If $x \in k\langle X\rangle$, we call $\mathrm{cf}_u(x)$ the *coefficient of u in x*. Given a basic reduction $r = (u, \sigma, v)$, one can define an associated k-linear map $\hat{r} : k\langle X\rangle \to k\langle X\rangle$ such that, for every $x \in k\langle X\rangle$,

$$\hat{r}(x) = x - \mathrm{cf}_{u\sigma v}(x)u(w_\sigma - f_\sigma)v.$$

Thus, the map $\hat{r}$ replaces the word $uw_\sigma v$ with $uf_\sigma v$ and leaves the rest of the terms of x unchanged. The assignment $r \mapsto \hat{r}$ induces a monoid morphism $\langle B_S\rangle \to \mathrm{End}_k(k\langle X\rangle)$; given any $r \in \langle B_S\rangle$, we will refer to its image via this morphism as $\hat{r}$. We say that $\hat{r}$ *acts trivially on $x \in k\langle X\rangle$* if $\hat{r}(x) = x$.

An element $x \in k\langle X\rangle$ is said to be:

- *S-irreducible* if $\hat{r}(x) = x$ for any reduction $r \in \langle B_S\rangle$.
- *reduction-finite under S* if every time (r_n) is a sequence of reductions, there exists some i_0 such that $\hat{r}_i$ acts trivally on $(r_{i-1}\dots r_1)\hat{}(x)$ for all $i \geq i_0$.

- *reduction-unique under* S if it is reduction-finite under S and there exists some $r_S(x) \in k\langle X \rangle$ such that if $\hat{r}(x)$ is S-irreducible, then $\hat{r}(x) = r_S(x)$.

Consider a 5-uple $\alpha = (\sigma, \tau, u, v, w)$ such that $\sigma, \tau \in S$ and $u, v, w \in \langle X \rangle$. We say that α is an *overlap ambiguity of* S if u, v, w are words of positive length, $w_\sigma = uv$ and $w_\tau = vw$. Such an ambiguity is said to be *solvable* if there exist reductions $r, r' \in \langle B_S \rangle$ such that $\hat{r}(f_\sigma w) = \hat{r}'(u f_\tau)$. Otherwise, if $\sigma \neq \tau$, $w_\sigma = v$ and $w_\tau = uvw$ we say that α is an *inclusion ambiguity of* S, and call it *solvable* if there exist reductions $r, r' \in \langle B_S \rangle$ such that $\hat{r}(u f_\sigma w) = \hat{r}'(f_\tau)$.

A monomial order $\preceq$ over $\langle X \rangle$ is said to be compatible with a rewriting system S if for all $\sigma \in S$ we have that any monomial u appearing as a term in f_σ is such that $u \prec w_\sigma$.

After all these preliminary definitions, we are now able to formulate the main result in this section:

Theorem 2.3 (Bergman's diamond lemma)**.** *Let X be a set, S be a rewriting system for X and $\preceq$ a monomial order on $\langle X \rangle$ compatible with S and satisfying the descending chain condition. Let I_S be the ideal given by the relations induced by S, that is, $I_S = (f_\sigma - w_\sigma)_{\sigma \in S}$. Then the following conditions are equivalent:*

1. *All ambiguities of S are solvable.*

2. *All elements of $k\langle X \rangle$ are reduction-unique under S.*

3. *A set of representatives in $k\langle X \rangle$ for the elements of the algebra $k\langle X \rangle / I_S$ is given by the k-submodule $k\langle X \rangle_{\mathrm{irr}}$ spanned by the S-irreducible monomials of $\langle X \rangle$.*

If any of these conditions hold, the rewriting system S is said to be confluent. *In that case, there is a k-algebra isomorphism between $k\langle X \rangle / I_S$ and $k\langle X \rangle_{\mathrm{irr}}$, where the latter is a k-algebra with product defined as $x \cdot y = r_S(xy)$.*

Proof. See [Ber78, Theorem 1.2]. □

Let us illustrate how the diamond lemma is used in some examples:

Example 2.4. Consider the polynomial algebra $A = k[x, y]$, which is presented by generators and relations as $A = k\langle x, y \rangle / (xy - yx)$. If we order our variables x and y so that $x < y$, then the associated graded lexicographical order on $\langle x, y \rangle$ satisfies the descending chain condition by Lemma 2.2. Consider the terms in the unique relation $xy - yx$ and sort them using the grlex order. We have that $xy \preceq yx$, and so the rewriting rule $\sigma = yx \rightsquigarrow xy$ is compatible with our chosen monomial order. Thus, the rewriting system S consisting of the unique rewriting rule σ is such that:

- the grlex order $\preceq$ is compatible with S,

- the associated ideal I_S is $\langle xy - yx \rangle$,

- there are no ambiguities in S, since the monomial yx does not overlap with itself in any non-trivial way.

The diamond lemma guarantees that a basis for $k[x,y]$ is given by the S-irreducible monomials. Now, a monomial is S-irreducible iff it does not contain yx as a factor, and thus the set of S-irreducible monomials is exactly the set $\{x^i y^j : i,j \in \mathbb{N}_0\}$, that is, the set of ordered monomials. Moreover, taking an arbitrary element in $k\langle x,y\rangle$ into its corresponding normal form in A is trivial: it suffices to order the letters in each monomial term lexicographically.

Example 2.5. Consider the Weyl algebra $A_1 = k\langle x,y\rangle/(yx - xy - 1)$. The reasoning carried out in the previous example holds almost verbatim: this time, our unique rewriting rule is $yx \rightsquigarrow xy + 1$, and once again there are no ambiguities. Therefore, the set of ordered monomials is a basis for A_1. Notice that taking an element into its irreducible normal form is not as easy as in the previous example. For instance,

$$y^2x \rightsquigarrow y(xy+1) = (yx)y + y \rightsquigarrow (xy+1)y + y = xy^2 + 2y.$$

The same thing happens for the quantum polynomial algebra $k\langle x,y\rangle/(xy - qyx)$, where $q \in k^*$. In that case, the unique rewriting rule is $yx \rightsquigarrow q^{-1}xy$, and once again the set of ordered monomials forms a basis.

Example 2.6. Let us now consider an example in which ambiguities appear. Consider the polynomial algebra

$$A = k[x,y,z] = k\langle x,y,z\rangle/(xy - yx, xz - zx, yz - zy).$$

Once again we consider the standard lexicographical order $x < y < z$ and the induced grlex monomial order on $\langle x,y,z\rangle$, which satisfies the descending chain condition. We may now produce a rule from each relation by reducing the biggest term in it into the other term. In this case, we get the rules

$$\begin{aligned} \sigma_1 &= yx \rightsquigarrow xy \\ \sigma_2 &= zx \rightsquigarrow xz \\ \sigma_3 &= zy \rightsquigarrow yz \end{aligned}$$

The rewriting system $S = \{\sigma_1, \sigma_2, \sigma_3\}$ is compatible with our monomial order and its associated ideal I_S is exactly $(xy - yx, xz - zx, yz - zy)$. It remains to show that all of the ambiguities of S are solvable. There is in fact a unique overlap ambiguity, which is $(\sigma_3, \sigma_1, z, y, x)$, since the monomial zyx may be reduced using both the reduction associated to σ_1 and the one associated to σ_3. The following diagram shows that this ambiguity is indeed solvable and illustrates why the diamond lemma is called that way:

$$\begin{tikzcd}
 & zyx \arrow[dl, "\sigma_3"'] \arrow[dr, "\sigma_1"] & \\
yzx \arrow[d, "\sigma_2"'] & & zxy \arrow[d, "\sigma_2"] \\
yxz \arrow[dr, "\sigma_1"'] & & xzy \arrow[dl, "\sigma_3"] \\
 & xyz &
\end{tikzcd}$$

Now that we have checked that the only ambiguity is solvable, by the diamond lemma we know that the set of S-irreducible monomials form a basis for $k[x, y, z]$, and once again those turn out to be the set of ordered monomials $\{x^i y^j z^k : i, j, k \in \mathbb{N}_0\}$. Similar arguments hold for polynomial algebras with an arbitrary number n of variables, although the amount of ambiguities that one needs to deal with grows with n.

Example 2.7. Consider the k-algebra $A = k\langle x, y, z\rangle/(xy - yx, yz - zy)$. As usual, we consider the usual lexicographical order $x < y < z$ and the grlex monomial order it induces on $\langle x, y, z\rangle$, which satisfies the descending chain condition. Once again, the relations $xy - yx$ and $yz - zy$ naturally induce the following rewriting rules:

$$\sigma_1 = yx \rightsquigarrow xy$$
$$\sigma_2 = zy \rightsquigarrow yz$$

The rewriting system $S = \{\sigma_1, \sigma_2\}$ is compatible with the grlex monomial order and the associated ideal I_S is precisely $(xy - yx, yz - zy)$. All that remains to check is that all ambiguities are solvable.

In fact, there is only one ambiguity, which is $(\sigma_2, \sigma_1, z, y, x)$. In other words, the monomial zyx may be reduced in two different ways, as the following diagram shows:

$$\begin{tikzcd}
 & zyx \arrow[dl, "\sigma_2"'] \arrow[dr, "\sigma_1"] & \\
yzx & & zxy
\end{tikzcd}$$

We now see that, since both yzx and zxy are S-irreducible, the ambiguity is in fact unsolvable. However, not all is lost: since the equality $yzx = zxy$ holds in A, we may include the rewriting rule $\sigma_3 = zxy \rightsquigarrow yzx$ in S and still have $I_S = (xy - yx, yz - zy, yzx - zxy) = (xy - yx, yz - zy)$. The addition of rule σ_3 now solves our previous ambiguity, since we have the diagram:

$$\begin{tikzcd}
 & zyx \arrow[dl, "\sigma_2"'] \arrow[dr, "\sigma_1"] & \\
yzx \arrow[dr, equal] & & zxy \arrow[dl, "\sigma_3"] \\
 & yzx &
\end{tikzcd}$$

However, there is a new overlap ambiguity, namely $(\sigma_3, \sigma_1, zx, y, x)$. We have

$$\begin{tikzcd}
 & zxyx \arrow[dl, "\sigma_3"'] \arrow[dr, "\sigma_1"] & \\
yzx^2 & & zx^2y
\end{tikzcd}$$

and once again we are stuck, since both yzx^2 and zx^2y are S-irreducible. One may try to enlarge the current set of rewriting rules as to include the new rule $zx^2y \rightsquigarrow yzx^2$, but this will in turn create another unsolvable ambiguity.

The problem is solved if we consider the rewriting system S' composed of the rewriting rules:

$$\sigma = yx \rightsquigarrow xy$$
$$\tau_n = zx^n y \rightsquigarrow yzx^n$$

for all $n \in \mathbb{N}_0$. As we can see, the rewriting system S' is compatible with the grlex monomial order, and since the identities $zx^n y = yzx^n$ hold in A, we have that $I_{S'} = (xy - yx, yz - zy, yzx^n - zx^n y) = (xy - yx, yz - zy)$. The rules τ_j and τ_k do not overlap or contain each other for any choice of j and k, so the only overlapping ambiguities are of the form $(\tau_n, \sigma, zx^n, y, x)$, for $n \in \mathbb{N}_0$. We now check that these ambiguities are solvable:

$$\begin{tikzcd}
& zx^nyx \arrow[dl, "\tau_n"'] \arrow[dr, "\sigma"] & \\
yzx^{n+1} \arrow[dr, equal] & & zx^{n+1}y \arrow[dl, "\tau_{n+1}"] \\
& yzx^{n+1} &
\end{tikzcd}$$

Therefore, the rewriting system is confluent, and so the set of S'-irreducible monomials form a basis for the algebra A. In this case, characterizing this set is not as easy as in the previous examples, since the rewriting system is considerably more complex. This, of course, reflects the fact that the basis has a richer combinatorial structure.

We now summarize our usual method to produce confluent rewriting systems:

Heuristic 2.8. Let $A = k\langle x_1, \ldots, x_n \rangle / (r_1, \ldots, r_m)$ be a k-algebra given by generators and relations. The lexicographic order $x_1 < \cdots < x_n$ induces a grlex monomial order $\preceq$ on $\langle x_1, \ldots, x_n \rangle$, which satisfies the descending chain condition by Lemma 2.2. We can produce a rewriting system following these steps:

1. Write each relation r_i as
$$r_i = \lambda_{i,1} w_{i,1} + \lambda_{i,2} w_{i,2} + \cdots + \lambda_{i,k_i} w_{i,k_i},$$
where $w_{i,j} \in \langle x_1, \ldots, x_n \rangle$, $\lambda_{i,j} \in k$ and $w_{i,1} \succeq w_{i,2} \succeq \cdots \succeq w_{i,k_i}$.

2. Consider the rules
$$\sigma_i = w_{i,1} \rightsquigarrow -\lambda_{i,1}^{-1} \lambda_{i,2} w_{i,2} - \cdots - \lambda_{i,1}^{-1} \lambda_{i,k_i} w_{i,k_i}$$
which are all compatible with the monomial order $\preceq$ and are such that the rewriting system $S = \{\sigma_i\}$ induces the ideal $I_S = (r_1, \ldots, r_m)$.

3. If all ambiguities arising from this rewriting system are solvable, we are done. Otherwise, given an unresovable ambiguity $(\sigma_i, \sigma_j, u, v, w)$, reduce both $r_i(uv)w$ and $ur_j(vw)$ into irreducible elements, which we will call a and b respectively. We have that $a - b \in I_S$, and so
$$A = k\langle x_1, \ldots, x_n \rangle / (r_1, \ldots, r_m) = k\langle x_1, \ldots, x_n \rangle / (r_1, \ldots, r_m, a - b).$$
Treating $a - b$ as another relation, we can follow the steps 1 and 2 to produce a new rule that will solve the previous ambiguity, but may generate new ones.

4. Repeat step 3 until the system is confluent.

This procedure, of course, does not always terminate in a finite number of steps, but is good enough to solve most of the cases that arise throughout this thesis.

2.2 A diamond lemma for path algebras

As seen in the previous section, Bergman's diamond lemma is a statement about certain quotients of free algebras. However, we will only apply it on a particular family of algebras, namely quotients of path algebras. Even though obviously path algebras are themselves quotients of free algebras, using Bergman's diamond lemma can be quite cumbersome in this case. For example, consider the following quiver, which we will name Q:

$$u \xrightarrow{\alpha} v$$

The path algebra $k\langle Q\rangle$ may be described as the quotient of the free algebra $k\langle u, v, \alpha\rangle$ by the ideal

$$(\underbrace{uv, vu, u^2 - u, v^2 - v, u + v - 1}_{\substack{u \text{ and } v \text{ are idempotent orthogonal} \\ \text{elements of sum } 1}},\ \underbrace{\alpha^2, \alpha u - \alpha, v\alpha - \alpha, \alpha v, u\alpha}_{\substack{\text{relations forced by the way arrows} \\ \text{and vertices concatenate}}})$$

Even in a quiver as simple as Q and without introducing further relations in the path algebra, many rewriting rules and ambiguities arise. Therefore, it makes sense to produce a specialized version of the diamond lemma in order to deal with this particular case in a more efficient manner, in a similar vein to [FFG93].

Given a quiver Q, a *rewriting system* on Q is a subset $S \subseteq Q_* \times k\langle Q\rangle$, where Q_* stands for the set of all paths on Q. A *path order* on Q_* is a partial order $\preceq$ on Q_* such that if v and v' are parallel paths (that is, they share the same source and target) and $v \preceq v'$, then $uvw \preceq uv'w$ for all $u, w \in Q_*$ with appropriate source and target. All other previous definitions translate almost verbatim to this new setting. We now are able to formulate our specialized version of the diamond lemma:

Theorem 2.9. *Let Q be a quiver, S be a rewriting system for Q and $\preceq$ a path order on Q_* compatible with S and satisfying the descending chain condition. Let I_S be the ideal given by the relations induced by S, that is, $I_S = (f_\sigma - w_\sigma)_{\sigma \in S}$. Then the following conditions are equivalent:*

1. *All ambiguities of S are solvable.*

2. *All elements of $k\langle Q\rangle$ are reduction-unique under S.*

3. *A set of representatives in $k\langle Q\rangle$ for the elements of the algebra $k\langle Q\rangle / I_S$ is given by the k-submodule $k\langle Q\rangle_{\mathrm{irr}}$ spanned by the S-irreducible paths of Q_*.*

If any of these conditions hold, the rewriting system S is said to be confluent. *In that case, there is a k-algebra isomorphism between $k\langle Q\rangle / I_S$ and $k\langle Q\rangle_{\mathrm{irr}}$, where the latter is a k-algebra with product defined as $x \cdot y = r_S(xy)$.*

Example 2.10. Consider the quiver Q given by the following diagram:

$$\begin{tikzcd}
 & v \arrow[dr, "\beta"] & & \\
u \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] & & x \arrow[r, "\varepsilon"] & y \\
 & w \arrow[ur, "\delta"'] & &
\end{tikzcd}$$

We will make use of the diamond lemma to produce a basis for the algebra $A = k\langle Q\rangle / (\delta\gamma - \beta\alpha, \varepsilon\beta, \varepsilon\delta)$. The usual grlex ordering induced by the lexicographical order suggests the rewriting rules:

$$\begin{aligned}
\sigma_1 &= \delta\gamma \rightsquigarrow \beta\alpha \\
\sigma_2 &= \varepsilon\beta \rightsquigarrow 0 \\
\sigma_3 &= \varepsilon\delta \rightsquigarrow 0
\end{aligned}$$

The rewriting system $S = \{\sigma_1, \sigma_2, \sigma_3\}$ presents a unique ambiguity, which is $(\sigma_3, \sigma_1, \varepsilon, \delta, \gamma)$. The following diagram shows that this ambiguity is solvable:

$$\begin{tikzcd}
 & \varepsilon\delta\gamma \arrow[dl, "\sigma_3"'] \arrow[dr, "\sigma_1"] & \\
0 \arrow[dr, equal] & & \varepsilon\beta\alpha \arrow[dl, "\sigma_2"] \\
 & 0 &
\end{tikzcd}$$

The rewriting system S is confluent, and our algebra A has a basis given by the S-irreducible paths, which are

$$\{e_u, e_v, e_w, e_x, e_y, \alpha, \beta, \gamma, \delta, \varepsilon, \beta\alpha\},$$

where e_a stands for the stationary path at the vertex a.

Example 2.11. Let Q be the following quiver:

$$\begin{tikzcd}
& v \arrow[dr, "\beta"] & & \\
u \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] & & x \arrow[r, shift left, "\varepsilon"] \arrow[r, shift right, "\zeta"'] & y \\
& w \arrow[ur, "\delta"'] & &
\end{tikzcd}$$

Consider the algebra $A = k\langle Q\rangle/(\delta\gamma - \beta\alpha, \varepsilon\beta, \zeta\delta)$. Following our usual procedure, we first consider the rewriting system S given by the rules

$$\begin{aligned}
\sigma_1 &= \delta\gamma \rightsquigarrow \beta\alpha \\
\sigma_2 &= \varepsilon\beta \rightsquigarrow 0 \\
\sigma_3 &= \zeta\delta \rightsquigarrow 0
\end{aligned}$$

In this case, the rewriting system is not confluent, since the ambiguity $(\sigma_3, \sigma_1, \zeta, \delta, \gamma)$ is unsolvable. However, after enlarging the system as to contain the rule

$$\sigma_4 = \zeta\beta\alpha \rightsquigarrow 0,$$

we achieve confluence. Thus, a basis is given by the S-irreducible monomials.

2.3 A topological diamond lemma

As Jacobian algebras are quotients of completed path algebras by closed ideals, one needs to further specialize the diamond lemma in order to account for issues of convergence, as carried out in the works [Hel02] and [SAV15].

Most of the terminology needed to introduce this last version of the lemma has already been presented; we will only need to provide a slight variation on the descending chain condition. We say that a monomial order $\preceq$ satisfies the *descending chain condition in norm* if every sequence (x_n) of monomials in $\langle X\rangle$ such that $x_{n+1} \prec x_n$ for all $n \geq 0$ converges to zero in $k\langle\langle X\rangle\rangle$.

Theorem 2.12. *Let X be a set, S be a rewriting system for X and $\preceq$ a monomial order on $\langle X\rangle$ compatible with S and satisfying the descending chain condition in norm. Let I_S be the closed ideal given by the relations induced by S, that is, $I_S = \overline{(f_\sigma - w_\sigma)}_{\sigma\in S}$. Then the following conditions are equivalent:*

1. *All ambiguities of S are solvable.*

2. *All elements of $k\langle\langle X\rangle\rangle$ are reduction-unique under S.*

3. *A set of representatives in $k\langle\langle X\rangle\rangle$ for the elements of the algebra $k\langle\langle X\rangle\rangle / I_S$ is given by the closed k-submodule $k\langle\langle X\rangle\rangle_{\mathrm{irr}}$ spanned by the S-irreducible monomials of $\langle X\rangle$.*

Once again, if any of these conditions hold, the rewriting system S is said to be confluent, *and in that case there is a k-algebra isomorphism between $k\langle\langle X\rangle\rangle / I_S$ and $k\langle\langle X\rangle\rangle_{\mathrm{irr}}$, where the latter is a k-algebra with product defined as $x \cdot y = r_S(xy)$.*

A similar statement for quotients of complete path algebras may be produced as well. Indeed, one may modify the statement of Theorem 2.9 replacing "descending chain condition" by "descending chain condition in norm" and all algebraic objects by their complete counterparts (algebras and their completions, ideals and closed ideals, submodules and closed submodules, etc.) to obtain the result.

Given a total order $\leq$ on a set X, the *reverse graded lexicographical order*, or *revglex* for short, is the monomial order $\preceq$ defined on $\langle X\rangle$ as follows: given $u, v \in \langle X\rangle$, we have $u \preceq v$ if

- $|u| > |v|$ or
- $|u| = |v|$ and $u = wau'$, $v = wbv'$, with $w, u', v' \in \langle X\rangle$, $a, b \in X$ and $a \leq b$.

Therefore, monomials are sorted first by length (but this time longer terms are smaller with respect to $\preceq$) and then lexicographically according to the total order $\leq$. Analogously one defines the revglex order for paths on a quiver. This will be our preferred order when dealing with completions, since we have that:

Lemma 2.13. *Let $(X, \leq)$ be a finite, totally ordered set. Then, the revglex order on $\langle X\rangle$ satisfies the descending chain condition in norm.*

Proof. Let (x_n) be a strictly decreasing sequence in $\langle X\rangle$ and k be the cardinality of X. There are exactly $j = \sum_{l=0}^{d} k^l$ words in $\langle X\rangle$ of length at most d, and so x_{j+1} must be of length at least $d+1$, since the sequence (x_n) is strictly decreasing. Thus, in $k\langle\langle X\rangle\rangle$ we have that $\|x_n\| \leq e^{-d-1}$ for all $n \geq j+1$ and so $x_n \to 0$ as we wanted. $\square$

Once again, an analogous result holds for quivers and complete path algebras.

Example 2.14. In general, given an ideal I, the algebras $k\langle X\rangle / I$ and its completed counterpart $k\langle\langle X\rangle\rangle / \overline{I}$ may be quite different. For instance, consider the ideal $I = (x^2 - x)$ in $k\langle x\rangle$. We may form the algebras $A = k\langle x\rangle / I$ and $B = k\langle\langle x\rangle\rangle / \overline{I}$.

Since we know that $(x^2 - x) = (x) \cap (x-1)$ and the ideals (x) and $(x-1)$ are coprime, by the chinese remainder theorem

$$A = \frac{k\langle x\rangle}{(x^2 - x)} \simeq \frac{k\langle x\rangle}{(x)} \times \frac{k\langle x\rangle}{(x-1)} \simeq k \times k.$$

We now study B using the diamond lemma. The rewriting system $S = \{x \rightsquigarrow x^2\}$ is compatible with the reverse graded path order and is obviously confluent, since the monomial x does not overlap with itself in any non-trivial way. We see that the only S-irreducible monomials are the constant ones, so a basis for B is given by $\{[1]\}$, and thus $B \simeq k$.

More generally, if $I = (x^n - x)$, then

$$\frac{k\langle x\rangle}{(x^n - x)} \simeq \frac{k\langle x\rangle}{(x)} \times \frac{k\langle x\rangle}{(x^{n-1} - 1)} \simeq k \times \frac{k\langle x\rangle}{(x^{n-1})},$$

which is an n-dimensional algebra, while in the completed case we have that $k\langle\langle x\rangle\rangle / \bar{I} \simeq k$ by the same reasoning as above, since the rewriting system $S = \{x \rightsquigarrow x^n\}$ is confluent.

Chapter 3

Jacobian algebras from polygonal subdivisions

We have now introduced all of the definitions and tools needed to start studying Jacobian algebras arising from triangulations of surfaces. Perhaps the simplest invariant one can study is their dimension. As shown in [Lad12], all Jacobian algebras arising from surfaces with empty boundary are finite-dimensional, with perhaps the exception of the sphere with 4 punctures, which is not discussed.

One may ask if the result does not hold in that case or if only Ladkani's proof does not apply but the result holds nevertheless. We start off studying the tetrahedron, which is a *nice* triangulation of the sphere with 4 punctures, in the sense that it does not involve self-folded triangles, and determine that its associated Jacobian algebra is infinite-dimensional.

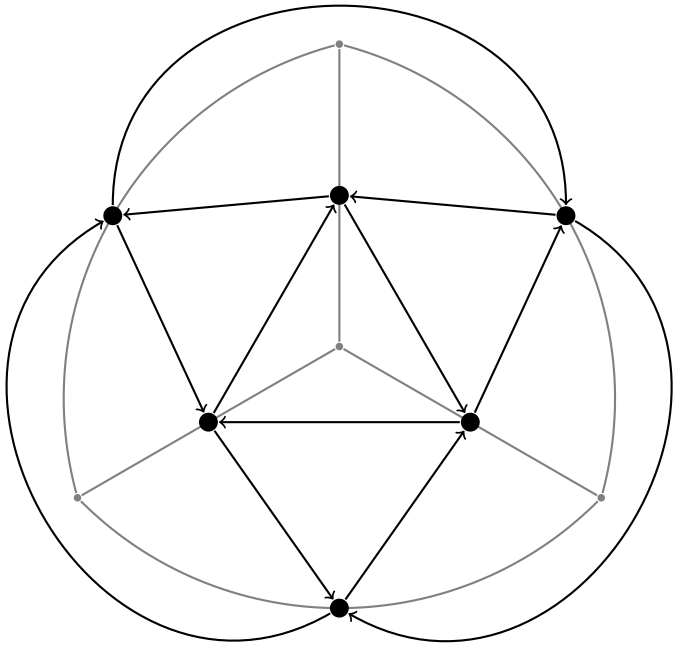
We then introduce a procedure to generate a QP, and correspondingly a Jacobian algebra, from an arbitrary polygonal subdivision of a surface. We find that in this situation, there are both infinite families of subdivisions with finite-dimensional Jacobian algebra and infinite families with infinite-dimensional ones, so the situation is considerably more complicated than in the triangulated case.

Another result from [Lad12] states that in the triangulated setting the finite-dimensionality of the Jacobian algebra does not depend on the choice of scalars for the potential. As we will show in this section, this does not generalize to the polygonal case.

Finally, we will make use of the diamond lemma to produce confluent rewriting system for three families of polyhedra: pyramids, prisms and antiprisms. These examples will be further studied in the next section.

3.1 The tetrahedron

We start by constructing the quiver associated to the tetrahedron, which looks like this:



In the figure, the edges of the triangulation are gray, while arrows in the quiver are black. The outermost 3-cycle corresponds to the bottom face of the tetrahedron. We observe that this quiver appears in [VD14, Example 1.4.4], as an example of what the author calls a quiver with an infinite *cyclic sequence*. The potential, as we may recall from Section 1.5, is the sum of the four 3-cycles coming from the triangular faces of the tetrahedron and the four 3-cycles around each puncture of the sphere.

While this triangulation is very simple, studying its associated Jacobian algebra can be quite troublesome. Our strategy is to name all arrows, sort paths using the revglex order and try to apply the diamond lemma in order to produce a basis for it. However, the fact that there are 12 edges in the quiver makes calculation by hand difficult. Since every edge induces a different cyclic derivative, and therefore a different relation in the Jacobian algebra, we have 12 rewriting rules that overlap themselves in various ways. It is not clear at all if the system is confluent or not, and checking that would imply a sizeable number of verifications. Moreover, if it turns out *not* to be confluent, we may apply Heuristic 2.8 as to enlarge the rule set and try to achieve confluence, but this only increases the number of verifications one needs to perform.

The problem is then too large to deal with by hand, but it looks reasonable enough to try to solve it using a computer. Perhaps the most suitable existing software package for this is `bergman` (freely available at http://servus.math.su.se/bergman/), which is a general purpose Gröbner basis calculator for the non-commutative setting. However, while `bergman` operates with arbitrary quotients of free algebras, we are dealing with a very specialized problem: our algebra is a quotient of a complete path algebra. Imitating the process we described in Section 2.2, we can present complete path algebras as quotients of the completion of free algebras, but this introduces a huge number of relations that greatly slows down computation. Thus it makes sense to program our own piece of software to deal with this particular situation.

We chose `SageMath` (freely available at http://www.sagemath.org/) as a framework to develop our program, since it has an extensive general purpose library. On a more particular note, it also includes a comprehensive directed graphs library that makes it easier for us to input and work with our quiver.

A description of our algorithms and the code itself are presented in detail in Appendix A. We will now make use of our software to study the Jacobian algebra associated to the tetrahedron.

We start by inputting our triangulation; the program will then produce the associated QP, label the vertices of the quiver and then generate the rewriting system induced by the revglex order. Then, the system will be tested for confluence and, if that is not the case, will enlarge the set of rules as described in Heuristic 2.8 until confluence is achieved. The output reads:

```
sage: tetrahedron = Rewriting_System(graphs.TetrahedralGraph())
Total rules: 12
```

As we anticipated, there are 12 rewriting rules, but the program finishes its execution without adding more of them, which means the obvious rewriting system is indeed confluent. We now display the labeling of the vertices of the quiver the program used:

```
sage: tetrahedron.quiver.show()
```

The output of this line of code is shown on Figure 3.1. Notice that it is exactly the same quiver as we have drawn before. We will call it Q. Let us now display the set of rewriting rules produced:

```
sage: tetrahedron.rules
{(0, 2, 1): -(0, 3, 1),
 (0, 2, 4): -(0, 3, 4),
 (1, 0, 2): -(1, 5, 2),
 (1, 0, 3): -(1, 5, 3),
 (2, 1, 0): -(2, 4, 0),
 (2, 1, 5): -(2, 4, 5),
 (3, 1, 0): -(3, 4, 0),
 (3, 1, 5): -(3, 4, 5),
 (4, 0, 2): -(4, 5, 2),
```

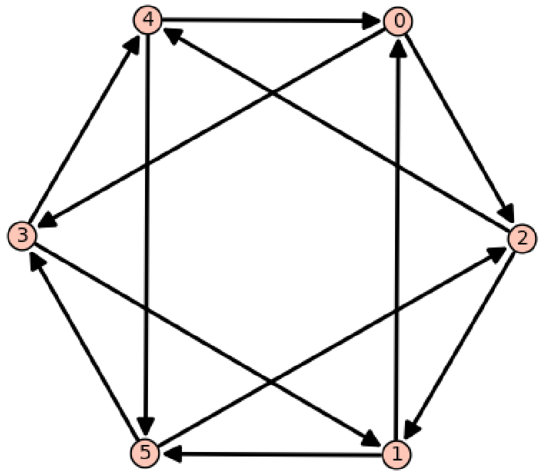
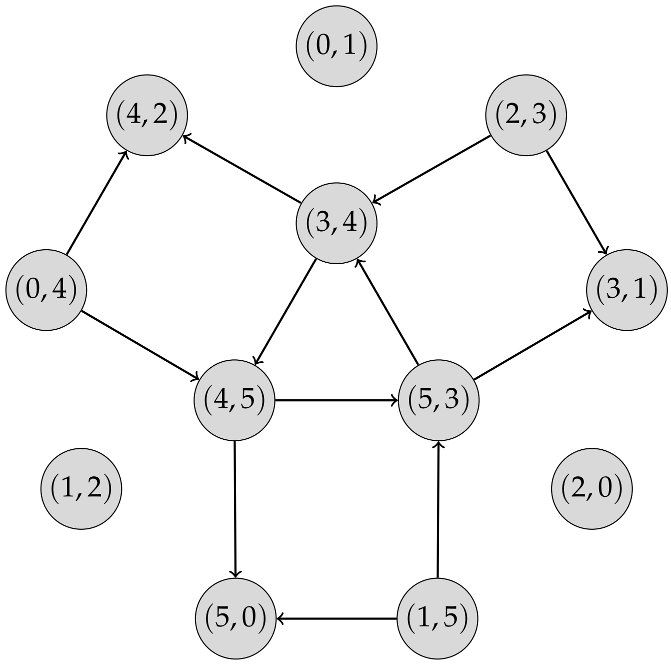


Figure 3.1

```
(4, 0, 3): -(4, 5, 3),
(5, 2, 1): -(5, 3, 1),
(5, 2, 4): -(5, 3, 4)}
```

Paths are described as lists of vertices, meant to be read from left to right. For example, $(0, 2, 1)$ is the path of length 2 obtained by concatenating the arrow with source 0 and target 2 to the arrow with source 2 and target 1. Rewriting rules are described as pairs of paths, where the leftmost path is the one that is replaced by the rightmost. For instance, following our previous notation, the rule $(0, 2, 1) : -(0, 3, 1)$ may be represented as $(0, 2, 1) \rightsquigarrow -(0, 3, 1)$.

Since the rewriting system is confluent, a basis for the Jacobian algebra is given by the set of irreducible paths, which are exactly the ones that do not contain any of the 12 paths that appear in the left side of the rewriting system. We will now draw a different quiver, called Q', having the arrows of our old quiver as vertices and we will place an arrow joining two of them if the path of length 2 formed by concatenating those arrows is irreducible:



Any path in this new quiver represents a path in the original quiver Q obtained by concatenating irreducible paths of length 2. Now, since the rewriting rules affect paths of length 2 only, we see that any path in the new quiver Q' represents an irreducible element in the Jacobian algebra. As Q' is not acyclic, we conclude that there are an infinite number of irreducible elements, and therefore the Jacobian algebra is infinite-dimensional.

In fact, we can calculate a finer invariant. Given a graded algebra $A = \bigoplus_{n=0}^{\infty} A_n$, its *Hilbert series* $h_A(t)$ is the ordinary generating function induced by the sequence $(\dim_k(A_n))$. In other words,

$$h_A(t) = \sum_{n=0}^{\infty} \dim_k(A_n)\, t^n.$$

Any path algebra is obviously graded by path length. Now, the Jacobian algebra associated to the tetrahedron (which we will call A for the remainder of this section) is a quotient of a path algebra by an *homogeneous* ideal, since the Jacobian ideal is generated by the 12 homogeneous binomials of degree 2 given by the rewriting rules shown above. Therefore, the original grading of the path algebra passes to the quotient and so A itself is graded.

In the light of this fact, it makes sense to calculate its Hilbert series. We know that a basis for the homogeneous component of degree n of A is given by the set of irreducible paths of length n, so we just have to count them. Since the rewriting rules involve paths of length 2 only, all paths of length 0 and 1 are irreducible, and so $\dim_k(A_0) = 6$

(the number of vertices in Q) and $\dim_k(A_1) = 12$ (the number of arrows in Q). We know that $\dim_k(A_2) = 12$ as well, since this is the number of arrows in the quiver Q'. In general, for $n \geq 3$, there are as many irreducible paths of length n in Q as paths of length $n-1$ in Q'. An easy counting argument shows that there are exactly 12 of them for all $n \geq 3$, since there are exactly 2 of them starting at each of the vertices $(0,4), (1,5), (2,3), (3,4), (4,5)$ and $(5,3)$ and none starting at any other vertices. Therefore, we conclude that the Hilbert series is

$$h_A(t) = 6 + \sum_{n=1}^{\infty} 12t^n = -6\frac{t+1}{t-1}.$$

Our program can calculate this Hilbert series up to a specified maximum degree, by exhaustively generating the list of all paths of length n and checking how many of them are irreducible. The program outputs

```
sage: tetrahedron.generating_function(15)
[6, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12]
```

which agrees with our observations.

We remark that, in general, the Jacobian ideal I is not homogeneous, since relations may involve paths of different lengths, and so the Jacobian algebra $A = k\langle\langle Q \rangle\rangle / I$ has no obvious grading. Nevertheless, it is still a *filtered* algebra (with the filtration induced, once again, by path length).

3.2 Polygonal subdivisions of surfaces

With the sole exception of Ladkani's paper covered, the problem of determining whether a Jacobian algebra arising from a triangulation of a closed surface is finite-dimensional or not is now settled. We now propose a natural generalization of the problem.

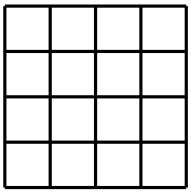
Given a surface Σ, we will say that a non-necessarily maximal family of compatible arcs in Σ is a *polygonal subdivision* of the surface. Such a subdivision divides the surface up into *polygons* or *faces*. Just as we did in the previous scenario, we will call a polygon *self-folded* if some of its sides are identified. We will restrict ourselves only to the case where the surface has empty boundary and no self-folded polygons appear in the subdivision.

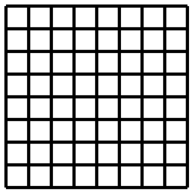
The exact same process discussed in Section 1.5 translates to this situation and lets us produce a QP out of a polygonal subdivision, so it now makes sense to speak of the Jacobian algebra associated to the subdivision.

As we will now see, the situation is not as clear-cut as when dealing with triangulations only, where there was only one example of an infinite-dimensional algebra. In the following two sections, we will produce infinite families of polygonal subdivisions of closed surfaces inducing infinite and finite-dimensional Jacobian algebras, respectively.

3.3 A family of infinite-dimensional examples

Let $n > 2$. We say that a polygonal subdivision of a surface with empty boundary Σ is *n-regular* if all of its faces have exactly n sides and exactly n faces meet at every vertex. For example, a tetrahedron is a 3-regular subdivision of the sphere, and the following figures induce 4-regular subdivisions of the torus after identifying the opposing sides of each square:





Obviously, in this way one can produce an infinite number of these examples on the torus.

The existence of an n-regular subdivision imposes strong conditions on the surface. In fact, suppose we have an n-regular subdivision of a surface and let v, e and f denote its number of vertices, edges and faces respectively. Since every face contains exactly n vertices and every vertex belongs to exactly n faces, we have that $v = f$. Moreover, every face contains n edges and any edge belongs to exactly 2 faces, so $e = nf/2$. Recalling the fact that the Euler characteristic of a surface of genus g is $2 - 2g$, we have that $2 - 2g = v - e + f = (2 - n/2)f$, or equivalently $4g - 4 = (n - 4)f$. If $n = 4$, the right hand side of the equation vanishes, and so $g = 1$. Therefore, the only surface that can admit a 4-regular subdivision is the torus (and in fact we have already shown there are an infinite number of them).

Suppose now that $n \neq 4$. Then

$$f = \frac{4g - 4}{n - 4}, \tag{1}$$

and so g must be such that this quotient is integral. Moreover, since n faces meet at every vertex, there must be at the very least n faces, and so $n(n - 4) \leq 4(g - 1)$. Therefore, we have that $g \in \Omega(n^2)$.

All of these are *necessary* conditions for an n-regular subdivision to exist. We have already shown an infinite number of examples of these objects, but we do not know if these subdivisions exist for any genus (or even for arbitrarily high genera).

Example 3.1. Consider the complete graph K_8 with vertex set $\{1, 2, \ldots, 8\}$. One may glue eight heptagons having the rows of the following matrix as edges:

1	2	3	4	5	6	7
2	1	3	7	5	8	6
3	2	4	1	6	8	7
4	3	5	2	7	8	1
5	4	6	3	1	8	2
6	5	7	4	2	8	3
7	6	1	5	3	8	4
1	7	2	6	4	8	5

In this way, we obtain an orientable, closed surface of genus seven, such that exactly seven heptagonal faces meet at every vertex; in other words, a 7-regular surface.

Our main interest in the concept of n-regularity lies in the following fact:

Theorem 3.2. *The Jacobian algebra associated to an n-regular subdivision of a surface is infinite-dimensional.*

Proof. We will first prove the case where $n > 3$. Let (Q, P) be the quiver with potential associated to the subdivision, I the ideal in $k\langle Q\rangle$ spanned by the cyclic derivatives of the potential P and J the closure of I in $k\langle\langle Q\rangle\rangle$.

Any arrow x in the quiver is a factor of exactly two cycles in P: one of them is contained in the same face of the subdivision as x and the other surrounds a vertex and is composed of arrows lying in different faces. The condition of n-regularity forces these two cycles to be of length n, and so $\partial_x(P) = a + b$, with a and b paths of length $n-1$. Therefore, I is an homogeneous ideal, since it is spanned by homogeneous elements, and so $k\langle Q\rangle/I$ inherits a grading from $k\langle Q\rangle$. In fact, we introduced the notion of n-regularity solely to force the ideal I to be homogeneous.

Since I is homogeneous, the quotient $k\langle Q\rangle/I$ turns out to be $\bigoplus A_n/(I \cap A_n)$, where A_n stands for the graded component of degree n. The completion is then carried out componentwise, so $k\langle\langle Q\rangle\rangle/J$ is actually $\prod A_n/(I \cap A_n)$. We have a natural inclusion $i : k\langle Q\rangle/I \to k\langle\langle Q\rangle\rangle/J$, which is obviously not the case if the ideal is non-homogeneous, as one can infer from Example 2.14. In the light of this fact, it suffices to show that $k\langle Q\rangle/I$ is infinite-dimensional.

The path algebra $k\langle Q\rangle$, regarded as a k-vector space, admits a direct sum decomposition $A \oplus B$, where A is the subspace spanned by paths

> *containing either $n-1$ consecutive arrows belonging to the same face or $n-1$ consecutive arrows surrounding the same vertex of the subdivision as factors* $\quad$ ($\clubsuit$)

and B is the subspace spanned by paths not satisfying ($\clubsuit$). One immediately checks that A is actually an ideal in $k\langle Q\rangle$. Even more, since every cyclic derivative is the sum of two paths satisfying ($\clubsuit$), we have that $I \subseteq A$. Therefore, any non-zero path $c \in B$ is not contained in I, and in turn is not zero in $k\langle Q\rangle/I$.

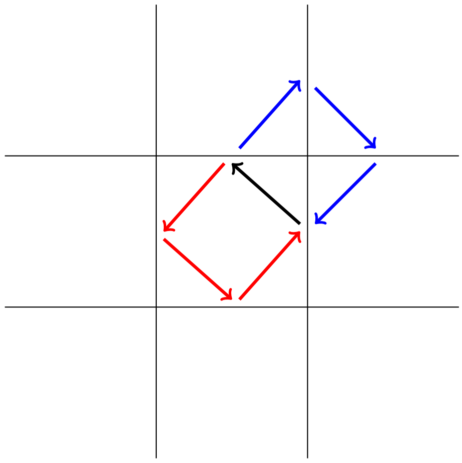


Figure 3.2: Any cyclic derivative of the potential P associated to a 4-regular subdivision turns out to be of the form $a + b$, with both a and b of length 3.

Suppose now that X is an infinite subset of B composed of non-zero paths, all of different lengths. Since paths are homogeneous elements of $k\langle Q\rangle$ and I is an homogeneous ideal, the projection of a path into $k\langle Q\rangle/I$ is an homogeneous element too. Therefore, the projection of X into $k\langle Q\rangle/I$ is an infinite linearly independent set, since it is composed of non-zero homogeneous elements, each belonging to a different homogeneous component. So finally, all that remains to show is that one can actually produce such a set X.

Choose any path of length 2 contained in a single face and name it x_2. One may concatenate any path with exactly two arrows: one which lies on the same face as the last one and one which does not. If the two last arrows of x_k lie on the same face, let x_{k+1} be the path obtained from x_k by concatenating the arrow lying on a different face. Otherwise, let x_{k+1} be the path obtained from x_k by concatenating the arrow lying on the same face as x_k's last arrow. Since this process may be carried out indefinitely, we obtain a sequence $X = (x_k)_{k\geq 2}$ of paths, all of them of different lengths. Notice that every path x_k is contained in B, since by construction it contains at most 2 consecutive arrows on the same face or around the same vertex, and $n - 1 > 2$ by hypothesis. In fact, this construction can not be carried out if $n = 3$, since in that case B is the k-vector space spanned by paths of length 0 and 1.

This settles the case $n > 3$ and now we turn to the study of the case $n = 3$. By Equation (1), we know that $f = (4g - 4)/(3 - 4) = 4 - 4g$. Since the number of faces in the subdivision must be positive, it follows that $g = 0$, and so a 3-regular division of a surface must necessarily be a triangulation of a sphere consisting of 4 faces, or in other words a tetrahedron. We have already determined that the Jacobian algebra associated to the tetrahedron is infinite-dimensional in Section 3.1, and so the proof is now complete. □

3.4 A family of finite-dimensional examples

We now turn to the study of another family of polygonal subdivisions, this time inducing finite-dimensional algebras. Let us start by showing that closed surfaces of any genus admit a particular decomposition.

A *loop* f on a surface Σ is a smooth embedding $f : S^1 \to \Sigma$. We will identify a loop with its image on Σ. A *handle* is a torus with a disk removed. We recall that, by the classification theorem, a closed surface of genus $g > 0$ can be obtained starting with a disk with $g - 1$ holes and gluing handles along each hole and the boundary of the disk.

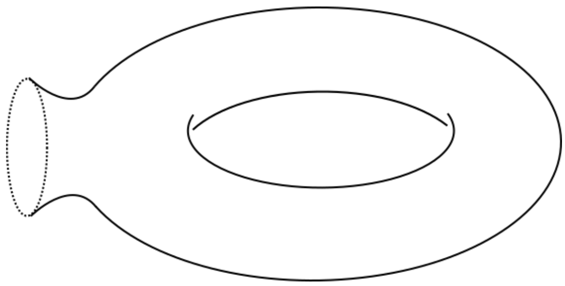


Figure 3.3: A handle.

Proposition 3.3. *Let Σ be a closed surface of genus g. There exist finite families $R = \{r_i\}$ and $B = \{b_j\}$ of loops on Σ such that:*

1. *Two loops in the same family are disjoint.*

2. *Any point of the surface belongs to at most two loops.*

3. *The loops divide the surface into a disjoint family of regions, each of them a disk (up to homeomorphism).*

Proof. We give an explicit construction of such families.

The case in which $g = 0$ (the sphere) is easily dealt with by choosing a single loop r_1 as the equator, since both hemispheres are homeomorphic to a disk and the other conditions are vacuously true.

For the cases in which $g > 0$, we will make use of the handle decomposition we mentioned previously. We will draw a handle as a square (with opposite edges identified) having a gray hole in its center.

We first deal with the case in which $g = 1$. If we glue a handle decomposed as in Figure 3.4 to the boundary of the disk pictured in Figure 3.5, we obtain a torus with a single red loop r_1 and a single blue loop b_1.

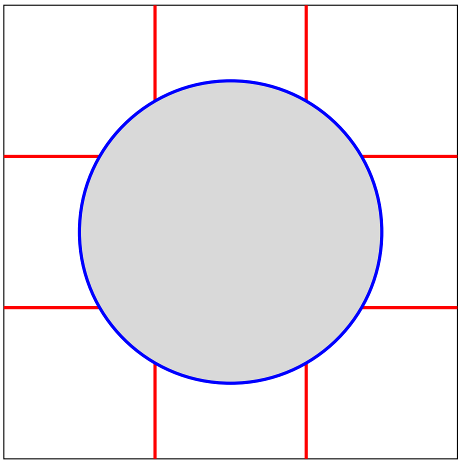


Figure 3.4: A decomposed handle.

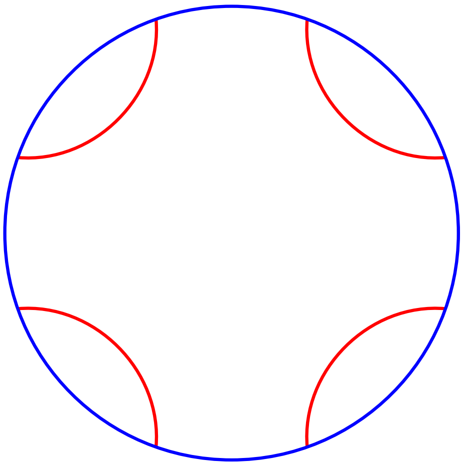


Figure 3.5: A decomposed disk.

Once again, the intersection conditions are vacuous. Finally, we observe that the blue loop cuts the torus into a disk and a handle, so one can check condition 3 in both objects separately. One sees at once that the disk gets cut up into 5 smaller disks, while the handle is divided into 3 disks, as we wanted.

While the solution to the case $g = 1$ may be easier to draw directly on a square instead of considering a handle decomposition, the latter helps to illustrate the general case, which we now consider. For the case $g > 1$, we regard our surface as a disk with $g - 1$ holes, with a handle divided as in Figure 3.4 attached to its boundary and to each of its holes. We exemplify the configuration on the disk with holes for $g = 4$ in Figure 3.6.

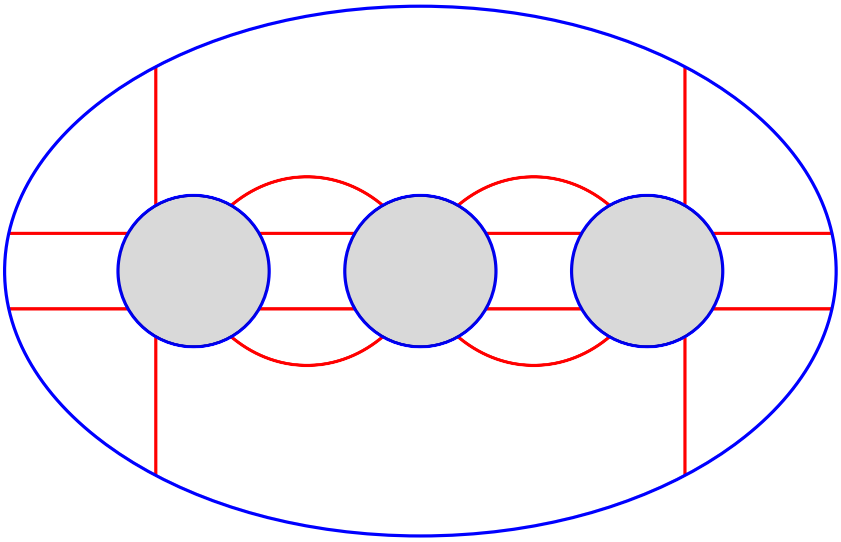


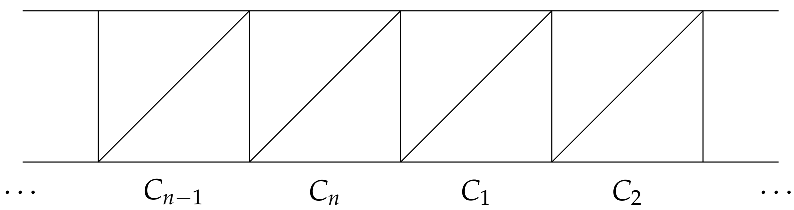
Figure 3.6: A decomposed disk with holes.

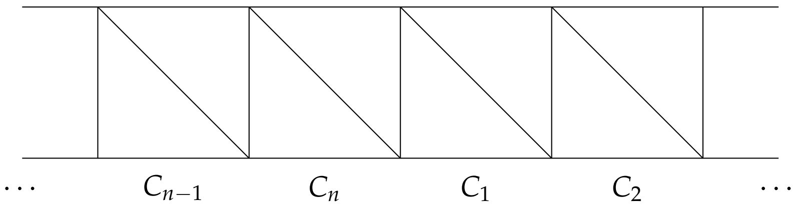
The following is an outline for the construction of the previous figure for general g:

1. Place the $g-1$ holes in a straight line inside the disk.
2. Connect the boundary of the left and rightmost holes to the boundary of the disk using 4 red arcs, mimicking the figure.
3. Cycle through the holes from left to right. If the current hole has a neighboring hole to its right, draw 4 red arcs connecting their boundaries.

Notice that after gluing the handles, both the red and the blue arcs are now loops. We now consider the families $\{r_i\}$ and $\{b_j\}$, consisting of the red and blue loops respectively. Inspecting the figures, we see that conditions 1 and 2 are both satisfied. Finally, it remains to check condition 3. Since we have already seen that each handle is split up into disks in the discussion of the case $g=1$, it suffices to check this for the disk with holes, and once again this is easily seen in the drawing. $\square$

A *band* on a surface Σ is a finite sequence $\{C_1, \ldots, C_n\}$, where each S_i is a square on Σ subdivided by one of its main diagonals, arranged as in one of the following two figures:





Notice that in both cases the choice of diagonal is consistent throughout the band. We will say a band is *positively oriented* if it is arranged as in the first figure and *negatively oriented* otherwise (we stress that this makes sense since the surface is oriented). We require that the only adjacency relations between squares in a band are the ones expressed by the figures, so in particular bands have well-defined top and bottom sides. We remark that two differently oriented bands may intersect as the following figure shows:

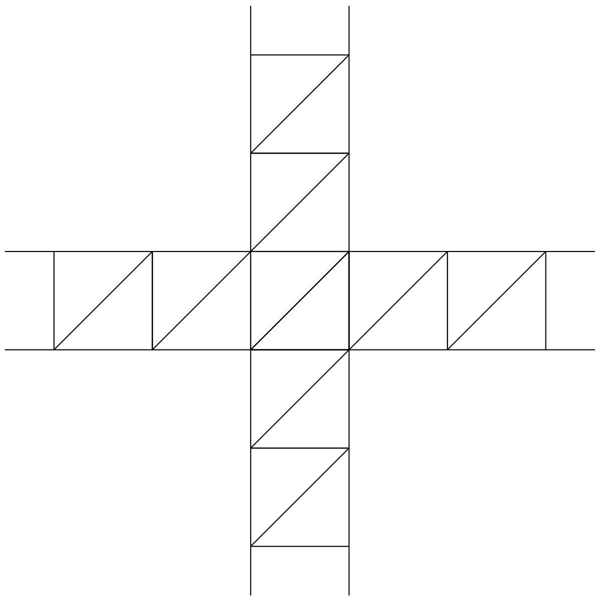


Figure 3.7: Two intersecting bands.

Given a surface Σ of arbitrary genus, we consider families of loops $R = \{r_i\}$ and $B = \{b_j\}$ on Σ satisfying the conditions stated in Proposition 3.3. We can now pick suitably small tubular neighborhoods of each loop and place a positively (resp. negatively) oriented band over each neighborhood corresponding to a loop in R (resp. B). Conditions 1 and 2 of Proposition 3.3 guarantee that all intersections between bands resemble that of Figure 3.7. Condition 3 guarantees that each connected component of the complement of the bands is a polygon. We will refer to these polygons as *regions*. Therefore, the collection of bands and regions actually define a polygonal subdivision of Σ. For technical reasons that will be clearer later, we will require each band to

$$\textit{have at least 5 squares between any pair of intersections with other bands.} \qquad (\diamond)$$

This can be easily achieved by refining the division of the band if necessary. From now on we fix a subdivision as described above for each genus g and call its associated QP (Q_g, P_g). The corresponding Jacobian algebra will be denoted A_g.

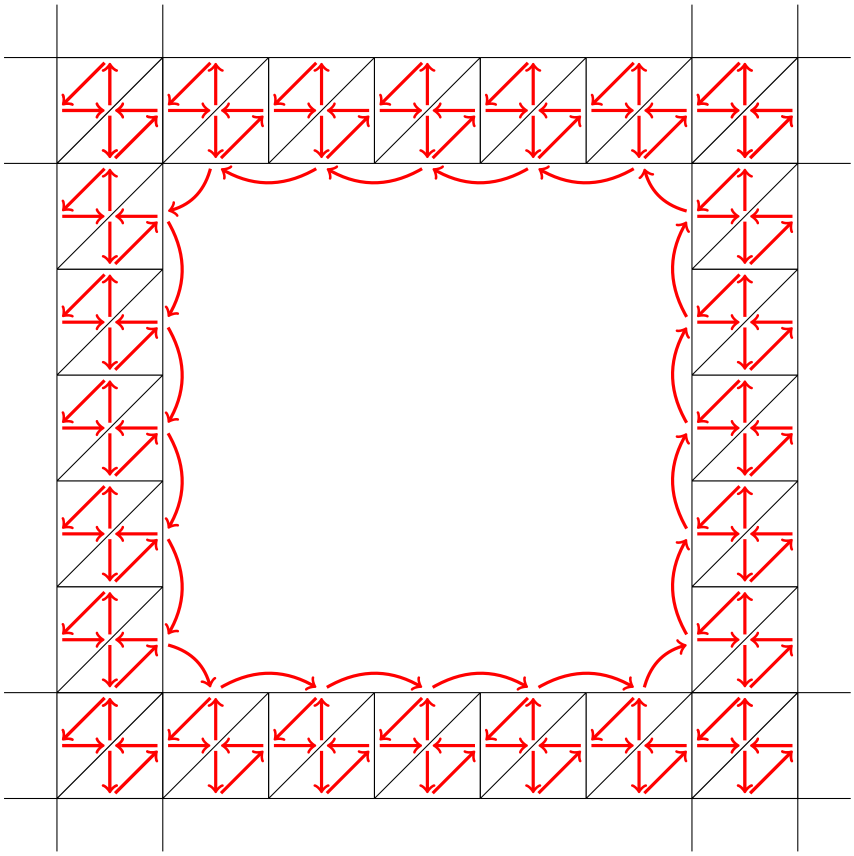
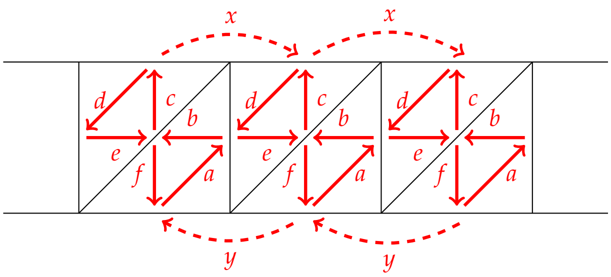


Figure 3.8: The configuration of the associated quiver Q_g around a region satisfying $(\diamond)$.

We now turn to the study of some relations that hold in A_g, that will enable us to prove its finite-dimensionality.

Lemma 3.4. *Any path in A_g passing through vertices of both sides of a band is zero.*

Proof. Throughout this and the following proofs, we will only consider our bands to be positively oriented, since the analogous statements for negatively oriented bands are proved in a similar fashion. We name the arrows in the quiver as in the following figure:



Since our quiver arises from a polygonal subdivision of a surface with *empty* boundary, we know that there is a cycle around each vertex, and so each x and each y are paths of length at least 1. We mark those paths in the figure with a dashed arrow. We have given the same name to arrows occupying the same position in different squares, since this abuse of notation will be useful for calculation.

Inspecting the figure we see that any path passing through vertices of both sides of the band must contain a path of the form cba or fed as factors. It suffices to check that both of them are zero in A_g.

We have that $\partial_f(P_g) = ba + eay$, and so $ba = -eay$ in A_g. Therefore, we have that $cba = -ceay$. Moreover, $\partial_d(P_g) = ce + xcb$, from which we deduce that $ce = -xcb$ in A_g. Putting all of this together, we get that $cba = -ceay = xcbay$ (one should note that the cba factor in the right hand side of the equality consists of arrows placed on the square immediately left from the one where we started). Proceeding inductively we get $cba = x^n cbay^n$ for all positive n. Since x and y are paths of length at least 1, this shows that cba is equal to paths of arbitrarily high length. Therefore, by Observation 1.8, we conclude that $cba = 0$ in A_g.

An analogous argument shows that $fed = 0$ in A_g as well, concluding the proof. □

Lemma 3.5. *Any sufficiently long path contained entirely in bands is zero in A_g. More precisely, any non-zero path contained in a single band (resp. several bands) is of length at most 5 (resp. 9).*

Proof. We first prove that any sufficiently long path contained entirely in a single band is zero. As we have already seen, any path passing through vertices of both sides of the band is zero, so we will suppose without loss of generality that our path is placed on the upper part of the band. Maintaining the notation used on the proof of the previous lemma, this is equivalent to saying our path only has arrows b, c, d or e as factors.

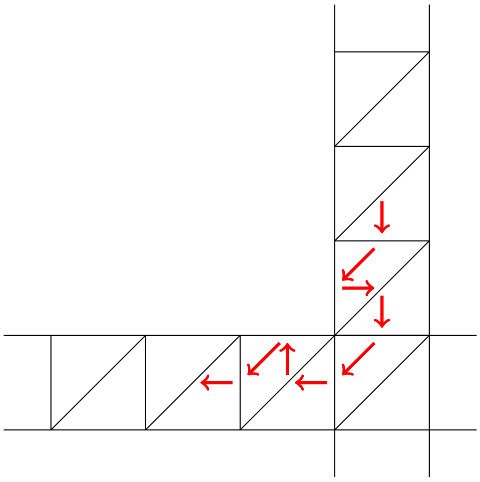
We start by studying paths containing a 3-cycle as a prefix. The 3-cycles edc and ced only prefix two paths of length 4, namely $cedc$ and $dced$. Since $\partial_e(P_g) = dc + ayf$, the relation $dc = -ayf$ holds, and so $cedc = -ceayf = 0$ and $dced = -ayfed = 0$ by Lemma 3.4. Moreover, the 3-cycle dce only prefixes two paths of length 5, which are $cbdce$ and $cedce$. Clearly, $cedce = 0$ since we have already shown that $cedc = 0$, and using the relation $dc = -ayf$ we get $cbdce = -cbayfe = 0$ once again by Lemma 3.4.

Now we turn to paths containing a 3-cycle as a suffix. Once again, the 3-cycles ced and dce only suffix two paths of length 4, which are $cedc$ and $dced$, already shown to be zero. The 3-cycle edc is a suffix to only two paths of length 5, which are $edcbd$ and $edced$. We see that $edced = 0$ since $dced = 0$ and $edcbd = -eayfbd = 0$ as we wanted.

Therefore, any path in a band of length greater than 5 containing a 3-cycle is zero, since we have shown these cycles only admit prefixes and suffixes of length at most 1.

Now, a path of length greater than 5 not containing a 3-cycle is either of the form $dcbdcb$, $bdcbdc$ or $cdbcbd$. All possibilites have $cbdc$ as a factor, which is clearly zero since $cbdc = -cbayf = 0$ by Lemma 3.4. Therefore, we conclude that any path of length greater than 5 contained in a single band is zero.

Finally, consider a path of length greater than 9 contained in possibly different bands. If its first six arrows lie on the same band, the path is zero as seen previously. Otherwise, at most five of its first arrows lie on the same band and the sixth is then placed on a different band intersecting the original one, as seen in the following figure:

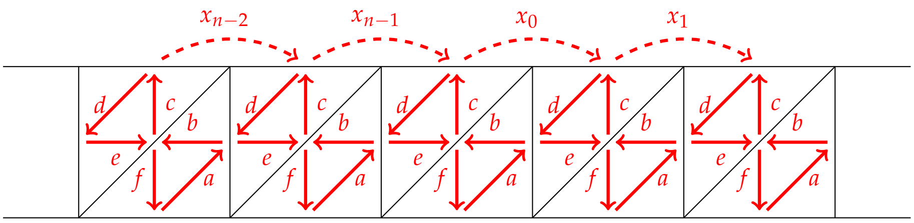


Since by hypothesis ($\diamond$) two intersections in the same band are distanced by at least five squares, we conclude that at least the next six arrows belong to the same band. Therefore, any path of length greater than 9 contained entirely in bands is zero in A_g, as we wanted. $\square$

Lemma 3.6. *Any non-zero path entirely contained in an n-sided region is of length at most $3n - 4$.*

Proof. We fix an n-sided region. Consider any band neighboring our region and pick any of its squares that is at least two squares away from an intersection with another band, which is always possible by hypothesis ($\diamond$). Recalling the fact that our region induces an n-cycle in the quiver, we name the arrow starting at the square we picked as x_0. In general, given $j \in \mathbb{Z}/n\mathbb{Z}$ we call x_j the arrow starting at the target of x_{j-1}.

Keeping the previous notation for arrows contained in bands, our current situation is illustrated by this figure:



We now prove that the path $L = x_{n-1}x_{n-2}\dots x_2x_1x_0x_{n-1}\dots x_3x_2$ is zero. We have that $\partial_{x_0}(P_g) = x_{n-1}x_{n-2}\dots x_2x_1 + cbd$ and $\partial_{x_1}(P_g) = x_0x_{n-1}\dots x_3x_2 + cbd$. Therefore, the relations

$$x_{n-1}x_{n-2}\dots x_2x_1 = -cbd$$
$$x_0x_{n-1}\dots x_3x_2 = -cbd$$

hold in A_g. We stress that cbd denotes a different path in each one of the two equations: they are similar paths contained in different squares. Using these relations, we see that

$$x_{n-1}x_{n-2}\dots x_2x_1x_0x_{n-1}\dots x_3x_2 = -cbdx_0x_{n-1}\dots x_3x_2 = cbdcbd,$$

and the latter is zero by Lemma 3.5 since $cbdcbd$ is a path of length 6 lying on a single band.

Finally, since the longest path entirely contained in our region not having L as a factor is

$$\underbrace{x_{n-2}x_{n-1}\dots x_0}_{n-1 \text{ arrows}}\underbrace{x_{n-1}x_{n-2}\dots x_0}_{n \text{ arrows}}\underbrace{x_{n-1}x_{n-2}\dots x_3}_{n-3 \text{ arrows}},$$

which is of length $3n-4$, the result follows. □

Lemma 3.7. *Any sufficiently long path not having factors from two different regions is zero in A_g.*

Proof. We write our path as $B_{k+1}R_k\dots R_2B_2R_1B_1$, where the paths R_i are contained in a same region and the paths B_j are non-trivial (except possibly for B_1 and B_{k+1}) and contained in bands.

Let $1 < j < k+1$. The path B_j starts and ends at the boundary of our region of interest. Therefore, it must pass through both of the endpoints of a same arrow, which we will call x_0, belonging to the cycle associated to the region, since otherwise it would be sufficiently long as to reduce to zero, by Lemma 3.5. Using the relation induced by $\partial_{x_0}(P_g)$, we can replace some of the arrows in B_j with a path entirely contained in the region. After applying this argument repeatedly, we can then suppose our path is of the form $B_2R_1B_1$. The proof now follows from Lemmas 3.5 and 3.6, since if our path is non-zero, then $|B_1| < 10$, $|B_2| < 10$ and $|R_1| < 3n-3$, where n is the number of sides of the region. □

After proving all these lemmas, the main result of this section now follows easily:

Theorem 3.8. *The algebra A_g is finite-dimensional.*

Proof. By Lemma 3.7, any sufficiently long path is zero if it does not have factors from two different regions, but any path that does must be zero (regardless of length) by Lemma 3.4, since it must cross a band. Since any sufficiently long path is zero, we conclude that A_g is finite-dimensional. □

3.5 Pyramids

A *pyramid* is a polyhedron formed by placing a regular polygon (called the *base*) and a single point (called the *apex*) in different parallel planes, and then taking their convex hull. One immediately checks that if the base is an n-sided polygon, then the pyramid has exactly $n + 1$ faces, n of them triangles and the last one being the base itself. Any pyramid is a polygonal subdivision of the sphere, so we now study the properties of its associated Jacobian algebra. We first label the arrows in the associated quiver as shown in Figure 3.9.

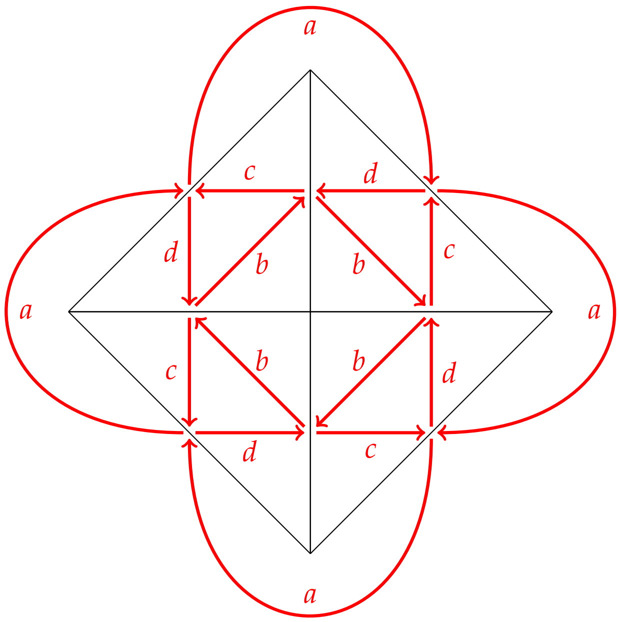


Figure 3.9: The quiver arising from a pyramid with a square base.

The 4-cycle placed outside of the square is the cycle corresponding to the base, and each 3-cycle inside of the square corresponds to a triangular face of the pyramid. We will name the arrows of the base as a and the arrows surrounding the apex as b. The rest of the arrows corresponding to triangular faces are named c and d in such a way that dcb is the only path of length 3 taking place in a single triangle.

Theorem 3.9. *Let $n \geq 3$. The Jacobian algebra associated to a pyramid with an n-sided base is finite-dimensional iff n is even.*

Proof. The case $n = 3$ has already been covered in Section 3.1, since a pyramid with a 3-sided base is just a tetrahedron. We will then suppose that $n > 3$.

We call (Q, P) the QP associated to the pyramid and A its Jacobian algebra. We will study A making heavy use of the diamond lemma. To do so, we will order terms according first to length (longer terms being smaller) and then lexicographically from right to left. For instance, $a^5 \prec bd$ and $bda \prec b^3$. This is a minor modification of the usual revglex order (since words are sorted lexicographically but from right to left), and the same argument as in Lemma 2.13 proves that it satisfies the descending chain condition in norm.

The careful reader will note that, a priori, we are not under the hypotheses of the diamond lemma, since we have not specified a term order for the entire set of paths, but rather we have given an equivalence relation on the set of paths (where two paths are considered to be equal if they are given the same label) and then ordered the set of equivalence classes. This is of course no obstruction, since we may number the arrows carrying the same label and sort them first by their labels and then by their numbering. For instance, two different paths of the form a^3 may be numbered as, say, $a_2a_1a_0$ and $a_3a_2a_1$.

We now have to find a confluent reduction system compatible with our term order. We start off by computing the cyclic derivatives of the potential P, which turn out to be:

$$\begin{aligned}
\partial_a(P) &= a^{n-1} + cd \\
\partial_b(P) &= b^{n-1} + dc \\
\partial_c(P) &= bd + da \\
\partial_d(P) &= ac + cb
\end{aligned}$$

This suggests the following reduction system, which we will call Γ:

$$cd \rightsquigarrow -a^{n-1} \tag{∂_a}$$
$$dc \rightsquigarrow -b^{n-1} \tag{∂_b}$$
$$bd \rightsquigarrow -da \tag{∂_c}$$
$$ac \rightsquigarrow -cb \tag{∂_d}$$

Every reduction rule is compatible with our term order, since the right hand side terms are either longer (since $n > 3$) or of the same length but lexicographically smaller.

Notice that the ideal I_Γ generated by this reduction system is exactly the Jacobian ideal, so if the system turns out to be confluent, then the set of irreducible paths forms a basis for the Jacobian algebra A. The system Γ presents only a few ambiguities, namely the ones arising from the monomials cdc, dcd, bdc and acd. We now check if they are solvable:

$$\begin{tikzcd}
 & -a^{n-1}c \arrow[r, "\partial_d\ (n-1\text{ times})"] & (-1)^n cb^{n-1} \\
cdc \arrow[ur, "\partial_a"] \arrow[dr, "\partial_b"'] & & \\
 & -cb^{n-1} &
\end{tikzcd}$$

$$\begin{tikzcd}
 & -b^{n-1}d \arrow[r, "\partial_c\ (n-1\text{ times})"] & (-1)^n da^{n-1} \\
dcd \arrow[ur, "\partial_b"] \arrow[dr, "\partial_a"'] & & \\
 & -da^{n-1} &
\end{tikzcd}$$

$$\begin{tikzcd}
 & -dac \arrow[r, "\partial_d"] & dcb \arrow[r, "\partial_b"] & -b^n \\
bdc \arrow[ur, "\partial_c"] \arrow[dr, "\partial_b"'] & & & \\
 & -b^n & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & -cbd \arrow[r, "\partial_c"] & cda \arrow[r, "\partial_a"] & -a^n \\
acd \arrow[ur, "\partial_d"] \arrow[dr, "\partial_a"'] & & & \\
 & -a^n & &
\end{tikzcd}$$

As we can see, the rewriting system Γ turns out to be confluent iff n is odd, since our ground field k is of characteristic zero. In that case, we know that the set of irreducible terms forms a basis of the Jacobian algebra. Since paths of the form a^k and b^k are irreducible for any k, we conclude that the Jacobian algebra is infinite-dimensional for odd values of n.

As for the even case, we may try to enlarge the set of rewriting rules as to make the system confluent. From our previous computation using our old rewriting system Γ, we know that $cb^{n-1} = da^{n-1} = 0$. Moreover, $0 = dcb^{n-1} = -b^{2n-2}$ and $0 = cda^{n-1} = -a^{2n-2}$. We may consider the rewriting system Γ', composed of the old set of rewriting rules from Γ and the following new rules:

$$\begin{aligned} da^{n-1} &\rightsquigarrow 0 \\ cb^{n-1} &\rightsquigarrow 0 \\ a^{2n-2} &\rightsquigarrow 0 \\ b^{2n-2} &\rightsquigarrow 0 \end{aligned}$$

We remark that this system is compatible with our term order. Moreover, the ideal $I_{\Gamma'}$ coincides with the Jacobian ideal as well, since the new rules come from identities holding in the Jacobian algebra A. It suffices to check that Γ' is indeed a confluent system in order to be in the hypotheses of the diamond lemma. The set of ambiguities is now slightly larger: it contains all of the old ones (which are now solvable, thanks to the rules $da^{n-1} \rightsquigarrow 0$ and $cb^{n-1} \rightsquigarrow 0$) plus the ambiguities arising from the following set of monomials:

$$bda^{n-1}, cda^{n-1}, acb^{n-1}, dcb^{n-1}, da^{n-1}c, cb^{n-1}d, da^{2n-2}, cb^{2n-2}, a^{2n-2}c, b^{2n-2}d$$

After a straightforward but somewhat tedious check one sees that all of these ambiguities are solvable (in fact, rather easily, since every term reduces to zero). Once again by the diamond lemma, we know that the set of irreducible paths forms a basis for the Jacobian algebra A. We will now prove that there are no irreducible paths of length greater than $2n-2$, which in turn implies the finite-dimensionality of A.

Indeed, suppose x is an irreducible path starting with an a arrow. As we see in Figure 3.9, an a arrow is only concatenable with another a arrow or with a d arrow. Since da is a reducible path, it follows that any irreducible path starting with an a arrow is of the form a^k, and since a^{2n-2} is reducible, that $k < 2n-2$. The same argument holds for paths starting with a b arrow.

Finally, we notice that an irreducible path starting with a c arrow must be c itself, since the only possible successors are a or d, and both ac and dc are reducible. Using the same idea we see that d is the only irreducible path starting with a d arrow, and thus there is no irreducible path of length greater than $2n-2$, concluding the proof. □

Since we used the same heuristic as our software does, we may now check our computation against it. We will try to see if our rewriting system produces the same number of irreducible monomials of each length. If n is odd, the only reducible monomials are ac, bd, cd and dc, so the irreducible monomials of positive length are of the form a^k, b^k, da^{k-1} or cb^{k-1} for $k \geq 1$, and there are exactly n paths with each of these names. Therefore, there are $4n$ paths of each possible positive length (and $2n$ stationary paths). Notice that this fits with the Hilbert series corresponding with the tetrahedral case studied previously by setting $n = 3$, even though the rewriting system we obtained for general pyramids is a different one.

As for the case where n is even, one has to consider that the monomials da^{n-1}, bc^{n-1}, a^{2n-2} and b^{2n-2} are reducible as well. Therefore, paths a^k and b^k are irreducible iff $1 \leq k \leq 2n-3$ and paths da^j and cb^j are irreducible iff $0 \leq j \leq n-2$. An easy counting argument then shows that there are

- $2n$ stationary paths,
- $4n$ irreducible paths of length k, where k ranges from 1 to $n-1$,
- $2n$ irreducible paths of length j, where j ranges from n to $2n-3$.

We may now check our computations against our piece of software:

```
sage: Rewriting_System(pyramid(6)).generating_function(12)
15 rules added.
21 rules added.
9 rules added.
4 rules added.
4 rules added.
Total rules: 77
[12, 24, 24, 24, 24, 24, 12, 12, 12, 12, 0, 0]

sage: Rewriting_System(pyramid(7)).generating_function(12)
13 rules added.
8 rules added.
2 rules added.
2 rules added.
Total rules: 53
[14, 28, 28, 28, 28, 28, 28, 28, 28, 28, 28, 28]
```

As we can see, these results agree with our computations.

3.6 Scalars in the polygonal case

We recall that, so far, we have only considered potentials in which all scalars involved were set to 1. One of the main results from Sefi Ladkani's paper [Lad12] states:

Theorem 3.10. *Let (Σ, M) be a surface with marked points and empty boundary.*

1. *If (Σ, M) is not a sphere with 4 punctures, then for any choice of scalars the Jacobian algebra associated to the QP arising from any ideal triangulation of (Σ, M) is finite-dimensional.*
2. *If (Σ, M) is a sphere with 4 punctures, then the same conclusion holds provided that the product of the scalars is not equal to 1.*

The family of pyramids studied in the previous section shows that this result does not hold for polygonal subdivions in general. Keeping the notation previously used, consider the potential P which arises from assigning the scalar 1 to cycles of the form a^{n+1} and bdc, and the scalar -1 to cycles of the form b^{n+1} and acd. The cyclic derivatives of this potential are then

$$\begin{aligned}\partial_a(P) &= a^{n-1} - cd\\ \partial_b(P) &= -b^{n-1} + dc\\ \partial_c(P) &= bd - da\\ \partial_d(P) &= -ac + cb\end{aligned}$$

Therefore, the obvious rewriting system

$$cd \rightsquigarrow a^{n-1} \qquad (\partial_a)$$
$$dc \rightsquigarrow b^{n-1} \qquad (\partial_b)$$
$$bd \rightsquigarrow da \qquad (\partial_c)$$
$$ac \rightsquigarrow cb \qquad (\partial_d)$$

turns out to be confluent. An easy way to show this is to carry out the exact same computation that proved that this system is confluent for odd n, but deleting all the minus signs from it. In the light of this fact, we conclude that the Jacobian algebra associated to this potential is infinite-dimensional, since for instance cycles of the form a^{kn} and b^{kn} are irreducible for all natural k. Nevertheless, we have already seen that for even values of n, the Jacobian algebra arising from the standard potential is finite-dimensional. This shows that, when dealing with polygonal subdivisions, the finite-dimensionality of the Jacobian algebra is highly dependent on the choice of scalars for the potential.

3.7 Prisms and antiprisms

In this section we will introduce two families of convex polyhedra, which are polygonal subdivisions of the sphere. We will then show confluent rewriting systems for their respective Jacobian algebras, so as to have more examples in which we may compute invariants in the following chapter.

A *prism* is a polyhedron composed of two parallel copies of an n-sided polygon, which we will call *base faces*, joined by parallelograms.

We will label the arrows in the quiver associated to a prism in a similar way as we did for pyramids, in order to help us simplify the description of the rewriting system. This is illustrated in Figure 3.11, where we have drawn the prism as a planar figure in which the left and right sides are identified. In the drawing, the black edges represent the edges of the prism and the red arrows are the ones from the associated quiver. Arrows labeled a (resp. b) correspond to the n-cycle inside the top (resp. bottom) base face. The arrows corresponding to 4-cycles inside parallelograms are labeled as c, d, e

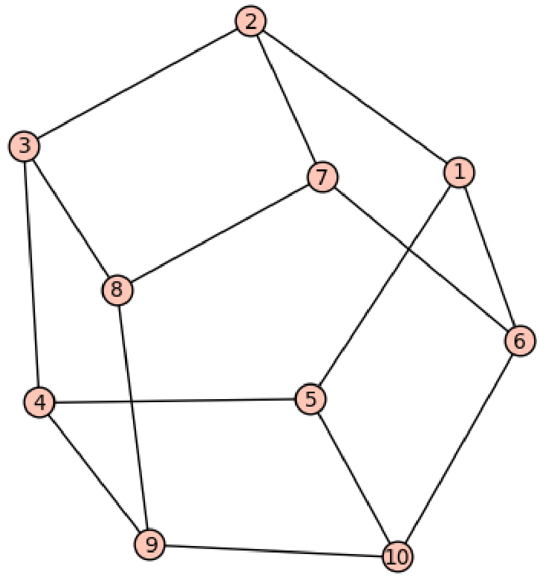


Figure 3.10: A prism with a pentagonal base.

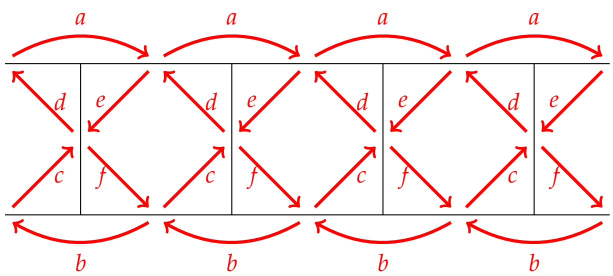


Figure 3.11: The quiver associated to a prism.

or f, in such a way that a 4-cycle having a vertex from the top base as source is labeled $dcfe$.

After naming the arrows in this way, a confluent rewriting system for the associated Jacobian algebra in the case $n > 3$ is given by the rules:

$$\begin{aligned} de &\rightsquigarrow -a^{n-1} & dcfe &\rightsquigarrow a^n \\ fc &\rightsquigarrow -b^{n-1} & fedc &\rightsquigarrow b^n \\ ad &\rightsquigarrow -dcf & cfed &\rightsquigarrow edcf \\ bf &\rightsquigarrow -fed & a^{n+1} &\rightsquigarrow 0 \\ ea &\rightsquigarrow -cfe & b^{n+1} &\rightsquigarrow 0 \\ cb &\rightsquigarrow -edc & & \end{aligned}$$

As one may easily check, all cycles in the quiver are reducible, except for a^n, b^n and $edcf$. Nevertheless, all powers of these cycles are reducible, and so the Jacobian algebra is finite-dimensional. In fact, once again we may count the number of irreducible monomials of each length, which are

- $3n$ stationary paths,
- $6n$ paths of lengths k, for $1 \leq k \leq 3$,
- $3n$ paths of length 4,
- $2n$ paths of length j, for $5 \leq j \leq n$.

Our second family of polyhedra is the family of *antiprisms*, which are once again composed of two parallel copies of an n-sided polygon, but this time they are joined by triangles.

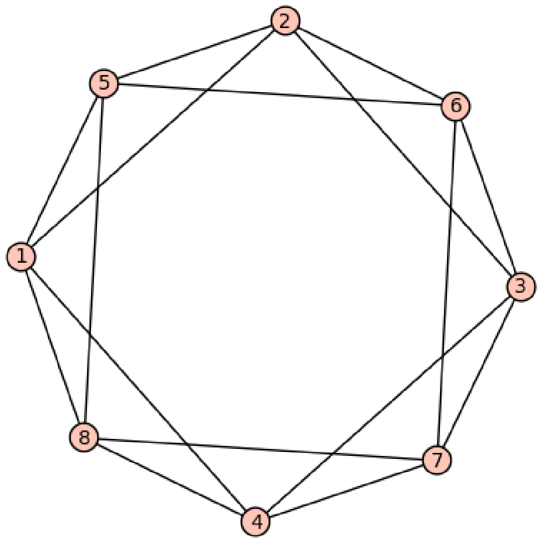
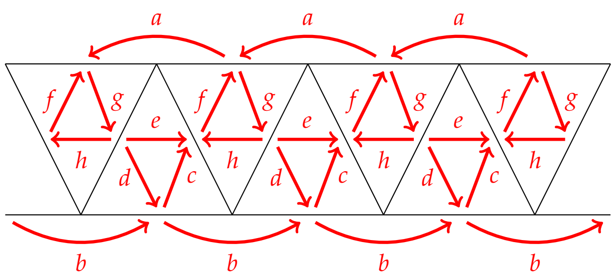


Figure 3.12: Top view of an antiprism with square base.

Our labeling of the arrows in the quiver will be:



Once again, the polyhedron is assembled by identifying the left and right sides of the figure. Arrows labeled a (resp b) correspond to the top (resp. bottom) n-cycle. Arrows inside a triangle sharing a side with the top (resp. bottom) face are labeled f, g or h (resp. c, d, e), in such a way that a 3-cycle having a vertex from the top (resp. bottom) face as source is labeled fhg (resp. ced).

If $n > 3$, a confluent rewriting system for the Jacobian algebra associated to an antiprism with an n-sided base is given by:

$$\begin{aligned}
feg &\rightsquigarrow -a^{n-1} & hdbc &\rightsquigarrow egaf \\
chd &\rightsquigarrow -b^{n-1} & gafe &\rightsquigarrow dbch \\
ed &\rightsquigarrow -hbd & ga^{n-1} &\rightsquigarrow 0 \\
ce &\rightsquigarrow -bch & a^{n-1}f &\rightsquigarrow 0 \\
dc &\rightsquigarrow -gaf & db^{n-1} &\rightsquigarrow 0 \\
hg &\rightsquigarrow -ega & b^{n-1}c &\rightsquigarrow 0 \\
fh &\rightsquigarrow -afe & a^{2n-2} &\rightsquigarrow 0 \\
gf &\rightsquigarrow -dbc & b^{2n-2} &\rightsquigarrow 0
\end{aligned}$$

Although it is not as easy as with prisms, it is still straightforward to check that all squares of cycles are reducible, and therefore the Jacobian algebra turns out to be finite-dimensional as well. It is easy, although a bit tedious, to enumerate all of the irreducible monomials. It turns out that there are

- $4n$ stationary paths,
- $(6+2k)n$ irreducible paths of length k, for $1 \leq k \leq 6$,
- $18n$ irreducible paths of lengths 6 to $n-1$,
- $14n$ irreducible paths of length n,
- $8n$ irreducible paths of length $n+1$,
- $4n$ irreducible paths of length $n+2$,
- $2n$ irreducible paths of lengths $n+3$ to $2n-3$.

As usual, we check this computation against the output of our software:

```
sage: Rewriting_System(prism(7), True).generating_function(9)
105 rules added.
Total rules: 231
[21, 42, 42, 42, 21, 14, 14, 14, 0]
```

```
sage: Rewriting_System(antiprism(7), True).generating_function(13)
98 rules added.
48 rules added.
Total rules: 230
[28, 56, 70, 84, 98, 112, 126, 98, 56, 28, 14, 14, 0]
```

Note that we set the parameter `zero_rules_flag` as `True`, since there are rules in the rewriting system which are induced by the topology (namely, there are paths that reduce to zero since they may be extended to arbitrarily high lengths).

Chapter 4

Cohomological properties

In the previous chapter, we produced confluent rewriting systems for several Jacobian algebras. By the diamond lemma, such a rewriting system provides a basis for the algebra as a k-vector space, which is given by the irreducible monomials. In this chapter, we will make use of these bases to compute some invariants of the families of algebras considered previously. These invariants are closely related to the first two Hochschild cohomology modules of those algebras, which we will now introduce.

Given an associative k-algebra A, we define its *enveloping algebra* as $A^e = A \otimes_k A^{\mathrm{opp}}$. The product on A^e is given by $(a \otimes b)(c \otimes d) = ac \otimes db$. There is a natural equivalence between the category of A-A-bimodules and the category of left A^e-modules, which we will consider an identification. Therefore, if M is an A-A-bimodule, it makes sense to compute $\mathrm{Ext}^{\bullet}_{A^e}(A, M)$, which we will call the *Hochschild cohomology of A with coefficients in M* and denote $H^{\bullet}(A, M)$. If $M = A$, this module will be plainly called the *Hochschild cohomology of A*, and will be denoted as $HH^{\bullet}(A)$.

We refer the reader to [Wei94, Chapter 9] for a detailed introduction to Hochschild cohomology.

4.1 The center

Let A be an associative k-algebra. The *center* of A, denoted as $\mathcal{Z}(A)$, is the subset of all elements $x \in A$ such that $xa = ax$ for all $a \in A$, and is in fact a subalgebra of A. An algebra is called *central* if $\mathcal{Z}(A) = k$. For example, the matrix algebra $M_n(k)$ is a central k-algebra.

Lemma 4.1. *Let A be an associative k-algebra. Then $HH^0(A) = \mathcal{Z}(A)$.*

Proof. See [Red01]. □

We will now compute the center of the Jacobian algebras associated to pyramids, prisms and antiprisms. In order to do so, we first make a useful observation:

Remark 4.2. *Let x be a central element in a Jacobian algebra A. Then x is a linear combination of* loops, *that is, paths with identical source and target.*

Proof. First of all, we stress that two paths which are equivalent in the Jacobian algebra must share the same endpoints, since the Jacobian ideal is generated by sums of paths satisfying this property. Therefore, it makes sense to speak of endpoints and loops in A.

Suppose now that x is central in A. We write x as a combination of linearly independent classes of paths $\sum_{i=1}^{n} \lambda_i p_i$. Let v_j be the stationary path corresponding to the source of p_j. Since x is central, we have that

$$\sum_{i=1}^{n} \lambda_i p_i v_j = (\sum_{i=1}^{n} \lambda_i p_i) v_j = x v_j = v_j x = v_j (\sum_{i=1}^{n} \lambda_i p_i) = \sum_{i=1}^{n} \lambda_i v_j p_i.$$

The scalar corresponding to p_j in the left hand side of the equation is λ_j, since $p_j v_j = p_j$. However, the scalar corresponding to p_j on the right hand side is λ_j if p_j has the vertex at v_j as target, and 0 otherwise. Therefore, if p_j is not a loop, then $\lambda_j = 0$, and so x is a linear combination of loops, as we wanted. □

We remark that, by definition, stationary paths are loops. Moreover, it is easy to see that any loop commutes with all stationary paths.

As a consequence of the diamond lemma, we know that given a confluent rewriting system for a Jacobian algebra, the set of irreducible monomials forms a basis for it. This will be an essential tool to compute its center. For the remainder of this chapter, we will use the labeling of arrows presented in the previous chapter for the quivers associated to each polygonal subdivision.

We will start by proving a fact about the quiver associated to a polygonal subdivision that will be particularly useful in this situation:

Lemma 4.3. *The quiver Q associated to a polygonal subdivision of a surface Σ is* strongly connected, *that is, given any pair of vertices v_1, v_2 in Q there exists a path with source v_1 and target v_2.*

Proof. Given two faces X, Y of the subdivision of Σ, let $d(X, Y)$ be the minimum $n \in \mathbb{N}$ such that there exists a sequence $Z_0, \ldots, Z_n$ with $X = Z_0$, $Y = Z_n$ and such that Z_i shares an edge with Z_{i+1} for all $0 \leq i \leq n-1$. The fact that the surface is connected guarantees that $d(X, Y)$ is well defined.

Let v_1, v_2 be vertices in Q and pick F_1, F_2 faces of the subdivision such that v_1 (resp. v_2) is a vertex which corresponds to an edge of F_i (resp. F_2). If $d(F_1, F_2) = 0$, then $F_1 = F_2$ and so there is a path going from v_1 to v_2, since there is a cycle in the quiver joining all of the vertices arising from the same face. If $d(F_1, F_2) = 1$ such a path exists as well, since there is an arrow in Q with source v_1 which is part of the cycle corresponding to F_2, since F_1 and F_2 are contiguous. Since $d(F_1, F_2)$ is a finite number and we have already shown that we can connect vertices from neighbouring faces, an easy inductive argument finishes the proof. $\square$

Proposition 4.4. *The Jacobian algebra associated to a pyramid with n-sided base, where $n > 3$, is central.*

Proof. Suppose first that n is even. The confluent rewriting system found previously induces the basis of irreducible paths

$$B = \{e_v, a^k, b^k, cb^j, da^j\},$$

where v runs through the set of vertices Q_0, $1 \leq k \leq 2n-3$ and $0 \leq j \leq n-2$. Notice that the label a^k denotes n different cycles, each starting at a different vertex. We will enumerate those vertices from 0 to $n-1$, in such a way that if $i+1 = j \mod n$ then there exists an a arrow such that $a : i \to j$. We will denote the path starting at vertex k consisting of j arrows labeled a as a_k^j. For example, using this notation, we have that $a_1 a_0 = a_0^2$. Notice that a_k^n both starts and ends at k, since it is a cycle. We will use the same idea to name the n different cycles of the form b^n.

Now, the only irreducible loops are of the form e_v, a^n or b^n, and so if we write a central element x in terms of our basis B, by Observation 4.2 we get that

$$x = \sum_{v \in Q_0} \lambda_v e_v + \sum_{i \in \mathbb{Z}/n\mathbb{Z}} (\mu_i a_i^n + \eta_i b_i^n),$$

where all greek letters are scalars. Since x is central, it must commute with a_i, and so:

$$\lambda_{s(a_i)} a_i + \mu_{i-1} a_{i-1}^{n+1} = a_i x = x a_i = \lambda_{t(a_i)} a_i + \mu_{i+1} a_i^{n+1}.$$

Exactly the same identity holds if we replace a_i with b_i throughout. Running down through all possible values of i, this set of equalities imply that all μ (resp. η) scalars are zero and that $\lambda_u = \lambda_v$ if u and v are vertices which are sources of a (resp. b) arrows.

We now observe that every vertex of the quiver is either the source of an a or a b arrow. Therefore $x = \lambda_1 y + \lambda_2 z$, where y (resp. z) denotes the sum of all stationary paths corresponding to vertices which are sources of a (resp. b) arrows. Now, since x is central, it must commute with any c arrow, and thus

$$\lambda_2 c = cx = xc = \lambda_1 c,$$

from where we obtain $\lambda_1 = \lambda_2$. Therefore, any central element must be a scalar multiple of the sum of all stationary paths, which is the identity element of the algebra. Thus, we conclude that the Jacobian algebra is central, as we wanted.

The case where n is odd is entirely analogous. The only minor difficulty is that cycles of different lengths appear in the generic expression of a central element, since cycles of the form a^{kn} or b^{kn} are irreducible even if $k > 1$. Nevertheless, the only difference this introduces is that the equations we used are somewhat more cumbersome to write down. $\square$

Proposition 4.5. *Let A be the Jacobian algebra associated to a prism with n-sided base, where $n > 3$. Then, $\mathcal{Z}(A) = \langle 1, a^n, b^n, edcf \rangle$, where 1 stands for the identity element of A.*

Proof. First of all, we notice that any cycle x of the form a^n, b^n or $edcf$ is central. Indeed, they commute with stationary paths since they are loops, and if y is any path of positive length then the rewriting system shows that $yx = xy = 0$.

Since the only irreducible loops are either stationary or of the form a^n, b^n or $edcf$, an arbitrary central element x can be written down as

$$x = \sum_{v \in Q_0} \lambda_v e_v + y,$$

where $y \in \langle a^n, b^n, edcf \rangle$ and is thus central. Considering that $\mathcal{Z}(A)$ is a subspace and both x and y are central, we get that $x - y = \sum_{v \in Q_0} \lambda_v e_v$ is as well. Now, let p be any arrow in the quiver. Then, by centrality we have that

$$\lambda_{s(p)} p = p \sum_{v \in Q_0} \lambda_v e_v = \sum_{v \in Q_0} \lambda_v e_v p = \lambda_{t(p)} p,$$

and so $\lambda_{s(p)} = \lambda_{t(p)}$. But then, since the quiver is strongly connected, an easy induction on the distance between any pair of paths shows that $\lambda_u = \lambda_v$ for all $u, v \in Q_0$. Thus, $x = \lambda 1 + y$, proving our claim. □

Proposition 4.6. *Let A be the Jacobian algebra associated to an antiprism with n-sided base, where $n > 3$. Then, $\mathcal{Z}(A) = \langle 1, egaf, dbch \rangle$, where 1 stands for the identity element of A.*

Proof. We will only sketch the proof, since it uses the same arguments as the previous two propositions. The loops $egaf$ and $dbch$ are easily seen to be central, because any product involving them and a path of positive length is zero, just as what happened in the proof of the previous proposition.

Now, in this case, the irreducible loops are stationary or of the form a^n, b^n, $egaf$ or $dbch$. Thus, a generic central element x is of the form

$$x = \sum_{v \in Q_0} \lambda_v e_v + y + z,$$

where $y \in \langle a^n, b^n \rangle$ and $z \in \langle egaf, dbch \rangle \subseteq \mathcal{Z}(A)$. Using the same argument as in the pyramidal case, namely multiplying by a, b on the left and right, we see that $y = 0$.

Since x and z are central, so is $x - z = \sum_{v \in Q_0} \lambda_v e_v$, and once again a connection argument shows that $x - z$ must be a scalar multiple of the identity. Therefore, $x = \lambda 1 + z$, as we wanted. □

4.2 Derivations

Let A be a k-algebra. A *derivation* of A is a k-linear morphism $f : A \to A$ satisfying the *Leibniz rule*

$$f(ab) = f(a)b + af(b).$$

The set of derivations of A, which we will denote as $D(A)$, is a Lie algebra with Lie bracket given by the commutator

$$[f, g] = f \circ g - g \circ f.$$

If $x \in A$, the map $f_x(y) = xy - yx$ is a derivation. The set of such maps, which we call *inner derivations*, is denoted $\operatorname{Inn}(A)$, and is actually a k-subspace of $D(A)$. Our main interest in derivations relies on the following fact:

Lemma 4.7. *Let A be an associative k-algebra. Then $HH^1(A) = D(A)/\operatorname{Inn}(A)$.*

Proof. See [Red01] or [Wei94, Lemma 9.2.1]. □

Recall that, given a quiver Q, we denote its vertex span k^{Q_0} as R. Since a Jacobian algebra A is an R-bimodule, it makes sense to consider the Lie subalgebra of R-linear derivations of A, which we will denote $D_R(A)$. The following easy observations will greatly simplify our work:

Remark 4.8. *Let A be a k-algebra and $I \subseteq A$ an ideal. If f is a derivation of A such that $f(I) \subseteq I$, then the induced linear map $\hat{f} : A/I \to A/I$ is a derivation of A/I. Moreover, if R is a set of generators for I, it suffices to see that $f(R) \subseteq I$ to show that $f(I) \subseteq I$.*

Proof. Since $f(I) \subseteq I$, the ideal I is contained in the kernel of $\pi \circ f : A \to A/I$, where π denotes the natural projection. Thus there is a well defined linear map $\hat{f} : A/I \to A/I$ which obviously satisfies the Leibniz rule, since f does.

As for the second assertion, suppose $x \in I$. Then $x = \sum_{i=1}^{n} a_i r_i b_i$, where $r_i \in R$ and $a_i, b_j \in A$. Therefore, the Leibniz rule implies that

$$f(x) = \sum_{i=1}^{n} f(a_i r_i b_i) = \sum_{i=1}^{n} (f(a_i) r_i b_i + a_i f(r_i) b_i + a_i r_i f(b_i)) \in I$$

since by hypothesis $f(R) \subseteq I$. □

Remark 4.9. *Let $v \in R$ be a stationary path in a Jacobian algebra A. If f is an R-linear derivation of A, then $f(v) = 0$.*

Proof. Using the Leibniz rule, the R-linearity of f and the fact that $v = v^2$, we have that

$$f(v) = f(v^2) = f(v)v + vf(v) = f(v^2) + f(v^2) = 2f(v^2) = 2f(v)$$

and so $f(v) = 0$. □

Remark 4.10. *Let B be a path basis for a Jacobian algebra A. If f is an R-linear derivation of A and $p \in B$, then $f(p)$ is a linear combination of elements of B only involving paths sharing the same endpoints as p.*

Proof. Since B is a basis for A, we have that

$$f(p) = \sum_{q \in B} \lambda_q q$$

for some scalars $\lambda_q \in k$. Let $x, y \in R$ be the source and the target of p, respectively. Then

$$\sum_{q \in B} \lambda_q q = f(p) = f(ypx) = yf(p)x = \sum_{q \in B} \lambda_q yqx,$$

and so $\lambda_q = 0$ if q has different endpoints than p. □

From now on, we will refer to R-linear derivations plainly as derivations.

4.2.1 Pyramids with an even-sided base

Let n be an even number greater than 3. In Section 3.5, we proved that the Jacobian algebra A associated with a pyramid having an n-sided base is finite-dimensional. Following the notation used in that section, a basis of irreducible monomials for the corresponding rewriting system we found is given by

$$B = \{e_v, a^k, b^k, cb^j, da^j\},$$

where v runs through the set of vertices Q_0, $1 \leq k \leq 2n-3$ and $0 \leq j \leq n-2$. Once again, we are abusing notation, since for instance a denotes several different paths of length 1. For simplicity, we will only study derivations f that assign the same value to all paths sharing the same name. Thus, we may speak of the value of $f(a)$ in an unambigous manner.

If f is such a derivation of A, then by Observation 4.10 we have that

$$\begin{aligned} f(a) &= \alpha a + \hat{\alpha} a^{n+1} \\ f(b) &= \beta b + \hat{\beta} b^{n+1} \\ f(c) &= \gamma c \\ f(d) &= \delta d, \end{aligned} \tag{1}$$

where all greek letters are scalars in k. We recall that the following relations generate the Jacobian ideal I:

$$a^{n+1} + cd \tag{2}$$
$$b^{n+1} + dc \tag{3}$$
$$bd + da \tag{4}$$
$$ac + cb \tag{5}$$

By relation (2) and the Leibniz rule, we have that

$$\begin{aligned}
-(\gamma+\delta)cd &= -f(c)d - cf(d) \\
&= f(-cd) \\
&= f(a^{n+1}) \\
&= \sum_{m=0}^{n} a^m f(a) a^{n-m} \\
&= \sum_{m=0}^{n} a^m (\alpha a + \hat{\alpha} a^{n+1}) a^{n-m} \\
&= (n+1)(\alpha a^{n+1} + \hat{\alpha} a^{2n+2}) \\
&= (n+1)\alpha a^{n+1} \\
&= -(n+1)\alpha cd,
\end{aligned}$$

and so $\gamma + \delta = (n+1)\alpha$. Reasoning analogously using relation (3), we conclude that $\gamma + \delta = (n+1)\beta$, and thus $\alpha = \beta$.

Relation (4) implies

$$\begin{aligned}
-(\beta+\delta)bd &= -(\beta+\delta)bd - \hat{\beta} bd^{n+1} \\
&= -f(b)d - bf(d) \\
&= f(-bd) \\
&= f(da) \\
&= f(d)a + df(a) \\
&= (\delta+\alpha)da + \hat{\alpha} da^{n+1} \\
&= (\delta+\alpha)da \\
&= -(\delta+\alpha)bd,
\end{aligned}$$

and thus we get the equation $\alpha = \beta$ again. Relation (5) implies the same identity as well.

Now, if g is a derivation of $k\langle Q\rangle$ satisfying the set of equations (1), its values on a, b, c and d completely determine it, since any path is either a stationary path (which is mapped to zero by Observation 4.9) or equal to a unique product of a, b, c and d, and thus its image is uniquely determined by the Leibniz rule. Moreover, as we have just seen, if $\alpha = \beta$ and $\gamma + \delta = (n+1)\alpha$ then the image of a set of generators of I by g is contained in I, and so g induces a derivation of the Jacobian algebra by Observation 4.8.

We now write the values of a generic derivation of the form we described in terms of the basis B:

$$\begin{aligned}
g(e_v) &= 0 \\
g(a^k) &= \sum_{m=0}^{k-1} a^m g(a) a^{k-1-m} = k(\alpha a^k + \hat{\alpha} a^{n+k}) \\
g(b^k) &= \sum_{m=0}^{k-1} b^m g(b) b^{k-1-m} = k(\alpha b^k + \hat{\beta} b^{n+k}) \\
g(cb^j) &= g(c)b^j + cg(b^j) = (\gamma + j\alpha)cb^j \\
g(da^j) &= g(d)a^j + dg(a^j) = (\delta + j\alpha)da^j = ((n+j+1)\alpha - \gamma)\, da^j
\end{aligned}$$

It is clear that these derivations make up a vector space V of dimension 4. A basis for this space is given by $\{\alpha^*, \gamma^*, \hat{\alpha}^*, \hat{\beta}^*\}$, where α^* stands for the map defined by setting $\alpha = 1$ and $\hat{\alpha} = \beta = \gamma = 0$, and the other maps are defined analogously. We now write down the image of the basis B by these maps to ease computation:

	α^*	γ^*	$\hat{\alpha}^*$	$\hat{\beta}^*$
e_v	0	0	0	0
a^k	ka^k	0	ka^{n+k}	0
b^k	kb^k	0	0	kb^{n+k}
cb^j	jcb^j	cb^j	0	0
da^j	$(n+j+1)da^j$	$-da^j$	0	0

From this table we see that B is a basis of eigenvectors for both α^* and γ^*, and thus these two maps commute. Moreover, γ^*, $\hat{\alpha}^*$ and $\hat{\beta}^*$ commute with each other as well, since they act trivially outside of $\langle cb^j, da^j \rangle$, $\langle a^k \rangle$ and $\langle b^k \rangle$ respectively, and these three spaces are in direct sum. These facts imply the vanishing of the brackets $[\alpha^*, \gamma^*]$, $[\gamma^*, \hat{\alpha}^*]$, $[\gamma^*, \hat{\beta}^*]$ and $[\hat{\alpha}^*, \hat{\beta}^*]$. By direct computation using the table we see that $[\alpha^*, \hat{\alpha}^*] = n\hat{\alpha}^*$ and $[\alpha^*, \hat{\beta}^*] = n\hat{\beta}^*$. Therefore, V is actually closed under the Lie bracket and so is a 4-dimensional Lie subalgebra of the algebra of derivations of A.

In fact, we may further characterize the Lie structure on V. Given Lie algebras L_1 and L_2 and an action by derivations $\cdot$ of L_1 on L_2, the *semi-direct product* $L_1 \ltimes L_2$ is the k-vector space $L_1 \oplus L_2$ with Lie bracket given by

$$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2] + x_1 \cdot y_2 - y_1 \cdot x_2).$$

It is now easy to see that V is a semi-direct product of the abelian Lie algebras $\langle \alpha^* \rangle$ and $\langle \gamma^*, \hat{\alpha}^*, \hat{\beta}^* \rangle$, where α^* acts trivially over γ^* and by multiplication by n over $\hat{\alpha}^*$ and $\hat{\beta}^*$.

Following the exact same modus operandi, we will now produce non-trivial derivations for the Jacobian algebras arising from prisms and antiprisms. Since in these cases there are more relations generating the Jacobian ideal than in the pyramidal case, the derivations will have to satisfy more constraints.

4.2.2 Prisms

As in the previous subsection, any derivation ψ of the Jacobian algebra A associated to a prism with an n-sided base must send paths to paths sharing the same endpoints. Therefore, by writing the image of the paths of length 1 by ψ in the basis of irreducible monomials, we find that ψ must satisfy:

$$
\begin{aligned}
\psi(a) &= \alpha a \\
\psi(b) &= \beta b \\
\psi(c) &= \gamma c \\
\psi(d) &= \delta d \\
\psi(e) &= \varepsilon e \\
\psi(f) &= \zeta f
\end{aligned}
$$

Notice that in this case there are no pairs of different paths sharing the same endpoints, which will make the rest of the computation quite easier. Mimicking the process carried out in the pyramidal case, we obtain constraints (in the right column) for the greek scalars using the relations that span the Jacobian ideal (in the left column) and the Leibniz rule:

$$
\begin{array}{rc}
de + a^{n-1} & \delta + \varepsilon = (n-1)\alpha \\
fc + b^{n-1} & \zeta + \gamma = (n-1)\beta \\
ad + dcf & \alpha + \delta = \delta + \gamma + \zeta \\
bf + fed & \beta + \zeta = \zeta + \varepsilon + \delta \\
ea + cfe & \varepsilon + \alpha = \gamma + \zeta + \varepsilon \\
cb + edc & \gamma + \beta = \varepsilon + \delta + \gamma
\end{array}
$$

Solving the linear system of equations on the right column, we find that $\alpha = \beta = 0$, $\delta = -\varepsilon$ and $\gamma = -\zeta$. Once again, any set of scalars satisfying these constraints induces a derivation of the path algebra that passes to the quotient and induces a bonafide derivation of A. We thus obtain two linearly independent derivations of A, γ^* and δ^*, which are defined on paths of length 1 as

	γ^*	δ^*
a	0	0
b	0	0
c	c	0
d	0	d
e	0	$-e$
f	$-f$	0

4.2.3 Antiprisms

We follow the usual procedure, this time for antiprisms. We start by writing down the image of the paths of length 1 by an eventual derivation ψ in the basis of irreducible monomials, and find that ψ must satisfy:

$$\begin{aligned}
\psi(a) &= \alpha a + \hat{\alpha} a^{n+1} \\
\psi(b) &= \beta b + \hat{\beta} b^{n+1} \\
\psi(c) &= \gamma c \\
\psi(d) &= \delta d \\
\psi(e) &= \varepsilon e \\
\psi(f) &= \zeta f \\
\psi(g) &= \eta g \\
\psi(h) &= \theta h
\end{aligned}$$

Once again, we have two pairs of paths sharing the same endpoints. We now find the constraints the scalars must verify using the relations in the Jacobian ideal:

$$\begin{array}{rr}
feg + a^{n-1} & \zeta + \varepsilon + \gamma = (n-1)\alpha \\
chd + b^{n-1} & \gamma + \theta + \delta = (n-1)\beta \\
ed + hbd & \varepsilon + \delta = \theta + \beta + \delta \\
ce + bch & \gamma + \varepsilon = \beta + \gamma + \theta \\
dc + gaf & \delta + \gamma = \eta + \alpha + \zeta \\
hg + ega & \theta + \eta = \varepsilon + \eta + \alpha \\
fh + afe & \zeta + \theta = \alpha + \zeta + \varepsilon \\
gf + dbc & \eta + \zeta = \delta + \beta + \gamma
\end{array}$$

The linear system of equations on the right column imposes the following constraints:

$$\begin{aligned}
\alpha &= \beta = 0 \\
\delta &= -\gamma - \varepsilon \\
\eta &= -\varepsilon - \zeta \\
\theta &= -\varepsilon
\end{aligned}$$

Any solution for this system of equations induces a derivation of the Jacobian algebra of the antiprism in the usual manner. We thus obtain five linearly independent derivations: $\hat{\alpha}^*$, $\hat{\beta}^*$, γ^*, ε^* and ζ^*, which values on the set of paths of length 1 we present on the following table:

	$\hat{\alpha}^*$	$\hat{\beta}^*$	γ^*	ε^*	ζ^*
a	a^{n+1}	0	0	0	0
b	0	b^{n+1}	0	0	0
c	0	0	c	0	0
d	0	0	$-d$	$-d$	0
e	0	0	0	e	0
f	0	0	0	0	f
g	0	0	0	$-g$	$-f$
h	0	0	0	$-h$	0

Appendix A

A path algebra SageMath class

We have developed a `SageMath` class to deal with the long and tedious process of checking the hypotheses of the diamond lemma, and to automatically generate bases and compute some invariants of the Jacobian algebra associated to a polygonal subdivision, with the only input being the subdivision itself. We now present the source code, which we will then discuss in more detail:

```
def n_slices(n, list_):
    for i in xrange(len(list_) + 1 - n):
        yield list_[i:i+n]

def is_sublist(list_, sub_list):
    for slice_ in n_slices(len(sub_list), list_):
        if slice_ == sub_list:
            return True
    return False

def grlex_sort(list_):
    return sorted(sorted(list_), key=len)

class Term:
    def __init__(self, body, sign):
        self.body = body
        self.sign = sign

    def __eq__(self, other):
        if self.body == ():
            return (self.body == other.body)
        else:
            return (self.body == other.body) and (self.sign == other.sign)

    def __repr__(self):
        if self.sign == -1:
            return '-' + str(self.body)
        else:
```

```
            return str(self.body)

    def __len__(self):
        return len(self.body)

class Rewriting_System:
    def __init__(self, triangulation, zero_rules_flag=False):
        self.triangulation = triangulation
        self.generate_qp()
        self.ambiguity_depth = 20
        self.zero_depth = 50
        self.rules = {}
        self.zero_rules_flag = zero_rules_flag
        self.rewriting_rules()

    def generate_qp(self):
        dict = {}
        edges = self.triangulation.edges(labels=False)
        faces = [map(tuple, map(sorted, face)) for face in self.triangulation
            .faces()]

        for i, edge in enumerate(edges):
            key = []
            for face in faces:
                if edge in face:
                    ind = face.index(edge)
                    if ind == len(face)-1:
                        key.append(edges.index(face[0]))
                    else:
                        key.append(edges.index(face[ind+1]))
            dict[i] = key

        self.quiver = DiGraph(dict)
        self.potential = []

        #cycles from faces
        for face in faces:
            self.potential.append(map(edges.index, face + [face[0]]))

        #cycles from punctures
        for puncture in self.triangulation:
            unordered_cycle = map(edges.index, self.triangulation.
                edges_incident(puncture, labels=False))
            cycle = [unordered_cycle[0]]
            for _ in xrange(len(unordered_cycle)):
                cycle.append([vertex for vertex in dict[cycle[-1]] if vertex
                    in unordered_cycle][0])
            self.potential.append(cycle)

    def cyclic_derivative(self, edge):
```

```
        derivative = []
        for cycle in self.potential:
            if is_sublist(cycle, edge):
                if edge[1] == cycle[0]:
                    derivative.append(cycle[:-1])
                else:
                    i = cycle.index(edge[1])
                    derivative.append(cycle[i:] + cycle[1:i])
        return grlex_sort(derivative)

    def reduce(self, term):
        term_length = len(term)
        body = term.body
        for i in xrange(0, term_length):
            for j in self.rules.keys():
                rule_length = len(j)
                if (term_length - i >= rule_length) and body[i:i+rule_length]
                     == j:
                    rep = self.rules[j]
                    if rep:
                        return Term(body[:i] + rep.body + body[i+rule_length
                            :], rep.sign * term.sign)
                    else:
                        return Term((), 1)
        return term

    def n_paths(self, n):
        return [path for path in self.quiver.all_paths_iterator(max_length=n
            +1, trivial=True) if len(path)==n+1]

    def is_zero(self, term):
        for _ in xrange(2*self.zero_depth):
            next_term = self.reduce(term)
            if next_term == term:
                break
            term = next_term
        return (len(next_term) > self.zero_depth) or (len(next_term) == 0)

    def null_n_paths(self, n):
        return [tuple(path) for path in self.n_paths(n) if self.is_zero(Term(
            tuple(path), 1))]

    def ambiguities(self):
        forbidden_terms = self.rules.keys()
        length = max(map(len, forbidden_terms))-1
        ambs = []
        for n in xrange(1,length):
            for x in forbidden_terms:
                for y in forbidden_terms:
                    if x != y and x[-n-1:] == y[:n+1]:
```

```
                    ambs.append([x, y, x + y[n+1:]])
        return ambs

    def is_unsolvable(self, amb):
        rule0 = self.rules[amb[0]]
        rule1 = self.rules[amb[1]]
        if rule0.body == ():
            leaf0 = Term((), 1)
        else:
            leaf0 = Term(rule0.body + amb[2][len(amb[0]):], rule0.sign)
        if rule1.body == ():
            leaf1 = Term((), 1)
        else:
            leaf1 = Term(amb[2][:-len(amb[1])] + rule1.body, rule1.sign)

        for _ in xrange(self.ambiguity_depth):
            leaf0 = self.reduce(leaf0)
            leaf1 = self.reduce(leaf1)
            if leaf0 == leaf1:
                return False

        if leaf0.body == ():
            return [leaf1.body, Term((), 1)]
        elif leaf1.body == ():
            return [leaf0.body, Term((), 1)]
        elif (leaf0.body == leaf1.body) and (leaf0.sign != leaf1.sign):
            return [leaf0.body, Term((), 1)]
        else:
            r1, r2 = grlex_sort([leaf0.body] + [leaf1.body])
            return [r1, Term(r2, leaf0.sign * leaf1.sign)]


    def needed_rules(self):
        new_rules = {}
        for amb in self.ambiguities():
            new_rule_needed = self.is_unsolvable(amb)
            if new_rule_needed:
                new_rules[new_rule_needed[0]] = new_rule_needed[1]
        return new_rules

    def rewriting_rules(self):
        for vertex in self.quiver.vertices():
            for neighbor in self.quiver.neighbor_out_iterator(vertex):
                derivative = self.cyclic_derivative([vertex,neighbor])
                self.rules[tuple(derivative[0])] = Term(tuple(derivative[1]),
                    -1)
        if self.zero_rules_flag:
            zero_paths = []
            n = 3
            while not zero_paths:
```

```
                zero_paths = self.null_n_paths(n)
                n += 1
            self.rules.update(dict.fromkeys(zero_paths, Term((), 1)))

        missing_rules = self.needed_rules()
        while missing_rules:
            self.rules.update(missing_rules)
            print len(missing_rules), 'rules added.'
            missing_rules = self.needed_rules()
        self.rules.update(missing_rules)
        print 'Total rules:', len(self.rules)

    def is_admissible(self, path):
        for forbidden in self.rules.keys():
            if is_sublist(path, list(forbidden)):
                return False
        return True

    def basis(self, max_degree):
        homogeneous_bases = [self.n_paths(n) for n in xrange(2)]
        for _ in xrange(2, max_degree):
            cur_degree_paths = []
            for j in homogeneous_bases[-1]:
                cur_degree_paths += [[k]+j for k in self.quiver.
                    neighbor_in_iterator(j[0])]
            admissible_paths = filter(lambda path: self.is_admissible(path),
                cur_degree_paths)
            homogeneous_bases.append(sorted(admissible_paths))
        return homogeneous_bases

    def generating_function(self, max_degree):
        return map(len, self.basis(max_degree))
```

We will explain how the program works by following an example execution in which we will compute some invariants of pyramids, as studied in Section 3.5.

An instance of the `Rewriting_System` class is constructed from an undirected graph representing a polygonal subdivison of the sphere. The associated quiver and standard potential will be automatically generated. We only implemented this feature for the spherical case since an enumeration of the faces of the polygonal subdivision is carried out by finding a planar embedding of the graph on the surface, and `SageMath` only provides such an algorithm for the sphere.

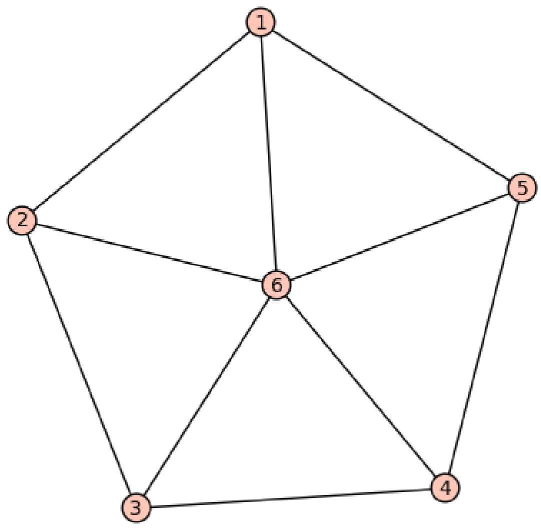
We would like to generate the rewriting system associated to a pyramid with an odd-sided base. In order to do that, we use the following snippet, which produces a pyramid with an n-sided base:

```
def pyramid(n):
  edges = {}
  for i in xrange(1, n):
    edges[i] = [i+1, n+1]
```

```
    edges[n] = [1, n+1]
    return Graph(edges)
```

We now instantiate our desired triangulation:

```
sage: odd_pyramid = pyramid(5)
sage: odd_pyramid.show()
```



Before producing the rewriting system, let us explain how the class constructor works. As mentioned previously, the QP will be automatically generated. If the `zero_rules_flag` is set to `False`, the `rewriting_rules()` method will produce the set of usual rewriting rules (the ones associated to the revglex order) and perform Heuristic 2.8 indefinitely until confluence is achieved. In order to test confluence, for every ambiguity we reduce both of its branches a maximum of `ambiguity_depth` times and compare if both branches eventually reduce to the same element.

If the `zero_rules_flag` is set to `True`, the following heuristic will be executed just after producing the first set of rewriting rules, which are the ones that arise from the Jacobian relations:

Heuristic A.1. As we have seen in Observation 1.8, any path that may be prolonged to a path of arbitrarily high length is zero. Reductions of the form $x \rightsquigarrow 0$ are highly desirable, since the ambiguities they generate are usually simple to solve. Obviously, if $x = 0$, then $yxz = 0$ for all y, z, and so we are interested in finding only the shortest paths that reduce to zero. Therefore, we perform the following steps:

1. Let $n = 2$.

2. Produce a list of all paths of length n.

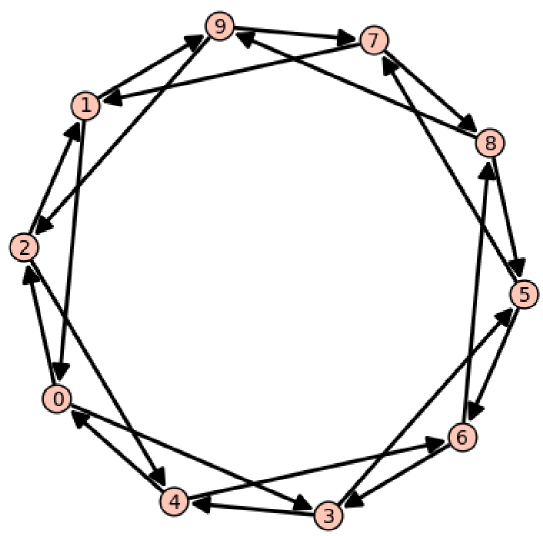
3. Apply rewriting rules to each path until either they are irreducible or they are longer than `zero_depth`.
4. If a path x is equal to a path of length greater than `zero_depth`, add the rewriting rule $x \rightsquigarrow 0$.
5. If no path was found to reduce to zero, increase n by 1 and repeat steps 2 to 5.

We now generate the rewriting system associated to our pyramid:

```
sage: odd_rs = Rewriting_System(odd_pyramid)
9 rules added.
4 rules added.
Total rules: 33
```

As we can see, the program had to enlarge the set of rules twice before achieving confluence. Let us examine the quiver and its associated potential, which both were generated automatically out of the triangulation:

```
sage: odd_rs.quiver.show()
```




```
sage: odd_rs.potential
[[8, 5, 6, 8],
 [3, 4, 6, 3],
 [0, 3, 5, 7, 1, 0],
 [7, 8, 9, 7],
 [1, 9, 2, 1],
 [4, 0, 2, 4],
 [0, 2, 1, 0],
 [0, 3, 4, 0],
 [3, 5, 6, 3],
```

```
 [5, 7, 8, 5],
 [1, 9, 7, 1],
 [2, 4, 6, 8, 9, 2]]
```

As the output shows, there are five 3-cycles coming from the triangular faces, a 5-cycle from the pentagonal face, five 3-cycles from punctures where three faces meet and another 5-cycle from the puncture corresponding to the apex of the pyramid.

The confluent rewriting system found by the software is given by this set of 33 rules:

```
sage: odd_rs.rules
{(0, 2, 1): −(0, 3, 5, 7, 1),
 (0, 2, 4): −(0, 3, 4),
 (1, 0, 2): −(1, 9, 2),
 (1, 0, 3, 4): (1, 9, 2, 4),
 (1, 0, 3, 5, 6): −(1, 9, 2, 4, 6),
 (1, 0, 3, 5, 7, 8): (1, 9, 2, 4, 6, 8),
 (1, 9, 2, 4, 6, 8, 9, 7): (1, 0, 3, 5, 7, 1, 0, 3, 5, 7),
 (1, 9, 7): −(1, 0, 3, 5, 7),
 (2, 1, 0): −(2, 4, 0),
 (2, 1, 9): −(2, 4, 6, 8, 9),
 (3, 4, 0): −(3, 5, 7, 1, 0),
 (3, 4, 6): −(3, 5, 6),
 (4, 0, 2): −(4, 6, 8, 9, 2),
 (4, 0, 3): −(4, 6, 3),
 (5, 6, 3): −(5, 7, 1, 0, 3),
 (5, 6, 8): −(5, 7, 8),
 (5, 7, 1, 0, 3, 4): −(5, 7, 8, 9, 2, 4),
 (6, 3, 4): −(6, 8, 9, 2, 4),
 (6, 3, 5): −(6, 8, 5),
 (6, 8, 9, 2, 4, 0): (6, 8, 9, 7, 1, 0),
 (7, 1, 0, 3, 5, 6): (7, 8, 9, 2, 4, 6),
 (7, 1, 9): −(7, 8, 9),
 (7, 8, 5): −(7, 1, 0, 3, 5),
 (7, 8, 9, 7): (7, 1, 0, 3, 5, 7),
 (8, 5, 6): −(8, 9, 2, 4, 6),
 (8, 5, 7): −(8, 9, 7),
 (8, 9, 2, 4, 6, 3): −(8, 9, 7, 1, 0, 3),
 (9, 2, 1): −(9, 7, 1),
 (9, 2, 4, 0): (9, 7, 1, 0),
 (9, 2, 4, 6, 3): −(9, 7, 1, 0, 3),
 (9, 2, 4, 6, 8, 5): (9, 7, 1, 0, 3, 5),
 (9, 2, 4, 6, 8, 9, 7): −(9, 7, 1, 0, 3, 5, 7),
 (9, 7, 8): −(9, 2, 4, 6, 8)}
```

Rules are stored as a dictionary, where keys are reducible monomials. The value associated to a key is a `Term` object, which consists of a monomial and a sign, representing the term into which the key reduces.

We may now compute the set of all irreducible paths up to a certain length. This is performed by just filtering the list of all paths. For instance, we may produce the list of all irreducible paths of length 4:

```
sage: odd_rs.basis(5)[-1]
[[0, 3, 5, 7, 1],
 [0, 3, 5, 7, 8],
 [1, 0, 3, 5, 7],
 [1, 9, 2, 4, 6],
 [2, 4, 6, 8, 5],
 [2, 4, 6, 8, 9],
 [3, 5, 7, 1, 0],
 [3, 5, 7, 8, 9],
 [4, 6, 8, 9, 2],
 [4, 6, 8, 9, 7],
 [5, 7, 1, 0, 3],
 [5, 7, 8, 9, 2],
 [6, 8, 9, 2, 4],
 [6, 8, 9, 7, 1],
 [7, 1, 0, 3, 5],
 [7, 8, 9, 2, 4],
 [8, 9, 2, 4, 6],
 [8, 9, 7, 1, 0],
 [9, 2, 4, 6, 8],
 [9, 7, 1, 0, 3]]
```

The `generating_function()` method counts the number of irreducible paths up to a specified length:

```
sage: odd_rs.generating_function(15)
[10, 20, 20, 20, 20, 20, 20, 20, 20, 20, 20, 20, 20, 20, 20]
```

Finally, we include a snippet that generates the triangulations corresponding to prisms and antiprisms:

```
def prism(n):
    edges = {}
    for i in xrange(1, n):
        edges[i] = [i+1, i+n]
        edges[i+n] = [i+n+1]
    edges[n] = [1, 2*n]
    edges[2*n] = [n+1]
    return Graph(edges)

def antiprism(n):
    edges = {}
    for i in xrange(1, n):
        edges[i] = [i+1, i+n]
        edges[i+n] = [i+n+1, i+1]
    edges[n] = [1, 2*n]
    edges[2*n] = [1, n+1]
    return Graph(edges)
```

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