UNIVERSIDAD DE BUENOS AIRES
Facultad de Ciencias Exactas y Naturales
Departamento de Matemática

Tesis de Licenciatura

3-Calabi-Yau Algebras from Steiner Systems

Julián Martín Eisenschlos

Director: Mariano Suárez-Alvarez

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Introducción

En este trabajo estudiamos una familia de potenciales de *Calabi-Yau* construidos a partir de objectos combinatorios muy ricos, los sistemas de *Steiner*. Estudiamos algunas de las propiedades homológicas de las álgebras obtenidas y la forma en la que se relacionan con la combinatoria del objecto inicial en consideración.

La construcción es una generalización de la hecha por Mariano Suárez-Alvarez para sistemas triples en [SA13], que a su vez se basa en un álgebra presentada por S. Paul Smith en [SPS11].

Para cada sistema de Steiner S de tipo $(s, s+1, n)$ tenemos una función de un espacio afín de parámetros P hacia álgebras conexas asociativas. La idea central de la construcción es considerar álgebras obtenidas a partir de potenciales, pero solo permitiendo polinomios con monomios de la forma $x_{i_1} \dots x_{i_{s+1}}$ donde $\{i_1, i_2, \dots, i_{s+1}\}$ es un bloque en S. De esa forma, el potencial considerado contiene toda la estructura del sistema. El conjunto de todos los potenciales con estas características es el espacio afín de parámetros al cual nos referimos. Este teorema resume algunos de los resultados que enunciamos y probamos en este trabajo.

Teorema. *Sea S un sistema de Steiner de tipo $(s, s+1, n)$. Para todos los potenciales Φ en un abierto* Zariski *del espacio de parámetros, el álgebra correspondiente $A(S, \Phi)$ satisface:*

(i) *Su serie de Hilbert es*

$$h_A(t) = \frac{1}{1 - nt + nt^s - t^{s+1}}.$$

(ii) *Se trata de un álgebra central, es decir, los únicos elementos en el centro son los escalares del cuerpo de base.*

(iii) *Es un álgebra* 3-Calabi-Yau, Gorenstein *y* Koszul *generalizado.*

Esto resume el contenido de los teoremas 2.3, 4.1 y 5.2 que probamos en las siguientes secciones. Podemos ilustrar este resultado en el diagrama:

$$\begin{tikzcd}
\left\{\begin{matrix}\text{Sistemas de \textit{Steiner}} \\ \text{de tipo } (s,s+1,n)\end{matrix} \times P\right\} \arrow[r, "A(-)"] & \left\{\begin{matrix}\text{Álgebras asociativas} \\ \text{graduadas conexas}\end{matrix}\right\} \\
\left\{\begin{matrix}\text{Sistemas de \textit{Steiner}} \\ \text{de tipo } (s,s+1,n)\end{matrix} \times U\right\} \arrow[u, hook] \arrow[r, mapsto] & \left\{\begin{matrix}\text{Álgebras 3-\textit{CY},} \\ \text{\textit{Gorenstein} y \textit{s-Koszul}}\end{matrix}\right\} \arrow[u, hook]
\end{tikzcd}$$

Este trabajo está organizado de la siguiente manera. Comenzamos introduciendo los conceptos preliminares principales que son usados en la tesis. En la sección 2 presentamos la construcción del álgebra $A(S, \Phi)$ en detalle y en las secciones subsiguientes continuamos estudiando algunas propiedades homológicas de las álgebras obtenidas. En la sección 9 presentamos algunos ejemplos diversos. En el apéndice A recordamos algunas propiedades combinatorias de los sistemas de Steiner que necesitamos y en el apéndice B incluimos un programa en *Python* para calcular algunos de los objetos que aparecen a los largo del trabajo.

Creemos que aún queda mucho para profundizar en el trabajo. Entre otras cosas, de la misma forma que sucede para sistemas triples, nos preguntamos si las álgebras obtenidas son coherentes y si son dominios en el caso general. También nos gustaría mejorar los algoritmos para encontrar más ejemplos de sistemas donde existan elecciones de coeficientes con muchas derivaciones. Mas allá de eso, continuar el estudio de las propiedades homológicas con el objetivo de encontrar invariantes de los sistemas que se puedan recuperar en el álgebra y por último, idealmente, buscar formulaciones algebraicas de los resultados negativos de existencia que se conocen para sistemas de ciertos tipos.

Introduction

In this work we study a family of *Calabi-Yau* potentials constructed from a very rich combinatorial object, Steiner systems. We study some homological properties of the algebras obtained and the way in which they relate to the combinatorics of the initial object in consideration.

The construction is a generalization of the one made by Mariano Suárez-Alvarez for triple systems in [SA13], which is in turn is a generalization of an algebra introduced by S. Paul Smith in [SPS11].

For each Steiner system S of type $(s, s+1, n)$ we have a map from an affine space P of parameters to graded connected associative algebras. The core idea of the construction is to consider algebras obtained from potentials, but only allowing polynomials with monomials of the form $x_{i_1} \dots x_{i_{s+1}}$ where $\{i_1, i_2, \dots, i_{s+1}\}$ is a block in S. In that way, the potential in question encodes the whole structure of the Steiner system. The set of all such potential is the affine space of parameters. The following theorem summarizes some of the results we state and prove in this work.

Theorem. *Let S be a Steiner system of type $(s, s+1, n)$. For all potentials Φ in a* Zariski *open set of the space of parameters, the corresponding algebra $A(S, \Phi)$ has the following properties:*

(i) *Its Hilbert series is*

$$h_A(t) = \frac{1}{1 - nt + nt^s - t^{s+1}}.$$

(ii) *It is a central algebra, that is, the only elements in the center are the scalars in the ground field.*

(iii) *It is a* 3-Calabi-Yau, Gorenstein *and generalized* Koszul *algebra.*

This summarizes the content of propositions 2.3, 4.1 and 5.2 that are proven in the thesis. We can illustrate this last result in the diagram:

$$\begin{tikzcd}
\left\{ \begin{matrix} \textit{Steiner}\text{ systems} \\ \text{of type } (s,s+1,n) \end{matrix} \times P \right\} \arrow[r, "A(-)"] & \left\{ \begin{matrix} \text{Graded connected} \\ \text{associative algebras} \end{matrix} \right\} \\
\left\{ \begin{matrix} \textit{Steiner}\text{ systems} \\ \text{of type } (s,s+1,n) \end{matrix} \times U \right\} \arrow[u, hook] \arrow[r, mapsto] & \left\{ \begin{matrix} \text{3-}\textit{CY}, \textit{Gorenstein} \\ s\textit{-Koszul}\text{ algebras} \end{matrix} \right\} \arrow[u, hook]
\end{tikzcd}$$

This work is organized in the following way. We begin by introducing the main concepts that are visited through this work. In section 2 we present the construction of the algebra $A(S, \Phi)$ in detail, and continue to study some homological properties in the subsequent sections. Finally, we present some diverse examples from section 9 until the end of the document. In appendix A we recall some combinatorial properties of Steiner systems that we need, and in appendix B we also include a *Python* program to compute some of the objects that arise in the construction.

We believe there is still much work to do in this area. Among other things, as it happens with triple systems, we wonder if the algebras we obtain are coherent, and if they are domains in the general case. We would also like to improve the algorithms we have, to find more examples of systems where we can choose coefficients so that we have lots of derivations. We want to continue the study of the homological properties, in order to find invariants of systems that we can recover in the algebra and finally, ideally, search for algebraic formulations of negative existence results that are known for systems of certain types.

Contents

1 Preliminaries

1.1 Steiner Systems

A (t, k, n)-*Steiner System* is a pair (E, S) in which E is a non-empty finite set of *points*, and S a set of k-subsets of E, the *blocks*, such that every t-subset of E is contained in exactly one block. The number $n = |E|$ is the *order* of the system; we will assume throughout that $n \geq t$. With a bit of abuse of language, we will refer to S as the Steiner system.

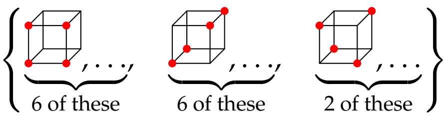
Special cases among these are $(2, 3, n)$ and $(3, 4, n)$ which are called Steiner triple and Steiner quadruple systems respectively. Those are the main source of our examples.

As Suárez-Alvarez [SA13] focuses on triple systems, our purpose is to generalize his construction to bigger systems, with the constraint that $t + 1 = k$: thus we are going to focus in the case $(s, s + 1, n)$ with $2 \leq s \leq n - 1$ and from now on (E, S) will be a Steiner system of that type. In that case we can define a unique operation $\star : \binom{E}{s} \to E$ such that for all $\{x_1, x_2, \dots x_s\} \in \binom{E}{s}$, we have:

$$\{x_1, x_2, \dots x_s, \star(x_1, x_2, \dots x_s)\} \in S.$$

1.1. *Example* (The unique cuadruple system of order 8). Let $E = \mathbb{F}_2{}^3$ which we can identify with the unit cube and

$$S = \left\{ \{v_i\}_{1 \leq i \leq 4} : \sum_{i=1}^{4} v_i = 0 \right\} =$$



This is an example of a cuadruple system. Let us check that it satisfies the condition. Let $v_1, v_2, v_3 \in E$ be three distinct points in E. There is a unique v_4, namely $v_1 + v_2 + v_3$, such that $\sum_{i=1}^{4} v_i = 0$, but it could coincide with one of the other three. Yet if, for example $v_4 = v_3$, then we have $v_1 + v_2 = 0$, which contradicts our original assumption.

1.2. *Example* (W_{12}: The unique $(5, 6, 12)$-system). Let $E = \mathbb{P}^2(\mathbb{F}_{11}) \cong \{0, \dots, 10, \infty\}$ and $B = \{0, 1, 3, 4, 5, 9\}$ the set of squares in $\mathbb{F}_{11}$. The group

$$G = \mathsf{PGL}_2(\mathbb{F}_{11}) = \left\{ z \mapsto \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{F}_{11},\ ad - bc \neq 0 \right\}$$

acts on E and we let S be the orbit of B under G. It can be proved by inspection that (E, S) is a Steiner system with automorphism group the *Mathieu* group M_{12}.

Very little is known about the existence of Steiner systems of a given type. In our case of interest $(s, s + 1, n)$ it is known that a necessary condition is $n \equiv s \pm 1 \mod 6$. In appendix A we discuss this and some other necessary conditions.

1.3. Theorem. [CD06] *For triple and cuadruple systems this condition is also sufficient.* $\square$

In the case of triple systems, this is a celebrated theorem of Reverend Thomas Kirkman [Kir47]. There are 1, 1, 2, 80, and 11 084 874 829 non-isomorphic Steiner triple systems of order 7, 9, 13, 15 and 19, respectively; and 1, 1, 4, and 1 054 163 non-isomorphic Steiner cuadruple systems of order 8, 10, 14 and 16; see sequences A030129 and A124119 in Sloane's database [Slo08] and the references therein.

However, as of 2013, it is an open problem in the field of Combinatorics to find sufficient conditions when $s \geq 4$ and, moreover, it is not known whether any system exists if $s \geq 6$. This is why we believe it may be useful to find connections between systems and algebraic objects which we know in depth.

1.2 Algebras derived from potentials

Throughout this work we construct a family of Calabi-Yau algebras. This sort of algebras arises naturally in the geometry of Calabi-Yau manifolds and it appears when one tries to do noncommutative geometry.

As shown by Ginzburg [Gin06], numerous concrete examples of Calabi-Yau algebras are found "in nature", and in most cases they arise as a certain quotient of the free associative algebra in the way described below. We refer the reader to Bocklandt's work in [Boc08] for a deeper approach on the subject.

Fix $\Bbbk$ a ground field of characteristic zero; all tensor products are over $\Bbbk$ unless we say otherwise. Let V be a finite dimensional vector space and let $X = \{x_1, x_2, \dots, x_n\}$ be a basis of V. We denote by $T(V)$ the tensor unital algebra of V. The basis of $T(V)$ consisting in non-commutative monomials in $x_1, \dots, x_n$ is denoted $\langle X \rangle$. The set $\langle X \rangle$ is a semigroup and we can see $T(V)$ as its semigroup algebra.

We have a $\mathbb{Z}^n$-grading in $\Bbbk\langle X \rangle$ such that x_i is in degree e_i, the ith canonical basis vector. A second, useful grading is the $\mathbb{Z}$-grading, which is such that x_i is in degree 1 for all $1 \leq i \leq n$. It is also known as the *length* grading. Unlike the first one, it does not depend on the choice of a basis and, unless said otherwise, when we talk of homogeneous elements, we will be referring to the length grading. If $w \in \langle X \rangle$ has length d, we will write $|w| = d$.

A polynomial is called *multi-linear* if it is homogeneous of degree $(1, \dots, 1) = \sum_{i=1}^{n} e_i$. Similarly we call a polynomial *sub-linear* if no monomials with repeated variables appear in it or, equivalently, if it lies in

$$\bigoplus_{0 \leq d_i \leq 1\ \forall i} \Bbbk\langle X \rangle_{(d_1, \dots, d_n)}.$$

Definition. *For each $x \in X$ the* cyclic derivative *with respect to x is the vector space morphism $\partial_x : \Bbbk\langle X \rangle \to \Bbbk\langle X \rangle$ such that if $\phi = x_1 x_2 \dots x_k$ is monomial,*

$$\partial_x \phi = \sum_{\substack{1 \leq i \leq k \\ x_i = x}} x_{i+1} \dots x_k x_1 \dots x_{i-1} = \sum_{\phi = uxv} vu.$$

While the cyclic derivative $\partial_x \Phi$ depends heavily on the choice of a basis and not only on the element $x \in V$, the space generated by all the cyclic derivatives does not. Indeed, we have the following lemma.

Lemma. *If $v \in T(V)$, and $\{x_i\}_{1\leq i\leq n}$ and $\{y_j\}_{1\leq j\leq n}$ are bases of V, then*

$$\langle \partial_{x_i} v\rangle_{1\leq i\leq n} = \langle \partial_{y_j} v\rangle_{1\leq j\leq n}$$

Proof. Assume for now $v = v_1 v_2 \dots v_k \dots v_n \in T(V)$ is a monomial. Consider $\{\phi_i\}$ and $\{\psi_j\}$, the dual bases of $\{x_i\}$ and $\{y_j\}$ respectively, and the matrix $g = (g_{i,j})$ such that $\psi_j = \sum_{i=1}^{n} g_{i,j}\phi_i$. With that notation we have

$$\partial_{x_i} v = \sum_{\substack{u\in V \\ w_1 u w_2 = v}} \phi_i(u) w_2 w_1,$$

which allows us to write

$$\partial_{y_j} v = \sum_{\substack{u\in V \\ w_1 u w_2 = v}} \psi_j(u) w_2 w_1 = \sum_{\substack{u\in V \\ w_1 u w_2 = v}} \sum_{i=1}^{n} g_{i,j}\phi_i(u) w_2 w_1 = \sum_{i=1}^{n} g_{i,j}\partial_{x_i} v.$$

This extends linearly to every element $v \in T(V)$ and proves the lemma. □

Let C_d be the cyclic group of order d, and let σ be a generator. We consider the action of C_d on $V^{\otimes d}$ such that $\sigma \cdot (x_1 x_2 \dots x_d) = (x_2 \dots x_d x_1)$. It is easy to see that $\partial_x \circ \sigma = \partial_x$ for each $x \in X$. If follows immediately from this that $\partial_x \Phi = 0$ if $\Phi \in [V, V^{\otimes(d-1)}]$, and therefore ∂_x induces a morphism

$$\left(\frac{T(V)}{[T(V), T(V)]}\right)_d = \frac{V^{\otimes d}}{[V, V^{\otimes(d-1)}]} \to T(V).$$

All this maps taken together give a map

$$\partial_x : \frac{T(V)}{[T(V), T(V)]} \to T(V).$$

Two homogeneous polynomials that lie in the same orbit of the action of C_d are called *conjugate*. Since $\Bbbk$ has characteristic zero, among the possible representatives for a d-homogeneous polynomial we can consider one which is fixed under the action of C_d. It can be obtained by means of the linear map $c : V^{\otimes d} \to V^{\otimes d}$ such that

$$c(w) = \frac{1}{d}\sum_{uv=w} vu = \frac{1}{d}\sum_{1\leq i\leq n} \sigma^i w$$

for each monomial w of degree d. Elements which are invariant under the action of C_d are called *cyclic*.

1.4. Lemma (Non-commutative Euler relation). *Let $\Phi \in \Bbbk\langle X\rangle$ be a homogeneous cyclic polynomial of degree d. Then*

$$\sum_{x\in X} x \cdot \partial_x \Phi = d\Phi = \sum_{x\in X} \partial_x \Phi \cdot x. \tag{1}$$

Proof. We may assume by linearity that $\Phi = c(v)$ for $v \in \langle X \rangle$ a monomial of length d. Let $e \geq 1$ and $u \in \langle X \rangle$ be such that $v = u^e$ and e is maximal. All conjugates of v appear exactly e times when considering all d possible rotations, and the coefficient of every monomial in $d\Phi$ is e. Every conjugate w of v starting (respectively, ending) with x also appears with coefficient e on the left (right) hand side, recalling that $\partial_x c(v) = \partial_x v$. Finally, every monomial in Φ starts (ends) with one and only one element in X. All this implies that the identity holds. □

The converse is also true, in the following sense.

1.5. Lemma. *If $(r_x)_{x \in X}$ are homogeneous polynomials of degree d such that*

$$\sum_{x \in X} [r_x, x] = 0$$

then $\Phi = \sum_{x \in X} x r_x$ is a cyclic polynomial, and $\partial_x \Phi = d r_x$ for all $x \in X$.

This can be viewed as a non-commutative version of the Poincare lemma.

Proof. Consider the unique linear map $\alpha : V \otimes \Bbbk\langle X \rangle \to \Bbbk\langle X \rangle$ such that

$$\alpha(x \otimes p) = [x, p], \text{ for all } x \in X.$$

Fix $d > 0$. At degree d the domain and codomain of α have the same dimension, so $\ker \alpha_d$ and $\operatorname{coker} \alpha_d$ must also have the same dimension. The dimension of $\operatorname{coker} \alpha_d$ corresponds to the dimension of the space of cyclic polynomials of degree d, because dividing out by commutators is the same as identifying conjugate monomials. But the Euler relation (1) implies that for each cyclic polynomial Φ of degree $d+1$, then $\sum_{x \in X} x \otimes \partial_x \Phi$ belongs to $\ker \alpha_d$. Note that two different cyclic polynomials yield different elements in the kernel, because the Euler relation allows us to reconstruct the polynomial from its cyclic derivatives.

As a consequence, we have one element in the kernel for every cyclic polynomial in degree $d+1$, but as dimensions match, those elements generate the whole kernel. This is exactly the statement of the lemma. □

Definition. *If $\Phi \in T(V)$ we put*

$$A_\Phi = \frac{T(V)}{\langle \partial_{x_i} \Phi \rangle_{1 \leq i \leq n}}$$

and we refer to Φ as a potential *of A_Φ and say that the algebra A_Φ is derived from Φ.*

As mentioned above, in the proper sense, "most" Calabi-Yau algebras of dimension 3 arise in this way from potentials. However, it is very difficult to tell *a priori* whether a potential Φ will generate such an algebra; when this is the case we say the potential is Calabi-Yau. The problem of determining if a potential is Calabi-Yau has been addressed in depth by Berger and Solotar in [BS13].

1.3 Monomial orders

We denoted $\langle X \rangle$ the set of monomials that generates $T(V)$ as a vector space. We can talk of monomials as *words* in the alphabet X, left and right divisors as *prefixes* and *suffixes* respectively. If a word u appears with non-zero coefficient in a polynomial $p \in \Bbbk\langle X \rangle$ we will write $u \in p$, or simply say that u *appears* in p.

Definition. *A* monomial order *is a total order on* $\langle X \rangle$ *satisfying:*

(i) *The multiplicative property:*

$$\text{For all } a, b, c \text{ and } d \in \langle X \rangle \text{ such that } a \succ b \text{ then } cad \succ cbd.$$

(ii) *The decreasing chain condition:*

$$\text{No infinite decreasing sequence } w_1 \succ w_2 \succ \dots \text{ exists.}$$

If $p \in \Bbbk\langle X \rangle$ we write $\mathsf{tip}(p)$ the maximum among the monomials that appear in p. If p and $q \in \Bbbk\langle X \rangle$ then $\mathsf{tip}(pq) = \mathsf{tip}(p)\mathsf{tip}(q)$; this is consequence of the multiplicative and transitive properties.

We use monomial orders in this work in order to find explicit bases of the algebras under study via the reduction systems in Bergman's Diamond Lemma [Ber78]. In fact, only one such order will be considered: the *length-lexicographic* order. The way to construct it is to fix a total order on the set of letters X, and then for any given two words in $v, w \in \langle X \rangle$ we put $v \prec w$ if

- $|v| < |w|$ or,
- $|v| = |w|$ and v is smaller in the lexicographical order obtained from the total order on X, that is, the first different letter (from left to right) is bigger in w.

The fact that $\prec$ is multiplicative can be seen by observing that when comparing two words, common prefixes (or suffixes) can be canceled out. The chain condition holds since the length grading in $\langle X \rangle$ is locally finite.

1.4 Koszulity

Our general reference in the field of homological algebra and the theory of derived functors is Weibel's book [Wei94].

Let A be a connected non-negatively graded algebra. Let $A_+ \subseteq A$ be the ideal generated by homogeneous elements of positive degree, and let us identify $\Bbbk$ with the quotient A/A_+. A graded left A-module is *bounded below* if for some n_0, $M_n = 0$ for all $n < n_0$. We write Mod_A^+ the full subcategory of the category of graded modules with degree preserving homomorphisms spanned by bounded below modules. This is an abelian category. We begin by going over some widely known facts about the category Mod_A^+, which can be traced back to Cartan seminar in [Car58]

1.6. Lemma (Nakayama). *If M is a module in Mod_A^+ and $A_+M = M$ or, equivalently, if $\Bbbk \otimes_A M = 0$, then $M = 0$.*

Proof. To reach a contradiction, let $n_0 = \min\{n \in \mathbb{Z} : M_n \neq 0\}$ and let $m \in M_{n_0} \setminus \{0\}$. Then $m \in A_+M = \sum_{n\in\mathbb{N}} A_n M_{n_0-n} = 0$ and this is absurd. $\square$

1.7. Lemma. *If M is a module in Mod_A^+ and $j : M/A_+M \to M$ is a homogeneous degree preserving $\Bbbk$-linear retraction of the quotient, then the map $\bar{\jmath} : A \otimes (M/A_+M) \to M$ such that $\bar{\jmath}(a \otimes m) = aj(m)$ is surjective.*

Proof. The map $\bar{\jmath}$ is surjective in sufficiently low degrees because M is bounded below. Let $d \in \mathbb{Z}$ and assume that $\bar{\jmath}_k$ is surjective for each $k < d$. Let us consider the following diagram of vector spaces:

$$\begin{tikzcd}
 & & (A \otimes (M/A_+M))_d \arrow[d, "\bar{\jmath}_d"] \arrow[dr, dotted] & & \\
0 \arrow[r] & (A_+M)_d \arrow[r, hook] & M_d \arrow[r, two heads] & (M/A_+M)_d \arrow[r] & 0
\end{tikzcd}$$

Any element in $(A_+M)_d$ is in the image of $\bar{\jmath}_d$ by the inductive hypothesis. Then $\bar{\jmath}_d$ is surjective if the dotted arrow is surjective, and it is by construction. $\square$

The shift $M(l)$ of a module M has the same module structure but the grading is such that $M(l)_n = M(l+n)$. A module M is said to be *graded free* if M has a basis of homogeneous elements or, equivalently, if M is isomorphic to a direct sum of shifts $A(l_i)$ where the l_i are bounded below. With this setting we can state the first fact.

Proposition. *A module M is projective in Mod_A^+ if and only if M is graded free.*

Proof. If M is graded free the fact that it is projective can be easily checked, just as in the non-graded setting. Let us prove the converse.

Let P be a projective object in the category. Let $j : P/A_+P \to P$ be a $\Bbbk$-linear retraction of the quotient, and let $\bar{\jmath}$ be the epimorphism constructed in the previous lemma. As P is projective it is then isomorphic to a direct summand of $A \otimes (P/A_+P)$. Let Q be a graded A-module such that

$$P \oplus Q \cong A \otimes (P/A_+P). \tag{2}$$

To prove that $Q = 0$, in view of Nakayama lemma, it suffices to see that $\Bbbk \otimes_A Q = 0$. Applying the functor $\Bbbk \otimes_A -$ to equation (2) and observing that the first projection $\Bbbk \otimes_A \bar{\jmath}$ is the identity we obtain the desired result. $\square$

A surjective morphism $f : N \to M$ is called *essential* if, for any morphism $g : X \to N$ with $f \circ g$ surjective, g is surjective. An essential surjective morphism $f : N \to M$ with N projective is called a projective cover of M in the graded category.

1.8. Lemma. *A morphism $f : N \to M$ is surjective (respectively, essential) if and only if the induced map $\hat{f} : \Bbbk \otimes_A N \to \Bbbk \otimes_A M$ is surjective (bijective).*

Proof. If f is surjective, then $\hat{f}$ is surjective since $\Bbbk \otimes_A (-)$ is right exact. The converse follows from the Nakayama lemma applied to $f(N)$, that satisfies $f(N) + A_+M = M$.

If f is essential it is surjective and, because of the first part, we can choose a homogeneous $\Bbbk$-linear retraction $i : \Bbbk \otimes_A M \to \Bbbk \otimes_A N$. As $A_+\Bbbk = 0$, i is actually A-linear and $i(\Bbbk \otimes_A M) \subseteq \Bbbk \otimes_A N$ is a submodule. If we denote $\pi : N \to \Bbbk \otimes_A N$ the natural projection, then $R = \pi^{-1}\left(i\left(\Bbbk \otimes_A M\right)\right)$ is a submodule of N that satisfies $f(R) + A_+M = M$. Again by the Nakayama lemma $f(R) = M$ and, since f is essential, we have $R = N$, which implies i is surjective and $\hat{f}$ is an isomorphism, as desired.

$$\begin{tikzcd} R \arrow[r, hook] & N \arrow[r, "f"] \arrow[d, "\pi"] & M \arrow[d, "\tau"] \\ & \Bbbk \otimes_A N \arrow[r, "\hat{f}"] & \Bbbk \otimes_A M \arrow[l, dotted, bend left, "i"] \end{tikzcd} \tag{3}$$

To prove the converse first we observe that the projection π is essential, once again applying the Nakayama lemma. Standard categoric diagram chasing proves that composition of essential morphisms is again essential and that whenever $p \circ q$ and p are essential, then q is essential as well. We can apply these facts to the commutative diagram of A-modules (3). As $\hat{f}$ is an isomorphism by hypothesis, $\hat{f} \circ \pi = \tau \circ f$ is essential and, since τ is also essential, the lemma is proved. □

A corollary of this lemma is that $\Bbbk \otimes_A (-)$ reflects exact sequences, indeed:

1.9. Corollary. *The complex of left A-modules in Mod_A^+*

$$P \xrightarrow{g} M \xrightarrow{f} N$$

is exact if

$$\Bbbk \otimes_A P \xrightarrow{\hat{g}} \Bbbk \otimes_A M \xrightarrow{\hat{f}} \Bbbk \otimes_A N$$

is exact.

Proof. By hypothesis, we know $\hat{g}$ co-restricted to $\ker \hat{f}$ is surjective. Also it is clear that $\mathrm{im}(\hat{g}) = \Bbbk \otimes_A \mathrm{im}(g)$ and $\Bbbk \otimes_A \ker f \subseteq \ker(\hat{f})$. Therefore $\mathrm{im}\,\hat{g} \subseteq \Bbbk \otimes_A \ker f$ and the corresponding co-rectriction is surjective as well. Finally lemma 1.8 implies that $\ker f \subseteq \mathrm{im}\, g$ and we are done. □

The second fact we need is:

Proposition. *Any module M has a projective cover in Mod_A^+, unique up to a non-unique isomorphism.*

Proof. According to lemma 1.7, there is a surjective morphism $\bar{\jmath} : A \otimes (M/A_+M) \to M$. Using lemma 1.8, $\bar{\jmath}$ is essential because $\Bbbk \otimes \bar{\jmath}$ is the identity. Since the $A \otimes (M/A_+M)$ is graded free, it is a projective cover of M.

For the uniqueness part, if $f : P \to M$ and $g : Q \to M$ are projective covers, then as P is projective and g surjective, there exists a map $\bar{f} : P \to Q$ such that $g \circ \bar{f} = f$. Then $\bar{f}$ is also a projective cover. By symmetry, we can construct a projective cover $\bar{g} : Q \to P$. Hence $q = \bar{f} \circ \bar{g}$ and $p = \bar{g} \circ \bar{f}$ are projective covers of Q and P respectively.

The fact that $\mathrm{id}_{\Bbbk} \otimes_A q$ is a bijection and identifying Q with $A \otimes (\Bbbk \otimes_A Q)$ we find that q is also a bijection; the same applies to p. This implies $\bar{f}$ and $\bar{g}$ are isomorphisms. $\square$

An immediate consequence of the last result is that any M has a projective resolution in the graded category.

$$P_\bullet \xrightarrow{d_\bullet} M \twoheadrightarrow 0$$

which is minimal, that is, each surjective morphism $d_i : P_i \to \mathrm{im}(d_i)$ induced by d_i is essential. Every two minimal projective resolutions of M are isomorphic, but the isomorphism is non-unique, and any projective resolution of M contains a minimal one as direct summand.

1.10. Definition. *A module M such that there exists l satisfying $M = M_l$ is said to be* concentrated *in degree l. Similarly, a module M such that there exists l satisfying $M = A \cdot M_l$ is said to be* pure *in degree l.*

In both cases, l is uniquely determined if $M \neq 0$. Concentrated in degree l implies pure in degree l. Any module concentrated in degree l is isomorphic to a direct sum of shifts $\Bbbk(-l)$. Any projective module M pure in degree l is isomorphic to a direct sum of shifts $A(-l)$ and is isomorphic to $A \otimes M_l$, where M_l is considered as a module concentrated in degree l. Note that the objects which are simple (respectively, indecomposable projective) in the graded category are exactly the modules $\Bbbk(-l)$ (respectively, $A(-l)$, $l \in \mathbb{Z}$. The next result is a criterion of essentiality in the pure situation.

1.11. Proposition. *Let $f : M \to M'$ a surjective morphism in the graded category. Assume that M is pure in degree l. Then M' is pure in degree l. Moreover, f is essential if and only if the linear map $f_l : M_l \to M'_l$ induced by f is bijective.*

Proof. The first assertion is clear. Let us prove the second assertion. As stated in lemma 1.8, f is essential if and only if the linear map $\hat{f} : \Bbbk \otimes_A M \to \Bbbk \otimes_A M'$ naturally defined by f is bijective. But purity implies that $\Bbbk \otimes_A M$ and $\Bbbk \otimes_A M'$ are canonically identified to M_l and M'_l respectively, so that $\hat{f}$ becomes f_l in this identification. $\square$

Priddy introduced *Koszul* complexes in [Pri70] for quadratic augmented algebras which turn out to be projective resolutions for a large class of such algebras. If the Koszul complex is in fact exact, the algebra is said to be Koszul. Berger generalized this concept in [Ber01] to homogeneous augmented algebras generated in degree one, not necesarilly quadratic, as algebras having a pure resolution of the ground field, the same thing that happens in the quadratic case. We need not say that having a simple and small projective resolution comes in handy for the computation of the Hochschild homology and cohomology of the algebra.

Let V be a finite dimensional vector space, $s \geq 2$ and $R \subseteq V^{\otimes s}$, a space which we call the space of relations. Then $A = T(V)/(R)$ is an augmented homogeneous $\Bbbk$-algebra. We say that A is an s-homogeneous algebra. We write $I = (R)$, the two sided ideal in $T(V)$ generated by R, which is graded by the subspaces I_n given by $I_n = 0$ if $0 \leq n \leq s-1$ and

$$I_n = \sum_{i+j+s=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}, \quad n \geq s.$$

The algebra A is graded by the subspaces $A_n = V^{\otimes n}/I_n$; clearly $A_n = V^{\otimes n}$ for $n < s$. We define vector spaces J_n concentrated in degree n by $J_n = V^{\otimes n}$ if $0 \leq n \leq s-1$ and

$$J_n = \bigcap_{i+j+s=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}, \quad n \geq s.$$

Now we want to know if it is possible to construct a pure resolution of $\Bbbk$, the trivial left A-module. The natural projection $\varepsilon : A \to \Bbbk$ is a projective cover of $\Bbbk$ and its kernel is pure in degree 1 with $(\ker \varepsilon)_1 = V$. A projective cover of $\ker \varepsilon$ is the morphism $A \otimes V \to \ker \varepsilon$ induced by the inclusion of V in $\ker \varepsilon$. Including $\ker \varepsilon$ in A, we get $\delta_1 : A \otimes V \to A$ defined by $\bar{a} \otimes v \mapsto \bar{a}\bar{v}$, where $\bar{a}$ denotes the class of an element a of $T(V)$ in A and $v \in V$. Clearly $\ker \delta_1$ vanishes in degree $< s$ and is exactly R in degree s. For $n \geq s$, the equality

$$(\ker \delta_1)_n = \frac{I_{n-1} \otimes V + V^{\otimes n-s} \otimes R}{I_{n-1} \otimes V}$$

shows that $\ker \delta_1$ is pure in degree s. A projective cover of $\ker \delta_1$ is $A \otimes R \to \ker \delta_1$ induced by the inclusion of R in $\ker \delta_1$. From the inclusion $\ker \delta_1 \subseteq A \otimes V$, we get $\delta_2 : A \otimes R \to A \otimes V$ defined by the restriction of the linear map $\bar{a} \otimes v \otimes w \mapsto \bar{a}\bar{v} \otimes w$, where $v \in V^{\otimes(s-1)}$ and $w \in V$.

We can thus begin a pure resolution

$$A \otimes J_s \xrightarrow{\delta_2} A \otimes J_1 \xrightarrow{\delta_1} A \otimes J_0 \overset{\varepsilon}{\twoheadrightarrow} \Bbbk \tag{4}$$

We consider the function $n : \mathbb{N} \to \mathbb{N}$ such that $n(2j) = js$ and $n(2j+1) = js+1$ for $j \geq 2$, and we define $K_i = A \otimes J_{n(i)}$ for each $i \geq 0$. There is a pure complex $(K_i)_{i \geq 0}$ with differentials δ_i induced by the inclusions

$$J_{n(i+1)} \subseteq V^{\otimes n(i+1)-n(i)} \otimes J_{n(i)} \subseteq A \otimes J_{n(i)} = K_i.$$

It is indeed a complex: $\delta^2(1_A \otimes J_{n(i+2)})$ vanishes since $J_{n(i+2)} \subseteq R \otimes J_{n(i)}$. This complex is called the (left) *Koszul* complex of the trivial left module ${}_A\Bbbk$.

1.12. Definition. *An s-homogeneous algebra A is said to be (generalized)* Koszul *if the graded vector space* $\mathrm{Tor}_i^A(\mathbb{k}, \mathbb{k})$ *is pure in degree* $n(i)$ *for all* $i \geq 3$.

1.13. Theorem. [Ber01] *Let* $A = T(V)/I$ *be an s-homogeneous algebra on V, with R as space of relations. Then A is Koszul if and only if the Koszul complex* $(K_\bullet, \delta)$ *is exact.*

Observe that if $P^\bullet$ is a minimal complex of left bounded A-modules, then the differentials of $\mathbb{k} \otimes_A P^\bullet$ vanish. Hence $P^\bullet$ is pure if and only if $H^\bullet(P)$ is pure. This is because whenever applying the functor $\mathbb{k} \otimes_A (-)$ to an exact sequence

$$0 \longrightarrow P \overset{i}{\hookrightarrow} M \overset{\pi}{\twoheadrightarrow} N \longrightarrow 0$$

we have $\mathrm{im}(\mathrm{id}_\mathbb{k} \otimes i) \subseteq \ker(\mathrm{id}_\mathbb{k} \otimes \pi)$, so $\mathrm{id}_\mathbb{k} \otimes i = 0$ if π is essential. All differentials of $P^\bullet$ factor through an inclusion such as the one described above.

Proof. If the Koszul complex is exact, it is a resolution of the ground field $\mathbb{k}$, and applying the functor $\mathbb{k} \otimes_A (-)$ we obtain a graded complex $\mathbb{k} \otimes_A K_\bullet$. Because K_i is generated in degree $n(i)$, the complex $\mathbb{k} \otimes_A K_\bullet$ is concentrated in degree $n(i)$, and the homology of that complex also has to be concentrated in the same degrees. This means that A is Koszul.

Conversely, assume $\mathrm{Tor}_i^A(\mathbb{k}, \mathbb{k})$ is pure in degree $n(i)$ for any $i \geq 3$. We can begin a pure resolution of $\mathbb{k}$ as in (4) and if $1 \leq m \leq s-1$, we have

$$(\ker \delta_2)_{s+m} = (V^{\otimes m} \otimes R) \cap (R \otimes V^{\otimes m} + \cdots + V^{\otimes m-1} \otimes R \otimes V)$$

which obviously contains $V^{\otimes(s-1)} \otimes J_{s+1}$. As $\mathrm{Tor}_3^A(\mathbb{k}, \mathbb{k})$ is pure in degree $s+1 = n(3)$, as observed above, we necessarily have that $\ker \delta_2$ is generated by its homogeneous component $(\ker \delta_2)_{s+1} = J_{s+1}$. One particular consequence of this is that for $2 \leq m \leq s-1$,

$$(V^{\otimes m} \otimes R) \cap (R \otimes V^{\otimes m}) \subseteq V^{\otimes m-1} \otimes R \otimes V. \tag{5}$$

We will proceed by induction to prove the exactness of the Koszul complex $(K_\bullet, \delta)$. Assume that it is exact for all degrees $1 \leq j < i$ for some $i \geq 2$. We know that $\mathbb{k} \otimes_A \ker \delta_i \cong \mathrm{Tor}_{i+1}^A(\mathbb{k}, \mathbb{k})$, and by assumption this is pure in degree $n(i+1)$: it follows that $\ker \delta_i$ is generated by

$$(\ker \delta_i)_{n(i+1)} = R \otimes V^{\otimes n(i-1)} \cap V^{\otimes n(i+1)-n(i)} \otimes J_{n(i)}. \tag{6}$$

When i is even, $n(i+1) - n(i) = 1$ and the intersection (6) equals $J_{n(i+1)}$; however, when i is odd, this is not so clear. Using equation (5), one can prove inductively that for all $2 \leq m \leq s-1$

$$(\ker \delta_i)_{n(i+1)} \subseteq R \otimes V^{\otimes n(i-1)} \cap V^{\otimes m} \otimes R \otimes V^{\otimes n(i-1)-m} \subseteq V^{\otimes m-1} \otimes R \otimes V^{\otimes n(i-1)+1-m}$$

and we conclude that $(\ker \delta_i)_{n(i+1)} = J_{n(i+1)}$, the same as in the even case. Again by the hypothesis on the Tor group, $\ker \delta_i = A \cdot (\ker \delta_i)_{n(i+1)} = A \cdot J_{n(i+1)}$, and this coincides with $\mathrm{im}\, \delta_{i+1}$: this means that the Koszul complex is exact in degree i, completing the induction, and therefore we are done. □

We have talked long about resolutions of the ground field but, in order to compute the Hochschild homology and cohomology, we need a bimodule projective resolution of A. We will devote the last part of this section to the construction of such a resolution when A is Koszul.

We will now consider the category of left-bounded graded A-A-bimodules, naturally identified, as usual, with the category of left-bounded graded left $A^e = A \otimes A^{\mathrm{op}}$ modules. The definition of pure, concentrated and graded-free modules are the ones inherited from this identification. The graded Nakayama lemma for this category states:

1.14. Lemma. *If M is bimodule and $\Bbbk \otimes_A M \otimes_A \Bbbk$ vanishes, then so does M.* □

As proved in 1.8, a direct corollary of this lemma is:

1.15. Corollary. *A morphism of bimodules f is surjective (respectively, essential) if and only if the induced map $\hat{f} = \mathrm{id}_{\Bbbk} \otimes_A f \otimes_A \mathrm{id}_{\Bbbk}$ is surjective (bijective).* □

In order to construct a bimodule complex of A let us first look a little deeper into the construction of the left Koszul complex (K_L, δ_L). Consider the left A-linear map $\gamma_{L,n} : A \otimes J_n \to A \otimes J_{n-1}$ for each $n \in \mathbb{N}$ defined as before by the natural inclusion $J_n \subseteq A \otimes J_{n-1}$. It is clear that $\gamma_L^s = 0$ and $\delta_{L,i} = \gamma_L^{s-1}$ or γ_L whether i is even or odd, where the sub-indexes of γ_L are chosen in the correct degrees.

The tool used to construct δ_L from γ_L is a generalization of complexes denoted in the literature as s-complexes. We refer the reader to [BDVW02] for more information on the subject.

Symmetrically, we can construct a map of right A-modules $\gamma_{R,n} : J_n \otimes A \to J_{n-1} \otimes A$ for each $n \in \mathbb{N}$ defined by the natural inclusion $J_n \subseteq J_{n-1} \otimes A$. In the same way, we have that $\gamma_R^s = 0$.

The morphisms of bimodules $\gamma_L \otimes \mathrm{id}_A$ and $\mathrm{id}_A \otimes \gamma_R$ define maps which we will denote δ'_L and δ'_R respectively. Observe that $\delta'_L \otimes_A 1_{\Bbbk} = \gamma_L \otimes \mathrm{id}_A \otimes_A \mathrm{id}_{\Bbbk} \cong \gamma_L$ and that $\delta'_R \otimes_A \mathrm{id}_{\Bbbk} = \mathrm{id}_A \otimes \gamma_R \otimes_A \mathrm{id}_{\Bbbk} = 0$. This will be useful in the proof of out next theorem.

We can now introduce the augmented bimodule Koszul complex, which extends the multiplication $\mu : A \otimes A \to A$ and is defined as:

$$K_{LR,i} = A \otimes J_{n(i)} \otimes A,$$

with differential

$$\delta_i = \delta'_L - \delta'_R$$

if i is odd and

$$\delta_i = {\delta'_L}^{s-1} + {\delta'_L}^{s-2}\delta'_R + \ldots \delta'_L{\delta'_R}^{s-2} + {\delta'_R}^{s-1},$$

if i is even. We obtained a pure projective complex in the category of bimodules. The fact that δ'_L and δ'_R commute implies that δ^2 is indeed 0.

1.16. Theorem. *Let A be an s homogeneous algebra on V. The augmented bimodule Koszul complex*

$$\dots \xrightarrow{\delta_3} K_{LR,2} \xrightarrow{\delta_2} K_{LR,1} \xrightarrow{\delta_1} K_{LR,0} \xrightarrow{\mu} A$$

is exact if and only if A is Koszul.

Proof. Assume that A is Koszul. The complex $K_{LR} \otimes_A \Bbbk$ is isomorphic to K_L as left A-modules, which is exact by Theorem 1.13. Using lemma 1.9 we can conclude that K_{LR} itself is exact.

Let us prove the converse. We know that the complex K_{LR} is projective resolution of A in the category of right A-modules, but as A is free as a right A-module, this projective resolution has to be null-homotopic and its exactness is preserved by any functor. In particular $K_L \cong K_{LR} \otimes_A \Bbbk$ is exact and, by Theorem 1.13, A is Koszul. □

As a corollary, we obtained a symmetric condition for an algebra to be Koszul, so the left Koszul complex is exact iff the right Koszul complex is exact.

2 The algebra $A(S)$

We fix a Steiner system (E, S) of type $(s, s+1, n)$; we assume whenever it is convenient that $E = \{1, \dots, n\}$. We fix a ground field $\Bbbk$ of characteristic zero and consider the vector space V freely spanned by a set $X = \{x_i : i \in E\}$ of formal variables indexed by the elements of E, let $T(V)$ be the free algebra on V, and for each block $B = \{i_1, i_2, \dots, i_{s+1}\} \in S$ we choose a non-zero cyclic polynomial ϕ_B multi-linear in the variables $x_{i_1}, x_{i_2}, \dots, x_{i_{s+1}}$ homogeneous of degree $s+1$. Let

$$\Phi = \Phi_S = \sum_{B \in S} \phi_B \in T^{s+1}(V). \tag{7}$$

For every tuple $(i_1, i_2, \dots, i_{s+1}) \in E^{s+1}$ we define $\varepsilon_{i_1, i_2, \dots, i_{s+1}}$ as the coefficient of the monomial $x_{i_1} x_{i_2} \dots x_{i_{s+1}}$ in Φ. Note that it can only be non-zero if $\{i_1, i_2, \dots, i_{s+1}\}$ is a block and, as Φ, it is cyclic, so that $\varepsilon_{i_1, i_2, \dots, i_{s+1}} = \varepsilon_{i_{s+1}, i_1, i_2, \dots, i_s}$. For simplicity we can also define:

$$\varepsilon_{i_1, \dots, i_s} = \begin{cases} 0, & \text{if } |\{i_1, \dots, i_s\}| \leq s; \\ \varepsilon_{i_1, \dots, i_s, \star(i_1, \dots, i_s)}, & \text{in any other case.} \end{cases}$$

In some wide sense the choice of the $\varepsilon_{\_}$ corresponds to a "coloring" of the Steiner system, by assigning a number to the cyclic permutations of each block. The main object studied in this work is the algebra $A = A(S, \Phi)$ which is the quotient of $T(V)$ by the ideal $I = I(S, \Phi)$ generated by the scaled cyclic derivatives

$$r_k = r_k^S = \frac{1}{s+1} \partial_{x_k} \Phi = \sum_{i_1, \dots, i_s \in E} \varepsilon_{k, i_1, \dots, i_s} x_{i_1} x_{i_2} \dots x_{i_s}, \qquad k \in E,$$

as in [Kon93] or [Gin06]. We let $R = R_S \subseteq T^s(V)$ be the subspace spanned by $r_1, \dots, r_n$, so that $I = (R)$. The defining relations are homogeneous of degree s, so A is an $\mathbb{N}_0$-graded algebra.

By construction a sub-linear monomial $x_{i_1} x_{i_2} \dots x_{i_s}$ of length s may only appear in the relation $r_{\star(i_1, \dots, i_s)}$. This is the key reason for choosing Steiner systems: the support of r_i and r_j in terms of the basis $\langle X \rangle$ of $T(V)$, are disjoint whenever $i \neq j$. In particular, this and the fact that they are non-zero, imply that the set of the r_i is linearly independent.

We want to obtain a basis of $A(S)$ to compute the Hilbert series explicitly. For the algebra $A(S)$ to have the properties we are looking for, we are going to need some genericity conditions.

2.1. Definition. *A cyclic potential $\Phi \in \Bbbk\langle X \rangle$ will be called t-*sincere *if for every tuple of distinct elements $i_1, \dots, i_t \in E$ there is a monomial $w \in \Phi$ with $x_i \dots x_t$ as a prefix.*

Observe that as the potential in consideration is cyclic, the definition remains equivalent when changing prefixes for suffixes.

In the case of Steiner triple systems, when $s = 2$, requiring potentials to be 2-sincere is the same as stating that $x_i x_j x_k$ appears in Φ for every block $\{i, j, k\} \in S$. In other words, for each block $B \in S$ the polynomial ϕ_B has all six monomials.

2.2. Proposition. *Assume either $s > 2$ and Φ is 3-sincere, or $s = 2$ and Φ is 2-sincere. Fix a pair of distinct elements $i_1, i_2 \in E$. Then there exist words $w_i \in r_i$, $1 \leq i \leq n$ such that*

(i) *There is only one overlap between them, namely $x_{i_1} w_{i_1} = x_{i_1} \dots x_{i_{s+1}} = w_{i_{s+1}} x_{i_{s+1}}$, where the set $\{i_1, i_2, \dots, i_{s+1}\}$ is a block in S.*

(ii) *If we name α_i the coefficient of w_i in r_i, the only ambiguity in the rewriting system in $T(V)$ with rules*

$$w_i \rightsquigarrow -\frac{1}{\alpha_i} r_i + w_i$$

is resolvable.

(iii) *The set of words on $\{x_i : i \in E\}$ which do not contain the words w_i as sub-words is a basis of A.*

Proof. There is a monomial of the form $x_{i_1} x_{i_2} \dots x_{i_{s+1}}$ in Φ, because of the 2-sincerity property. We fix a total order $\prec$ on the set E such that the $s+1$ largest elements are $i_1 \succ i_2 \succ \dots \succ i_{s+1}$, and extend it to the set of words on E length-lexicographically. Let $w_i = \mathsf{tip}(r_i)$ and let $\alpha_i \in \Bbbk$ be the coefficient of w_i in r_i. Observe that the w_i are all distinct, since the variables that appear in each one belong to a unique block together with x_i. By construction $w_{i_{s+1}} = x_{i_1} \dots x_{i_s}$ and $w_{i_1} = x_{i_2} \dots x_{i_{s+1}}$.

In the case $s > 2$, because Φ is 3-sincere, for every $j \notin \{i_1, i_2, i_{s+1}\}$ there exists a word in Φ starting with $x_j x_{i_1} x_{i_2}$. Therefore $r_j = \partial_{x_j} \Phi$ has a monomial starting in $x_{i_1} x_{i_2}$, so w_j also starts with $x_{i_1} x_{i_2}$. In the same way, w_{i_2} starts in $x_{i_1} x_{i_3}$.

If however $s = 2$, as observed above, this means there is a single monomial starting in x_{i_1} in ∂_{x_j} for every $j \in E \setminus \{i_1, i_3\}$, namely $x_{i_1} x_{\star(i_1, j)} = w_j$.

We apply Bergman's Diamond Lemma [Ber78] to the rewriting system in the free algebra $T(V)$ with rules

$$w_i \rightsquigarrow -\frac{1}{\alpha_i} r_i + w_i.$$

There are evidently no inclusion ambiguities. Assume $v \in \langle X \rangle$ is an overlap ambiguity. First observe that sub-linear monomials with leading x_{i_1} cannot overlap, so w_{i_1} is necessarily part of v. If the other word is w_j with $j \neq i_2$ the position of x_{i_2} forces $v = x_{i_1} w_1$ and $j = i_{s+1}$. If $j = i_2$ then both w_{i_1} and w_{i_2} have x_{i_3} in the second position, so they cannot overlap either.

Then as wanted, the only overlap is the monomial $x_{i_1} x_{i_2} \dots x_{i_{s+1}}$ and we have to check that this is resolvable. From lemma 1.4, the difference between the two reductions in the ambiguity can be written as:

$$x_{i_1} r_{i_1} - r_{i_{s+1}} x_{i_{s+1}} = x_{i_1} r_{i_1} - \Phi + \Phi - r_{i_{s+1}} x_{i_{s+1}} = \sum_{i \neq i_{s+1}} r_i x_i - \sum_{j \neq i_1} x_j r_j.$$

The fact that $x_{i_1} x_{i_2} \dots x_{i_{s+1}}$ is the highest sub-linear word of length $s+1$ in the chosen order means that the right hand side lies in $\langle r_i x_i : i \neq i_{s+1},\ x_j r_j : j \neq i_1 \rangle_{\Bbbk} \subseteq I_{x_{i_1} \dots x_{i_{s+1}}}$, according to the notation defined in [Ber78]. The reason for this is that the leading monomial in any of the $r_i x_i$ or $x_j r_j$ involved is a sub-linear word of length $s+1$, thus smaller than $x_1 \dots x_{s+1}$. $\square$

As the sincerity conditions also hold for suffixes, we could repeat the proof but ordering words from right to left, to get a different set of reduction rules, the transposed of the original.

For a given reduction system $\{w_\sigma \rightsquigarrow f_\sigma\}_\sigma$, we say that the set of w_σ are the *forbidden* words and we call *admissible* polynomials, those that do not contain any forbidden word.

2.3. Proposition. *Assume either $s > 2$ and Φ is 3-sincere, or $s = 2$ and Φ is 2-sincere, then the Hilbert series of A is*

$$h_A(t) = \frac{1}{1 - nt + nt^s - t^{s+1}}. \tag{8}$$

Proof. Define $c_0 = 1$ and for $\ell \in \mathbb{N}$ define $c_\ell = |W_\ell|$, where W_ℓ is the set of words on length ℓ which do not contain any w_i as a sub-word, thus admissible according to the rewriting system defined in Proposition 2.2. Clearly $c_\ell = n^\ell$ for $\ell < s$ and $c_s = n^s - n$.

If $\ell > s$, to find the value of c_ℓ, we can add an element of X to an admissible word of length $\ell - 1$;

$$xw \quad \text{with } x \in X,\ w \in W_{\ell-1}$$

So we got nc_{l-1} possibly admissible words. Of those, we have to subtract words that should not be counted, namely those of length ℓ with only one forbidden word as a prefix.

An upper bound for that are the $nc_{\ell-s}$ words that arise concatenating a forbidden word and an admissible word of length $\ell - s$;

$$w_i w, \quad \text{with } i \in E,\ w \in W_{\ell-s}.$$

However, not all such words are valid, because when there is an overlap as a prefix forbidden words appear which are not prefixes. This is exactly the case when the word is the result of concatenating the overlap (of length $s+1$) with any admissible word of length $\ell - s - 1$.

This process would continue with overlap of overlaps, but in our case, the only overlap does not overlap with itself, and we obtain the following recursion

$$c_\ell = nc_{\ell-1} - nc_{\ell-s} + c_{\ell-s-1}.$$

A simple induction based on this recursive relation shows that

$$h_A(t) = \sum_{\ell \geq 0} c_\ell t^\ell = \left(1 - nt + nt^s - t^{s+1}\right)^{-1}. \qquad \square$$

Definition. *Let* (E, S) *be a Steiner system. A subset* $F \subseteq E$ *is a* subsystem *of* S *if the point* $\star(i_1, \ldots, i_s) \in F$ *for each* $i_1, \ldots, i_s \in F$. *Set* $T = \{B \in S : B \subseteq F\}$. *The subsystem* F*, together with the blocks* T *it contains, is a Steiner system in its own right. We often write* $(F, T) \subseteq (E, S)$ *to represent a subsystem* F *with induced set of blocks* T. *If* $F \subsetneq E$ *we say that the subsystem is* proper.

Our construction behaves well with respect to subsystems. Indeed, if $(F, T) \subsetneq (E, S)$ is a proper subsystem of our fixed Steiner system (E, S), we may consider $(\phi_B)_{B \in T}$ as a family of multi-linear cyclic polynomials for each block of T, needed to construct $A(T)$. Let $\Psi = \sum_{B \in T} \phi_B$ the potential associated to the induced choice of coefficients. In that case we have

2.4. Proposition. *The algebra* $A(T)$ *is quotient of* $A(S)$ *by an homogeneous ideal.*

Proof. Consider the ideals

$$J = \langle x_k : k \in E \setminus F \rangle \subset \Bbbk\langle x_i : i \in E \rangle \text{ and } \bar{J} = \langle \bar{x}_k : k \in E \setminus F \rangle \subset A(S).$$

Every block B not contained in F has at least two points outside. Therefore, we have that $r_k \in J$ if $k \in E \setminus F$, and $r_j = \partial_{x_j} \Phi \equiv \partial_{x_j} \Psi \mod J$ if $j \in F$. It follows from this that we have an isomorphism of graded algebras

$$\frac{A(S)}{\bar{J}} \cong \frac{\Bbbk\langle x_i : i \in E \rangle}{(\partial \Phi) + J} \cong \frac{\Bbbk\langle x_i : i \in F \rangle}{(\partial \Psi)} = A(T).$$

□

3 Quotients by sub-linear polynomials

In this section we want to study algebras that arise when taking a quotient by a homogeneous sub-linear polynomial, because this algebras appear as quotients or subalgebras of the algebras we are considering. We will need the following result from Berger [Ber09, Theorem 1.1].

Proposition (Berger). *Let V be a $\Bbbk$ vector space, and $R \subseteq V^{\otimes N}$ a one dimensional space. The algebra $T(V)/(R)$ is N-Koszul iff the following condition holds*

$$R \otimes V^{\otimes \ell} \cap V^{\otimes \ell} \otimes R \subseteq V^{\otimes(\ell-1)} \otimes R \otimes V, \quad \textit{for each } 1 \leq \ell \leq N-1.$$

This is a powerful tool to prove Koszulity. Indeed, we have the following proposition.

3.1. Proposition. *Fix V a $\Bbbk$-vector space of dimension m and fix X a basis of V. Let $p \in T(V)$ be a homogeneous sub-linear polynomial of degree N and let R be the vector space generated by p. Define $B = T(V)/(R)$. Then B is a N-Koszul algebra of global dimension 2 whose Hilbert series is*

$$h_B(t) = \frac{1}{1 - mt + t^N}.$$

Proof. It follows from Berger's theorem that it suffices to prove that for each $\ell \in \mathbb{N}$ such that $1 \leq \ell \leq N-1$:

$$R \otimes V^{\otimes \ell} \cap V^{\otimes \ell} \otimes R \subseteq V^{\otimes(\ell-1)} \otimes R \otimes V.$$

Fix a total order on X and extend it lexicographically to $V^{\otimes(\ell+N)}$; recall that lexicographical orders are multiplicative. If $\alpha \in R \otimes V^{\otimes \ell} \cap V^{\otimes \ell} \otimes R$ we can write: $\alpha = p \otimes v = u \otimes p$ with $v, u \in V^{\otimes \ell}$. The prefix and sufix of length N of $w = \mathsf{tip}(\alpha)$ are equal to $\mathsf{tip}(p)$, because of the multiplicativity of the order. The fact $|w| < 2N$ implies that $\mathsf{tip}(p)$ overlaps with itself in w. But as p is sub-linear, so is its tip, and it cannot self overlap. Hence α is in fact 0, and the intersection considered is trivial.

Now that we know that B is N-Koszul, it remains to find the corresponding Koszul complex. If we set $R_{N+1} = R \otimes V \cap V \otimes R$ we know that it equals 0 since it is one of the intersections considered above and therefore the Koszul complex is of the form:

$$0 \longrightarrow B \otimes R \longrightarrow B \otimes V \longrightarrow B \twoheadrightarrow \Bbbk$$

Since this is exact, we deduce that the Hilbert series $h_B(t)$ of B is

$$h_B(t) = \left(1 - mt + t^N\right)^{-1}. \qquad \square$$

4 Normal elements

An element $\alpha \in A$ is *normal* if the left and right ideals αA and $A\alpha$ coincide. When A is graded (as in our case) homogeneous components of normal elements are again normal. As a first step towards studying A let us try and calculate the set of normal elements. The technique used hereafter is that if a polynomial $p \in T(V)$ is such that $\mathsf{tip}(p)$ is admissible, then it is not zero in $A = T(V)/R$. The reason for this is that any reduction replaces a monomial for a sum of smaller monomials, so $\mathsf{tip}(p)$ can never be canceled out.

4.1. Proposition. *If Φ is 3-sincere and $s \geq 3$, the normal elements of $A = T(V)/(\partial\Phi)$ are the scalars.*

Proof. Let $\bar{\alpha} \in A$, the class of $\alpha \in T(V)$, be a d-homogeneous normal element with $d > 0$. Using Proposition 2.2 we can choose two elements $1, 2 \in E$ such that there is a total order on X starting with x_1, x_2 that constructs a resolvable reduction system. We may take α to be admissible. Let $w = \mathsf{tip}(\alpha)$ be the leading monomial that appears in α.

For each $i \in E$ we have $x_i\bar{\alpha} \in A_1\bar{\alpha} = \bar{\alpha}A_1$, so there exist scalars $u_{i,j}$ for $1 \leq j \leq n$ such that $x_i\bar{\alpha} = \sum_{j=1}^{n} \bar{\alpha}x_j u_{i,j}$ in A. This means that $\beta_i = x_i\alpha - \sum_{j=1}^{n} \alpha x_j u_{i,j}$ belongs to $(R) \subseteq T(V)$.

Observe β_i is a difference between a polynomial with tip $x_i w$ and a second polynomial with tip of the form wx_j for some j. If $w = x_1^d$ from $i = 2$ we see all possible values for $\mathsf{tip}(\beta_2)$ are distinct (so they may not cancel out) and admissible, since (because $s > 2$) every s sub-word has x_1 more than one time. Thus β_2 cannot reduce down to zero, and we get a contradiction. Then $x_1^d \notin \alpha$.

Now set $i = 1$. As w is not a power of x_1, all candidates for $\mathsf{tip}(\beta_1)$ are different, the higher among them being $x_1 w$. So we deduced that $\mathsf{tip}(\beta_1) = x_1 w$ and as it reduces to zero, it cannot be admissible. Hence $x_1 w$ has w_i as a prefix for some $i \neq 1$. Observe that this implies $w = x_{i_1}x_{i_2}\dots$ with $1 \succ i_1 \succ i_2$.

Finally, set $i = i_1$ and as $x_{i_1}x_{i_1}\cdots \succ x_{i_1}x_{i_2}\dots$ we have $\mathsf{tip}(\beta_{i_1}) = x_{i_1}w$ which should again be non-admissible by a prefix. But now it starts with a repeated letter, and all our reduction rules are sub-linear. So we get a contradiction, which arises from assuming $\alpha \neq 0$. □

In the case $s = 2$ and Φ 2-sincere we have to divide in two cases. When $n = 3$ we get a really different scenario.

4.2. Proposition. *Assume $s = 2$, $n = 3$ and Φ is 2-sincere. Then*

(i) *There exists a scalar $q \in \Bbbk^\times$ such that $A \cong \frac{\Bbbk\langle x,y,z\rangle}{(xy-qyx,\ yzy-qzy,\ zx-qxz)}$,*

(ii) *All monomials are normal,*

(iii) *The element $x^a y^b z^c$ belongs to the center whenever $a \equiv b \equiv c \mod d$ where d is the possibly infinite order of q in $\Bbbk^\times$.*

Proof. It is clear that Φ is multiple of $c(xyz - qxzy)$ for a non-zero scalar q and we get the q-commutative algebra in x, y, z, described in the proposition. The rest of the items are checked by direct computation. $\square$

When $n > 3$ the same results as when $s > 2$ hold assuming sufficiently general conditions for the coefficients ε, but the approach changes radically, to resemble the proofs in [SA13]. In this case we shall see that A is an Ore extension of another algebra, and the lack of normal elements in the second one will allow us to prove the same for the first one in most cases.

To construct an Ore extension of an algebra B we need an injective endomorphism σ, and a σ-derivation δ. A σ-derivation is a linear morphism δ that satisfies

$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b), \qquad \text{for each } a, b \in B.$$

Then we define $B[x; \sigma, \delta]$ as the sum $\bigoplus_{i \in \mathbb{N}_0} Bx^i$ with the multiplication structure given by $xa = \sigma(a)x + \delta(a)$ for each $a \in B$.

Among σ-derivations, lies the subspace of inner σ-derivations $\{[\xi, -]_\sigma : \xi \in B\}$ that satisfy

$$[\xi, a]_\sigma = \xi a - \sigma(a)\xi, \qquad \forall a \in B.$$

4.3. Proposition. *Assume $s = 2$ and Φ is 2-snicere. Let $a \in E$ and let B_a be the algebra generated by $\{x_i : i \in E \setminus \{a\}\}$ subject to the single relation*

$$\sum_{\{i,j,a\} \in S} \varepsilon_{i,j} x_i x_j = 0.$$

Then B_a is a quadratic Koszul algebra of global dimension 2 whose Hilbert series is

$$h_{B_a}(t) = \frac{1}{1 - (n-1)t + t^2}.$$

Moreover, there is a unique algebra automorphism $\sigma_a : B_a \to B_a$ such that

$$\sigma_a(x_i) = -\frac{\varepsilon_{i,a}}{\varepsilon_{a,i}} x_i$$

and a unique σ-derivation $\delta_a : B_a \to B_a$ such that

$$\delta_a(x_i) = -\frac{1}{\varepsilon_{a,i}} \sum_{\substack{\{u,v,i\star a\} \in S \\ u,v \neq a}} \varepsilon_{u,v} x_u x_v$$

for each $i \in E \setminus \{a\}$.

Proof. Let $E' = E \setminus \{a\}$, let V' be the vector space with basis $\{x_i : i \in E'\}$ and let $R' \subseteq V' \otimes V'$ be the space spanned by $\rho = \sum_{i,j \in E} \varepsilon_{i,j,a} x_i \otimes x_j$, so that $B_a = T(V')/(R')$. It follows from 3.1 and the fact that ρ is sub-linear that B_a is 2-Koszul with Hilbert series

$$h_{B_a}(t) = \left(1 - (n-1)t + t^2\right)^{-1}.$$

First observe that the formula in the statement defines a unique algebra automorphism $\overline{\sigma}_a : T(V') \to B_a$. We have to see that it factors through B_a, so we have to check $\sigma_a(\rho) = 0$ in B_a. Indeed,

$$\begin{aligned}
\overline{\sigma}_a(\rho) &= \overline{\sigma}_a\left(\sum_{\{i,j,a\} \in S} \varepsilon_{i,j} x_i x_j\right) \\
&= \sum_{\{i,j,a\} \in S} \varepsilon_{i,j} \overline{\sigma}_a(x_i) \overline{\sigma}_a(x_j) \\
&= \sum_{\{i,j,a\} \in S} \varepsilon_{i,j} \left(-\frac{\varepsilon_{i,a}}{\varepsilon_{a,i}}\right) x_i \left(-\frac{\varepsilon_{j,a}}{\varepsilon_{a,j}}\right) x_j \\
&= \sum_{\{i,j,a\} \in S} \varepsilon_{i,j} \left(-\frac{\varepsilon_{i,a,j}}{\varepsilon_{j,a,i}}\right) x_i \left(-\frac{\varepsilon_{j,a,i}}{\varepsilon_{i,a,j}}\right) x_j \\
&= \sum_{\{i,j,a\} \in S} \varepsilon_{i,j} x_i x_j = \rho.
\end{aligned}$$

Again, the formula in the statement of the lemma defines a unique $\overline{\sigma}_a$-derivation $\overline{\delta}_a : T(V') \to B_a$, and to see that it descends to a σ_a-derivation $\delta_a : B_a \to B_a$ it is enough to check that $\overline{\delta}_a(\rho) = 0$ in B_a. Observe that we can write

$$\delta_a(x_i) = \frac{1}{\varepsilon_{a,i}} \partial_{i \star a}(\phi_{\{i,a,i \star a\}} - \Phi).$$

Using that, we can now compute:

$$\begin{aligned}
\overline{\delta}_a(\rho) &= \overline{\delta}_a\left(\sum_{\{i,j,a\} \in S} \varepsilon_{i,j} x_i x_j\right) \\
&= \sum_{\{i,j,a\} \in S} \varepsilon_{i,j} \overline{\delta}_a(x_i) x_j + \sum_{\{i,j,a\} \in S} \varepsilon_{i,j} \overline{\sigma}_a(x_i) \overline{\delta}_a(x_j) \\
&= \sum_{\{i,j,a\} \in S} \varepsilon_{i,j} \frac{1}{\varepsilon_{a,i}} \partial_{i \star a}(\phi_{\{i,a,i \star a\}} - \Phi) x_j \\
&\qquad + \sum_{\{i,j,a\} \in S} \varepsilon_{i,j} \left(-\frac{\varepsilon_{i,a}}{\varepsilon_{a,i}}\right) x_i \frac{1}{\varepsilon_{a,j}} \partial_{j \star a}(\phi_{\{j,a,j \star a\}} - \Phi)
\end{aligned}$$

We know the coefficients $\varepsilon_{i,a} = \varepsilon_{a,j} = \varepsilon_{j,i}$ and $\varepsilon_{a,i} = \varepsilon_{j,a} = \varepsilon_{i,j}$ in every term, becuase ε is cyclic. Therefore, interchanging the summation variables i and j in the first sum, we find that $\overline{\delta}_a(\rho)$ is equal to

$$-\Big(\sum_{\{i,j,a\}\in S} \partial_{j\star a}(\phi_{\{j,a,j\star a\}} - \Phi)x_i\Big) + \Big(\sum_{\{i,j,a\}\in S} x_i\partial_{j\star a}(\phi_{\{j,a,j\star a\}} - \Phi)\Big)$$

If we define $\phi_a = \frac{1}{2}\sum_{i\in E'} \phi_{\{a,i,i\star a\}}$ we have $\partial_i\phi_a = \partial_i\phi_{\{a,i,i\star a\}}$ because there is a single block containing both a and i. The sum can be rewritten as

$$-\Big(\sum_{\{i,j,a\}\in S} \partial_i(\phi_a - \Phi)x_i\Big) + \Big(\sum_{\{i,j,a\}\in S} x_i\partial_i(\phi_a - \Phi)\Big).$$

This is zero because it is the Euler relation (1) applied to $\phi_a - \Phi$. □

4.4. Proposition. *Let $a \in S$, let B_a, σ_a and δ_a be as in Proposition 4.3, and let $B_a[x_a;\sigma_a;\delta_a]$ be the Ore extension of B_a with respect to σ_a and δ_a. Then there is an isomorphism $A \cong B_a[x_a;\sigma_a;\delta_a]$ of graded algebras.*

As a consequence, A is a free right (or left) B_a-module on the set $\{x_a^i : i \in \mathbb{N}_0\}$.

Proof. The existence of an isomorphism $A \cong B_a[x_a;\sigma_a;\delta_a]$ is immediate, since commutation relations in the Ore extension are:

$$\sigma_a(x_i)x_a + \delta_a(x_i) - x_a x_i = -\frac{1}{\varepsilon_{a,i}} \sum_{\{u,v,i\star a\}\in S} \varepsilon_{u,v}x_u x_v = -\frac{1}{\varepsilon_{a,i}} r_{i\star a} \quad \text{for } i \in E'.$$

□

The next proofs are exactly the same as in [SA13], slightly adapted, since in our case the monomials that appear in relations do not change, only scalars.

4.5. Proposition. *Assume $s = 2$, $n \geq 5$ and Φ is 2-sincere. If $a \in E$, the only normal elements in the algebra B_a of Proposition 4.3 are the scalars.*

Proof. Let $m = (n-1)/2$; this is an integer because n is congruent to 1 or 3 modulo 6. Renaming the elements of E, we can assume that $E = \{1,\dots,n\}$, that $a = n$, and that $(1, m+1, n)$, $(2, m+2, n)$, $\dots$, $(m, 2m, n)$ are the blocks of S that contain n. Then $B = B_a$ is the free algebra generated by variables $x_1, \dots, x_{2m}$ subject to the relation

$$\varepsilon_{1,m+1}x_1x_{m+1} + \varepsilon_{m+1,1}x_{m+1}x_1 + \cdots + \varepsilon_{m,2m}x_m x_{2m} + \varepsilon_{2m,m}x_{2m}x_m = 0.$$

Notice that $m \geq 2$ because $n \geq 5$. It is clear that B is a graded algebra for the grading that has all the generating variables in degree 1. Moreover, it is evident from Bergman's diamond lemma [Ber78] that the set of words in the variables $x_1, \dots, x_{2m}$ that do not contain the subword $x_{2m}x_m$ is a basis of B; recall that these are the admissible words. In particular, we remark that $x_1 \neq x_m$ in B.

For each $d \geq 0$ let $F_d \subseteq B$ be the subspace spanned by all words of the form wx_{2m}^i with $i \leq d$ and w a word not ending in x_{2m}; notice such a word is admissible iff w is. We have $F_d \subseteq F_{d+1}$ for all $d \geq 0$ and clearly $B = \bigcup_{d\geq 0} F_d$.

Let $u \in B$ be an homogeneous normal element of degree $k \geq 1$. There exists an integer $d \geq 0$ and homogeneous elements $u_0, \dots, u_d \in B$, each a linear combination of admissible words not ending in x_{2m}, such that $u = \sum_{i=0}^{d} u_i x_{2m}^i$ and $u_d \neq 0$. As u is normal, there exist $\alpha_1, \dots, \alpha_{2m} \in \Bbbk$ such that $x_1 u = \sum_{j=1}^{2m} \alpha_j u x_j$. Let us consider the elements

$$s = x_1 u = \sum_{i=0}^{d} x_1 u_i x_{2m}^i$$

and

$$\begin{aligned} t &= \sum_{j=1}^{2m} \alpha_j u x_j = \sum_{j=1}^{2m} \sum_{i=0}^{d} \alpha_j u_i x_{2m}^i x_j \\ &= \alpha_m u_d x_{2m}^d x_m + \alpha_{2m} u_{d-1} x_{2m}^d + \alpha_{2m} u_d x_{2m}^{d+1} \\ &\qquad + \underbrace{\sum_{\substack{j=1 \\ j \neq m, 2m}}^{2m} \sum_{i=0}^{d} \alpha_j u_i x_{2m}^i x_j + \alpha_m \sum_{i=0}^{d-1} u_i x_{2m}^i x_m + \alpha_{2m} \sum_{i=0}^{d-2} u_i x_{2m}^{i+1}}_{\in F_{d-1}}. \end{aligned}$$

We must have $\alpha_{2m} = 0$: if that were not the case, we would have $F_d \ni s = t \in F_{d+1} \setminus F_d$, which is absurd. Using this, we see that

$$t \equiv \alpha_m u_d x_{2m}^d x_m \equiv \alpha_m u_d \sigma_a^{-d}(x_m) x_{2m}^d \mod F_{d-1}$$

while $s \equiv x_1 u_d x_{2m}^d \mod F_{d-1}$, so in fact $(\alpha_m u_d x_m - x_1 u_d) x_{2m}^d \in F_{d-1}$. This is only possible if $\alpha_m u_d \sigma_a^{-d}(x_m) = \alpha_m u_d \left(\frac{\varepsilon_{n,m}}{\varepsilon_{m,n}}\right)^{-d} x_m = x_1 u_d$, so that $\alpha_m \neq 0$ and, moreover, $x_1 = x_m$. This is a contradiction. It follows that there are no homogeneous normal elements of positive degree in B. Since elements of degree 0 are normal, the lemma is proved. □

4.6. Lemma. *No vector in V' is zero divisor in B_a.*

Proof. We can write $v = \sum_{j \in E'} x_j c_j$ for some $c_j \in \Bbbk$. Fix $i \in E'$ such that $c_i \neq 0$. By choosing any total order on E' with x_i as maximum, it is evident from Bergman's diamond lemma [Ber78] that the set of words in the variables $(x_j)_{j \in E'}$ that do not contain the sub-word $x_i x_{i \star a}$ is a basis of B. If $\alpha \in T(V')$ is such that $\alpha v \in (R')$ we know that $\mathsf{tip}(\alpha v) = \mathsf{tip}(\alpha) x_i$ must be non-admissible. We may assume α was admissible to start with, so $\mathsf{tip}(\alpha) x_i$ must have one of the $x_i x_{i \star a}$ as a suffix, but this is clearly a contradiction. By symmetry we deduce v is not a left zero divisor either. □

Actually, with a little bit more work we can prove that it is domain.

4.7. Corollary. *The algebra B_a is a domain.*

Proof. Suppose by contradiction that for some homogeneous irreducible polynomials $\alpha, \beta \in T(V')$, that do not lie in (R') we have $\alpha\beta \in (R')$. By the previous lemma we know that both α and β live in degrees greater than 1.

Again by choosing a monomial order with x_i as the first letter, and assuming α and β are admissible $\mathsf{tip}(\alpha\beta) = \mathsf{tip}(\alpha)\mathsf{tip}(\beta)$ must be non-admissible. Then $\mathsf{tip}(\alpha)$ ends with x_i and $\mathsf{tip}(\beta)$ starts with $x_{i\star a}$. We can rewrite $\alpha = \alpha_1 x_i + \alpha_2$ and $\beta = x_{i\star a}\beta_1 + \beta_2$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in T(V')$, α_2 does not have x_i as a right divisor and β_2 does not have $x_{i\star a}$ as a left divisor. As α and β are irreducible polynomials, neither α_2 nor β_2 are zero.

Necessarily we have $\mathsf{tip}(\alpha) = \mathsf{tip}(\alpha_1 x_i) = \mathsf{tip}(\alpha_1)x_i$. In particular $\mathsf{tip}(\alpha_1)x_i \succ \mathsf{tip}(\alpha_2)$. By applying the reduction rule to $\alpha_1 x_i x_{i\star a}\beta_1$ we obtain

$$\alpha\beta = (\alpha_1 x_i + \alpha_2)(x_{i\star a}\beta_1 + \beta_2) \equiv \alpha_1 x_i \beta_2 + \alpha_2\beta + \alpha_1 r(x_i x_{i\star a})\beta_1 \mod (R)$$

where $r(x_i x_{i\star a}) = x_i x_{i\star a} - \varepsilon_{a,i}^{-1}\rho$. But the previous observations imply that the tip of the reduced polynomial is $\mathsf{tip}(\alpha_1 x_i \beta_2)$ which is admissible, since no word in β_2 starts with $x_{i\star a}$. Thus $\alpha\beta$ cannot lie inside (R') and we get a contradiction. □

4.8. Lemma. *Let Λ be a graded locally finite connected 1-generated algebra that has no zero divisors in degree 1 and whose only normal elements are the scalars. Let $\sigma : \Lambda \to \Lambda$ be a graded automorphism and $\delta : \Lambda \to \Lambda$ a homogeneous σ-derivation of degree 1. Assume for all $d > 0$ that $\sum_{e=0}^{d-1} \sigma^e \delta \sigma^{-e}$ is not σ-inner, then the only normal elements in the Ore extension $\Lambda[x; \sigma, \delta]$ are the scalars.*

Proof. Every non-zero element $u \in \Lambda[x; \sigma, \delta]$ can be written in a unique way as a sum $u = \sum_{i=0}^{d} \lambda_i x^i$ with $d \geq 0$, $\lambda_0, \ldots, \lambda_d \in \Lambda$ and $\lambda_d \neq 0$. Let us say that the *weight* of u is then $w(u) = d$. Also, as all commutation relations that arise from the Ore extension are homogeneous of degree 2, the algebra $\Lambda[x; \sigma, \delta]$ with x in degree 1 is naturally graded, connected and generated in degree 1.

Suppose now that $u = \sum_{i=0}^{d} \lambda_i x^i$, with $d \geq 0$, $\lambda_0, \ldots, \lambda_d \in \Lambda$ and $\lambda_d \neq 0$, is a non-zero e-homogeneous normal element in $\Lambda[x; \sigma, \delta]$. Since u is normal and Λ has no zero divisors in degree 1, there is an injective morphism $\phi : \Lambda_1 \to \Lambda[x; \sigma, \delta]_1 = \Lambda_1 + \langle x \rangle$ of vector spaces such that $vu = u\phi(v)$ for all $v \in \Lambda_1$. However, if $\phi(v) \notin \Lambda_1$, we would have $w(u\phi(v)) = d + 1$ which would be a contradiction. In particular, ϕ restricts to an isomorphism of vector spaces $\phi : \Lambda_1 \to \Lambda_1$.

If $d = 0$, then $u \in \Lambda$ is a normal element in Λ and therefore a scalar. We may then suppose that $d > 0$. Let $\mu \in \Lambda_1$. We have that

$$\mu u = \sum_{i=0}^{d} \mu \lambda_i x^i = \mu \lambda_d x^d + \mu \lambda_{d-1} x^{d-1} + \cdots$$

and this is equal, by definition of ϕ, to $u\phi(\mu)$ or, equivalently, to

$$\sum_{i=0}^{d} \lambda_i x^i \phi(\mu) = \lambda_d \sigma^d \phi(\mu) x^d + \left(\sum_{e=0}^{d-1} \lambda_d \sigma^{d-1-e} \delta \sigma^e \phi(\mu) + \lambda_{d-1} \sigma^{d-1} \phi(\mu) \right) x^{d-1} + \cdots,$$

where the omitted terms all involve powers of x smaller than $d - 1$.

It follows from this that $\mu\lambda_d = \lambda_d \sigma^d \phi(\mu)$ for all $\mu \in \Lambda_1$, so that λ_d is a normal element of Λ, because Λ_1 generates Λ. The hypothesis, then, implies that λ_d is a scalar. By eventually substituting u by $\lambda_d^{-1}u$, we can assume that $\lambda_d = 1$. Then, in fact, we see that $\phi(\mu) = \sigma^{-d}\mu$ for all $\mu \in \Lambda$ and, looking at the coefficient of x^{d-1} in μu and in $u\phi(\mu)$, that $\mu\lambda_{d-1} = \sum_{e=0}^{d-1} \sigma^e \delta \sigma^{d-1-e} \sigma^{-d}(\mu) + \lambda_{d-1}\sigma^{d-1}\sigma^{-d}\mu$ for all $\mu \in \Lambda$ or, equivalently, by setting $\nu = \sigma^{-1}(\mu)$, that

$$\sum_{e=0}^{d-1} \sigma^e \delta \sigma^{-e}(\nu) = \sigma(\nu)\lambda_{d-1} - \lambda_{d-1}\nu = [-\lambda_{d-1}, \nu]_\sigma, \qquad \forall \mu \in \Lambda.$$

This is impossible, because by hypothesis $\sum_{e=0}^{d-1} \sigma^e \delta \sigma^{-e}$ is not an σ-inner derivation. $\square$

The condition that $\sum_{e=0}^{d-1} \sigma^e \delta \sigma^{-e}$ is not an σ-inner derivation may seem difficult to proof, but we have the following lemma.

4.9. Lemma. *If $\Delta_d = \sum_{e=0}^{d-1} \sigma^e \delta \sigma^{-e}$ is a σ-inner derivation for some $d > 0$, then $\Delta_d = 0$.*

Proof. Suppose Δ_d is σ-inner, so that there exists $\xi \in B$ such that $\Delta_d = [\xi, -]_\sigma$. Since Δ_d is homogeneous of degree 1, we can assume that ξ itself is of degree 1, and then there exist $\xi_i \in \Bbbk$ for $i \in E'$ such that $\xi = \sum_{i \in E'} \xi_i x_i$.

Let $i \in E'$ and let $\alpha_{u,v} \in \Bbbk$ be the coefficient of $x_u x_v$ in $\Delta_d(x_i)$. Then we have that

$$\Delta_d(x_i) = \sum_{\substack{\{u,v,i\star a\} \in S \\ u,v \neq a}} \alpha_{u,v} x_u x_v$$

is the same as

$$[\xi, x_i]_\sigma = \sum_{k \in E'} \xi_k [x_k, x_i]_\sigma,$$

so there exists a scalar $\lambda \in \Bbbk$ such that

$$\sum_{\substack{(u,v,i\star a) \in S \\ u,v \neq a}} \alpha_{u,v} x_u x_v - \sum_{k \in E'} \xi_k [x_k, x_i]_\sigma = \lambda \sum_{(u,v,n) \in S} \varepsilon_{u,v} x_u x_v.$$

in the free algebra $T(V')$. We can rewrite this equality as

$$\sum_{\substack{(u,v,i\star a) \in S \\ u,v \neq a}} \alpha_{u,v} x_u x_v - \sum_{\substack{k \in E' \\ k\star i \neq a}} \xi_k [x_k, x_i]_\sigma = \lambda \sum_{(u,v,a) \in S} \varepsilon_{u,v} x_u x_v + \xi_{i\star a}[x_{i\star a}, x_i]_\sigma.$$

On the left we have a possibly empty sum of monomials $x_s x_t$ with $s \star t \neq a$, while on the right only monomials $x_s x_t$ with $s \star t = a$ appear. It follows that both sides of the equality vanish, so that

$$\lambda \sum_{(u,v,a) \in S} \varepsilon_{u,v} x_u x_v + \xi_{i\star a}[x_{i\star a}, x_i]_\sigma = 0.$$

If $j \in E \setminus \{a, i, i \star a\}$, then the coefficient of $x_j x_{j\star a}$ on the left hand side of this last equation is $\varepsilon_{j,j\star a}\lambda$, so we see that $\lambda = 0$ and, as a consequence, that $\xi_{i\star a} = 0$. Every element of E' is of the form $i \star a$ for some $i \in E'$, so we have shown that $\xi = 0$. □

4.10. Corollary. *Fix $a \in E$ and define for every $i \in E'$, $c_i = -\frac{\varepsilon_{i,a}}{\varepsilon_{a,i}}$. Assume that for some block $B = \{i, j, k\} \in S$ with $a \notin B$ we have $c_i c_j c_k$ is not a nontrivial root of the unity. Then the only normal elements in A are the scalars and, in particular, the center $\mathsf{Z}(A)$ is spanned by 1.*

Proof. If B and $\sigma, \delta : B \to B$ are as in Proposition 4.3, the algebra A can be identified with the Ore extension $B[x_n; \sigma; \delta]$. In view of lemma 4.8, to prove the proposition it is enough to show that $\Delta_d = \sum_{e=0}^{d-1} \sigma^e \delta \sigma^{-e}$ is not a σ-inner derivation for any value of d greater than zero, and in view of the last lemma, it suffices to prove that it is not zero.

Let us first calculate $\Delta_d(x_i)$ for some $i \in E'$. Observe that by definition $\sigma(x_i) = c_i x_i$, and also $c_i c_{i\star a} = 1$. For a given $e \in \mathbb{N}_0$, we have

$$\sigma^e \delta \sigma^{-e}(x_i) = c_i^{-e} \sigma^e \delta(x_i) = -\frac{1}{\varepsilon_{a,i}} \sum_{\substack{\{u,v,i\star a\} \in S \\ u,v \neq a}} \varepsilon_{u,v} (c_{i\star a} c_u c_v)^e x_u x_v$$

This means that if $\{i, j, k\}$ is a block, the coefficient of $x_j x_k$ in $\Delta_d(x_{i\star a})$ is a fixed multiple of $\sum_{e=0}^{d-1} (c_i c_j c_k)^e$ which equals d if $c_i c_j c_k = 1$ or $\frac{(c_i c_j c_k)^d - 1}{c_i c_j c_k - 1}$, in other case. The hypothesis tells us exactly that Δ_d is non-zero, and we are done. □

It remains to think whether the converse is true. That is the existence of non trivial normal elements when $c_i c_j c_k$ is a non-trivial root of the unity for every $\{i, j, k\} \in S$.

Another desirable property we can prove in the same spirit is the following

4.11. Proposition. *Assume either $s > 2$ and Φ is 3-sincere, or $s = 2$ and Φ is 2-sincere, then v is not a zero divisor for each $v \in V$.*

Proof. We can write $v = \sum_{i=1}^{n} x_i c_i$ for some $c_i \in \Bbbk$. Fix i such that $c_i \neq 0$. Again choose any total order on X with x_i as maximum as in 2.2. If $\alpha \in T(V)$ is such that $\alpha x_i \in (R)$ we know that $\mathsf{tip}(\alpha x_i) = \mathsf{tip}(\alpha) x_i$ must be non-admissible. We may assume α was admissible to start with, so $\mathsf{tip}(\alpha) x_i$ must have one of the w_j as a suffix, but non of them end with the biggest letter x_i. By symmetry we deduce v is not a left zero divisor either. □

As Ore extensions of domains are domains, we actually know that A is a domain for $s = 2$, the question remains whether A can have zero divisors at all for $s > 2$.

5 Homological properties

In the section we are going to try to obtain a homological identikit of $A(S)$. Assume throughout the whole section either $s > 2$ and Φ is 3-sincere, or $s = 2$ and Φ is 2-sincere. This are the hypothesis that allowed as to compute most results in the previous section.

5.1. Proposition. *The space*

$$R_{s+1} = (R \otimes V) \cap (V \otimes R) \subseteq T^{s+1}(V)$$

is 1*-dimensional and generated by the element* Φ *defined in equation* (7) *on page* 13*, and*

$$R_{2s} = \bigcap_{i+j=s} (V^{\otimes i} \otimes R \otimes V^{\otimes j}) \subseteq T^{2s}(V)$$

is the trivial subspace.

Proof. Let $\alpha \in T^{s+1}(V)$ be an element in R_{s+1}, so that there exist scalars $u_{i,j}, v_{i,j} \in \Bbbk$ for each $i, j \in E$, such that

$$\alpha = \sum_{i,j \in E} u_{i,j} r_i x_j \tag{9}$$

and

$$\alpha = \sum_{i,j \in E} v_{i,j} x_i r_j. \tag{10}$$

If for some $i \neq j$, $u_{i,j} \neq 0$ we know from by 2-sincerity that r_i has a monomial ending in x_j. So in (9) we see that there is a monomial ending in x_j^2 in α. It cannot be canceled out, since all the r_i have disjoint support in term of the basis of words of length s in the alphabet X. But it is clear from (10) that there cannot be such a monomial. In that way $u_{i,j} = 0$ when $i \neq j$. By a similar argument $v_{i,j} = 0$ when $i \neq j$, and we are left with

$$\alpha = \sum_{i \in E} u_i r_i x_i \tag{11}$$

and

$$\alpha = \sum_{j \in E} v_j x_j r_j. \tag{12}$$

At this point we have to make the key observation that again by 2-sincerity, there is a monomial in Φ with non-zero coefficient starting with x_j and ending with x_i for all $i \neq j \in E$. This is $x_i w x_j \in \Phi$ with coefficient β for some monomial w.

In (11), w appears with coefficient βu_i but in (12), w appears with coefficient βv_j. Hence $u_i = v_j$ for every $i \neq j$, which clearly implies $u_i = v_j$ for all $i, j \in E$. Then $\alpha = u \sum_{i \in E} r_i x_i = u\Phi$ by 1.4. Again from 1.4 follows the fact that $\Phi \in R_{s+1}$.

For a simple proof of the second part we are going to have to jump a little ahead of us by considering one of the morphisms in the Koszul complex in disguise.

Let $\alpha \in T^{2s}(V)$ be an element in R_{2s}, then we can view it as an element $a \otimes \Phi$ of $A_{s-1} \otimes \Bbbk\Phi$. We can also write $\alpha = \sum_{i \in E} a \otimes x_i \otimes r_i$, and by definition of R_{2s} this means $a \otimes b \in R$ if we take $b = \sum_{i \in E} x_i$. We know by Proposition 4.11 that A has no zero divisors in degree 1. Hence $a = 0$, which implies $\alpha = 0$, recalling that the projection to A is an isomorphism in degree $s - 1$. □

A connected graded algebra Λ is *Gorenstein of dimension d and parameter ℓ* if we have an isomorphism $\mathrm{Ext}^d_\Lambda(\Bbbk, \Lambda) \cong \Bbbk(\ell)$ and $\mathrm{Ext}^p_\Lambda(\Bbbk, \Lambda) = 0$ for all $p \neq d$. In a similar vein, Λ is *Calabi-Yau of dimension d* if $\mathrm{Ext}^d_\Lambda(\Lambda, \Lambda \otimes \Lambda) \cong \Lambda$ as a bimodule and $\mathrm{Ext}^p_\Lambda(\Lambda, \Lambda \otimes \Lambda) = 0$ for all $p \neq d$, and it has finite global dimension.

5.2. Proposition. *The algebra A is a s-Koszul algebra of global dimension* 3. *It is a Gorenstein algebra of dimension* 3 *and parameter $s + 1$ and Calabi-Yau of dimension* 3.

The following theorem from Berger and Solotar [BS13, Theorem 2.7] will save as lots of computations in our proof.

Proposition (Berger, Solotar). *Let $A = T(V)/(R)$ be a N-Koszul algebra of* gldim $A = 3$, *with Hilbert series $h_A(t) = 1 - nt + nt^N - t^{N+1}$ for some $n \in \mathbb{N}$. Then A is* 3*-Calabi-Yau.*

The proof is omitted. Now we can proceed and prove 5.2.

Proof. For each $i,j \in E$ we define the polynomials $r_{k,i}$ and $\tilde{r}_{i,k}$ such that

$$r_k = \sum_{i \in E} r_{k,i} x_i = \sum_{i \in E} x_i \tilde{r}_{i,k}.$$

It is easy to see that $r_{k,i}$ equals the sum of monomials in Φ that end in $x_i x_k$ with corresponding coefficient. And in the same manner $\tilde{r}_{i,k}$ equals the sum of monomials in Φ that end in $x_k x_i$ with corresponding coefficient. So $r_{i,k} = \tilde{r}_{i,k}$ for each $i, k \in E$.

It follows from Proposition 5.1 that the Koszul complex [Ber01] K_L for the homogeneous algebra A as defined in section 1.4 is of the form

$$0 \longrightarrow A \otimes R_{s+1} \xrightarrow{d_3} A \otimes R \xrightarrow{d_2} A \otimes V \xrightarrow{d_1} A \twoheadrightarrow \Bbbk \tag{13}$$

with the canonical augmentation $A \to \Bbbk$ and differentials given by

$$\begin{aligned} d_1(a \otimes x_k) &= a x_k, \\ d_2(a \otimes r_k) &= \sum_{i \in E} a r_{k,i} \otimes x_i, \\ d_3(a \otimes \Phi) &= \sum_{i \in E} a x_i \otimes r_i \end{aligned} \tag{14}$$

for all $a \in A$ and all $k \in E$.

This complex is exact at $\Bbbk$, at A and at $A \otimes V$. Since A has no zero divisors in degree 1, we see from (14) that the differential d_3 is an injective map, so the complex (13) is also exact at $A \otimes R_3$. Let $\eta(t)$ be the Hilbert series for the cohomology space of (13) at $A \otimes R$.

Since that is the only possible non-zero cohomology space of the complex, and since the Euler characteristic does not change when passing to homology, we see that

$$\begin{aligned}\eta(t) &= 1 - h_A(t) + nth_A(t) - nt^s h_A(t) + t^{s+1} h_A(t) \\ &= 1 - (1 - nt + nt^s - t^{s+1}) h_A(t) = 0.\end{aligned}$$

In view of the expression (8) for $h_A(t)$, we conclude that the complex (13) is also exact at $A \otimes R$, so that A is a Koszul algebra, plainly of global dimension 3.

In order to prove A is Gorenstein we will compute $\mathsf{Ext}^*_A(\Bbbk, A)$ by hand. However, we know it is Gorenstein since we will prove it is Calabi-Yau, but we find this computation illustrative.

Let V^*, R^* and R^*_{s+1} be the spaces dual to V, R and R_{s+1}, and let $\{\hat{x}_1, \dots, \hat{x}_n\}$, $\{\hat{r}_1, \dots, \hat{r}_n\}$ and $\{\hat{\Phi}\}$ be the bases of these spaces which are dual to $\{x_1, \dots, x_n\}$, to $\{r_1, \dots, r_n\}$ and to $\{\Phi\}$, respectively. The complex obtained by applying the functor $\mathsf{hom}_A(-, A)$ to the Koszul complex (13) is, up to standard identifications,

$$A \xrightarrow{d_1^*} V^* \otimes A \xrightarrow{d_2^*} R^* \otimes A \xrightarrow{d_3^*} R^*_{s+1} \otimes A \tag{15}$$

with differentials given by

$$\begin{aligned} d_1^*(a) &= \sum_{i \in E} \hat{x}_i \otimes x_i a, \\ d_2^*(\hat{x}_k \otimes a) &= \sum_{i \in E} \hat{r}_i \otimes r_{i,k} a, \\ d_3^*(\hat{r}_k \otimes a) &= \hat{\Phi} \otimes x_k a \end{aligned}$$

for all $a \in A$ and all $k \in E$. We can also write down the differentials of the Koszul complex K_R of $\Bbbk$ as right A-module. We know it is exact for the same reason the left complex is.

$$0 \longrightarrow R_{s+1} \otimes A \xrightarrow{d_3} R \otimes A \xrightarrow{d_2} V \otimes A \xrightarrow{d_1} A \twoheadrightarrow \Bbbk \tag{16}$$

with the canonical augmentation $A \to \Bbbk$ and differentials given by

$$\begin{aligned} d_1'(x_k \otimes a) &= x_k a, \\ d_2'(r_k \otimes a) &= \sum_{i \in E} x_i \otimes \tilde{r}_{i,k} a, \\ d_3'(\Phi \otimes a) &= \sum_{i \in E} r_i \otimes x_i a \end{aligned}$$

for all $a \in A$ and all $k \in E$.

Comparing these formulas with those of the differential in the Koszul dual complex we see at once that the complexes (16) and (15) are isomorphic as right A-module complexes, since $r_{i,k} = \tilde{r}_{i,k}$. It follows from this that

$$\mathsf{Ext}^p_A(\Bbbk, A) \cong \begin{cases} 0, & \text{if } p \neq 3; \\ \Bbbk(s+1), & \text{if } p = 3. \end{cases}$$

By definition, then, A is Gorenstein of dimension 3 and parameter $s+1$.

Let us now consider the diagram of A-bimodules

$$0 \longrightarrow A \otimes R_{s+1} \otimes A \xrightarrow{d_3} A \otimes R \otimes A \xrightarrow{d_2} A \otimes V \otimes A \xrightarrow{d_1} A \otimes A \xrightarrow{\mu} A \tag{17}$$

where $\mu : A \otimes A \to A$ is the multiplication map, and

$$\begin{aligned}
d_1(a \otimes x_k \otimes b) &= ax_k \otimes b - a \otimes x_k b, \\
d_2(a \otimes r_k \otimes b) &= \sum_{\{i_1,\dots i_s,k\} \in S} \varepsilon_{i_1,i_2,\dots,i_s} \sum_{1 \le j \le s} a \dots x_{i_{j-1}} \otimes x_{i_j} \otimes x_{i_{j+1}}, \dots x_{i_s} b \\
d_3(a \otimes \Phi \otimes b) &= \sum_{i \in E} \bigl(ax_i \otimes r_i \otimes b - a \otimes r_i \otimes x_i b \bigr)
\end{aligned}$$

for all $a, b \in A$ and all $k \in E$. A straightforward computation, which we leave to the reader, shows that this is a complex, which we write K_{LR}. Moreover, K_{LR} is known as the bimodule Koszul complex of A and it follows from [Ber01, Theorem 5.6] that it is exact iff A is Koszul, which we have already proven.

In order to compute $\mathrm{Ext}^\bullet_{A^e}(A, A \otimes A)$ from the complex obtained from (17) one has to apply the functor $\hom_{A^e}(-, A \otimes A)$. We could compute the dual complex and check by hand that is isomorphic to the original one, in the same manner as with K_L^* (16) and K_R (13), but we are able to skip the computation recalling [BS13, Theorem 2.7] which allows us to deduce from the fact that A is Koszul, $\mathrm{gldim}\, A = 3$ and the knowledge of the Hilbert series that K_{LR} (17) is indeed self dual as A bimodule complex.
This tells us that A is 3-Calabi-Yau. □

5.3. Corollary. *For each A-bimodule M, there is a natural isomorphism*

$$HH^\bullet(A, M) \cong HH_{3-\bullet}(A, M).$$

This follows immediately from Proposition 5.2 using the main result of van den Bergh [vdB98]. Now that we know that A has finite global dimension, we can also prove the following.

5.4. Proposition. *If $(n, s) \neq (3, 2)$, the algebra A is neither left nor right noetherian and has infinite Gel′fand-Kirillov dimension.*

Proof. Let $i_1, i_2, \dots, i_s \in E$ be s distinct points, and consider the ideal

$$I = \langle x_k : k \in E \setminus \{i_1, \dots, i_s\} \rangle \subset A.$$

Set $j = \star(i_1, \dots, i_s)$ and let $B = \{i_1, \dots, i_s, j\} \in S$. We have $r_k \in I$ if $k \in E$ is different from j, and $r_j \equiv \partial_{x_j} \phi_B \mod I$, a certain sub-linear polynomial of degree s. From 3.1 it follows that the Hilbert series of the left A-module A/I is $h_{A/I}(t) = (1 - st + t^s)^{-1}$.

Suppose now that A is left noetherian. Since $\mathrm{gldim}\, A < \infty$, A/I has a free resolution of finite length by finitely generated graded modules and, in particular, there is a polynomial $p \in \mathbb{Z}[t]$ such that $h_{A/I} = h_A p$. In the case $s = 2$ the order of the pole 1 at both rational functions brings a contradiction. If $s > 2$, it suffices to find a pole in $h_{A/I}$ that does not appear in h_A. As it is easy to check that $h_{A/I}$ has only simple poles, so it is equivalent to prove that $q = 1 - st + t^s$ does not divide $f = 1 - nt + nt^s - t^{s+1}$. But the remainder $f + tq - nq$ has non-zero constant term so it is non-zero.

By symmetry, looking at the Hilbert series of the right A-module A/I, we prove the first claim.

The second claim, that the Gel′fand-Kirillov dimension of A is infinite, follows from [SZ97, Corollary 2.2] and the fact that the Hilbert series of A computed in Proposition 2.3 has poles at points which are not roots of unity. Indeed,

$$1 - nt + nt^s - t^{s+1} = (1-t)\Big(\sum_{i=0}^{s} t^i - nt\big(\sum_{i=0}^{s-2} t^i\big)\Big),$$

and the second factor has a real root between 0 and 1 by Bolzano's theorem. □

6 Derivations

Let $\mathfrak{gl}(V)$ be the Lie algebra of endomorphisms of V, which we identify with $\mathfrak{gl}(n, \Bbbk)$ by fixing the basis $X = \{x_i : i \in E\}$ of V. Considering $V^{\otimes(s+1)}$ as a $\mathfrak{gl}(V)$-module with its natural diagonal action, we let

$$\mathfrak{s} = \mathfrak{s}(S) = \{g \in \mathfrak{gl}(V) : g \cdot \Phi = 0\}. \tag{18}$$

This is a Lie subalgebra of $\mathfrak{gl}(V)$.

6.1. Proposition. *Suppose that $n \geq s+1$ and Φ is 2-sincere.*

(i) *The matrix $g = (g_{i,j})_{i,j \in E} \in \mathfrak{gl}(V)$ is in $\mathfrak{s}$ iff*

$$\sum_{j=1}^{s+1} \varepsilon_{i_{j+1},\dots,i_{s+1},i_1,\dots,i_{j-1}} g_{\star(i_1,\dots,\hat{i}_j,\dots,i_{s+1}),i_j} = 0 \tag{$\mathbf{E}^1_{i_1,i_2,\dots,i_{s+1}}$}$$

for all choices of $(i_1, \dots, i_{s+1}) \in E^{s+1}$ with $|\{i_1, \dots, i_{s+1}\}| = s+1$ and

$$(\varepsilon_{i_{j+1},\dots,i_{s+1},i_1,\dots,i_{j-1}} + \varepsilon_{i_{k+1},\dots,i_{s+1},i_1,\dots,i_{k-1}}) g_{l,i} = 0 \tag{$\mathbf{E}^2_{i_1,i_2,\dots,i_{s+1}}$}$$

for all choices of $(i_1, \dots, i_{s+1}) \in E^{s+1}$ with $|\{i_1, \dots, i_{s+1}\}| = s$ where $i_j = i_k = i$ are the only repeated elements in the tuple and the s-set points to l in the Steiner system. If that is the case, then g has zero diagonal.

(ii) *If we let $\mathfrak{gl}(V)$ act on $V^{\otimes s}$ diagonally, then $\mathfrak{s} \oplus \Bbbk\,\mathrm{id}$ is precisely the subalgebra of $\mathfrak{gl}(V)$ of elements that preserve the subspace R spanned by $\{r_1, \dots, r_n\}$ and $\mathfrak{s} = (\mathfrak{s} \oplus \Bbbk\,\mathrm{id}) \cap \mathfrak{sl}(V)$. The $\mathfrak{s}$-modules R and V^* are isomorphic and $\mathfrak{s}$ is an* algebraic *Lie subalgebra of $\mathfrak{gl}(V)$.*

(iii) *The Lie algebra $\mathfrak{s} \oplus \Bbbk\,\mathrm{id}$ acts faithfully by homogeneous derivations of degree 0 on A, and this action provides an isomorphism $\mathfrak{s} \oplus \Bbbk\,\mathrm{id} \to \mathsf{Der}^0(A)$ to the space of all such derivations.*

Proof. As exposed in the proof of Proposition 5.1, 2-sincerity implies $(V \otimes R) \cap (R \otimes V)$ is spanned by Φ.

(i) We can compute

$$\begin{aligned} g \cdot \Phi &= \sum_{i_1,\dots,i_{s+1},l \in E} \Big(\sum_{j=1}^{s+1} \varepsilon_{i_1,\dots,i_{s+1}} g_{i_j,l}\, x_{i_1} \dots x_{i_{j-1}} x_l x_{i_{j+1}} \dots x_{i_{s+1}} \Big) \\ &= \sum_{i_1,\dots,i_{s+1} \in E} \sum_{l \in E} \Big(\sum_{j=1}^{s+1} \varepsilon_{l,i_{j+1},\dots,i_{s+1},i_1,\dots,i_{j-1}} g_{l,i_j} \Big) x_{i_1} \dots x_{i_{s+1}} \end{aligned}$$

so $g \in \mathfrak{s}$ iff for all $(i_1, \dots, i_{s+1}) \in E^{s+1}$ we have

$$\sum_{l \in E} \Big(\sum_{j=1}^{s+1} \varepsilon_{l,i_{j+1},\dots,i_{s+1},i_1,\dots,i_{j-1}} g_{l,i_j} \Big) = 0. \tag{21}$$

The statement follows from this since, for example,

$$\sum_{l\in E} \varepsilon_{l,i_{j+1},\dots,i_{s+1},i_1,\dots,i_{j-1}} g_{l,i_j} = \varepsilon_{i_{j+1},\dots,i_{s+1},i_1,\dots,i_{j-1}} g_{\star(i_1,\dots,\hat{i}_j,\dots,i_{s+1}),i_j},$$

whenever the elements $i_1,\dots,\hat{i}_j,\dots,i_{s+1}$ are all distinct and the fact that equation (21) holds trivially when $|\{i_1,\dots,i_{s+1}\}| < s$. This is because if $|\{i_1,\dots,i_{s+1}\}| < s$ when we switch one of the elements for a different one we get a tuple of $s+1$ points but at most with s different elements, and $\varepsilon = 0$ whenever the indexes are not distinct.

Let us suppose now that $g \in \mathfrak{s}$. If $B = \{i_1,\dots,i_{s+1}\} \in S$, labeled in a way such that $\varepsilon_{i_1,\dots,i_{s+1}} \neq 0$, which is possible since $\phi_B \neq 0$. Then the condition ($\mathbf{E}^1_{i_1,i_2,\dots,i_{s+1}}$) tells us that $\sum_{i\in B} g_{i,i} = 0$. so summing over all blocks and using the fact that each element appears in $\binom{n-1}{s-1}/s$ blocks due to A.1, we see that $\binom{n-1}{s-1}/s \operatorname{tr} g = 0$.

Moreover if we now sum over all the blocks that contain a given element $i \in E$, again applying lemma A.1 we see that

$$0 = \frac{\binom{n-1}{s-1}}{s} g_{i,i} + \frac{\binom{n-2}{s-2}}{s-1} \sum_{j\in E\setminus\{i\}} g_{j,j} = \left(\frac{\binom{n-1}{s-1}}{s} - \frac{\binom{n-2}{s-2}}{s-1} \right) g_{i,i} + \frac{\binom{n-2}{s-2}}{s-1} \operatorname{tr} g.$$

The coefficient of $g_{i,i}$ in the right hand side is non-zero because it equals the necessarily positive amount of blocks that contain a given point, but do not contain a second given point. As a consequence of this and the fact that $\operatorname{tr} g = 0$, we see that $g_{i,i} = 0$.

(ii) At this point it is clear that $\mathfrak{s} \cap \Bbbk \operatorname{id} = 0$, and it is obvious that id preserves R. Let $g = (g_{i,j})_{i,j\in E} \in \mathfrak{s}$. To see that g preserves R, it is enough to show that for all $k \in E$ we have

$$g \cdot r_j = -\sum_{i\in E} g_{i,j} r_i. \tag{22}$$

One readily computes that

$$0 = g \cdot \Phi = g \cdot \sum_{i\in E} x_i r_i = \sum_{i\in E} (x_i(g\cdot r_i) + (g\cdot x_i) r_i) = \sum_{i\in E} \left(x_i(g\cdot r_i) + \left(\sum_{j\in E} g_{i,j} x_j \right) r_i \right)$$

and interchanging the summation variables i and j in the first sum and regrouping, we get

$$\sum_{j\in E} x_j \left(g\cdot r_j + \sum_{i\in E} g_{i,j} r_i \right) = 0.$$

So $g \cdot r_j + \left(\sum_{i\in E} g_{i,j} r_i\right) = 0$, and equation (22) holds. We see that the algebra $\mathfrak{s} \oplus \Bbbk \operatorname{id}$ preserves R.

Conversely, suppose that $g = (g_{i,j})_{i,j\in E} \in \mathfrak{gl}(V)$ preserves R. Then g preserves the subspaces $V \otimes R$ and $R \otimes V$ of $V^{\otimes(s+1)}$ and, as a consequence, also their intersection

$R_{s+1} = (V \otimes R) \cap (R \otimes V)$, which we know is spanned by Φ. It follows that $g \cdot \Phi = \lambda\Phi$ for some scalar $\lambda \in \Bbbk$. We can write $g = (g - \frac{\lambda}{s+1}\mathrm{id}) + \frac{\lambda}{s+1}\mathrm{id} \in \mathfrak{s} \oplus \Bbbk\,\mathrm{id}$.

That $\mathfrak{s} = (\mathfrak{s} \oplus \Bbbk\,\mathrm{id}) \cap \mathfrak{sl}(V)$ is clear, because the elements of $\mathfrak{s}$ have zero diagonal. Since $\mathfrak{s} \oplus \Bbbk\,\mathrm{id}$ is precisely the subalgebra of $\mathfrak{gl}(V)$ which preserves $R \subseteq V^{\otimes s}$, the criterion given by Claude Chevalley in [Che47, Lemma 1] implies at once that $\mathfrak{s} \oplus \Bbbk\,\mathrm{id}$ is an algebraic subalgebra of $\mathfrak{gl}(V)$ and therefore $\mathfrak{s}$, which is its intersection with $\mathfrak{sl}(V)$, is also algebraic.

(iii) As usual, $\mathfrak{gl}(V)$ acts on the tensor algebra $T(V)$ by homogeneous derivations of degree 0, and by restriction so does $\mathfrak{s}$. Since $\mathfrak{s} \oplus \Bbbk\,\mathrm{id}$ preserves R, it preserves the ideal I generated by R, and then there is an induced action of $\mathfrak{s}$ on the quotient $A = T(V)/I$, which is evidently homogeneous of degree 0. This action is faithful, because its restriction to the degree one component $A_1 = V$ of $T(V)$ is the tautological representation.

Finally, suppose that $d : A \to A$ is an homogeneous derivation of degree 0. Identifying the homogeneous component A_1 with V, we get by restriction a linear map $g = d|_V : V \to V$. The diagonal action of g on $V^{\otimes s}$ clearly preserves R, so *(ii)* implies that $g \in \mathfrak{s} \oplus \Bbbk\,\mathrm{id}$. This proves the last statement. $\square$

It is to be expected, in view of the description given in Proposition 6.1, that the structure of the Lie algebra $\mathfrak{s}$ will strongly depend, in general, on the combinatorial information of the Steiner system under consideration. One possible way to approach the this problem is to focus first on *local* combinatorial information. Examples of this are presented in [SA13].

However, in the examples exposed there ε is always anti-symmetric, making condition ($\mathbf{E}^2_{i_1,i_2,\ldots,i_{s+1}}$) trivial. If ε being anti-symmetric fails for "enough" values of indexes we find $\mathfrak{s} = 0$.

6.2. Proposition. *Suppose that $n \geq s+1$ and Φ is 2-sincere. If for each distinct $a, b \in E$ there exists a block $B = \{i_1, \ldots, i_{s+1}\} \in S$ containing a and b such that anti-symmetry fails when interchanging a and b, namely*

$$\varepsilon_{i_1,\ldots,a,\ldots,b,\ldots,i_{s+1}} + \varepsilon_{i_1,\ldots,b,\ldots,a,\ldots,i_{s+1}} \neq 0$$

then $\mathfrak{s} = 0$.

Proof. Fix a matrix $g \in \mathfrak{s}$. For each distinct $a, b \in E$ let $I_{a,b} = (i_1, \ldots, i_{s+1})$ be the ordered block we get by hypothesis. If $I'_{a,b}$ is the tuple $(i_1, \ldots, i_{s+1})$ but replacing a for b and $I''_{a,b}$ is the result of replacing b for a, the equation ($\mathbf{E}^2_{i_1,i_2,\ldots,i_{s+1}}$) $\mathbf{E}^2_{I'_{a,b}}$ and $\mathbf{E}^2_{I''_{a,b}}$ imply $g_{a,b} = g_{b,a} = 0$. We also know from 6.1 that g has zero diagonal, then $g = 0$. $\square$

We have so far studied the structure of the degree preserving derivations, and another group we would like to study is the vector space of derivations of degree -1. However, the structure of $HH^1(A(S))_{-1}$ does not depend at all on the structure of the Steiner system, but on the nature of the potential considered.

6.3. Proposition. *If $\{\hat{x}_i\}_{i\in E}$ is the dual basis of our fixed basis $\{x_i\}_{i\in E}$ of V, there exists an isomorphism*

$$HH^1(A(S))_{-1} \cong \langle \hat{x}_i : r_i \in [V, V^{\otimes(s-1)}] \rangle_{\Bbbk}$$

Proof. As in the previous example, V^* acts on $T(V)$ by homogeneous derivations of degree -1. For a lineal form $\alpha \in V^*$ to act on $A(S)$, we need that $\alpha(r_i) \in (R)$ for each $i \in E$, but as $\alpha(R) \in A(S)_{s-1} = V^{\otimes(s-1)}$, this implies $\alpha(r_i) = 0$ for all $i \in E$.

We can write $\alpha = \sum_{i\in E} \alpha_i \hat{x}_i$ for some scalars $\alpha_i \in \Bbbk$. By evaluating in monomials, one can check that the following identity holds:

$$\hat{x}_j \cdot r_i = \hat{x}_j \cdot \partial_{x_i}\Phi = \partial_{x_i}\partial_{x_j}\Phi = \partial_{x_i} r_j.$$

This implies that $\alpha(R) = 0$ is equivalent to $\partial_{x_j}\left(\sum_{i\in E} \alpha_i r_i\right) = 0$ for all $j \in E$. Applying the Euler relation (1) this means that $c(\sum_{i\in E} \alpha_i r_i) = 0$, so $\sum_{i\in E} \alpha_i r_i$ and zero are conjugates by the cyclic action, which happens if and only if $\sum_{i\in E} \alpha_i r_i \in [V, V^{\otimes(s-1)}]$. As the space of commutators is homogeneous using the $\mathbb{Z}^n$ grading, and all the r_i lie in disjoint homogeneous spaces, this is again equivalent to the fact that $r_i \in [V, V^{\otimes(s-1)}]$ for each $i \in E$ such that $\alpha_i \neq 0$, which completes the proof. $\square$

7 Automorphisms

In this section we will study the group of homogeneous automorphisms of $A(S)$, denoted $\mathsf{Aut}\,(A(S))$, locally around the identity. Observe that one can think of $\mathsf{Aut}\,(A(S))$ as the subgroup of $\mathsf{GL}(V)$ that fixes the vector space R. Throughout this section, the ground field $\Bbbk$ will be the field of complex numbers $\mathbb{C}$.

Let $\mathsf{GL}(V)$ be the Lie group of automorphisms of V, which we identify with $\mathsf{GL}(n,\Bbbk)$ by fixing the basis $X = \{x_i : i \in E\}$ of V. Considering $V^{\otimes(s+1)}$ as a $\mathsf{GL}(V)$-module with its natural diagonal action, we let

$$\mathbb{S} = \mathbb{S}(S) = \{g \in \mathsf{GL}(V) : g \rightharpoonup \Phi = \Phi\}.$$

This is a subgroup of $\mathsf{GL}(V)$. We use the notation $\rightharpoonup$ to distinguish between the Lie algebra and Lie group actions. A second trivial observation is that the actions commute when the corresponding matrices commute.

7.1. Proposition. *Suppose that $n \geq s+1$ and Φ is 2-sincere.*

(i) *The group $\mathbb{S}$ is a algebraic Lie subgroup of $\mathsf{GL}(V)$.*

(ii) *The Lie algebra $\mathbf{Lie}(\mathbb{S})$ of $\mathbb{S}$ is $\mathfrak{s}$, as defined in equation (18).*

(iii) *The Lie group $\Bbbk^{\times}\mathbb{S}$ acts faithfully by homogeneous automorphisms of $A(S)$, and this action provides a group isomorphism $\Bbbk^{\times}\mathbb{S} \to \mathsf{Aut}\,(A\,(S))$.*

Proof. (i) The equations that determine if a matrix g belongs to $\mathbb{S}$, namely $g \rightharpoonup \Phi = \Phi$, are algebraic in the coefficients of the matrix g. Thus $\mathbb{S}$ is an algebraic subgroup of $\mathsf{GL}(V)$, hence it is closed. It follows that it is a Lie subgroup.

(ii) First observe that if $g \in \mathfrak{s}$, we can construct the map $\alpha : \mathbb{R} \to V^{\otimes(s+1)}$ such that $\alpha(t) = \exp(tg) \rightharpoonup \Phi$. Clearly $\alpha(0) = \Phi$. Also, one can compute the derivative $\alpha'(t)$ which equals

$$\alpha'(t) = g \cdot (\exp(tg) \rightharpoonup \Phi) = \exp(tg) \rightharpoonup (g \cdot \Phi) = 0.$$

As a consequence, $\exp(tg)$ belongs to $\mathbb{S}$ for each $t \in \mathbb{R}$. This implies that $\mathfrak{s} \subseteq \mathbf{Lie}(\mathbb{S})$.

For the converse, if $g \in \mathbf{Lie}(\mathbb{S})$ is a tangent vector, then one can define the same curve α, and as $\exp(tg)$ falls inside $\mathbb{S}$, then α is constant and equal to Φ. This implies that $0 = \alpha'(0) = g \cdot \Phi$. Hence $\mathbf{Lie}(\mathbb{S}) \subseteq \mathfrak{s}$.

(iii) Let $g \in \mathsf{GL}(V)$ be an automorphism that fixes R. Using the same argument we used for derivations, one can deduce that g fixes $V \otimes R$ and $R \otimes V$ and as a consequence their intersection $R_{s+1} = \langle \Phi \rangle_{\Bbbk}$. Therefore $g \rightharpoonup \Phi = \lambda\Phi$ for some scalar $\lambda \in \Bbbk$. It cannot be zero, since $g^{-1} \rightharpoonup (g \rightharpoonup \Phi) = \Phi$. As $\Bbbk$ is algebraically closed, there exists a scalar $\mu \in \Bbbk^{\times}$ such that $(\mu g) \rightharpoonup \Phi = \Phi$. This proves that every element of $\mathsf{Aut}\,(A\,(S))$ is included in $\Bbbk^{\times}\mathbb{S}$. For the converse, if $g \in \mathbb{S}$ we will prove that $g \rightharpoonup r_i = \sum_{j \in E} g^{j,i} r_j$ if $(g^{i,j})_{1 \leq i,j \leq n}$ is the inverse matrix of g. Indeed we have:

$$\sum_{j \in E} x_j r_j = \Phi = g \rightharpoonup \Phi = \sum_{i \in E} (g \rightharpoonup x_i)(g \rightharpoonup r_i) = \sum_{i,j \in E} g_{i,j} x_j (g \rightharpoonup r_i)$$

which implies

$$r_j = \sum_{i \in E} g_{i,j}(g \rightharpoonup r_i).$$

If we fix $k \in E$ and sum over $j \in E$, multiplying by $g^{j,k}$ we obtain:

$$\sum_{j \in E} g^{j,k} r_j = \sum_{i,j \in E} g_{i,j} g^{j,k}(g \rightharpoonup r_i) = \sum_{i \in E} \delta_i^k (g \rightharpoonup r_i) = g \rightharpoonup r_k.$$

This proves that $g \rightharpoonup R \subseteq R$, as we wanted. $\square$

Note that the knowledge of the Lie algebra $\mathfrak{s}$ allows us to describe the Lie group $\mathbb{S}$ in the connected component of the identity. Although this gives us a good characterization of the group of automorphisms locally, we can expect to have automorphisms in different connected components. For example if we have an automorphism σ of the Steiner system that preserves the choice of scalars, clearly σ will induce an automorphism of V that preserves the space of relations.

8 About $HH^2(A)$ and $HH^3(A)$

In order to compute higher cohomology groups, the first step would be to calculate the complex that generates those groups. We have already constructed a free bimodule resolution of $A(S)$ in (17), which we called K_{LR}, assuming either $s > 2$ and Φ is 3-sincere, or $s = 2$ and Φ is 2-sincere. We will maintain those assumptions throughout this section as well. Applying the functor $\hom_{A^e}(-, A)$ and modulo standard identifications, we obtain the following complex of vector graded spaces.

$$0 \longleftarrow A \otimes R^*_{s+1} \xleftarrow[d_3^*]{} A \otimes R^* \xleftarrow[d_2^*]{} A \otimes V^* \xleftarrow[d_1^*]{} A \tag{23}$$

with differentials given by

$$\begin{aligned} d_1^*(a) &= \sum_{k \in E} [x_k, a] \otimes \hat{x}_k, \\ d_2^*(a \otimes \hat{x}_i) &= \sum_{k \in E} \left(\sum_{\{i_1, \dots, i_j = i, \dots i_s, k\} \in S} \varepsilon_{i_1, i_2, \dots, i_s} x_{i_1} \dots x_{i_{j-1}} a x_{i_{j+1}}, \dots x_{i_s} \otimes \hat{r}_k \right) \\ d_3^*(a \otimes \hat{r}_i) &= [x_i, a] \otimes \hat{\Phi} \end{aligned}$$

for all $a \in A$ and all $i \in E$.

The dual spaces should be regarded as concentrated in the opposite degree of their pre-duals, so that all differentials preserve degrees. Observe that the rather complicated definition of d_2^* can be very much simplified by considering an element of $A \otimes V^* \cong \hom_{\Bbbk}(V, A)$ acting as a derivation on $T(V)$ with the diagonal action. Then $d_2^*(a \otimes \hat{x}_i)$ can actually be written as $\sum_{k \in E} (a \otimes \hat{x}_i) \cdot r_k \otimes \hat{r}_k$.

A second observation is that we can easily compute the Euler characteristic of $HH^\bullet(A)$.

8.1. Lemma. *The Euler characteristic $\chi(t)$ of $HH^\bullet(A)$ is*

$$\chi(t) = -t^{-s-1}.$$

Therefore it is -1 in degree $-s-1$ and zero in all the other degrees.

Proof. As $HH^\bullet(A)$ is the homology of the complex (23), their Euler characteristic coincide. We can now compute the Euler characteristic of (23) using the knowledge of the Hilbert series of A and find that

$$\begin{aligned} \chi(t) &= h_A(t) - t^{-1} n h_A(t) + t^{-s} n h_A(t) - t^{-s-1} h_A(t) \\ &= -t^{-s-1} h_A(t)(1 - nt + nt^s - t^{s+1}) \\ &= -t^{-s-1}. \end{aligned}$$

□

Before we can continue our calculations, we need to recall some combinatoric facts. The number of equivalence classes of conjugate words of length ℓ in n letters is exactly the same as the amount of necklaces with ℓ circularly connected beads of up to n different colors, taking all rotations as equivalent. This number, denoted $\mathcal{N}_n(\ell)$ is known to be

$$\mathcal{N}_n(\ell) = \frac{1}{\ell} \sum_{d|\ell} \varphi(d) n^{\ell/d}.$$

where φ is Euler's totient function.

Now we can begin making some computations. Note that structure of the space of relations does not appear at all in the lowest degrees.

8.2. Lemma. *In degree $-s-1$ we have that*

$$HH^p(A)_{-s-1} = \begin{cases} \Bbbk, & \text{if } p = 3 \\ 0, & \text{otherwise,} \end{cases} \tag{24}$$

and if $e \in \mathbb{Z}$ is such that $-(s+1) < e < -1$,

$$HH^p(A)_e = \begin{cases} \frac{V^{\otimes(s+1+e)}}{[V, V^{\otimes(s+e)}]} \cong \Bbbk^{\mathcal{N}_n(s+1+e)}, & \text{if } p = 2, 3 \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

Proof. First observe that when looking at the complex (23) at degree $-(s+1)$, only one of the vector spaces is non zero, which proves equation (24).

For the second part let $-(s+1) < e < -1$. Then it is evident that all the graded components of $A(S)$ that appear in the complex at degree e are in a degree smaller than the degree of the relations, so everything happens on $T(V)$. Also, there are only non-zero spaces at homological degrees 2 and 3. Hence, we have

$$HH^3(A)_e = \operatorname{coker}(d_3^*)_e = \frac{V^{\otimes(s+1+e)}}{[V, V^{\otimes(s+e)}]}.$$

This is isomorphic to the space of cyclic polynomials of degree $s+1+e$, which has dimension $\mathcal{N}_n(s+1+e)$, identifying the cyclic polynomial $c(m)$ associated to a monomial m with the corresponding necklace. Since the Euler characteristic is zero at degree e, which follows from lemma 8.1, we know that $HH^2(A)_e = \ker(d_3^*)_e$ will have the same dimension
In view of lemmas 1.4 and 1.5 we know that

$$\ker(d_3^*)_e = \langle \sum_{i \in E} \partial_{x_i} \alpha \otimes \hat{r}_i : \alpha \in V^{\otimes(s+1+e)} \rangle_{\Bbbk}.$$

and if we only allow α to be a cyclic polynomial we get a basis of $\ker(d_3^*)_e$. The result from equation (25) follows from that. $\square$

At degree -1 things begin to get more interesting. The space R of relations appears in the quotient that computes $HH^3(A)_{-1}$, since one needs to consider the homogeneous space $A(S)_s$. Indeed, we can prove the following.

8.3. Proposition. *The dimension of $HH^3(A)_{-1}$ is*

$$\mathsf{dim}\, HH^3(A)_{-1} = \mathscr{N}_n(s) - \#\left\{i : r_i \notin [V, V^{\otimes(s-1)}]\right\}$$

and the dimension of $HH^2(A)_{-1}$ equals

$$\mathsf{dim}\, HH^2(A)_{-1} = \mathscr{N}_n(s) + 2\#\left\{i : r_i \in [V, V^{\otimes(s-1)}]\right\} - n.$$

Proof. First observe that

$$HH^3(A)_{-1} = \mathsf{coker}(d_3^*)_{-1} = \frac{V^{\otimes s}}{[V, V^{\otimes(s-1)}] + R}.$$

If we want to know the dimension of the quotient, it suffices to compute the dimension of the intersection $[V, V^{\otimes(s-1)}] \cap R$. Observe that $[V, V^{\otimes(s-1)}]$ is homogeneous in the $\mathbb{Z}^n$ grading that counts the degree of each $x \in X$. As no pair of monomials $u \in r_i$ and $v \in r_j$ have the same degree, when $i \neq j$, we deduce that

$$[V, V^{\otimes(s-1)}] \cap R = \langle r_i : r_i \in [V, V^{\otimes(s-1)}]\rangle_{\Bbbk}.$$

Thus the dimension of $HH^3(A)_{-1}$ is $\mathscr{N}_n(s) - \#\left\{i : r_i \notin [V, V^{\otimes(s-1)}]\right\}$, as we wanted. The morphism c identifies this space with the space of cyclic polynomials modulo $\langle c(r_i) : r_i \notin [V, V^{\otimes(s-1)}]\rangle_{\Bbbk}$. We already know the dimension of $HH^1(A)_{-1}$ from Proposition 6.3. Finally the fact that Euler characteristic is zero allows us to deduce that

$$\begin{aligned}\mathsf{dim}_{\Bbbk}\, HH^2(A)_{-1} &= \mathscr{N}_n(s) - \#\left\{i : r_i \notin [V, V^{\otimes(s-1)}]\right\} + \#\left\{i : r_i \in [V, V^{\otimes(s-1)}]\right\} \\ &= \mathscr{N}_n(s) + 2\#\left\{i : r_i \in [V, V^{\otimes(s-1)}]\right\} - n,\end{aligned}$$

which proves the proposition. $\square$

To understand the nature of the space $HH^2(A)_{-1}$ we may first observe that $\mathsf{ker}(d_3^*)_{-1}$ has dimension

$$\begin{aligned}\mathsf{dim}\,\mathsf{ker}(d_3^*)_{-1} &= n\,\mathsf{dim}(A_{s-1}) - \mathsf{dim}\,\mathsf{im}(d_3^*)_{-1} \\ &= \mathsf{dim}\,\mathsf{coker}(d_3^*)_{-1} + n\,\mathsf{dim}(A_{s-1}) - \mathsf{dim}(A_s) \\ &= \mathscr{N}_n(s) - \#\left\{i : r_i \notin [V, V^{\otimes(s-1)}]\right\} + n \\ &= \mathscr{N}_n(s) + \#\left\{i : r_i \in [V, V^{\otimes(s-1)}]\right\}.\end{aligned}$$

As before, this corresponds with cyclic polynomials of length s that generate elements of the kernel, but this time we need to add more elements. For each $i \in E$ such that r_i is a commutator, we can find $(a_j)_{j\in E} \in V^{\otimes(s-1)}$ such that $\sum_{j\in E}[x_j, a_j] = r_i$, and in that way construct an element $\sum_{j\in E} a_j \otimes \hat{r}_j$ inside the kernel. Similarly, we know that $\mathrm{im}(d_2^*)_{-1}$ has a basis consisting of $\sum_{j\in E} \hat{x}_i \cdot r_j \otimes \hat{r}_j$, where $i \in E$ is such that $c(r_i) \neq 0$ and $\hat{x}_i$ is regarded as a derivation of degree -1 acting on $V^{\otimes s}$.

For the last part of this section we will study the cohomology space at degree 0.

8.4. Proposition. *The dimension of $HH^3(A)_0$ is*

$$\dim HH^3(A)_0 = \mathscr{N}_n(s+1) - n^2 + \dim \mathfrak{s}.$$

and the dimension of $HH^2(A)_0$ equals

$$\dim HH^2(A)_0 = \mathscr{N}_n(s+1) - n^2 + 2\dim \mathfrak{s}.$$

Proof. As in degree -1, we find that

$$HH^3(A)_0 = \mathrm{coker}(d_3^*)_0 = \frac{V^{\otimes(s+1)}}{[V, V^{\otimes s}] + V \otimes R + R \otimes V} = \frac{V^{\otimes(s+1)}}{[V, V^{\otimes s}] + R \otimes V}.$$

The last equality holds because $x_i \otimes r_j = r_j \otimes x_i + [x_i, r_j] \in R \otimes V + [V, V^{\otimes s}]$ for each $i, j \in E$. Then we need to compute the intersection $[V, V^{\otimes s}] \cap (R \otimes V)$. An element $\sum_{i,j\in E} g_{i,j} r_i x_j$ belong to $[V, V^{\otimes s}]$ if and only if the sum of the coefficients of all the rotations of a single monomial is zero. Namely, if and only if

$$\sum_{k=1}^{s+1} \varepsilon_{i_{k+1},\dots,i_{s+1},i_1,\dots,i_{k-1}} g_{\star(i_1,\dots,\hat{i}_k,\dots,i_{s+1}),i_k} = 0, \quad \text{for all tuples } i_1, \dots, i_{s+1} \in E.$$

Observe that these are exactly the equations that a matrix $(g_{i,j}) \in \mathfrak{s}$ satisfies, according to equation (21). Hence we know that

$$\dim HH^3(A)_0 = \mathscr{N}_n(s+1) - n^2 + \dim \mathfrak{s}$$

as stated in the proposition. Finally, one can compute $\dim HH^2(A)_0$ using once again the knowledge of the Euler characteristic. $\square$

We can also compute the dimension of the space of 2-cocycles, that generates $HH^2(A)_0$.

$$\begin{aligned} \dim \ker(d_3^*)_0 &= n \dim(A_s) - \dim \mathrm{im}(d_3^*)_0 \\ &= \dim \mathrm{coker}(d_3^*)_0 + n \dim(A_s) - \dim(A_{s+1}) \\ &= \mathscr{N}_n(s+1) + \dim \mathfrak{s} - n^2 + n(n^s - n) - (n^{s+1} - 2n^2 + 1) \\ &= \mathscr{N}_n(s+1) + \dim \mathfrak{s} - 1. \end{aligned}$$

This space can be expressed as a sum of two parts. First we have, as always, the space of cyclic polynomials of degree $s+1$, which generate elements in the kernel via cyclic derivatives, but divided by $\langle \Phi \rangle_{\Bbbk}$, since all its derivatives are zero in A.

Secondly, we have for each g in a fixed basis of $\mathfrak{s}$ a commutator $\sum_{i \in E} g_{i,j} r_i x_j$ which can be expressed in a non-unique way as $\sum_{i \in E}[x_i, a_i]$ for some $a_i \in V^{\otimes s}$. So we can construct an element $\sum_{i \in E} a_i \otimes \hat{r}_i$ in the kernel. We can check that those are linearly independent and, by a dimension argument, that is all there is.

At this point, we can observe a relation between $HH^1(A)$ and $HH^3(A)$ at degrees -1 and 0. More concretely, we have shown that the dimensions of $[V, V^{\otimes(s-1)}] \cap R$ and that of $\mathrm{Der}^{-1}(A)$ coincide, and the dimensions of $[V, V^{\otimes s}] \cap R \otimes V$ and $\mathrm{Der}^0(A)$ differ only by one. So the question arises as to whether this relation occurs in higher degrees.

In that direction, we have found the following:

8.5. Lemma. *Fix $d \geq 0$. Let $\Gamma^k_{i_1,\dots,i_d} \in \Bbbk$ be a scalar for each sequence $(i_1, \dots, i_d)$ of length d and each k in E. We can construct a polynomial p in $V^{\otimes d} \otimes R$ defined as*

$$p = p_\Gamma = \sum_{i_1,\dots,i_d,k \in E} \Gamma^k_{i_1,\dots,i_d} x_{i_1} \dots x_{i_d} r_k.$$

We can also construct a morphism $g : V \to V^{\otimes d}$ which acts diagonally as a derivation of degree $d-1$ on $T(V)$, such that

$$g = g_\Gamma = \sum_{i_1,\dots,i_d,k \in E} \Gamma^k_{i_1,\dots,i_d} x_{i_1} \dots x_{i_d} \otimes \hat{x}_k.$$

Then for each $j \in E$ we have that

$$\partial_{x_j} p = g \cdot r_j + \sum_{k,i_1,\dots,i_{d-1} \in E} \left(\sum_{t=1}^{d} \Gamma^k_{i_1,\dots,i_{t-1},j,i_t,\dots,i_{d-1}} x_{i_t} \dots x_{i_{d-1}} r_k x_{i_1} \dots x_{i_{t-1}} \right).$$

We omit the proof because it requires new notation and is very technical. Recall that a derivation in A is simply a derivation in $T(V)$ that fixes R and that an element is a commutator if and only all its cyclic derivatives vanish. Then a corollary of this is that g_Γ defines a derivation on A if and only if p_Γ falls inside (R). In particular if p_Γ is cyclic, then g_Γ is a derivation.

In degrees -1 and 0, it was easy to determine the quotient space between Γs where ∂p_Γ falls in (R), and when p_Γ is commutator. However in higher degrees this does not seem to work so well. We believe that when A is Koszul the conditions imposed on the intersections of spaces that involve R may help to simplify the computations.

9 Examples

9.1 Lie Potentials

For one of our examples we are interested in the case when the generating relations are all Lie polynomials. We will show that in this case $A(S)$ is automatically a domain.

Fix a ground field $\Bbbk$ of characteristic zero, and a vector space V. We define *Lie polynomials* as the elements of the Lie-subalgebra in $T(V)$ generated by V or, equivalently, the minimal Lie algebra inside $T(V)$ containing V; from now on we denote it $\langle V\rangle_{Lie}$.

An alternate approach is the fact that Lie polynomials are a realization of the free Lie algebra on V. So we have a functor $\mathscr{L}$ from the category of vector spaces $\mathbf{Vect}_\Bbbk$ to the category of Lie algebras $\mathbf{Lie}_\Bbbk$, that is the left adjoint to the forgetful functor $\mathscr{F} : \mathbf{Lie}_\Bbbk \to \mathbf{Vect}_\Bbbk$. In the language of categories, we have an isomorphic natural transformation

$$\mathbf{Vect}_\Bbbk[V, \mathscr{F}(\mathfrak{g})] \cong \mathbf{Lie}_\Bbbk[\mathscr{L}(V), \mathfrak{g}] \quad \text{for every } \mathfrak{g} \in \mathbf{Lie}_\Bbbk.$$

Before we begin studying the functor $\mathscr{L}$, we need to prove its existence. For that we are going to invoke algebras with no structure at all, the linearization of magmas. A magma is a set M with a function $\mu : M \times M \to M$. Morphisms between magmas are functions that preserve the product.

The free magma M_X on a set X is the "most general possible" magma generated by the set X. It can be identified with the set of non-associative words on the alphabet X where μ is concatenation or, equivalently, as the magma of binary trees with leaves labeled by elements of X. The operation is that of joining trees at the root. A free magma has the universal property such that, if $f : X \to N$ is a function from the set X to any magma N, then there is a unique extension of f to a morphism of magmas $\bar{f} : M_X \to N$.

If X is a basis of V, then the formal $\Bbbk$-linearization $M(V) = \langle M_X\rangle_\Bbbk$ of M_X is the free (non-associative, non-Lie) algebra of V, with the multiplication structure inherited from M_X. The length of words in M_X induces a $\mathbb{N}$-grading on $M(V)$. This allow us to represent $M(V)$ recursively with $M(V)_1 = V$ and

$$M(V)_d = \bigoplus_{0<i<d} M(V)_i \otimes M(V)_{d-i}.$$

Finally we obtain a construction of the free Lie algebra $\mathscr{L}(V)$ by taking the quotient of $M(V)$ by all the anti-symmetric and Jacobi relations of Lie algebras. Namely,

$$\mathscr{L}(V) \cong \frac{M(V)}{\langle \mu(a,b) + \mu(b,a), \mu(\mu(a,b),c) + \mu(\mu(b,c),a) + \mu(\mu(c,a),b) : a,b,c \in M(V)\rangle}.$$

The universal property is deduced from the universal characterization of the quotient. This construction however is impossible to work with, so we shall prove it is isomorphic to the set of Lie polynomials.

As the composition of the forgetful functors from the category of associative algebras $\mathbf{Ass}_{\Bbbk} \to \mathbf{Lie}_{\Bbbk} \to \mathbf{Vect}_{\Bbbk}$ is isomorphic to the forgetful functor $\mathbf{Ass}_{\Bbbk} \to \mathbf{Vect}_{\Bbbk}$ and composition of left adjoint functors is again a left adjoint, we have that

$$\mathscr{U}(\mathscr{L}(V)) \cong T(V).$$

where $\mathscr{U}$ is the universal enveloping algebra functor, the free functor from $\mathbf{Lie}_{\Bbbk} \to \mathbf{Ass}_{\Bbbk}$. By applying the universal property of free objects, we can prove that the isomorphism fixes V.

This, together with the fact that $\mathscr{L}(V) \hookrightarrow \mathscr{U}(\mathscr{L}(V))$ is injective, because of Poincaré-Birkhoff-Witt's theorem, implies that $\mathscr{L}(V)$ can be identified with a Lie subalgebra of $T(V)$. Because of the minimality condition, we know $\langle V \rangle_{Lie} \subseteq \mathscr{L}(V)$, and using the universal property, the inclusion $\mathscr{L}(V) \hookrightarrow T(V)$ factors through $\langle V \rangle_{Lie}$. Hence, $\mathscr{L}(V) = \langle V \rangle_{Lie}$ as subspaces of $T(V)$.

The algebra $T(V)$ has a bi-algebra structure given by the co-multiplication Δ which is defined as:

$$\Delta(v) = v \otimes 1 + 1 \otimes v \quad \text{for } v \in V.$$

An element $\alpha \in T(V)$ satisfying $\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$ is called *primitive*. Of course all elements of V are primitive. Also, if α and $\beta \in T(V)$ are primitive, then

$$\Delta([\alpha, \beta]) = [\Delta(\alpha), \Delta(\beta)] = [\alpha \otimes 1 + 1 \otimes \alpha, \beta \otimes 1 + 1 \otimes \beta] = [\alpha, \beta] \otimes 1 + 1 \otimes [\alpha, \beta].$$

So primitive elements form a Lie subalgebra of $T(V)$, and includes $\mathscr{L}(V)$. Actually the converse is true and every primitive element is a Lie polynomial, as shown by Reutenauer in [Reu93, Theorem 1.4].

Therefore, we can think of Lie polynomials in two ways, as the vector space $L \subseteq T(V)$ which contains V, closed under taking commutators, or the *primitive* elements in $T(V)$ under the co-multiplication Δ. Since Δ is graded with the natural grading in $T(V) \otimes T(V)$, L is a graded subspace. We will fix $X = \{x_1, x_2, \ldots, x_n\}$ a basis of V so that

$$T(V) \cong \Bbbk\langle X \rangle = \Bbbk\langle x_1, x_2, \ldots, x_n \rangle.$$

Fix $d \in \mathbb{N}$ and $1 \leq j \leq d-1$. The second characterization means that if $p \in \Bbbk\langle X \rangle$ is an d-homogeneous primitive element and $(x_{i_1}, \ldots, x_{i_d}) \in X^d$, the coefficient of $x_{i_1} x_{i_2} \ldots x_{i_j} \otimes x_{i_{j+1}} \ldots x_{i_d}$ in $\Delta(p)$ is zero. That equals the sum of the $\binom{d}{j}$ coefficients in p that correspond to monomials that arise when mixing together $x_{i_1}, x_{i_2}, \ldots, x_{i_d}$ without changing the relative order of the first j variables, nor the order of the last $d-j$. This is commonly known as a *shuffle*.

Lie brackets preserve $\mathbb{Z}^n$-grading in $\Bbbk\langle X \rangle$. For most cases, we are going to focus in *sub-linear* polynomials, which means we will not allow monomials with repeated variables, although some of the results exposed are also true in general. Two monomials w_1, w_2 are called *disjoint* if they do not share common variables and two polynomials p_1, p_2 are called *disjoint* when every pair of monomials $w_1 \in p_1$, $w_2 \in p_2$ are disjoint.

First let us use the two descriptions we have, to obtain some useful combinatorial facts about Lie polynomials.

9.1. Lemma. *We set τ the anti-algebra automorphism in $T(V)$ such that $\tau(x) = -x$ for $x \in V$. Let $p \in T(V)$ be a Lie polynomial , then $\tau(p) = -p$.*

Proof. As τ is an anti-ring morphism, for any polynomials $u, v \in T(V)$, we have that $\tau([u, v]) = -[\tau(u), \tau(v)]$ so the vector space of elements that satisfy $\tau(p) = -p$ contains V and is closed under brackets, therefore it contains all Lie polynomials. □

9.2. Corollary. *If $p \in \Bbbk\langle X\rangle$ is a homogeneous Lie polynomial of degree d, then the coefficient of $x_{i_1}x_{i_2}\ldots x_{i_d}$ in p equals $(-1)^{d+1}$ times the coefficient of $x_{i_d}x_{i_{d-1}}\ldots x_{i_1}$.*

Proof. As $\tau(x_{i_1}x_{i_2}\ldots x_{i_d}) = (-1)^d x_{i_d}x_{i_{d-1}}\ldots x_{i_1}$ and the monomial on the left appears in p, then the monomial on the right appears with the same coefficient in $-p$, thus obtaining the result we wanted. □

We can combine Lie polynomials two at a time using brackets in several ways starting with elements in the set X, to obtain d-homogeneous brackets, which generate all Lie polynomials, as seen before. A good way to think about this is as a binary tree T with d leaves, with an element of X at every leaf and each fork represents a bracket. There are $d - 1$ forks. This is illustrated in figure 1. Let p be the polynomial obtained following the associations that T yields.

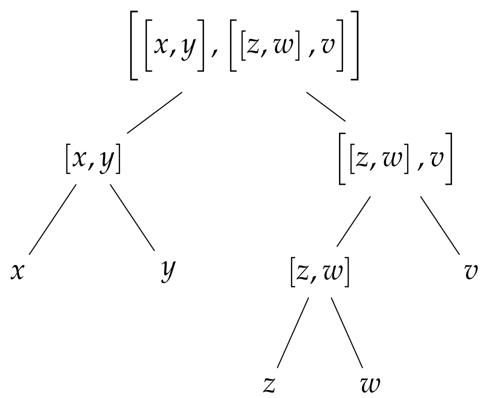


Figure 1. A binary tree with leaves labeled by elements of X, and its associated Lie polynomial in the root of the tree.

If we swap left and right children of any node we get a binary tree $s(T)$ that yields $-p$. We call $m(T)$ the monomial which arises when reading leaf elements from left to right. In the case of sub-linear polynomials (in other words, every leaf has a distinct element in X) we know that a given monomial q appears in p with coefficient 1 (respectively, -1) iff there exists a finite sequence of children swaps such that $m(s_1s_2\ldots s_l(T)) = q$ with l

even (odd). Thus, we obtain 2^{d-1} monomials and the coefficient of every monomial is either 1 or -1.

9.3. Lemma. *If $p \in \Bbbk\langle X\rangle$ is a homogeneous Lie polynomial of degree d, $x \in X$, such that x appears in p, then there is a word in $u \in p$ with x as a prefix.*

By symmetry, or using lemma 9.2, the same is valid for suffixes.

Proof. Without loss of generality assume p is $\mathbb{Z}^n$-homogeneous, so that x appears $1 \leq k \leq n$ times in every monomial in p. Now proceed by induction to prove x does not appear in p at all. By the inductive hypothesis we know no monomial in p has an x in its first $j \leq n-k-1$ places. Then choose any word $u = x_{i_1}x_{i_2}\dots x_{i_d}$ with the same $\mathbb{Z}^n$-grading, with the first x in the position $j+1$. We will see that $u \notin p$. Indeed, look at the coefficient of $x_{i_1}x_{i_2}\dots x_{i_j} \otimes x_{i_{j+1}}\dots x_{i_d}$ in $\Delta(p)$, as a sum of $\binom{d}{j}$ coefficients as described at the beginning of the section. All of them are zero by hypothesis, except the one of the word u we started with. Then u does not appear in p. □

When working with sub-linear d-homogeneous polynomials, any derivative $\partial_x\Phi$ determines Φ if it is cyclic, but the monomials that do not contain x. Simply take $\Phi = c(x\partial_x\Phi)$ So we would like to characterize potentials that generate derivatives which are primitive. The following lemma will come in handy with that purpose.

9.4. Lemma. *Let p, $q \in \Bbbk\langle X\rangle$ be two disjoint sub-linear Lie polynomials and $x \in X$, and $\Phi = pq$. Then $\partial_x\Phi$ is also Lie.*

Proof. If x does not appear in p or q, the result is trivial. Assume p and q are homogeneous. Again, proceed by induction in the degree of the polynomial that contains x, say q.
If $q = x$, the result follows immediately. If not, by linearity of ∂_x we may assume q is a bracket. Write $q = [q_1(x), q_2]$. Then

$$\partial_x(pq) = \partial_x(pq_1(x)q_2 - pq_2q_1(x)) = \partial_x(q_2pq_1(x) - pq_2q_1(x)) = \partial_x([q_2, p]q_1(x)).$$

But q_1 is bracket with lower degree, so we can use the inductive hypothesis. □

As a corollary of this lemma, if we want a homogeneous potential such that all its derivatives are Lie polynomials, it suffices to take any sub-linear Lie polynomial p, and multiplying it by another variable, disjoint from p. Using this, we know it is really easy to find Lie potentials, or at least as easy as to find Lie polynomials. For this family of examples we will ask ϕ_B to be a Lie potential for each block.

If for each $k \in E$ we have by construction that r_k is a Lie-polynomial in the elements of X, and we can consider the free Lie algebra $\mathfrak{g} = \mathfrak{g}(S)$ generated by the set X subject to the relations $\{r_i : i \in E\}$. Then $\mathfrak{g}$ is a $\mathbb{N}$-graded Lie algebra, and there is an obvious isomorphism $A \cong \mathscr{U}(\mathfrak{g})$ that extends the identity of V.

Observe that choosing Φ that way automatically ensures it is 2-sincere, using 9.3, however for $s > 2$, 3-sincerity needs be asked explicitly. A second consequence of Φ

being a Lie potential is that all its cyclic derivatives are commutators, a question that appeared when computing $HH^\bullet(A)_{-1}$ in Propositions 6.3 and 8.3.

Proposition. *The algebra A is an integral domain isomorphic to its opposite algebra A^{op}.*

Proof. All enveloping algebras of Lie algebras have these properties. □

If we want to study $\mathfrak{g}$ we should first look at its Hilbert series.

9.5. Proposition. *Assume Φ is a 3-sincere Lie potential. Let $\xi_0 = 1, \xi_1, \dots, \xi_s$ be the roots of the polynomial $1 - nt + nt^s - t^{s+1}$. The Hilbert series of $\mathfrak{g}$ is*

$$h_{\mathfrak{g}}(t) = \sum_{k\geq 1}\Big(\tfrac{1}{k}\sum_{d|k}\mu\big(\tfrac{k}{d}\big)\big(\sum_{i=0}^{s}\xi_i^d\big)\Big)t^k.$$

Using this formula we can readily compute the dimensions of the homogeneous components of $\mathfrak{g}$ for small values of n, since it depends only on the coefficients of h_A^{-1}.

Proof. For each $k \geq 1$ let $g_k = \dim \mathfrak{g}_k$, so that $h_{\mathfrak{g}}(t) = \sum_{k\geq 1} g_k t^k$. The Poincaré-Birkhoff-Witt theorem implies that the Hilbert series of the enveloping algebra $\mathscr{U}(\mathfrak{g})$ is then $h_{\mathscr{U}(\mathfrak{g})}(t) = \prod_{k\geq 1}(1-t^k)^{-g_k}$, and the isomorphism $A \cong \mathscr{U}(\mathfrak{g})$ implies that we have

$$\prod_{k\geq 1}\frac{1}{(1-t^k)^{g_k}} = \frac{1}{(1-nt+nt^s-t^{s+1})}.$$

Taking logarithms, we see that

$$\sum_{k\geq 1} g_k \ln(1-t^k) = \sum_{i=0}^{s}\ln(1-\xi_i t)$$

so

$$\sum_{k\geq 1} g_k \sum_{l\geq 1}\tfrac{1}{l}t^{kl} = \sum_{k\geq 1}\tfrac{1}{k}\big(\sum_{i=0}^{s}\xi_i^k\big)t^k.$$

Looking at the coefficient of t^k in both sides of this equality we find that

$$\sum_{d|k} d g_d = \sum_{i=0}^{s}\xi_i^k.$$

Using the classical Möbius inversion formula [Sta97, §3.7], then, we conclude that

$$g_k = \tfrac{1}{k}\sum_{d|k}\mu\big(\tfrac{k}{d}\big)\big(\sum_{i=0}^{s}\xi_i^d\big),$$

for each $k \geq 1$, as we wanted. □

9.2 Steiner Quadruple Systems

Through this section, we will focus on quadruple systems, and as in the previous example, on Lie potentials. In appendix A we discussed that $n \equiv 2$ or $4 \mod 6$ is a necessary and sufficient condition for the existence of quadruple systems.

If $B \in S$ is a block, we will consider the *dihedral* orders on its elements. Namely, two orders are the same if one can be obtained from the other by means of rotation and/or reflection. It is clear that there are only 3 possible orders, for instance if $\{i, j, k, l\} \in S$ is a block, the dihedral orders are:

$$i \prec j \prec k \prec l, \;\; i \prec k \prec l \prec j, \;\; i \prec l \prec j \prec k$$

We will call an *orientation* of a cuadruple system a choice of a scalar $\varepsilon_{i,j,k,l} \in \Bbbk$ for each dihedral-order of every block $B = \{i, j, k, l\} \in S$ such that the sum of the 3 scalars of orders associated with the same block is zero. We can construct a polynomial ϕ_B as

$$\phi_B = \sum \varepsilon_{i,j,k,l} x_i x_j x_k x_l,$$

where the sum is over all permutations of the elements in B. Summing over all blocks we get the cyclic potential Φ we need to construct $A(S, \Phi)$.

As always, we can extend $\varepsilon_{i,j,k,l} = 0$ for every non necessarily distinct i,j,k,$l \in E$ such that $B = \{i, j, k, l\} \notin S$. Also, as before, we write

$$\varepsilon_{i,j,k} = \begin{cases} 0, & \text{if } |\{i, j, k\}| \leq 2; \\ \varepsilon_{i,j,k,\star(i,j,k)}, & \text{in any other case.} \end{cases}$$

Observe that in this case we have $\varepsilon_{i,j,k,l} = \varepsilon_{i,l,k,j} = \varepsilon_{j,k,l,i}$ and $\varepsilon_{i,j,k,l} + \varepsilon_{i,k,l,j} + \varepsilon_{i,l,j,k} = 0$ by definition of ε. In the same way, $\varepsilon_{i,j,k} = \varepsilon_{k,j,i}$ and $\varepsilon_{i,j,k} + \varepsilon_{k,i,j} + \varepsilon_{j,k,i} = 0$. It is clear that any two of the last three determines the orientation of S.

The next proposition shows that all derivatives of Φ are Lie polynomials. As stated in the previous section, using Lie potentials simplifies some of the computations and, among other things, automatically proves that the algebra $A(S, \Phi)$ is an integral domain.

9.6. Proposition. *Every sub-linear 4-homogeneous potential in x_i, x_j, x_k, x_l such that all its cyclic derivatives are Lie polynomials is of the form*

$$\alpha(x_i x_j x_k x_l + x_k x_j x_i x_l) + \beta(x_i x_k x_j z_l + x_j x_k x_i x_l) + \gamma(x_j x_i x_k x_l + x_k x_i x_j x_l),$$

with $\alpha + \beta + \gamma = 0$, the equality being up to projection to the space of cyclic potentials.

Proof. Multi-linear Lie polynomials of three variables are generated by $[x_i, [x_j, x_k]]$ and $[[x_i, x_j], x_k]$, and so is a 2-dimensional space, and it is easy to check that it can be rewritten as the vector space

$$\{\alpha(x_i x_j x_k + x_k x_j x_i) + \beta(x_i x_k x_j + x_j x_k x_i) + \gamma(x_j x_i x_k + x_k x_i x_j), \quad \text{with } \alpha + \beta + \gamma = 0\}.$$

From lemma 9.4 we know that the only way to achieve our potential is multiplying a Lie polynomial by another variable. Hence we obtain the result we wanted. □

A next step would be to try and use the automorphism group of a small cuadruple system in order to search for natural orientations to consider.

9.3 Antisymmetrizers

In this family of examples we assume s is even. For each block B in S, we can define an orientation as an equivalence class of total orders on its elements, where two orders are said to be conjugate, if they differ by an even permutation. Thus if $B = (i_1 \prec i_2 \prec \cdots \prec i_{s+1})$ is an ordered block, we define

$$\phi_B = \sum_{\sigma \in S_{s+1}} sg(\sigma)\, x_{i_{\sigma(1)}} \dots x_{i_{\sigma(s+1)}}.$$

The fact that $s + 1$ is odd makes $\Phi = \sum_{B \in S} \phi_B$ a cyclic potential.

It is clear that Φ is 3-sincere whenever $s > 2$. A positive aspect of this family of examples is that there are very few choices to make in order to construct $A(S)$, only one out of the two possible orientations for each block.

A second positive aspect is that we get a much smaller system of linear equations when computing the Lie algebra of derivations, so one can expect to encounter bigger cohomology groups. Indeed ($\mathbf{E}^2_{i_1,i_2,\dots,i_{s+1}}$) becomes trivial for antisymmetric potentials and the equations of ($\mathbf{E}^1_{i_1,i_2,\dots,i_{s+1}}$) may differ only up to a factor when swapping two indexes. So we can index ($\mathbf{E}^1_{i_1,i_2,\dots,i_{s+1}}$) only by subsets of E of order $s + 1$, rather than ordered tuples.

A third consequence of Φ being antisymmetric of odd degree is that all its cyclic derivatives are commutators, the same that happened when considering Lie potentials. This question appeared when computing $HH^\bullet(A)_{-1}$ in Propositions 6.3 and 8.3.

For this family of examples we have developed an algorithm to compute the Lie algebra of homogeneous derivations $\mathfrak{s}$, that can be found in appendix B. The smallest instance of a system with $s > 2$ even is the quintuple system obtained by derivation from the Witt system of type $(5, 6, 12)$ presented in example 1.2. Using our algorithms, we have not been able to find orientations with large symmetry group and, in all those cases, $\mathfrak{s}$ is zero.

Appendix A Some results on Steiner systems

The first question to ask whenever encountering a Steiner system (E, S) of type (t, k, n) is the order of S.

If $X = \{(p, b) \in \binom{E}{t} \times S : p \subset b\}$, then the first projection $X \to \binom{E}{t}$ is a bijection, and the fibers of the second projection $X \to S$ have exactly $\binom{k}{t}$ elements, provided $|E| \geq k$. We see that

$$|S| = \binom{n}{t} \bigg/ \binom{k}{t} . \tag{26}$$

It follows that $\binom{n}{t} \equiv 0 \mod \binom{k}{t}$.

A common operation among Steiner systems is removing a point $i \in E$, and defining the blocks as $S' = \{b \setminus \{i\} : b \in S \text{ and } i \in b\}$, thus obtaining a system of type $(t-1, k-1, n-1)$. The resulting system S' is sometimes denoted in the literature as the *derived* system of S with respect to i.

Applying equation (26) to S' we see that the amount of blocks that contain a given point is constant and equals $|S'| = \binom{n-1}{t-1} / \binom{k-1}{t-1}$. A simple inductive argument proves that the amount of blocks that contain a given set of $0 \leq j \leq t$ given points is also constant and equals $\binom{n-j}{t-j} / \binom{k-j}{t-j}$.

In the particular kind of systems we are interested in, we have the following lemma.

A.1. Lemma. *If (E, S) is a Steiner system of type $(s, s+1, n)$ then the amount of blocks that contain a given set of $0 \leq j \leq s$ distinct elements is $\binom{n-j}{s-j} / (s+1-j)$.*

This of course yields some more divisibility conditions that n must satisfy for a system of that order to exist. For example setting $j = s-1$ we get $n - s$ is odd, and $j = s-2$ yields $(n-s+2)(n-s+1) \equiv 0 \mod 6$. From these two conditions together we can conclude that $n \equiv s \pm 1 \mod 6$.

Appendix B A python class to compute derivations

We have developed a *python* class that, given the list of blocks of the Steiner system of type $(s, s+1, n)$ with s even, it can construct the matrix of linear equations that determine the Lie algebra $\mathfrak{s}$, defined in equation (18).

It can either calculate the rank of the matrix for a given choice of antisymmetric coefficients $\varepsilon_{(-)}$, or generate the matrix with the unknown parameters corresponding to the orientation of each block, in a format that we can enter into *Mathematica*. There we can calculate the determinant of the principal minors in terms of the unknowns, to determine which choice of orientation will yield a bigger space of derivations. Note that when $s \geq 4$, if the matrix has maximum rank $n(n-1)$, the $\mathfrak{s}$ is zero, so we are looking for matrices with small rank.

We choose to tackle this sort of potentials because for the other cases the linear system is much bigger and harder to solve.

Listing 1. Steiner.py

```
import functools, operator, random, numpy
from itertools import combinations

def perm_parity(lst):
  '''\
  Given a permutation of the digits 0..N in order
  as a list, returns its sign.
  '''
  parity = 1
  for i in range(0,len(lst)-1):
    if lst[i] != i:
      parity *= -1
      mn = min(range(i,len(lst)), key=lst.__getitem__)
      lst[i],lst[mn] = lst[mn],lst[i]
  return parity

class Steiner:
  '''\
  The points of the system should be consecutive integers
  starting from 0,1,... Usage example:
    s = Steiner(blocks = [...])
    s.printGenericMatrix()
    s.tryRandomOrientations()
  '''
  def __init__(self, blocks):
```

```
    self.count = 0
    self.blockStrings = [
      '.'.join(map(str,sorted(block))) for block in blocks
    ]
    self.size = len(blocks[0])
    self.order = max([max(block) for block in blocks]) + 1
    self.orientation = [1 for block in blocks]
    self.reset()

  def setRandomOrientation(self):
    self.orientation = [
      random.choice([1,-1]) for block in blocks
    ]

  def reset(self):
    self.count += 1
    self.matrix = []

  def isBlock(self, block):
    key = '.'.join(map(str,sorted(block)))
    return key in self.blockStrings

  def blockSign(self, block):
    # assert(isBlock(l))
    sortedBlock = sorted(block)
    key = '.'.join(map(str,sortedBlock))
    ind = self.blockStrings.index(key)
    blockOrientation = self.orientation[ ind ]
    perm = [block.index(x) for x in sortedBlock]
    parity = perm_parity(perm)
    sign = blockOrientation * parity
    if self.useGenericCoefficients:
      ans = 'e['+str(ind)+']'
      if sign < 0:
        ans = '-'+ans
      return ans
    else:
      return sign

  def star(self, lst):
    '''\
    Given a list of size-1, calculates the point missing
```

```
  to complete a block.
  '''
  valid = list(
    x for x in range(self.order) if self.isBlock(lst+[x])
  )
  return valid[0]

def eqlist(self, gij):
  '''\
  Given the coefficients that multiply each matrix entry
  g_{i,j} it generates the associated row in the linear
  system, skipping diagonal entries, since they are zero.
  '''
  res = []
  for i in range(self.order):
    for j in range(self.order):
      if i == j:
        continue
      res.append(gij[i][j])
  return res

def matrixrow(self, pts):
  '''\
  Given a tuple of size points, populate a matrix g_{i,j}
  with the coefficients in the linear equation associated
  to that tuple.
  '''
  gij = [
    [0] * range(self.order) for x in range(self.order)
  ]
  for j in range(self.size):
    block = list(pts)
    s = self.star(block[j+1:] + block[:j] )
    block[j] = s
    gij[s][pts[j]] = self.blockSign(block)
  self.matrix.append( self.eqlist(gij) )

def generateMatrix(self):
  self.reset()
  for vec in combinations(range(self.order), self.size):
    l = list(vec)
    if self.isBlock(l):
```

```
      continue
    self.matrixrow(l)

def printGenericMatrix(self):
  '''\
  Print the matrix with generic coefficients, in a
  Mathematica compatible format.
  '''
  self.useGenericCoefficients = True
  self.orientation = [1 for block in blocks]
  self.generateMatrix()
  print('{')
  for i in range(len(self.matrix)):
    line = '{' + ','.join(map(str,self.matrix[i])) + '}'
    if i != len(self.matrix) - 1:
      line = line + ','
    print(line)
  print('}')

def getMatrixRank(self):
  '''\
  Computes the rank of a matrix if an assignment of
  signs was chosen for each block.
  '''
  self.useGenericCoefficients = False
  self.generateMatrix()
  nMat = numpy.matrix(self.matrix)
  return numpy.linalg.matrix_rank(nMat)

def tryRandomOrientations(self, times = -1):
  '''\
  Try random orientations, until a matrix with small
  rank is found.
  '''
  self.count = 0
  while self.count != times:
    self.setRandomOrientation()
    rank = self.getMatrixRank()
    print(self.count, rank)
    if (rank < self.order*(self.order-1)):
      print(orientation)
      break
```

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