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Construcción de MV-pairs y Boolean ambiguity algebras a partir de una MV-algebra y viceversa.

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Introducción

En los últimos años ha habido un gran desarrollo en el campo de las Lógicas Multivaluadas y, en consecuencia, de las estructuras matemáticas comprometidas en ese desarrollo, como lo son las MV-alebras, las effect algebras y las MV-effect algebras. En el año 2006 Gejza Jenča [12] y Thomas Vetterlein [17] partiendo de hipótesis distintas representaron MV-algebras a través del cociente de un álgebra de Boole B por un subgrupo del grupo de todos los automorfismos de B ($Aut(B)$). Esto es, ambos toman un par (B, G) (donde B es un álgebra de Boole y G es un subgrupo de $Aut(B)$), definen la relación de equivalencia sobre B $a \sim b$ si y solo si existe $f \in G$ tal que $f(a) = b$ y se define una operación $\oplus$ en el conjunto de las clases que lo hace una MV-algebra. En este trabajo se desarrolla una parte de la representación de Jenča (la otra está desarrollada en [12] y en la Tesis de Licenciatura de Guillermo Herrmann) y se da una relación entre las ideas de estos dos autores.

En la primera sección se dan todas las definiciones y se demuestran todos los resultados que son necesarios para el desarrollo de las secciones posteriores lo que, aparte de darle a este trabajo el caracter de "autocontenido", da una ordenada introducción a estructuras básicas en el álgebra de la lógica como reticulados, álgebras de Boole, etc. También aparecen aquí las estructuras claves usadas en el trabajo de Jenča, las MV-alebras, las effect algebras y las MV-effect algebras.

En la segunda y tercer sección se desarrolla parte del trabajo de Gejza Jenča en la que se define que es un $MV-par$ y se muestra que a partir de una MV-effect algebra M puede construirse un álgebra de Boole $B(M)$ y un subgrupo $G(M)$ de $Aut(B(M))$ de tal forma que el par $(B(M), G(M))$ resulta un MV-par. Además en [12] y en la Tesis de Licenciatura de Guillermo Herrmann se demuestra que a partir de un MV-par (B, G) se puede obtener una MV-effect algebra $\mathcal{A}(B, G)$. En [12] y en la sección tres de este trabajo se demuestra también que $M \cong \mathcal{A}(B(M), G(M))$.

En la sección cuatro se demuestra una correspondencia uno a uno entre las MV-álgebras y las MV-effect álgebras. La demostración es una adaptación de [5] y corrige la demostración dada en [7] Teorema 1.8.12 (página 75).

En el apéndice, se transcribe parte del trabajo de Thomas Vetterlein en el que se definen los conceptos de Complete Boolean ambiguity algebras y normal Boolean ambiguity algebras, y a partir de estas se construye una MV-algebra. Se muestra en esta sección que si el par (B, G) es una Complete Boolean ambiguity algebra o una normal Boolean ambiguity algebra entonces (B, G) es un MV-par y que la MV algebra obtenida usando el camino de Vetterlein y la MV algebra obtenida usando el camino de Jenča y el teorema de correspondencia coinciden y son semisimples. Por último se prueba que partiendo de una MV-algebra semisimple y obteniendo un MV-par (mediante el teorema de correspondencia y el Teorema 3.3.3) este último es una normal Boolean ambiguity algebra aunque no necesariamente es una Complete Boolean ambiguity algebra.

1 Definitions and basic results

1.1 Lattices [16] [10] [3]

A *partially ordered set (or poset)* $\langle A, \leq \rangle$ consist of a nonempty set A and a binary relation $\leq$ on A such that $\leq$ satisfies:

Reflexivity $\quad a \leq a$

Antisymmetry $\quad a \leq b, \ b \leq a$ imply that $\ a = b$

Transitivity $\quad a \leq b, \ b \leq c$ imply that $\ a \leq c$

A poset $\langle A, \leq \rangle$ that also satisfies $\ \forall a, b \in A \quad a \leq b \quad or \quad b \leq a$,
is called a *chain* (or *fully ordered set*).

Let P a poset, $H \subseteq P$ and $a \in P$. Then a is an *upper bound* of H iff $h \leq a$ for all $h \in H$. An upper bound a of H is the *supremum* of H iff, for any upper bound b of H, we have $a \leq b$ (a is the least upper bound of H).
We shall write $\ a = supH\ $ or $\ a = \bigvee H$. If $H = \{x, y\}$, we write $\ \bigvee H = x \vee y$.
Let P a poset, $H \subseteq P$ and $a \in P$. Then a is an *lower bound* of H iff $a \leq h$ for all $h \in H$. An lower bound a of H is the *infimum* of H iff, for any lower bound b of H, we have $b \leq a$ (a is the greatest lower bound of H).
We shall write $\ a = infH\ $ or $\ a = \bigwedge H$. If $H = \{x, y\}$, we write $\ \bigwedge H = x \wedge y$.
It is easy to check te uniqueness of the infimum and supremum.

A poset $\langle P, \leq \rangle$ is a *lattice* if $\ a \wedge b\ $ y $\ a \vee b\ $ exist, for all $a, b \ \in L$.

Example 1.1.1 The set $\mathcal{P}(X)$ of all subset of a set X is a lattice with the operations $\ a \vee b = a \cup b\ ,\ \ a \wedge b = a \cap b\ $.

Example 1.1.2 If C is a chain, then C is a lattice.

Example 1.1.3 Let $N_d = \{1, 2, \ldots\ldots\}$ where $\ n \leq m\ $ iff $\ \exists k \ / \ n.k = m$ (i.e $n \mid m$). Then N_d is a lattice with the operations $\ a \vee b = mcm(a, b)\ $ and $a \wedge b = MCD(a, b)\ $.

In every lattice the following hold:
$(L1)$ Idempotency:

$x \vee x = x = x \wedge x$

$(L2)$ Conmutativity:

$x \vee y = y \vee x$ $\qquad\qquad\qquad x \wedge y = y \wedge x$

$(L3)$ Associativity:

$x \vee (y \vee z) = (x \vee y) \vee z$ $\qquad x \wedge (y \wedge z) = (x \wedge y) \wedge z$

$(L4)$ Absorption identities:

$x \vee (x \wedge y) = x = x \wedge (x \vee y)$

Also $x \leq y \Leftrightarrow x = x \wedge y \Leftrightarrow y = x \vee y$.
Therefore $x \leq y \Rightarrow x \wedge z \leq y \wedge z$ and $x \vee z \leq y \vee z$.

Example 1.1.4 If L is a lattice, $a, b \in L$, $a \leq b$ and
$[a, b] = \{x \in L \ / \ a \leq x \leq b\}$, then $[a, b]$ is a lattice.

Example 1.1.5 Let $\langle L, \leq, \vee, \wedge \rangle$ be a lattice. If we put $a \leq_D b$ iff $b \leq a$, $a \wedge_D b = a \vee b$ and $a \vee_D b = a \wedge b$ then $\langle L, \leq_D, \wedge_D, \vee_D \rangle$ is a lattice.

A lattice can be characterized purely in terms of the properties
$(L1), (L2), (L3), (L4)$.

Theorem 1.1.6 *Let A be a nonempty set and "$+$", "$.$" two binary operations on A satisfying $(L1), (L2), (L3), (L4)$ and set $a \leq b$ iff $a = a.b$.*
Then A is a lattice with $a \vee b = a + b$ and $a \wedge b = a.b$.
(Remark. If $a = a.b$, then $a + b = a.b + b$ and, by $(L4)$, $a + b = b$.
Similarly $b = a + b \Rightarrow a = a.b$. Thus $a \leq b$ iff $a = a.b$ iff $b = a + b$).

Proof.

$\leq$ is an order:

- $a \leq a$ by $(L1)$
- If $a \leq b$ and $b \leq a$, then $a = a.b$ and $b = b.a$.

  Therefore by $(L2)$ $a = a.b = b.a = b$.
- If $a \leq b$ and $b \leq c$, then $a = a.b$ and $b = b.c$.

  Therefore by $(L3)$ $a.c = (a.b).c = a.(b.c) = a.b = a$ and $a \leq c$.

$a + b = a \vee b$:

By $(L4)$ $a = a.(a + b)$ and thus $a \leq a + b$. Similarly $b \leq a + b$.
Let z such that $a \leq z$ and $b \leq z$, then $z = a + z$ and $z = b + z$. Then $(a+b)+z = a+(b+z) = a+z = z$ and thus $a+b \leq z$. Therefore $a+b = a \vee b$.

$a.b = a \wedge b$:

Since $(a.b).a = a.(a.b) = (a.a).b = a.b$, we have $a.b \leq a$. Similarly $a.b \leq b$.
Let z such that $z \leq a$ and $z \leq b$, then $z = a.z$ and $z = b.z$. Then $z = a.z = a.(b.z) = (a.b).z$ and thus $z \leq a.b$. Therefore $a.b = a \wedge b$. □

A lattice A is said to be *distributive* if, for all $a, b, c \in A$

$$(L5) \qquad \begin{aligned} a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \end{aligned}$$

Example 1.1.7 *If C is a chain, then C is a distributive lattice.*

A *bounded lattice* is one that has both a smallest element (or "0") and a largest element (or "1"),that is, $\forall a$ in the lattice, $0 \leq a$ and $a \leq 1$. $(L6)$

Notation $a = x\dot{\vee}y$ means $a = x \vee y$ and $x \wedge y = 0$.

A *sublattice* $\mathcal{K} = \langle K; \wedge, \vee \rangle$ of the lattice $\mathcal{L} = \langle L; \wedge, \vee \rangle$ is a nonempty subset K of L with the property that $a, b \in K$ implies that $a \wedge b, a \vee b \in K$ (the operations $\wedge$, $\vee$ are taken in $\mathcal{K}$), and the $\wedge$ and the $\vee$ of $\mathcal{K}$ are restrictions to K of the $\wedge$ and the $\vee$ of $\mathcal{L}$.
To put this in simpler language, we take a nonempty subset $\mathcal{K}$ of the lattice $\mathcal{L}$ such that $\mathcal{K}$ is closed under $\wedge$ and $\vee$. Under the same $\wedge$ and $\vee$, $\mathcal{K}$ is a lattice; this is a sublattice of L.

A $\{0, 1\} -$ *sublattice* of a bounded lattice L is a sublattice containing the 0 and 1 of L.

An element $a \neq 0$ of a bounded lattice is called *atom* if the condition
$0 \leq x \leq a$ implies that either $x = 0$ or $x = a$.

A set I of elements of a bounded distributive lattice L is said to be an *ideal* provided that:
$0 \in I$
If $a, b \in L$, $a \in I$ and $b \leq a$, then $b \in I$
If $a, b \in L$, $a \in I$ and $b \in I$, then $a \vee b \in I$

A set F of elements of a bounded distributive lattice L is said to be a *filter* provided that:
$1 \in F$
If $a, b \in L$, $a \in F$ and $a \leq b$, then $b \in F$
If $a, b \in L$, $a \in F$ and $b \in F$, then $a \wedge b \in F$

It is easy to see that intersection of any number of ideals (filers) of a lattice L is a ideal (filter) of L. Thus, if a subset H of a lattice L is nonempty, we can define *the ideal (filter) generated by the set* H, it is the intersection of all ideals (filters) containing H, and the least ideal (filter) containing H.
The ideal generated by H will be denoted by $(H]$, and the filter generated by H will be denoted by $[H)$.

Lemma 1.1.8 Let L be a lattice and let H be a subset of L. Then
$(H] = \{ \ x \in L$ such that $\exists$ an integer $n \geq 1$ and
elements $h_1 \ldots\ldots h_n \in L$ with $x \leq h_1 \vee \ldots\ldots \vee h_n \ \}$.

Proof. Let $I = \{ \ x \in L$ such that $\exists$ an integer $n \geq 1$ and
elements $h_1 \ldots\ldots h_n \in L$ with $x \leq h_1 \vee \ldots\ldots \vee h_n \ \}$.
It is clear that I is an ideal, and obviously $H \subseteq I$. Finally, if $H \subseteq J$ and J is an ideal, then $I \subseteq J$, and thus I is the smallest ideal containing H; that is, $I = (H]$. $\square$

Similarly, we have:

Lemma 1.1.9 Let L be a lattice and let H be a subset of L. Then
$[H) = \{ \ x \in L$ such that $\exists$ an integer $n \geq 1$ and
elements $h_1 \ldots\ldots h_n \in L$ with $x \geq h_1 \wedge \ldots\ldots \wedge h_n \ \}$.

In particular if $a, b \in L$,
$(a] = \{x \in L$ such that $x \leq a\}$ is the *principal ideal generated by* a.
$[b) = \{y \in L$ such that $y \geq b\}$ is the *principal filter generated by* b.

An ideal (filter) A of a bounded lattice L is called *proper* if $A \neq L$.

Lemma 1.1.10 An ideal I of a bounded lattice L is proper if and only if $1 \notin I$.

Proof. If $1 \notin I$, then $I \neq L$ and I is proper.
Let I be a proper ideal of L. If $1 \in I$ then $a \in I$ for all element a in L (since $\forall a \in L\ a \leq 1$ and I is an ideal), thus $L = I$, which is a contradiction $\square$

Similarly we have,

Lemma 1.1.11 A filter F of a bounded lattice L is proper if and only if $0 \notin I$.

An ideal I of a bounded lattice L is called *prime* if it is proper and the condition $a \wedge b \in I$ implies that either $a \in I$ or $b \in I$.

A filter F of a bounded lattice L is called *prime* if it is proper and the condition $a \vee b \in F$ implies that either $a \in F$ or $b \in F$.

Lemma 1.1.12 Let L be a lattice, and let M be a prime filter (ideal) of L. Then $P = M^c$ $(M^c = L \setminus M)$ is a prime ideal (filter) of L.

Proof. We will verify one case only, the other require similar arguments. Let M be a prime filter of L we will see that $P = M^c$ is a prime ideal of L.

P is an ideal:
M prime $\Rightarrow M$ proper $\Rightarrow$(by lemma 1.1.11) $0 \notin M \Rightarrow 0 \in M^c = P$.
Let a be an element of P $(\Rightarrow a \notin M)$ and $b \leq a$.
Since M is a filter, if $b \in M$ and $b \leq a$ then $a \in M$ which is a contradiction.
Therefore $b \notin M$ and thus $b \in P$.
Since M is a prime filter, if $a \vee b \in M$, then either $a \in M$ or $b \in M$, hence $a \notin M$ and $b \notin M$ imply $a \vee b \notin M$, that is $a \in P$ and $b \in P$ imply $a \vee b \in P$.

P is an prime ideal:

$M \neq \emptyset$ $(1 \in M) \Rightarrow P = M^c$ is proper.

Since M is a filter $a \in M$ and $b \in M \Rightarrow a \wedge b \in M$, hence

$a \wedge b \notin M \Rightarrow$ either $a \notin M$ or $b \notin M$,

that is $a \wedge b \in P \Rightarrow$ either $a \in P$ or $b \in P$ $\square$

Theorem 1.1.13 (Birkhoff-Stone) Let L be a bounded distributive lattice. If J is an ideal and F is a filter of L such that $J \cap F = \emptyset$, then there exist a prime filter M such that $J \cap M = \emptyset$ and $F \subseteq M$.

Proof.

Let L be a bounded distributive lattice, and

$\mathcal{F} = \{G/$ G is a filter of L , $F \subseteq G$ and $G \cap J = \emptyset\}$

Since $F \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. The set $\mathcal{F}$ is ordered by inclusion. Let $\{G_i\}_{i \in I}$ be a family totally ordered of $\mathcal{F}$, then

$$H = \cup_{i \in I} G_i \text{ is a filter of } L,$$

$$F \subseteq H,$$

$$H \cap J = (\cup_{i \in I} G_i) \cap J = \cup_{i \in I}(G_i \cap J) = \cup_{i \in I} \emptyset = \emptyset,$$

thus $H \in \mathcal{F}$ and H is an upper bound of $\{G_i\}_{i \in I}$.

Therefore, by Zorn's Lemma, $\mathcal{F}$ has a maximal element M.

It only remains to show that M is a prime filter of L.

Now suppose $x \vee y \in M$ and let

$M_1 = \langle M, x \rangle = \{s \in L$ such that $s \geq m \wedge x$ for some $m \in M\}$,

$M_2 = \langle M, y \rangle = \{s \in L$ such that $t \geq m \wedge x$ for some $m \in M\}$.

We have, either $M_1 \cap J = \emptyset$ or $M_2 \cap J = \emptyset$.

If not, $\exists\, u, v$ in J and m_1, m_2 in M such that

$$u \geq m_1 \wedge x \qquad v \geq m_2 \wedge y$$

Let $m = m_1 \wedge m_2$, then

$$u \geq m \wedge x \qquad v \geq m \wedge y.$$

Therefore $u \vee v \geq (m \wedge x) \vee (m \wedge y) = m \wedge (x \vee y)$.
Since $m \in M$, $x \vee y \in M$ and M is a filter, then $m \wedge (x \vee y) \in M$, hence $u \vee v \in M$. Also $u \vee v \in J$, then $u \vee v \in M \cap J$ which is a contradiction.
Therefore, either $M_1 \cap J = \emptyset$ or $M_2 \cap J = \emptyset$. Suppoes that $M_1 \cap J = \emptyset$. Since $F \subseteq M \subseteq M_1$, then $M_1 \in \mathcal{F}$. Now, $M \subseteq M_1$, $M_1 \in \mathcal{F}$ and M is maximal in $\mathcal{F}$, imply $M = M_1$ and since $x \in M_1$, then $x \in M$. Similarly, if $M_2 \cap J = \emptyset$, then $y \in M$. Therefore M is a prime filter $\square$

Corollary 1.1.14 Let L be a bounded distributive lattice. If J is an ideal and F is a filter of L such that $J \cap F = \emptyset$, then there exist a prime ideal P such that $F \cap P = \emptyset$ and $J \subseteq P$.

Proof. By Theorem 1.1.13 there exist a prime filter M such that $J \cap M = \emptyset$ and $F \subseteq M$. Let $P = M^c$ (i.e. $P = L \setminus M$) then P is a prime ideal (by Lemma 1.1.12) and $P \cap F = \emptyset$ (since $F \subseteq M$) and $J \subseteq P$ (since $J \cap M = \emptyset$).$\square$

Corollary 1.1.15 Let L be a distributive lattice, $a, b \in L$ and $a \neq b$. Then there is a prime ideal of L containing exactly one of a and b.

Proof. Let $(a]$ be the ideal generated by a and $[b)$ the filter generated by b.
By Corollary 1.1.14 there exist a prime ideal P such that $(a] \subseteq P$ and $P \cap [b) = \emptyset$. Thus $a \in P$ and $b \notin P$. $\square$

A *homomorphism* φ of the lattice L_0 into the lattice L_1 is a map of L_0 into L_1, satisfying both

$$\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$$
$$\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$$

Remark: Let $\varphi : L_0 \to L_1$ be a homomorphism of lattices and $a_1, a_2 \in L_0$. If $a_1 \leq a_2$ in L_0 then $\varphi(a_1) \leq \varphi(a_2)$ in L_1. Indeed, $a_1 \leq a_2$ in $L_0 \Rightarrow a_1 = a_1 \wedge a_2 \Rightarrow \varphi(a_1) = \varphi(a_1 \wedge a_2) = \varphi(a_1) \wedge \varphi(a_2)$, and then $\varphi(a_1) \leq \varphi(a_2)$ in L_1.

Remark: Let $\varphi : L_0 \to L_1$ be a homomorphism of lattices, then $\varphi(L_0)$ is a sublattice of L_1.

A homomorphism of a lattice into itself is called an *endomorphism*, and a one-to-one homomorphism will also be called an *embedding*.

A *isomorphism* of lattices is a biyective homomorphism. It is easy to see that f^{-1} (the inverse function of f) is an isomorphism of lattices as well.
The notation $A \cong B$ means that there exist a isomorphism $\varphi : A \to B$.

Let A and B be two bounded lattices. A $\{0,1\}$-homomorphism is a homomorphism that preserves 0 and 1.

Let L_1, L_2 and L_3 be three lattices and let $g : L_1 \to L_2$ and $f : L_2 \to L_3$ be two homomorphisms of lattices. We write $f \circ g$ for the *composition* of the two operators, that is $\forall a \in L_1$ $f \circ g(a) = f(g(a))$ in L_3. It is easy to see that $f \circ g$ is a homomorphism.

Some of the next results will be used in Section 2.

Let L be a lattice. An element a of L is *joint-irreducible* iff $a = b \vee c$ implies that $a = b$ or $a = c$; it is *meet-irreducible* iff $a = b \wedge c$ implies that $a = b$ or $a = c$. The set of all nonzero joint-irreducible elements of a lattice L is denoted by $J(L)$ and the set of all non-unit meet-irreducible elements of a lattice L is denoted by $M(L)$.

In what follows, $\succ$ denotes the usual covering relation on a poset, that means, $a \succ b$ iff b is a maximal element of the set $\{x : x < a\}$.

Lemma 1.1.16 Let L be a finite distributive lattice, let C be a maximal chain in L and let $a \in J(L)$. We define $\pi_C(a) = \bigwedge \{x \in C : x \geq a\}$ (the smallest member of C containing a, see Figure 1 to the left) and $m(a) = \bigvee \{x \in L : x \ngeq a\}$. Let $x \in C$, $\pi_C(a) \succ x$. Then

(i) $a \vee x = \pi_C(a)$

(ii) $a \wedge x = a \wedge m(a)$.

Proof.
(i) We have $\pi_C(a) \wedge (a \vee x) = (\pi_C(a) \wedge a) \vee (\pi_C(a) \wedge x) = a \vee x$, so $\pi_C(a) \geq a \vee x \geq x$. Since $\pi_C(a) \succ x$, we have either $\pi_C(a) = a \vee x$ or $a \vee x = x$. However, $a \vee x = x$ contradits with $\pi_C(a) \neq x$ (since $a \vee x = x \Rightarrow a \leq x \Rightarrow$ $\Rightarrow \pi_C(a) \leq x \Rightarrow \pi_C(a) = x$), hence $\pi_C(a) = a \vee x$.
(ii) First note that

- $\pi_C(a) \succ x \Rightarrow a \succ a \wedge x$.

  Indeed, let $a \wedge x \leq z \leq a$, then $x \leq x \vee z \leq x \vee a = \pi_C(a)$. Since $\pi_C(a) \leq x$ we have either $x = x \vee z$ or $\pi_C(a) = x \vee z$. Now

  $x = x \vee z \Rightarrow z \leq x \Rightarrow z \leq a \wedge x \Rightarrow z = a \wedge x$, and

  $x \vee z = \pi_C(a) \Rightarrow$ (since $a \leq \pi_C(a)$) $\quad a = a \wedge \pi_C(a) = a \wedge (x \vee z) =$

  $= (a \wedge x) \vee (a \wedge z) = (a \wedge x) \vee z = z \quad$ (since $z \leq a$ and $a \wedge x \leq z$).

- $a \nleq m(a)$. Indeed, let A be the set $\{x \in L : x \ngeq a\}$, since L is a finite lattice $A = \{x_1, \ldots, x_n\}$. If $a \leq m(a)$ then $a = a \wedge m(a) = a \wedge (\bigvee A) =$ $= a \wedge (x_1 \vee \ldots \vee x_n) = (a \wedge x_1) \vee \ldots \vee (a \wedge x_n)$. Since $a \in J(L)$, then $\exists j$, $1 \leq j \leq n$ such that $a = a \wedge x_j$ and thus $a \leq x_j$ which is a contradiction since $x_j \in A$.

Now, since $x \ngeq a$, we have $x \leq m(a)$ and $a \wedge x \leq a \wedge m(a) \leq a$. Since $a \vee x = \pi_C(a) \succ x$, $a \succ a \wedge x$. Therefore, $a \wedge x = a \wedge m(a)$ or $a \wedge m(a) = a$. Since $a \nleq m(a)$, $a \wedge x = a \wedge m(a)$. $\square$

Lemma 1.1.17 Let L be a finite distributive lattice. Then

(i) Every element is the join of nonzero joint-irreducible elements of L.

(ii) Let $2^{J(L)}$ be the set of all subsets of $J(L)$. Then the mapping $r : L \to 2^{J(L)}$ given by $r(x) = \{a \in J(L) : a \leq x\}$ is a $\{0, 1\}$-embedding of L into $2^{J(L)}$.

(iii) For every maximal chain C of L, the mapping $\pi_C : J(L) \to C$ is a bijection from the set of all join-irreducible elements onto C .

Note that π_C maps nonzero elements onto nonzero elements.

(iv) $a \in J(L)$ iff $\{x \in L : x \not\geq a\}$ is a prime ideal, and then,

$$m(a) = \bigvee \{x \in L : x \not\geq a\} \in M(L).$$

Proof.

(i) Let x be an element of L. If $x \in J(L)$, x is the join of nonzero joint-irreducible elements of L.

If not, $x = y \vee z$ with $x \neq y$ and $x \neq z$

If $y \in J(L)$ and $z \in J(L)$ then x is the join of nonzero joint-irreducible elements of L. If not, if for example, $y \in J(L)$ and $z \notin J(L)$

then $z = r \vee t$ with $r \neq z$ and $t \neq z$. Therefore $x = y \vee r \vee t$.

The others case are similarly.

Since L is a finite lattice, the process comes to an end at a certain point.

(ii) r is a $\{0,1\}$-homomorphism of lattices:

$r(0) = \{a \in J(L) : a \leq 0\} = \emptyset \qquad (a \in J(L) \Rightarrow a \neq 0)$

$r(1) = \{a \in J(L) : a \leq 1\} = J(L)$

Since $a \leq x \wedge y \Leftrightarrow a \leq x$ and $a \leq y$, then $r(x \wedge y) = r(x) \cap r(y)$

If $a \leq x$ or $a \leq y$, then $a \leq x \vee y$. Thus $r(x) \cup r(y) \subseteq r(x \vee y)$.

If $a \leq x \vee y \Rightarrow a = a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$ and since $a \in J(L)$, we have either $a = a \wedge x$ or $a = a \wedge y$ (i.e. $a \leq x$ or $a \leq y$), then $a \in r(x)$ or $a \in r(y)$. Thus $r(x \vee y) \subseteq r(x) \cup r(y)$ and then $r(x \vee y) = r(x) \cup r(y)$.

(iii) Since L is a finite lattice and $\forall a$ in $J(L)$ $a \leq 1 \in C$, π_C is well defined.

π_C is injective:

Let $a, b \in J(L)$, $x \in C$, $x \prec \pi_C(a)$ (i.e. $x = \bigvee \{x \in C : x < \pi_C(a)\}$), and $\pi_C(a) = \pi_C(b)$ (see Figure 1).

Then $x \vee a = \pi_C(a) = \pi_C(b) = x \vee b$, and

$a = a \wedge \pi_C(a) = a \wedge (x \vee a) = a \wedge (x \vee b) = (a \wedge x) \vee (a \wedge b)$.

Therefore $a = (a \wedge x)$ or $a = (a \wedge b)$ (since $a \in J(L)$) and thus $a \leq x$ or $a \leq b$.

If $a \leq x$ then $\pi_C(a) = \bigwedge \{z \in C : z \geq a\} \leq x$ and thus $\pi_C(a) \leq x < \pi_C(a)$, which is a contradiction. Therefore $a \leq b$. Similarly we can prove $b \leq a$ and thus $a = b$.

π_C is surjective:

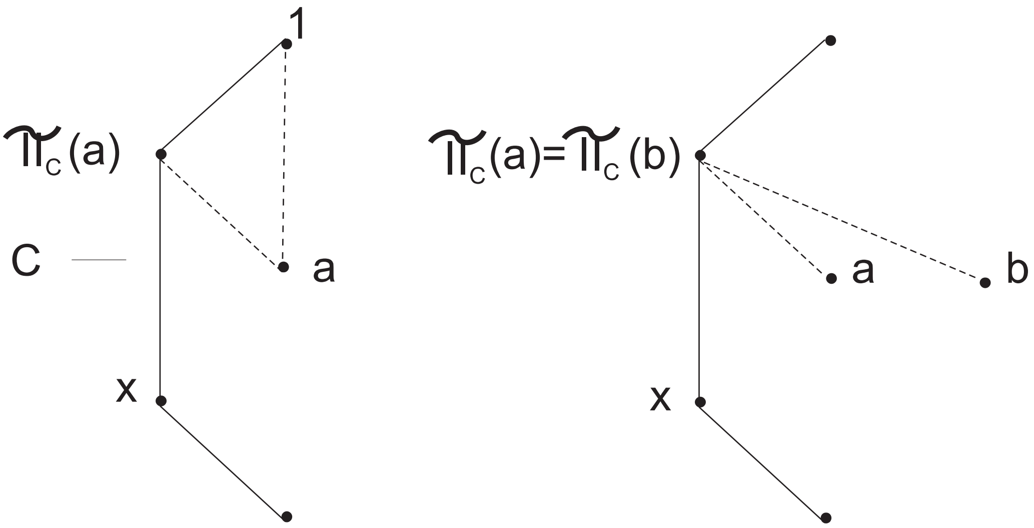

Figure 1:

Let $y \in C$ and $z \in C$, $z \prec y$. Since $z \prec y$ then $z < y$, therefore by (i) $\exists a \in J(L)$ such that $a \leq y$ and $a \nleq z$. Thus $\pi_C(a) \leq y$ (since $y \in \{x \in C : a \leq x\}$) and $z < \pi_C(a)$, (since $z \notin \{x \in C : a \leq x\}$. Therefore $y = \pi_C(a)$ and π_C is a surjective map.

(iv) Let A be the set $\{x \in L : x \ngeq a\}$ and $a \in J(L)$, then:
A is an ideal:
$a \in J(L) \Rightarrow 0 < a \Rightarrow 0 \ngeq a \Rightarrow 0 \in A$. If $x \in A$ and $y \leq x$ then $y \in A$ otherwise $a \leq y \leq x$ which is a contradiction since $x \in A$. If $x \in A$ and $y \in A$ then $x \vee y \in A$ otherwise $a \leq x \vee y$ then $a = a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$ and, since $a \in J(L)$, $a = a \wedge x$ or $a = a \wedge y$, therefore $a \leq x$ or $a \leq y$ which is a contradiction since $x \in A$ and $y \in A$.
A is a prime ideal:
If $x \notin A$ and $y \notin A$ then $a \leq x$ and $a \leq y$ hence $a \leq x \wedge y$ and thus $x \wedge y \notin A$. Therefore if $x \wedge y \in A$ then either $x \in A$ or $y \in A$.

Now suppose A is a prime ideal and $a = x \vee y$. Hence $x \vee y \notin A$ and, since A an ideal, $x \notin A$ or $y \notin A$ and then $a \leq x$ or $a \leq y$. Therefore either $x \leq x \vee y = a \leq x$ or $y \leq x \vee y = a \leq y$, i.e. $x = a$ or $y = a$ and thus $a \in J(L)$.

It only remains to show that $a \in J(L) \Rightarrow m(a) \in M(L)$. Suppose that $m(a) = = x \wedge y$. First note that $x \ngeq a$ or $y \ngeq a$. Indeed if $x \geq a$ and $y \geq a$ then $a \leq x \wedge y = m(a)$ which is a contradiction (see proof Lemma 1.1.16 (ii)),
i.e. either $x \in A$ or $y \in A$ hence $x \leq \bigvee A = m(a) = x \wedge y \leq x$ or
$y \leq \bigvee A = m(a) = x \wedge y \leq y$ then either $x = m(a)$ or $y = m(a)$ and thus $m(a) \in M(L)$. □

1.2 Boolean algebras [16] [10] [3] [15]

In a bounded lattice L, a is a *complement* of b iff

$$a \wedge b = 0$$
$$a \vee b = 1$$

Lemma 1.2.1 In a bounded distributive lattice, an element can have only one complement.

Proof. If b_0 and b_1 are both complements of a, then
$b_0 = b_0 \wedge 1 = b_0 \wedge (a \vee b_1) = (b_0 \wedge a) \vee (b_0 \vee b_1) = 0 \vee (b_0 \vee b_1) = b_0 \vee b_1$
similarly, $b_1 = b_0 \vee b_1$, thus $b_0 = b_1$ □

We denote to complement of an element a by a'. Note that $a'' = a$, $0' = 1$ and $1' = 0$.

A *complemented lattice* is a bounded lattice B in which every element has a complement, i.e. $\forall a \in B \;\; \exists a' \in B$ such that $a \wedge a' = 0$ and $a \vee a' = 1 \quad (L7)$.

A *Boolean algebra* is a distributive complemented lattice.
Thus a Boolean álgebra is a system: $\langle B, \wedge, \vee, {}', 0, 1\rangle$ where $\wedge, \vee$ are binary operations, $'$ is a unary operation, and 0, 1 are nullary operations.

As in lattices, we can define a Boolean algebra in terms of the properties of $\wedge, \vee, {}'$.

Theorem 1.2.2 Let B be a nonempty set and $+, .$ two binary operations and $'$ a unary operation on B satisfying $(L1), (L2), (L3), (L4), (L5), (L6)$ and $(L7)$ (see page $6, 8, 8, 17$). Set $a \leq b$ iff $a = a.b$.

Then B is a Boolean algebra and $a \vee b = a + b$ and $a \wedge b = a.b$.

Proof. Theorem 1.1.6. $\square$

Example 1.2.3 The set $\mathcal{P}(X)$ of all subset of a set X, is a Boolean algebra with the operations $a \vee b = a \cup b$, $a \wedge b = a \cap b$, $a' = a^c$, $0 = \emptyset$, $1 = X$. Its atoms are the subset with only one element.

Example 1.2.4 Let $\langle B, \wedge, \vee, \,', 0, 1\rangle$ be a Boolean algebra, let a be an element of B and the interval $[0, a] = \{x \in B / 0 \leq x \leq a\}$. Then $\langle [0, a], \wedge, \vee, \,^c, 0, a\rangle$ is a Boolean algebra, where $x^c := x' \wedge a$.
Indeed, $[0, a]$ is closed under $\vee$ and $\wedge$, and a is its largest element.
If $x \in [0, a]$ then $x^c \wedge x = (x' \wedge a) \wedge x = (x' \wedge x) \wedge a = 0 \wedge a = 0$ and
$x^c \vee x = (x' \wedge a) \vee x = (x' \vee x) \wedge (x \vee a) = 1 \wedge a = a$ (since B is a distributive lattice and $x \leq a$).

Example 1.2.5 Let $(A_j)_{j \in J}$ be a family of Boolean algebras. It is easy to see that the product $A = \prod_{j \in J} A_j$ is a Boolean algebra with the operations:
If $a, b \in A$ $\quad (a \vee b)_j = a_j \vee b_j \quad (a \wedge b)_j = a_j \wedge b_j \quad a' = (a'_j)_{j \in J}$
$1_A = (1_{A_j})_{j \in J}$ and $0_A = (0_{A_j})_{j \in J}$.

Lemma 1.2.6 (De Morgan's Identities) Let B be a Boolean algebra and let a, b in B. Then

(i) $(a \vee b)' = a' \wedge b'$ and

(ii) $(a \wedge b)' = a' \vee b'$.

Proof.
(i) $(a \vee b) \vee (a' \wedge b') = a \vee b \vee (a' \wedge b') \geq a \vee (b \wedge a') \vee (a' \wedge b') = a \vee (a' \wedge (b \vee b')) = a \vee (a' \wedge 1) = a \vee a' = 1$. Therefore $(a \vee b) \vee (a' \wedge b') = 1$. On the other hand, $(a \vee b) \wedge (a' \wedge b') = (a \wedge a' \wedge b') \vee (b \wedge a' \wedge b') = 0 \vee 0 = 0$. Thus $(a \vee b)' = a' \wedge b'$.
(ii) Replacing a by a' and b by b' in *(i)* and using that $\forall x \in B$ $x'' = x$, then $(a \wedge b)' = a' \vee b'$. $\square$

Let B be a Boolean algebra and $a, b \in B$. We define $a \setminus b = a \wedge b'$.

The next Lemma will be used in Section 3.

Lemma 1.2.7 Let B be a Boolean algebra and $a, b, c, d \in B$. Then:

(i) $(a \vee b) \setminus c = (a \setminus c) \vee (b \setminus c)$.

(ii) $(a \wedge b) \setminus c = (a \setminus c) \wedge (b \setminus c)$.

(iii) If $c \leq a$, $c \leq b$ and $a \setminus c = b \setminus c$ then $a = b$.

(iv) $a \setminus b \leq a$.

(v) If $a \leq b \leq c \setminus d$ then $b \setminus a = ((b \vee d) \wedge c) \setminus ((a \vee d) \wedge c)$.

(vi) $a \leq b \Leftrightarrow b' \leq a'$.

(vii) If $a \leq c$ then $(b \setminus c) \setminus (a \setminus c) = b \setminus a$.

(viii) Let $a_0, a_1, \ldots, a_n \in B$ be such that $0 = a_0 \leq a_1 \leq \ldots \leq a_n$, then

$a_n = (a_n \setminus a_{n-1}) \dot{\vee} \ldots \dot{\vee} (a_2 \setminus a_1) \dot{\vee} (a_1 \setminus a_0)$.

In particular, if $a_n = 1$ and we write $b_j = a_j \setminus a_{j-1}$, $1 \leq j \leq n$, we obtain $1 = b_n \dot{\vee} \ldots \dot{\vee} b_2 \dot{\vee} b_1$. Therefore for all $x \in B$,

$x = (x \wedge b_n) \dot{\vee} \ldots \dot{\vee} (x \wedge b_2) \dot{\vee} (x \wedge b_1)$. We say that $\{b_j\}_{j=1}^{n}$ is a decomposition of unit in the Boolean algebra B.

(ix) Let $a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n$ in B be such that $a_1, b_1 \leq c_1; \ldots$

$\ldots; a_n, b_n \leq c_n$ and $c_i \wedge c_j = 0$ for $i \neq j$ $1 \leq i, j \leq n$. Then

$(a_1 \vee \ldots \vee a_n) \wedge (b_1 \vee \ldots \vee b_n) = (a_1 \wedge b_1) \vee \ldots \vee (a_n \wedge b_n)$ and, if

$a_1 \vee \ldots \vee a_n = b_1 \vee \ldots \vee b_n$ then $a_1 = b_1, \ldots, a_n = b_n$.

Proof.

(i) $(a \vee b) \setminus c = (a \vee b) \wedge c' = (a \wedge c') \vee (b \wedge c') = (a \setminus c) \vee (b \setminus c)$.
(ii) $(a \wedge b) \setminus c = (a \wedge b) \wedge c' = (a \wedge c') \wedge (b \wedge c') = (a \setminus c) \wedge (b \setminus c)$.
(iii) $a = a \wedge 1 = a \wedge (c \vee c') = (a \wedge c) \vee (a \wedge c') = c \vee (a \setminus c) = c \vee (b \setminus c) = (b \wedge c) \vee (b \wedge c') = b \wedge (c \vee c') = b \wedge 1 = b$.
(iv) $a \setminus b = a \wedge b' \leq a$.
(v) $((b \vee d) \wedge c) \setminus ((a \vee d) \wedge c) = ((b \vee d) \wedge c) \wedge ((a \vee d) \wedge c)' =$
$= ((b \wedge c) \vee (d \wedge c)) \wedge ((a' \wedge d') \vee c') =$

$= (b \wedge c \wedge a' \wedge d') \vee (b \wedge c \wedge c') \vee (d \wedge c \wedge a' \wedge d') \vee (d \wedge c \wedge c') =$
$= (b \wedge c \wedge a' \wedge d') \vee 0 \vee 0 \vee 0 = (b \wedge a') \wedge (c \wedge d') = (b \setminus a) \wedge (c \setminus d) = b \setminus a$ since, by (iv) $b \setminus a \leq b$ and by hypothesis $b \leq c \setminus d$.
(vi) $a \leq b \Rightarrow a = a \wedge b$ then, by De Morgan's identities, $a' = a' \vee b'$ and thus $b' \leq a'$. Therefore $a \leq b$ imply $b' \leq a'$. In particular $b' \leq a'$ imply $a'' \leq b''$ that is $a \leq b$.
(vii) $(b\setminus c)\setminus(a\setminus c) = (b\wedge c')\wedge(a\wedge c')' = (b\wedge c')\wedge(a'\vee c) = (b\wedge c'\wedge a')\vee(b\wedge c'\wedge c) =$
$= b \wedge (c' \wedge a') \vee 0 = b \wedge a' = b \setminus a$ from $a \leq c$ and (vi).
$(viii)$ We use induction on n. If $n = 1$, we have $a_1 = a_1 \wedge 1 = a_1 \wedge 0' = a_1 \setminus 0 = a_1 \setminus a_0$. Let $a_0, a_1, \ldots, a_n, a_{n+1} \in B$ be such that $0 = a_0 \leq a_1 \leq \ldots \leq a_n \leq a_{n+1}$. Then, $a_{n+1} = a_{n+1} \wedge 1 = a_{n+1} \wedge (a_n \dot{\vee} a'_n) = (a_{n+1} \wedge a_n)\dot{\vee}(a_{n+1} \wedge a'_n) = a_n\dot{\vee}(a_{n+1} \setminus a_n) = (a_n \setminus a_{n-1})\dot{\vee} \ldots \dot{\vee}(a_2 \setminus a_1)\dot{\vee}(a_1 \setminus a_0)\dot{\vee}(a_{n+1} \setminus a_n)$ (by the induction hypothesis).
(ix) $(a_1\vee\ldots\vee a_n)\wedge(b_1\vee\ldots\vee b_n) = \bigvee_{i,j=1}^{n} a_i\wedge b_j$. If $i \neq j$, $a_i\wedge b_j \leq c_i\wedge c_j = 0$, thus $a_i\wedge b_j = 0$ and we obtain $(a_1\vee\ldots\vee a_n)\wedge(b_1\vee\ldots\vee b_n) = (a_1\wedge b_1)\vee\ldots\vee(a_n\wedge b_n)$. Now suppose $a_1\vee\ldots\vee a_n = b_1\vee\ldots\vee b_n$, then $a_j\wedge(a_1\vee\ldots\vee a_n) = a_j\wedge(b_1\vee\ldots\vee b_n)$ hence $a_j \wedge a_j = a_j \wedge b_j$ (since for $i \neq j$, $a_i \wedge a_j \leq c_i \wedge c_j = 0$ and
$a_i \wedge b_j \leq c_i \wedge c_j = 0$). Therefore $a_j = a_j \wedge b_j$ and thus $a_j \leq b_j$. Similarly $b_j \leq a_j$ and then $a_j = b_j$ $1 \leq j \leq n$. □

A *subalgebra* of a Boolean algebra B is a nonempty subset A of B satisfying the following conditions:

(i) $x \in A \Rightarrow x' \in A$,

(ii) $x, y \in A \Rightarrow x \wedge y \in A$ and $x \vee y \in A$.

Note that $0 \in A$, $1 \in A$ and A is a Boolean algebra.

Theorem 1.2.8 Every bounded distributive lattice can be embedded in a Boolean algebra.

Prof. Let L be a bounded distributive lattice and let X be the set of all prime ideals of L. For $a \in L$, let $r(a) = \{P/\ a \notin P,\ P \in X\}$.
Let ψ be the map of L into $\mathcal{P}(X)$, $\psi(a) = r(a)$.

We claim that ψ is a $\{0,1\}$-homomorphism of lattices of L into the lattice (the Boolean algebra) $\mathcal{P}(X)$.

Since $\forall P \in X,\ 0 \in P \quad$ then $r(0) = \emptyset$.

Since every P in X is proper and Lemma 1.1.10, then $r(1) = X$.

$r(a \wedge b) = r(a) \cap r(b)$:

$P \in r(a \wedge b)$ imply $a \wedge b \notin P$, since $a \wedge b \leq a$ and P is a ideal, if $a \in P$ then $a \wedge b \in P$, wich is a contradiction, then $a \notin P$. Similarly $b \notin P$, thus $P \in r(a)$ and $P \in r(b)$ that is $r(a \wedge b) \subseteq r(a) \cap r(b)$.

Conversely, $P \in r(a) \cap r(b)$ imply $a \notin P$ and $b \notin P$. Since that P is a prime ideal, $a \wedge b \in P \Rightarrow a \in P$ or $b \in P$,wich is a contradiction, then $a \wedge b \notin P$

Therefore $r(a) \cap r(b) \subseteq r(a \wedge b)$ and thus $r(a \wedge b) = r(a) \cap r(b)$.

$r(a \vee b) = r(a) \cup r(b)$:

$P \in r(a \vee b)$ imply $a \vee b \notin P$. Since P is a ideal, if $a \in P$ and $b \in P$ imply $a \vee b \in P$, therefore either $a \notin P$ or $b \notin P$. This is, either $P \in r(a)$ or $P \in r(b)$ and $r(a \vee b) \subseteq r(a) \cup r(b)$.

Since $a \leq a \vee b,\ b \leq a \vee b,$ and P is a ideal, $a \vee b \in P$ imply $a \in P$ and $b \in P$, then $a \notin P$ or $b \notin P$, imply $a \vee b \notin P$. This is $r(a) \cup r(b) \subseteq r(a \vee b)$. Thus $r(a \vee b) = r(a) \cup r(b)$.

ψ is an injective map:

Let $a, b \in L$, by Corollary 1.1.15 there exist a prime ideal P such that $a \in P$ and $b \notin P$, then $P \notin r(a)$ and $P \in r(b)$, thus $r(a) \neq r(b)$. $\square$

A *homomorphism* φ of Boolean algebras is a $\{0,1\}$-homomorphism of lattices that preseves the complement $'$.

Remark: Let A, B be two Boolean algebras and let $\varphi : A \to B$ be an $\{0,1\}$-homomorphism of lattices. Then φ preseves the complement $'$.

Indeed, let $a \in A$, $0_A = a \wedge a' \Rightarrow \varphi(0_A) = \varphi(a \wedge a') \Rightarrow 0_B = \varphi(a) \wedge \varphi(a')$.

$$1_A = a \vee a' \Rightarrow \varphi(1_A) = \varphi(a \vee a') \Rightarrow 1_B = \varphi(a) \vee \varphi(a').$$

Thus $(\varphi(a))' = \varphi(a')$.

Lemma 1.2.9 Let $\varphi : A \to B$ be a homomorphism of Boolean algebras. Let $a_1, a_2 \in A$, then

(i) If $a_1 \leq a_2 \Rightarrow \varphi(a_1) \leq \varphi(a_2)$ in B.

(ii) $\varphi(a_1 \setminus a_2) = \varphi(a_1) \setminus \varphi(a_2)$.

Proof. (i) Remark page 12.

$$(ii)\ \varphi(a_1 \setminus a_2) = \varphi(a_1 \wedge a_2') = \varphi(a_1) \wedge \varphi(a_2') = \varphi(a_1) \wedge (\varphi(a_2))' = $$
$$= \varphi(a_1) \setminus \varphi(a_2).$$ □

A homomorphism $\varphi : B_1 \to B_2$ of Boolean algebras is *onto* (or *surjective*) if for every $b_2 \in B_2$ there is a $b_1 \in B_1$ with $\varphi(b_1) = b_2$.
A homomorphism φ of Boolean algebras is *one-to-one* (or *injective*) if $\varphi(a) = \varphi(b) \Rightarrow a = b$.
An *isomorphism* of Boolean algebras is a biyective (one-to-one and onto) homohorphism.
The notation $A \cong B$ means that there exist an isomorphism $\varphi : A \to B$.
An isomorphism of a Boolean algebra with itself is called an *automorphism*.
Let B be a Boolean algebra and let $f : B \to B$ be an automorphisms on B, we write f^n for $f \circ \ldots \circ f$ (n times) and f^{-n} for $f^{-1} \circ \ldots \circ f^{-1}$ (n times) for all $n \in \mathbb{N}$.

Definition 1.2.10

- A *Group* $\langle A, +, 0\rangle$ is a non-empty set A with a binary operation $+$ and a constan 0 satisfying the following equations:

  for all $x, y, z \in A$ we have $x+(y+z) = (x+y)+z, \qquad x+0 = 0+x = x,$

  $\forall x \in A$ there is an element $-x \in A$ such that $x + (-x) = (-x) + x = 0$.

- Let $\langle A, +, 0\rangle$ be a group and $C \subseteq A$. We say that C is a *subgroup* of A if: $\quad 0 \in C \qquad x \in C \Rightarrow -x \in C \qquad$ and $\qquad \forall x, y \in C \quad x + y \in C$.

Example 1.2.11 It is easy to see that:

If B is a Boolean algebra, then $id : B \to B$ is an isomorphis on B where $id(b) = b$ for all $b \in B$ (the *identity* map).

If B_0, B_1 and B_2 are Boolean algebras, and $\varphi : B_0 \to B_1$, $\phi : B_1 \to B_2$, are two homomorphisms (isomorphisms) of Boolean algebras, then $\phi \circ \varphi : B_0 \to B_2$ is a homomorphism (isomorphism) of Boolean algebras.

If B_0, B_1 are Boolean algebras, and $\varphi : B_0 \to B_1$ is an isomorphism of Boolean algebras, then $\varphi^{-1} : B_1 \to B_0$ is an isomorphism of Boolean algebras.

Let B be a Boolean algebra. We write $Aut(B)$ for the set of all automorphisms of B. From Example 1.2.11 it is easy to see that $(Aut(B), \circ, id)$ is a group.

Example 1.2.12 Let $(A_j)_{j\in J}$ be a family of Boolean algebras. The map

$$p_k : A = \prod_{j\in J} A_j \to A_k$$

defined by

$$p_k((a_j)_{j\in J}) = a_k$$

is called the *projection map on the k th coordinate of* $\prod_{j\in J} A_j$.
It is easy to check that p_k is a surjective homomorphism of Boolean algebras.

1.3 Boolean algebras R-generated by a bounded distributive lattice [10]

Let A be a nonempty set with two binary operations " $+$ " and "." .
A is called a *ring* if $\forall a, b, c \in A$:
$(a + b) + c = a + (b + c)$ $\quad$ $\exists 0 \in A$ such that $\forall a \in A \;\; a + 0 = a$
$a + b = b + a$ $\quad$ $\forall a \in A \;\; \exists -a \in A$ such that $a + (-a) = 0$
$(a.b).c = a.(b.c)$ $\quad$ $a.(b + c) = a.b + a.c$ $\quad$ $(a + b).c = a.c + b.c$
A is called a *commutative ring* if A is a ring and $\forall a, b \in A \;\; a.b = b.a$.
A is called a *ring with a unit* if A is a ring and exists $1 \in A$ such that $\forall a \in A \;\; a.1 = a$.

Let A be a ring, and $C \subseteq A$.
C is a *subring* of A if:
$0 \in C$, $\quad a_1, a_2 \in C \Rightarrow a_1 + a_2 \in C$ and $a_1.a_2 \in C$, $\quad a \in C \Rightarrow -a \in C$

Let A be a commutative ring, and $I \subseteq A$.
I is an *ideal* of A if:

$C \neq \emptyset, \qquad x, y \in I \Rightarrow x + (-y) \in I, \qquad x \in A$ and $c \in I \quad \Rightarrow \quad x.c \in I.$

Let A and B be rings. A map $f : A \to B$ is called a *homomorphism* if
$\forall a_1, a_2 \in A \qquad f(a_1 + a_2) = f(a_1) + f(a_2), \qquad f(a_1.a_2) = f(a_1).f(a_2),$
and, furthermore, if A and B are rings with unit, then $f(1) = 1$.

The proofs of two next theorems are purely computational.

Theorem 1.3.1

(i) Let B be a Boolean algebra. We defined two binary operations in B:

$a + b = (a \wedge b') \vee (b \wedge a') = (a \setminus b) \vee (b \setminus a)$ *"symmetric difference"*

$a.b = a \wedge b$

them $B^{\mathcal{R}} = (B, +, 0, ., 1)$ is a conmutative ring satisfying $x^2 = x.x = x$.

Furthermore $\forall x \in B \quad x + x = 0$ and hence $x = (-x)$ and $x + y = x + (-y)$.

(ii) Conversely, let a $(B, +, 0, ., 1)$ conmutative ring with unit satisfying $x^2 = x.x = x$, (a *Boolean ring* with unit). If we defined $x \leq y$ iff $x = x.y$, then B become a Boolean algebra $B^{\mathcal{L}}$ in which $x \wedge y = x.y$ and $x \vee y = x + y + x.y$.

(iii) Let B be a Boolean algebra, then $(B^{\mathcal{R}})^{\mathcal{L}} = B$.

(iv) Let B be a Boolean ring with unit, then $(B^{\mathcal{L}})^{\mathcal{R}} = B$.

$\square$

Theorem 1.3.2 Let B_0 and B_1 be two Boolean algebras.

(i) Let $I \subseteq B_0$. Then I is an ideal of B_0 iff I is an ideal of $(B_0)^{\mathcal{R}}$.

(ii) Let $\varphi : B_0 \to B_1$. Then φ is a homomorphism of Boolean algebras of B_0 into B_1 iff φ is a homomorphism of $(B_0)^{\mathcal{R}}$ into $(B_1)^{\mathcal{R}}$.

(iii) B_0 is a subalgebra of B_1 iff $(B_0)^{\mathcal{R}}$ is a subring of $(B_1)^{\mathcal{R}}$.

$\square$

We will need the next Lemma.

Lemma 1.3.3 Let B be a Boolean algebra.

(i) If $a, b \in B$ then $a \wedge b = 0$ iff $a \leq b'$.

(ii) If $a, b \in B$ and $a \wedge b = 0$, then $a + b = a \dot{\vee} b$ (where $\dot{\vee}$ is the disjoint join).

(iii) If $a, b \in B$ and $a \leq b$, then $a + b = b \setminus a$.

(iv) If $a, b, c \in B$ and $a \leq b$, then $a \wedge (c \setminus b) = 0$.

(v) Let $a_1, a_2, \ldots, a_{2n} \in B$ be such that $a_1 \leq a_2 \leq \ldots \leq a_{2n}$. Then

$$a_1 + a_2 + \ldots + a_{2n} = (a_2 \setminus a_1) \dot{\vee} (a_4 \setminus a_3) \dot{\vee} \ldots \dot{\vee} (a_{2n} \setminus a_{2n-1}).$$

(vi) Let $a_1, a_2, \ldots, a_{2n-1} \in B$ be such that $0 < a_1 \leq a_2 \leq \ldots \leq a_{2n-1}$. Then

$$a_1 + a_2 + \ldots + a_{2n-1} = a_1 \dot{\vee} (a_3 \setminus a_2) \dot{\vee} \ldots \dot{\vee} (a_{2n-1} \setminus a_{2n-2}).$$

Proof. (i) If $a \wedge b = 0$ then $a = a \wedge 1 = a \wedge (b \vee b') = (a \wedge b) \vee (a \wedge b') =$
$= 0 \vee (a \wedge b') = a \wedge b'$, and thus $a \leq b'$. On the other hand $a \leq b' \Rightarrow a = a \wedge b'$ and then $a \wedge b = (a \wedge b') \wedge b = a \wedge (b' \wedge b) = a \wedge 0 = 0$.
(ii) By (i) $a \wedge b = 0 \Rightarrow a \leq b'$ and $b \leq a'$, hence $a + b = (a \wedge b') \dot{\vee} (b \wedge a') =$
$= a \dot{\vee} b$.
(iii) By (i) $a \leq b \Rightarrow a \wedge b' = 0$. Then $a + b = (a \wedge b') \vee (b \wedge a') = 0 \vee (b \wedge a') =$
$= b \wedge a' = b \setminus a$.
(iv) Since $a \leq b$, by (i), $a \wedge b' = 0$. Thus $a \wedge (c \setminus b) = a \wedge (c \wedge b') = (a \wedge b') \wedge c =$
$= 0 \wedge c = 0$.
(v) We proceed by induction on n. If $n = 1$ we have $a_1 \leq a_2$ and, by (iii), $a_1 + a_2 = a_1 \setminus a_2$. Now suppose $a_1 \leq a_2 \leq \ldots \leq a_{2n} \Rightarrow a_1 + a_2 + \ldots + a_{2n} =$
$= (a_2 \setminus a_1) \dot{\vee} (a_4 \setminus a_3) \dot{\vee} \ldots \dot{\vee} (a_{2n} \setminus a_{2n-1})$. Let $a_1 \leq a_2 \leq \ldots \leq a_{2n} \leq a_{2n+1} \leq$
$\leq a_{2n+2}$. From the induction hypothesis and $a_{2n+1} \leq a_{2n+2}$ and (iii) we have $a_1 + a_2 + \ldots + a_{2n} + a_{2n+1} + a_{2n+2} = (a_1 + a_2 + \ldots + a_{2n}) + (a_{2n+1} + a_{2n+2}) =$
$((a_2 \setminus a_1) \dot{\vee} (a_4 \setminus a_3) \dot{\vee} \ldots \dot{\vee} (a_{2n} \setminus a_{2n-1})) + (a_{2n+2} \setminus a_{2n+1})$. Note that
for all $1 \leq i \leq n$, $a_{2i} \setminus a_{2i-1} \leq a_{2i} \leq a_{2n+1}$ and thus
$(a_2 \setminus a_1) \dot{\vee} (a_4 \setminus a_3) \dot{\vee} \ldots \dot{\vee} (a_{2n} \setminus a_{2n-1}) \leq a_{2n+1}$. Therefore, by (iv),
$((a_2 \setminus a_1) \dot{\vee} (a_4 \setminus a_3) \dot{\vee} \ldots \dot{\vee} (a_{2n} \setminus a_{2n-1})) \wedge (a_{2n+2} \setminus a_{2n+1}) = 0$ and thus by (ii)

$((a_2 \setminus a_1)\dot{\vee}(a_4 \setminus a_3)\dot{\vee} \ldots \dot{\vee}(a_{2n} \setminus a_{2n-1})) + (a_{2n+2} \setminus a_{2n+1}) =$
$= (a_2 \setminus a_1)\dot{\vee}(a_4 \setminus a_3)\dot{\vee} \ldots \dot{\vee}(a_{2n} \setminus a_{2n-1})\dot{\vee}(a_{2n+2} \setminus a_{2n+1})$. Therefore
$a_1 + a_2 + \ldots + a_{2n} + a_{2n+1} + a_{2n+2} = (a_2 \setminus a_1)\dot{\vee}(a_4 \setminus a_3)\dot{\vee} \ldots \dot{\vee}(a_{2n} \setminus a_{2n-1})\dot{\vee}$
$\dot{\vee}(a_{2n+2} \setminus a_{2n+1})$.
(vi) Let $a_0 = 0$ then, by (v), $a_1 + a_2 + \ldots + a_{2n-1} = 0 + a_1 + a_2 + \ldots + a_{2n-1} =$
$a_0 + a_1 + a_2 + \ldots + a_{2n-1} = (a_1 \setminus a_0)\dot{\vee}(a_3 \setminus a_2)\dot{\vee} \ldots \dot{\vee}(a_{2n-1} \setminus a_{2n}) =$
$= (a_1 \setminus 0)\dot{\vee}(a_3 \setminus a_2)\dot{\vee} \ldots \dot{\vee}(a_{2n-1} \setminus a_{2n}) = a_1\dot{\vee}(a_3 \setminus a_2)\dot{\vee} \ldots \dot{\vee}(a_{2n-1} \setminus a_{2n})$. $\square$

Definition 1.3.4 Let L a $\{0,1\} -$ *sublattice* of the Boolean algebra B. Then L is said to *R-generate* B iff L generates B as a ring.

The next Lemma will be used in Section 2.

Lemma 1.3.5 Let L be a finite distributive lattice and $r : L \to 2^{J(L)}$ as Lemma 1.1.17. Then $r(L)$ R-generates $2^{J(L)}$.

Proof. From Remark page 12 $r(L)$ is a sublattice of $2^{J(L)}$.
Now, note that:

i) Let $a \in J(L)$, $z_1 \prec a$ and $z_2 \prec a$. Then $z_1 = z_2$.

   $z_1 \prec a$ and $z_2 \prec a$ imply $z_1 < a$, $z_2 < a$ and $z_1 \vee z_2 \leq a$. Thus $z_1 \leq z_1 \vee z_2 \leq a$. Since $z_1 \prec a$ we have either $z_1 \vee z_2 = a$ or $z_1 \vee z_2 = z_1$.

   $z_1 \vee z_2 = a \Rightarrow z_1 = a$ or $z_2 = a$ (since $a \in J(L)$) which is a contradiction.

   Then $z_1 \vee z_2 = z_1$ and thus $z_2 \leq z_1$. Similarly $z_1 \leq z_2$ and thus $z_1 = z_2$.

ii) Let $a \in J(L)$ and $z \prec a$. Then $z = \bigvee \{x \in L : x < a\}$.

   $0 \in \{x \in L : x < a\}$. Since L is finite, $\{x \in L : x < a\} = \{x_1, \ldots, x_n\}$ with $x_i < a$, $1 \leq i \leq n$, and thus $x_1 \vee \ldots \vee x_n \leq a$. Now

   $x_1 \vee \ldots \vee x_n = a \Rightarrow \exists j\ 1 \leq j \leq n$ such that $a = x_j$ (since $a \in J(L)$) which is a contradiction. Therefore $x_1 \vee \ldots \vee x_n < a$ and thus $x_1 \vee \ldots \vee x_n$ is a maximal element of $\{x \in L : x < a\}$ that is $\bigvee \{x \in L : x < a\} \prec a$ and thus ,from $i)$, $z = \bigvee \{x \in L : x < a\}$.

iii) Let $a \in J(L)$ and $z \prec a$. Then $r(a) + r(z) = \{a\}$.

$z \prec a \Rightarrow z < a \Rightarrow r(z) \subseteq r(a)$ and then $r(a) + r(z) = r(a) \setminus r(z)$.

$a \in r(a)$ and $a \notin r(z)$ (since $z < a$) then $\{a\} \in r(a) \setminus r(z) = r(a) + r(z)$ and thus $\{a\} \subseteq r(a) + r(z)$.

$x \in r(a) + r(z) = r(a) \setminus r(z) \Rightarrow x \in J(L)$, $x \leq a$ and $x \nleq z$. Also $x \leq a$ and $z < a \Rightarrow z \leq x \vee z \leq a$. Since $z \prec a$ either $x \vee z = z$ or $x \vee z = a$. $x \vee z = z \Rightarrow x \leq z$ which is a contradiction. Therefore $x \vee z = a$ and since $a \in J(L)$ either $z = a$ or $x = a$. Now, $z = a$ is a contradiction, then $x = a$ and $r(a) \setminus r(z) = r(a) + r(z) \subseteq \{a\}$. Therefore $r(a) + r(z) = \{a\}$.

Now, let $A \in 2^{J(L)}$, $A = \{a_1, \ldots, a_n\}$.
Let $z_1, \ldots, z_n$ such that $z_i \prec a_i$, $1 \leq i \leq n$. Then, from *iii)* $r(a_i) + r(z_i) = \{a_i\}$ $1 \leq i \leq n$ and thus $A = \{a_1, \ldots, a_n\} = \{a_1\} \dot{\cup} \ldots \dot{\cup} \{a_n\} = \{a_1\} + \ldots + \{a_n\} =$
$= (r(a_1) + r(z_1)) + \ldots + (r(a_n) + r(z_i)) = r(a_1) + r(z_1) + \ldots + r(a_n) + r(z_i)$.

□

Lemma 1.3.6 Let B be R-generated by L. Then every $a \in B$ can be expressed in the form $a_0 + a_1 + \ldots\ldots + a_{n-1}$ with $a_0 \leq a_1 \leq \ldots\ldots \leq a_{n-1}$ and $a_0, a_1, \ldots\ldots a_{n-1} \in L$.

Proof. Let B_1 denote the set of all elements that can be represented in the form $\quad a_0 + a_1 + \ldots\ldots + a_{n-1}, \quad a_0, a_1, \ldots\ldots a_{n-1} \in L$.
Then $L \subseteq B_1$, and B_1 is closed under $+$ and $-$ (since $x - y = x + y$).
Furthermore,

$$(a_0 + \ldots\ldots + a_{n-1}).(b_0 + \ldots\ldots + b_{n-1}) = \sum a_i b_j \tag{1}$$

and each term $a_i b_j = a_i \wedge b_j \in L$, so B_1 is closed under multiplication. We conclude that $B_1 = B$.
Note that L is a sublattice of B; therefore, for $a, b \in L$, $a \vee b$ in L is the same as $a \vee b$ in B. Thus $a \vee b = a + b + ab$, and so

$$a + b = ab + (a \vee b) = (a \wedge b) + (a \vee b).$$

Take $a_0 + a_1 + \ldots\ldots + a_{n-1} \in B$. We prove by induction on n that the summands can be made to form an increasing sequence. We will prove that $a_0 + a_1 + \ldots\ldots + a_{n-1} = b_0 + b_1 + \ldots\ldots + b_{n-1}$, where

$$b_j = \bigvee\left(\bigwedge_{k=0}^{n-1-j} a_{i_k} \;\; {}_{0 \leq i_0 < i_1 < \ldots < i_{n-1-j} \leq n-1}\right) \tag{2}$$

and thus $b_0 \leq \ldots\ldots \leq b_{n-1}$ and $b_j \in L \;\; 0 \leq j \leq n-1$.
For example if $n = 3$ the formula 2 is:

$$a_0+a_1+a_2 = \underbrace{(a_0 \wedge a_1 \wedge a_2)}_{b_0} + \underbrace{((a_0 \wedge a_1) \vee (a_0 \wedge a_2) \vee (a_1 \wedge a_2)}_{b_1} + \underbrace{(a_0 \vee a_1 \vee a_2)}_{b_2}.$$

For $n = 2$ we have $a_0 + a_1 = (a_0 \wedge a_1) + (a_0 \vee a_1)$.
Let $a_0 + a_1 + \ldots\ldots + a_{n-1} + a_n$.
$a_0 + a_1 + \ldots\ldots + a_{n-1} + a_n = a_0 + (a_1 + \ldots\ldots + a_{n-1} + a_n)$. By the induction hypothesis, $a_1 + \ldots\ldots + a_{n-1} + a_n = d_1 + \ldots\ldots + d_{n-1} + d_n$, where

$$d_j = \bigvee\left(\bigwedge_{k=0}^{n-1-j} a_{i_k} \;\; {}_{1 \leq i_1 < i_2 < \ldots < i_{n-1-j} \leq n}\right)$$

Now,
$a_0 + a_1 + \ldots\ldots + a_{n-1} + a_n = a_0 + (a_1 + \ldots\ldots + a_{n-1} + a_n) =$
$= a_0 + (d_1 + \ldots\ldots + d_{n-1} + d_n) = a_0 + d_1 + \ldots\ldots + d_{n-1} + d_n =$
$= (a_0 \wedge d_1) + (a_0 \vee d_1) + d_2 + \ldots\ldots + d_{n-1} + d_n =$
$= (a_0 \wedge d_1) + ((a_0 \vee d_1) \wedge d_2) + ((a_0 \vee d_1) \vee d_2) + d_3 + \ldots\ldots + d_{n-1} + d_n =$
$= (a_0 \wedge d_1) + ((a_0 \vee d_1) \wedge d_2) + (a_0 \vee d_2) + d_3 + \ldots\ldots + d_{n-1} + d_n =$
$= (a_0 \wedge d_1) + ((a_0 \vee d_1) \wedge d_2) + ((a_0 \vee d_2) \wedge d_3) + ((a_0 \vee d_2) \vee d_3) + d_4 \ldots\ldots + d_{n-1} + d_n =$
$= (a_0 \wedge d_1) + ((a_0 \vee d_1) \wedge d_2) + ((a_0 \vee d_2) \wedge d_3) + (a_0 \vee d_3) + d_4 \ldots\ldots + d_{n-1} + d_n =$
$\vdots$
$= (a_0 \wedge d_1) + ((a_0 \vee d_1) \wedge d_2) + \ldots\ldots + ((a_0 \vee d_{n-1}) \wedge d_n) + (a_0 \vee d_n)$,
and
$a_0 \wedge d_1 = a_0 \wedge (a_1 \wedge \ldots \wedge a_{n-1} \wedge a_n) = a_0 \wedge a_1 \wedge \ldots \wedge a_{n-1} \wedge a_n = b_0$.
$a_0 \vee d_n = a_0 \vee (a_1 \vee \ldots \vee a_{n-1} \vee a_n) = a_0 \vee a_1 \vee \ldots \vee a_{n-1} \vee a_n = b_n$.
and, for $j = 1, 2, \ldots, n-1$,
$(a_0 \vee d_j) \wedge d_{j+1} = (a_0 \wedge d_{j+1}) \vee (d_j \wedge d_{j+1}) = (a_0 \wedge d_{j+1}) \vee d_j =$

$$= (a_0 \wedge (\bigvee (\bigwedge_{k=0}^{n-j-1} a_{i_k} \ {}_{1\le i_0<i_1<\ldots<i_{n-j-1}\le n}))) \vee (\bigvee (\bigwedge_{k=0}^{n-j} a_{i_k} \ {}_{1\le i_0<i_1<\ldots<i_{n-j}\le n})) =$$

$$= (\bigvee (\bigwedge_{k=0}^{n-j-1} a_0 \wedge a_{i_k} \ {}_{1\le i_1<i_1<\ldots<i_{n-j-1}\le n})) \vee (\bigvee (\bigwedge_{k=0}^{n-j} a_{i_k} \ {}_{1\le i_0<i_1<\ldots<i_{n-j}\le n})) =$$

$$= (\bigvee (\bigwedge_{k=0}^{n-j} a_{i_k} \ {}_{0=i_0<i_1<\ldots<i_{n-j-1}\le n})) \vee (\bigvee (\bigwedge_{k=0}^{n-j} a_{i_k} \ {}_{1\le i_1<i_2<\ldots<i_{n-j}\le n})) =$$

$$= \bigvee (\bigwedge_{k=0}^{n-j} a_{i_k} \ {}_{0\le i_0<i_1<\ldots<i_{n-j-1}\le n}) = b_j$$

$\square$

Lemma 1.3.7 Let L be a bounded distributive lattice. Then there exist a Boolean algebra R-generated by L.

Proof. By Lemma 1.2.8 L can be embedded in a Boolean algebra A.
Let $[L]$ denote the set of all elements that can be represented in the form $a_0 + a_1 + \ldots\ldots + a_{n-1}$, $\quad a_0, a_1, \ldots\ldots a_{n-1} \in L$. Then
If $a \in [L] \Rightarrow a' \in [L]$ since $1 \in L$ and $a' = a + 1$ $(a + 1 = (a \wedge 1') \vee (a' \wedge 1) = (a \wedge 0) \vee (a' \wedge 1) = 0 \vee a' = a')$.
If $x, y \in [L]$, then by formula (1) page 27, $x \wedge y \in [L]$ and since $x \vee y = x + y + x \wedge y$, $x \vee y \in [L]$.
Thus $[L]$ is a subalgebra of the Boolean algebra A, in particular $[L]$ is a Boolean algebra. Furthermore, by definition, $L \subseteq [L]$ and $[L]$ is R-generate by L. $\square$

Lemma 1.3.8 Let B be a Boolean algebre R-generated by L.
Then $|B| \le |L| + \aleph_0$.

Proof. By Lemma 1.3.6, every element of B can be associated with a finite sequence of elements of $L \cup \{+\}$, and there are no more than $|L| + \aleph_0$ such sequences. $\square$

Definition 1.3.9 Let L be a bounded distributive lattice. B is a *Boolean algebra freely R-generated by* L if:

(i) B is a Boolean algebra.

(ii) B is R-generated by L.

(iii) If B_1 is R-generated by L, then there is a homomorphism φ of B onto B_1 that is the identity map on L.

Theorem 1.3.10 Let L be a bounded distributive lattice. Then, there exist a Boolean algebra B freely R-generated by L.

Proof. Let $(B_j)_{j\in J}$ the family of all Boolean algebras R-generated by L (from Lemma 1.3.7 this family is not empty). For any B_j there exist
$i_j : L \to B_j$ the inclusion. B has the property that, for any B_j ($j \in J$) there exist a homomorphism φ_j of B onto B_j that is the identity map on L. To construct B, we have to construct a Boolean algebra R-generated having this property for *all* B_j.
How would we construct such a Boolean algebra R-generated for two (B_1 and B_2)? Form the Boolean algebra $B_1 \times B_2$ (see example 1.2.5), and define a map $\phi : L \to B_1 \times B_2$ by $\phi(l) = (i_1(l), i_2(l))$. Then
ϕ is a $\{0,1\}$-homomorphism of lattices and ϕ is an injective map.
Thus, $\phi(L) \cong L$, and $\phi(L)$ is a bounded distributive lattice.
We identify $l \in L$ with $\phi(l) = (i_1(l), i_2(l)) \in \phi(L) \subseteq B_1 \times B_2$.
Let $N = [\phi(L)]$ ($N \subseteq B_1 \times B_2$) as Lemma 1.3.7.
By construction, N is R-generated by L (by $\phi(L)$).
Let $\varphi_j : N \to B_j$ ($j = 1,2$) $\varphi_j(b_1, b_2) = b_j$,
then $\varphi_j(\text{“}l\text{”}) = \varphi_j(\phi(l)) = \varphi_j(i_1(l), i_2(l)) = i_j(l) = l$ in B_j, and
φ_j is a homomorphism of Boolean algebras (see example 1.2.12) of N onto B_j.

If we are given any number of Boolean algebras B_i R-generated by L, we can proceed as before. There is only one problem. All B_i do not form a set, so their direct product cannot be formed. Observe that, by lemma 1.3.8, in every B_j we have $|B_j| \leq |L| + \aleph_0$. Thus, by choosing a large enough set S and taking only those B_j that satisfy $B_j \subseteq S$, we can solve our problem.

Now we are ready to proceed with the formal proof. Choose a set S satisfying $|S| = |L| + \aleph_0$. Note that for each B_j, since $|B_j| \leq |S|$, we have a injective map $\alpha_j : B_j \to S$.
Let $S_j = \alpha_j(B_j)$, and make S_j into a Boolean algebra by defining
$0_{S_j} = \alpha_j(0_{B_j})$, $\quad 1_{S_j} = \alpha_j(1_{B_j})$, $\quad \alpha_j(b_1) \wedge \alpha_j(b_2) = \alpha_j(b_1 \wedge b_2)$,
$\alpha_j(b_1) \vee \alpha_j(b_2) = \alpha_j(b_1 \vee b_2) \quad$ and $\quad (\alpha_j(b))' = \alpha_j(b')$.

Then α_j is an isomorphism of Boolean algebras, and $B_j \cong S_j$.
Let A be the Boolean algebra $A = \prod_{j \in J} S_j$ and $\phi : L \to A$ with
$\phi(l) = (\alpha_j(i_j(l)))_{j \in J}$. ϕ is a injective $\{0,1\}$-homomorphism of lattices.
As before, let $B = [\phi(L)] \subseteq A$. Then, B is a Boolean algebra and B is R-generated by L (by $\phi(L)$). Also, let $\varphi_k : B \to B_k$, $\varphi_k = \alpha_k^{-1} \circ p_k$ $(k \in J)$.
φ_k is a homomorphism of Boolean algebras of B onto B_k (see Example 1.2.11) and, if we identify $l \in L$ with $\phi(l)$ in B, then $\varphi_k(\text{“}l\text{”}) = \varphi_k(\phi(l)) =$
$= \varphi_k((\alpha_j(i_j(l)))_{j \in J}) = \alpha_k^{-1} \circ p_k((\alpha_j(i_j(l)))_{j \in J}) = \alpha_k^{-1}(\alpha_k(i_k(l))) = i_k(l) = l$ in B_k □

Lemma 1.3.11 Let $a_0, a_1, \ldots\ldots, a_{n-1}$ be elements of L such that $a_0 \leq a_1 \leq \ldots\ldots \leq a_{n-1}$. Let B be a Boolean algebra R-generated by L.
Then $\quad a_0 + a_1 + \ldots\ldots + a_{n-1} \leq a_{n-1} \quad$ in $\quad B$.

Proof. We proceed by induction on “n”.
The case $n = 1$ is trivial $(a_0 \leq a_0)$.
Let $a_0 + a_1 + \ldots\ldots + a_{n-1} + a_n$, $\quad a_0 \leq a_1 \leq \ldots\ldots \leq a_{n-1} \leq a_n$.
The induction hypothesis is $a_0 + a_1 + \ldots\ldots + a_{n-1} \leq a_{n-1}$.
Thus $a_0 + a_1 + \ldots\ldots + a_{n-1} \leq a_{n-1} \leq a_n$, and then
$(a_0 + a_1 + \ldots\ldots + a_{n-1}) \wedge a_n' = 0$
Therefore,
$a_0 + a_1 + \ldots\ldots + a_{n-1} + a_n = (a_0 + a_1 + \ldots\ldots + a_{n-1}) + a_n =$
$= ((a_0 + a_1 + \ldots\ldots + a_{n-1})' \wedge a_n) \vee ((a_0 + a_1 + \ldots\ldots + a_{n-1}) \wedge a_n') =$
$= ((a_0 + a_1 + \ldots\ldots + a_{n-1})' \wedge a_n) \vee 0 =$
$= (a_0 + a_1 + \ldots\ldots + a_{n-1})' \wedge a_n \leq a_n$ □

Lemma 1.3.12 Let B be a Boolean algebra freely R-generated by L and let B_1 be a Boolean algebra R-generated by L. Then $B \cong B_1$.

Proof. Since B a Boolean algebra freely R-generated by L, then there exist a homomorphism φ of B onto B_1 that is the identity map on L.
We will see that φ is one-to-one.
If $a, b \in B$ and $\varphi(a) = \varphi(b)$ then,
$0 = \varphi(a) \wedge (\varphi(a))' = \varphi(b) \wedge (\varphi(a))' = \varphi(b) \wedge \varphi(a') = \varphi(b \wedge a')$.
Let $c = b \wedge a' \in B$. By Lemma 1.3.6, if $c \neq 0$ then
$c = l_0 + l_1 + \ldots\ldots + l_{n-1}$,
where $l_0, l_1, \ldots\ldots, l_{n-1} \in L$ and $0 < l_0 \leq l_1 \leq \ldots\ldots \leq l_{n-1}$.
De todas las posibles escrituras de c, tomo
$c = l_0 + l_1 + \ldots\ldots + l_{n-1}, \quad l_0, l_1, \ldots\ldots, l_{n-1} \in L, \quad 0 < l_0 \leq l_1 \leq \ldots\ldots \leq l_{n-1}$
con n mínimo.
We have $\varphi(c) = 0$
$\varphi(l_0 + l_1 + \ldots\ldots + l_{n-2} + l_{n-1}) = 0$
$\varphi(l_0) + \varphi(l_1) + \ldots\ldots + \varphi(l_{n-2}) + \varphi(l_{n-1}) = 0$ (by Theorem 1.3.2 (ii)).
$\varphi(l_0) + \varphi(l_1) + \ldots\ldots + \varphi(l_{n-2}) = -\varphi(l_{n-1})$
$\varphi(l_0) + \varphi(l_1) + \ldots\ldots + \varphi(l_{n-2}) = \varphi(l_{n-1})$ (by Theorem 1.3.1 (i)).
Thus, by Lemma 1.3.11 and Lemma 1.2.9,
$\varphi(l_{n-2}) \leq \varphi(l_{n-1}) = \varphi(l_0) + \varphi(l_1) + \ldots\ldots + \varphi(l_{n-2}) \leq \varphi(l_{n-2})$.
Hence $\varphi(l_{n-2}) = \varphi(l_{n-1})$ and since φ is the identity map on L, $l_{n-2} = l_{n-1}$.
Therefore, $c = l_0 + l_1 + \ldots\ldots + l_{n-2} + l_{n-1} = l_0 + l_1 + \ldots\ldots + l_{n-1} + l_{n-1} =$
$= l_0 + l_1 + \ldots\ldots + (l_{n-1} + l_{n-1}) = l_0 + l_1 + \ldots\ldots + l_{n-3} + 0 =$
$= l_0 + l_1 + \ldots\ldots + l_{n-3}$. Absurdo pues n era mínimo.
Then we have $c = 0$ that is $b \wedge a' = 0$.
With the same argument $a \wedge b' = 0$.
Now, $b \wedge a' = 0 \Rightarrow a \leq b$ (by Lemma 1.3.3 (i)), and $a \wedge b' = 0 \Rightarrow b \leq a$.
Hence $a = b$ $\square$

Theorem 1.3.13 Let B_1 and B_2 be two Boolean algebras R-generated by L. Then $B_1 \cong B_2$.

Proof. By lemma 1.3.12 there exist $\varphi_1 : B \to B_1$ and $\varphi_2 : B \to B_2$ isomorphisms of Boolean algebras, where B is a Boolean algebra freely R-generated by L. Therefore $\varphi : B_1 \to B_2, \quad \varphi = \varphi_2 \circ (\varphi_1)^{-1}$ is a isomorphism of B_1 into B_2 $\square$

Remark: For a bounded distributive lattice L, we shall denote by $B(L)$ a Boolean algebra R-generated by L.

Example 1.3.14 For a bounded chain C an explicit representation of $B(C)$ is given as follows:

Let $B\left[C\right]$ be the set of all subsets of C of the form

$(a_0] + (a_1] + \ldots\ldots + (a_{n-1}]\,,$
$0 < a_0 \leq a_1 \leq \ldots\ldots \leq a_{n-1}, \quad a_0, a_1, \ldots\ldots, a_{n-1} \in C,$

where $+$ is the symmetric difference and $(a] = \{c \in C / c \leq a\}$. We consider $B\left[C\right]$ as a poset (partially ordered by $\subseteq$). We identify $a \in C$ with $(a]$ for $a \neq 0$, and 0 with $\emptyset$. Thus $C \subseteq B\left[C\right]$.
Note that,

- $a_0 \leq a_1 \Rightarrow (a_0] + (a_1] = (a_0, a_1]$,
- if A, B are disjoint sets, then $A + B = (B^c \cap A) \cup (B \cap A^c) = A \dot{\cup} B$,
- $a_0 \leq a_1 \leq a_2 \leq a_3 \Rightarrow (a_0, a_1]$ and $(a_2, a_3]$ are disjoint sets.

Then

$$\begin{aligned} &(a_0] + (a_1] + \ldots\ldots + (a_{2n-2}] + (a_{2n-1}] = \\ &= ((a_0] + (a_1]) + \ldots + ((a_{2n-1}] + (a_{2n-2}]) = \\ &= (a_0, a_1] + \ldots\ldots\ldots + (a_{2n-2}, a_{2n-1}] = \\ &= (a_0, a_1] \,\dot{\cup} \ldots\ldots\ldots \dot{\cup}\, (a_{2n-2}, a_{2n-1}] \end{aligned}$$

or

$$\begin{aligned} &(a_0] + (a_1] + (a_2] + \ldots\ldots + (a_{2n-1}] + (a_{2n}] = \\ &= (a_0] + ((a_1] + (a_2]) + \ldots + ((a_{2n-1}] + (a_{2n}]) = \\ &= (a_0] + (a_1, a_2] + \ldots\ldots\ldots + (a_{2n-1}, a_{2n}] = \\ &= (a_0] \,\dot{\cup}\, (a_1, a_2] \,\dot{\cup} \ldots\ldots\ldots \dot{\cup}\, (a_{2n-1}, a_{2n}] = \\ &= (0, a_0] \,\dot{\cup}\, (a_1, a_2] \,\dot{\cup} \ldots\ldots\ldots \dot{\cup}\, (a_{2n-1}, a_{2n}]. \end{aligned}$$

Lemma 1.3.15 Let C a bounded chain. Then $(\{\emptyset\} \cup B\left[C\right] \cup \{C\}\,, \cup, \cap, \emptyset, C)$ is the Boolean algebra R-generated by C.

Proof. The proof is obvious by construction and by Theorem 1.3.13. □

Lemma 1.3.16 If $[0,a]_L$ is an interval in a bounded distributive lattice L, then $B([0,a]_L)$ is naturally isomorphic to the interval $[0,a]_{B(L)}$.

Proof. We note that:
$[0,a]_{B(L)}$ is a Boolean algebra (see example 1.2.4),
$[0,a]_L$ is a sublattice of $[0,a]_{B(L)}$.
Moreover if $x \in [0,a]_{B(L)}$ then $x = l_1 + l_2 + \ldots + l_n$ with $l_1, l_2, \ldots, l_n \in L$ and $0 \le x \le a$. Then $x = x \wedge a = (l_1 + l_2 + \ldots + l_n) \wedge a = (l_1 + l_2 + \ldots + l_n).a =$
$= l_1.a + l_2.a + \ldots + l_n.a = l_1 \wedge a + l_2 \wedge a + \ldots + l_n \wedge a$, and since $l_j \wedge a \in [0,a]_L$, $j = 1, \ldots, n$, we have that $[0,a]_{B(L)}$ is R-generate by $[0,a]_L$.
Thus, by Theorem 1.3.13, $B([0,a]_L) \cong [0,a]_{B(L)}$.

Proposition 1.3.17 Let L_1 and L_2 be two bounded distributive lattices, and let $\varphi : L_1 \to L_2$ be a $\{0,1\}$ −homomorphism of lattices. Then φ uniquely extends to a homomorphism of Boolean algebras $\tilde{\varphi} : B(L_1) \to B(L_2)$.

Proof. Let $a \in B(L_1)$, then $a = a_0 + a_1 + \ldots + a_{n-1}$. We define
$\tilde{\varphi}(a) = \varphi(a_0) + \varphi(a_1) + \ldots + \varphi(a_{n-1})$. We shall see that $\tilde{\varphi}$ is well defined.
First suppose that $a_0 + a_1 + \ldots + a_{n-1} = 0$, then by Lemma 1.3.6, $a_0 + a_1 + \ldots + a_{n-1}$ can be expressed in the form

$$b_0 + b_1 + \ldots + b_{n-1} \;\; with \;\; b_0 \le b_1 \le \ldots \le b_{n-1} \;\; and$$

$$b_j = \bigvee \Big(\bigwedge_{k=0}^{n-1-j} a_{i_k} \;\; {}_{0 \le i_0 < i_1 < \ldots < i_{n-1-j} \le n-1} \Big)$$

Thus $b_0 + b_1 + \ldots + b_{n-1} = 0$ with $b_0 \le b_1 \le \ldots \le b_{n-1}$.
If n is even, by Lemma 1.3.3 (v), $b_0 = b_1; b_2 = b_3; \ldots; b_{n-4} = b_{n-3}; b_{n-2} = b_{n-1}$, and thus $\varphi(b_0) + \varphi(b_1) + \ldots + \varphi(b_{n-1}) =$
$= \varphi(b_0) + \varphi(b_0) + \varphi(b_2) + \varphi(b_2) + \ldots + \varphi(b_{n-1}) + \varphi(b_{n-1}) =$
$= (\varphi(b_0) + \varphi(b_0)) + (\varphi(b_2) + \varphi(b_2)) + \ldots + (\varphi(b_{n-1}) + \varphi(b_{n-1})) = 0 + 0 + \ldots + 0 = 0.$
If n is odd, by Lemma 1.3.3 (vi), $b_0 = 0; b_1 = b_2; b_3 = b_4; \ldots; b_{n-4} = b_{n-3};$
$b_{n-2} = b_{n-1}$, and thus $\varphi(b_0) + \varphi(b_1) + \ldots + \varphi(b_{n-1}) =$
$= \varphi(0) + \varphi(b_1) + \varphi(b_1) + \varphi(b_3) + \varphi(b_3) + \ldots + \varphi(b_{n-1}) + \varphi(b_{n-1}) =$
$= \varphi(0) + (\varphi(b_1) + \varphi(b_1)) + (\varphi(b_3) + \varphi(b_3)) + \ldots + (\varphi(b_{n-1}) + \varphi(b_{n-1})) =$

$= 0 + 0 + \ldots + 0 = 0.$

Therefore, since φ a homomorphism of lattices and formula (2) page 28,

$0 = \varphi(b_0) + \varphi(b_1) + \ldots + \varphi(b_{n-1}) =$

$$= \varphi(\bigvee(\bigwedge_{k=0}^{n-1} a_{i_k\ 0\leq i_0<\ldots<i_{n-1}\leq n-1})) + \varphi(\bigvee(\bigwedge_{k=0}^{n-2} a_{i_k\ 0\leq i_0<\ldots<i_{n-2}\leq n-1})) + \ldots + \varphi(\bigvee(\bigwedge_{k=0}^{0} a_{i_k\ 0\leq i_0\leq n-1})) =$$

$$= \bigvee(\bigwedge_{k=0}^{n-1} \varphi(a_{i_k})_{\ 0\leq i_0<\ldots<i_{n-1}\leq n-1}) + \bigvee(\bigwedge_{k=0}^{n-2} \varphi(a_{i_k})_{\ 0\leq i_0<\ldots<i_{n-2}\leq n-1}) + \ldots + \bigvee(\bigwedge_{k=0}^{0} \varphi(a_{i_k})_{\ 0\leq i_0\leq n-1}) =$$

$= \varphi(a_0) + \varphi(a_1) + \ldots + \varphi(a_{n-1})$ (again from formula (2)).

Thus, we have prove that

$a_0 + a_1 + \ldots + a_{n-1} = 0 \ \Rightarrow\ \varphi(a_0) + \varphi(a_1) + \ldots + \varphi(a_{n-1}) = 0.$

Now let $a_0 + a_1 + \ldots + a_{n-1} = c_0 + c_1 + \ldots + c_{m-1}$, then

$a_0 + a_1 + \ldots + a_{n-1} + c_0 + c_1 + \ldots + c_{m-1} = 0$ hence

$\varphi(a_0) + \varphi(a_1) + \ldots + \varphi(a_{n-1}) + \varphi(c_0) + \varphi(c_1) + \ldots + \varphi(c_{m-1}) = 0$ and thus

$\varphi(a_0) + \varphi(a_1) + \ldots + \varphi(a_{n-1}) = \varphi(c_0) + \varphi(c_1) + \ldots + \varphi(c_{m-1})$. Therefore $\tilde{\varphi}$ is well defined.

Moreover $\tilde{\varphi}(0) = \varphi(0) = 0 \qquad \tilde{\varphi}(1) = \varphi(1) = 1 \qquad$ and if

$a = a_0 + a_1 + \ldots + a_{n-1}$ and $c = c_0 + c_1 + \ldots + c_{m-1}$, then

$\tilde{\varphi}(a + c) = \tilde{\varphi}(a_0 + a_1 + \ldots + a_{n-1} + c_0 + c_1 + \ldots + c_{m-1}) =$

$= \varphi(a_0) + \varphi(a_1) + \ldots + \varphi(a_{n-1}) + \varphi(c_0) + \varphi(c_1) + \ldots + \varphi(c_{m-1}) =$

$= (\varphi(a_0)+\varphi(a_1)+\ldots+\varphi(a_{n-1}))+(\varphi(c_0)+\varphi(c_1)+\ldots+\varphi(c_{m-1})) = \tilde{\varphi}(a)+\tilde{\varphi}(c).$

$\tilde{\varphi}(a.c) = \tilde{\varphi}((a_0 + \ldots + a_{n-1}).(c_0 + \ldots + c_{m-1})) = \tilde{\varphi}(\sum a_i c_j) = \sum \varphi(a_i c_j) =$

$= \sum(\varphi(a_i).\varphi(c_j)) = (\varphi(a_0)+\ldots+\varphi(a_{n-1})).(\varphi(c_0)+\ldots+\varphi(c_{m-1})) = \tilde{\varphi}(a).\tilde{\varphi}(c).$

Therefore, by Theorem 1.3.2 (ii), $\tilde{\varphi}$ is a homomorphism of Boolean algebras.

Let ψ be another extension of φ, then if $a = a_0 + a_1 + \ldots + a_{n-1}$,

$\psi(a) = \psi(a_0 + a_1 + \ldots + a_{n-1}) = \psi(a_0) + \psi(a_1) + \ldots + \psi(a_{n-1}) =$

$= \varphi(a_0) + \varphi(a_1) + \ldots + \varphi(a_{n-1}) = \tilde{\varphi}(a)$ and thus the extension φ is unique. □

Corollary 1.3.18 Let L_1 and L_2 be two bounded distributive lattices, and let $\varphi : L_1 \to L_2$ be an isomorphism of lattices. Then φ uniquely extends to an isomorphism of Boolean algebras $\tilde{\varphi} : B(L_1) \to B(L_2)$.

Proof. By Theorem 1.3.17 there is an extension $\tilde{\varphi} : B(L_1) \to B(L_2)$ where $\tilde{\varphi}(a) = \varphi(a_0) + \varphi(a_1) + \ldots + \varphi(a_{n-1})$ if $a = a_0 + a_1 + \ldots + a_{n-1}$, $a_j \in L_1$ $0 \leq j \leq n-1$.
$\tilde{\varphi}$ is surjective:
Let $c \in B(L_2)$, then $c = c_0 + c_1 + \ldots + c_{n-1}$, $c_j \in L_2$, $j = 0, 1, \ldots, n-1$.
Since φ is an isomorphism of L_1 onto L_2, there exists $a_0, a_2, \ldots, a_{n-1} \in L_1$ such that $\varphi(a_j) = c_j$, $j = 0, 1, \ldots, n-1$. Therefore
$c = c_0 + c_1 + \ldots + c_{n-1} = \varphi(a_0) + \varphi(a_1) + \ldots + \varphi(a_{n-1}) = \tilde{\varphi}(a)$,
with $a = a_0 + a_1 + \ldots + a_{n-1}$.
$\tilde{\varphi}$ is injective:
Let $a_0, a_1, \ldots, a_{n-1}, c_0, c_1, \ldots, c_{m-1} \in L_1$ be such that
$\tilde{\varphi}(a_0 + a_1 + \ldots + a_{n-1}) = \tilde{\varphi}(c_0 + c_1 + \ldots + c_{m-1})$, that is
$\varphi(a_0) + \varphi(a_1) + \ldots + \varphi(a_{n-1}) = \varphi(c_0) + \varphi(c_1) + \ldots + \varphi(c_{m-1})$ in $B(L_2)$.
Since $\varphi^{-1} : L_2 \to L_1$ a homomorphism of lattices ,by Theorem 1.3.17, there is an extension $\tilde{\varphi^{-1}} : B(L_2) \to B(L_1)$, $\tilde{\varphi^{-1}}(d_0 + d_1 + \ldots + d_{n-1}) =$
$= \varphi^{-1}(d_0) + \varphi^{-1}(d_1) + \ldots + \varphi^{-1}(d_{n-1})$, $d_0, d_1, \ldots, d_{n-1} \in L_2$.
Since $\tilde{\varphi^{-1}}$ is well defined, if $d_0 + d_1 + \ldots + d_{n-1} = e_0 + e_1 + \ldots + e_{m-1}$ in $B(L_2)$, then $\tilde{\varphi^{-1}}(d_0 + d_1 + \ldots + d_{n-1}) = \tilde{\varphi^{-1}}(e_0 + e_1 + \ldots + e_{m-1})$ in $B(L_1)$.
In particular,
$\tilde{\varphi^{-1}}(\varphi(a_0) + \varphi(a_1) + \ldots + \varphi(a_{n-1})) = \tilde{\varphi^{-1}}(\varphi(c_0) + \varphi(c_1) + \ldots + \varphi(c_{m-1}))$.
Thus $a_0 + a_1 + \ldots + a_{n-1} = \varphi^{-1}(\varphi(a_0)) + \varphi^{-1}(\varphi(a_1)) \ldots + \varphi^{-1}(\varphi(a_{n-1})) =$
$= \tilde{\varphi^{-1}}(\varphi(a_0) + \varphi(a_1) + \ldots + \varphi(a_{n-1})) = \tilde{\varphi^{-1}}(\varphi(c_0) + \varphi(c_1) + \ldots + \varphi(c_{m-1})) =$
$= \varphi^{-1}(\varphi(c_0)) + \varphi^{-1}(\varphi(c_1)) \ldots + \varphi^{-1}(\varphi(c_{m-1})) = c_0 + c_1 + \ldots + c_{m-1}$. $\square$

1.4 Effect algebras [3]

An *effect algebra* is a partial algebra $\boldsymbol{E}= (E, \oplus, 0, 1)$ such that $\oplus$ is a binary partial operation and 0, 1, are nullary operations satisfying the following conditions, where x, y, z denote arbitrary elements of E.

- E_1 If $x \oplus y$ is defined, then $y \oplus x$ is defined and $x \oplus y = y \oplus x$.
- E_2 If $x \oplus y$ and $(x \oplus y) \oplus z$ are defined, then $y \oplus z$ and $x \oplus (y \oplus z)$ are defined, and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

E_3 For every $x \in E$, there exist a *unique* $x' \in E$ such that $x \oplus x' = 1$.

E_4 If $x \oplus 1$ is defined, then $x = 0$.

We denot "there exist $a \oplus b$" by $a \perp b$.

Example 1.4.1 Let $E = [0, 1]$ be the real unit interval, or $E = \mathcal{Q} \cap [0, 1]$, or $E = \mathcal{L}_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right\}$ $(n \in N, n \geq 2)$, and for all $x, y \in E$, $x \oplus y$ is defined iff $x \leq 1 - y$. In this case we defined $x \oplus y := x + y$.
It is easy to see that $(E, \oplus, 0, 1)$ is an effect algebras, where $x' = 1 - x$.
Also, if $(n_1 - 1) \mid (n_2 - 1)$, then $\mathcal{L}_{n_1} \subset \mathcal{L}_{n_2} \subset \mathcal{Q} \cap [0, 1] \subset [0, 1]$, where $\subset$ is a subalgebra inclusion.

Example 1.4.2 Let $\langle B, \wedge, \vee, {}^c, 0, 1\rangle$ be a Boolean algebra. For a, b in B we say $a \perp b$ iff $a \wedge b = 0$ and, if $a \perp b$, we define $a \oplus b := a \vee b$.
Then $(B, \oplus, 0, 1)$ is an effect algebra, where $x' = x^c$.

Lemma 1.4.3 The following properties hold in every effect algebra E:

(i) For every $x \in E$, $x'' = x$

(ii) $1' = 0$ and $0' = 1$

(iii) For each $x \in E$ $\quad x \oplus 0$ is defined and $x \oplus 0 = x$

(iv) If $x \oplus y$ is defined, then $y \oplus (x \oplus y)'$ is defined, and $x = (y \oplus (x \oplus y)')'$

(v) If $x \oplus y$ and $x \oplus z$ are defined and $x \oplus y = x \oplus z$, then $y = z$

(vi) If $x \oplus y = 0$, then $x = y = 0$

Proof. To prove (i), note that by E_1 and E_3, $x' \oplus x = x \oplus x' = 1$.
Hence $x'' = x$.

Since by E_3 $1 \oplus 1'$ is defined, E_4 implies that $1' = 0$, and by (i) we have that $0' = 1'' = 1$. This proves (ii).

To prove (iii), note first that by (ii) $1 \oplus 0 = 1$. Hence by E_3, E_1 and E_2:

$1 = 1 \oplus 0 = (x' \oplus x) \oplus 0 = x' \oplus (x \oplus 0)$. Then by E_3 and (i) we conclude that $x \oplus 0 = x'' = x$, and (iii) is proved.

If $x \oplus y$ is defined, then by E_3 and E_2 we have that
$1 = (x \oplus y) \oplus (x \oplus y)' = x \oplus (y \oplus (x \oplus y)')$,
and then (iv) follows from E_3 and (i).

To show the cancellative property, suppose that $x \oplus y = x \oplus z$. By (iv) and E_1 we have that $y = (x \oplus (y \oplus x)')' = (x \oplus (z \oplus x)')' = z$. This proves (v).

If $x \oplus y = 0$, then by (iv) and (ii), $y \oplus (x \oplus y)' = y \oplus 1$ is defined, and by E_4, $y = 0$. Hence by (iii), $0 = x \oplus 0 = x$. This completes the proof of (vi) $\square$

Let E be an effect algebra. The binary relation $\leq$ defined on E by the prescription $x \leq y$ if there is z such that $x \oplus z = y$ is a partial order on E, called the *natural order of* E. Indeed, reflexivity follows from (iii) of Lemma 1.4.3, transitivity from E_2, and antisymmetry from (v) in Lemma 1.4.3.

Example 1.4.4 In Example 1.4.1 $\leq$ is the usual order of numbers of E, and in Example 1.4.2 $\leq$ is the same as in B.

Lemma 1.4.5 Let E be an effect algebra and let $x, y, z \in E$. Then we have:

(i) $x \leq y$ if and only if $y' \leq x'$.

(ii) $x \oplus y$ is defined if and only if $x \leq y'$.

(iii) $\forall x \in E, 0 \leq x$.

(iv) $\forall x \in E, x \leq 1$.

(v) If $x \oplus y$ is defined and $z \leq x$ then $z \oplus y$ is defined
(if $x \oplus y$ is defined and $z \leq y$ then $z \oplus x$ is defined).

(vi) If $x \oplus y$ is defined then $x \leq x \oplus y$ and $y \leq x \oplus y$.

(vii) If $x \oplus z$ and $y \oplus z$ are defined and $x \leq y$, then $x \oplus z \leq y \oplus z$.

Proof. Suppose $x \leq y$, and take z such that $x \oplus z = y$. By (iv) and (i) in Lemma 1.4.3, $x' = z \oplus (x \oplus z)' = z \oplus y'$, and this shows that $y' \leq x'$. On the other hand, if $y' \leq x'$, by what we have just proved and (i) of Lemma 1.4.3, we have $x = x'' \leq y'' = y$. This completes the proof of (i).
To prove (ii), suppose first that $x \oplus y$ is defined. Then by (iv) in Lemma 1.4.3, $y' = x \oplus (x \oplus y)'$, hence $x \leq y'$. Suppose now that $x \leq y'$, i.e., that there is z such that $x \oplus z = y'$. Then $1 = y \oplus y' = y \oplus (x \oplus z)$, hence by E_2 and E_1, $x \oplus y$ is defined.
(iii) By Lemma 1.4.3 (iii) and definition of $\leq$.
(iv) By E_3 and definition of $\leq$.
(v) By (ii) $x \perp y \Rightarrow x \leq y'$, therefore $z \leq y'$ and then, by (ii), $z \perp y$. The rest follows by symmetry.
(vi) By definition of $\leq$.
(vii) $x \leq y \Rightarrow \exists s \in E$ such that $x \oplus s = y$. Then $y \oplus z = (x \oplus s) \oplus z =$
$= (s \oplus x) \oplus z = s \oplus (x \oplus z)$ (By E_1 and E_2), and thus $x \oplus z \leq y \oplus z$. □

Let E be an effect algebra, it is possible to introduce a new partial operation $\ominus$.
$b \ominus a$ exists and equals c if and only if $a \oplus c$ exists and equals b.
In other words, $b \ominus a$ is defined iff $a \leq b$ and then $a \oplus (b \ominus a) = b$
($\ominus$ is well defined by Lemma 1.4.3 (v)).

Example 1.4.6 In example 1.4.1 if $a \leq b$, $b \ominus a = b - a$, and in example 1.4.2 if $a \leq b$, $b \ominus a = b \setminus a$ (where $b \setminus a = b \wedge a' = b \wedge a^c$).

Remark: If $a \oplus b$ is defined and $a \oplus b = c$, then $a = c \ominus b$ and $b = c \ominus a$. Also, since $a \oplus b = a \oplus b$, we have $a = (a \oplus b) \ominus b$ and $b = (a \oplus b) \ominus a$.

Lemma 1.4.7 Let E be an effect algebra and let $a, b, c \in E$.

(i) If $a \leq b$, then $b \ominus a \leq b$.

(ii) If $a \leq b$ then $b \ominus (b \ominus a)$ is defined and $b \ominus (b \ominus a) = a$.

(iii) If $a \leq b \leq c$, then $b \ominus a \leq c \ominus a$.

(iv) $a \ominus 0$ is defined and $a \ominus 0 = a$.

(v) $a \ominus a$ is defined and $a \ominus a = 0$.

(vi) If $a \leq b \leq c$, then $(c \ominus a) \ominus (b \ominus a)$ is defined and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.

(vii) If $a \leq b \leq c'$, then $(b \oplus c), (a \oplus c)$ and $(b \oplus c) \ominus (a \oplus c)$ are defined and $(b \oplus c) \ominus (a \oplus c) = b \ominus a$.

(viii) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$.

(ix) If $b \leq c$ and $a \leq c \ominus b$ then $b \leq c \ominus a$ and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.

Proof.
(i) $a \leq b \Rightarrow b \ominus a$ is defined and $a \oplus (b \ominus a) = b \;\Rightarrow\; b \ominus a \leq b$.
(ii) If $a \leq b$ then $b \ominus a$ is defined and, by *(i)*, $b \ominus (b \ominus a)$ is defined. From $a \oplus (b \ominus a) = b$ and previous remark we have $b \ominus (b \ominus a) = a$.
(iii) $a \leq b \Rightarrow \exists t \in E$ such that $a \oplus t = b$ and, by previous remark, $t = b \ominus a$. $b \leq c \Rightarrow \exists s \in E$ such that $b \oplus s = c$, and $s = c \ominus b$. Therefore
$c = b \oplus s = (a \oplus t) \oplus s = a \oplus (t \oplus s)$ (by E_2), then $(t \oplus s) = c \ominus a$. Thus
(by Lemma 1.4.5 *(vi)*) $b \ominus a = t \leq t \oplus s = c \ominus a$.
(iv) and *(v)* follows from Lemma 1.4.3 (iii) and previous remark.
(vi) $b = (b \ominus a) \oplus a$ and $c = (c \ominus b) \oplus b$ imply $c = (c \ominus b) \oplus ((b \ominus a) \oplus a) =$
$= ((c \ominus b) \oplus (b \ominus a)) \oplus a$ then, by previous remark, $(c \ominus b) \oplus (b \ominus a) = c \ominus a$ and again by previous remark $b \ominus a = (c \ominus a) \ominus (c \ominus b)$.
(vii) If $a \leq b \leq c'$ then, by *(vii)* and Lemma 1.4.5 *(ii)*, $(b \oplus c), (a \oplus c)$ and $(b \oplus c) \ominus (a \oplus c)$ are defined. Since $a \leq b$, by definition of $\ominus$, we have $b = (b \ominus a) \oplus a$ then $b \oplus c = ((b \ominus a) \oplus a) \oplus c = (b \ominus a) \oplus (a \oplus c)$. Thus by previous remark $b \ominus a = (b \oplus c) \ominus (a \oplus c)$.
(viii) Since $a \leq c$, $b \leq c$ and $a \leq b$, $\exists e \in E$ such that $b = a \oplus e$ therefore
$a \oplus (c \ominus a) = c = b \oplus (c \ominus b) = (a \oplus e) \oplus (c \ominus b) = a \oplus (e \oplus (c \ominus b))$. Then , by the cancellative property, $c \ominus a = e \oplus (c \ominus b)$ and thus $c \ominus b \leq c \ominus a$.
(ix) Since, by *(i)*, $a \leq c \ominus b \leq c$ then, by *(viii)*, $c \ominus (c \ominus b) \leq c \ominus a$ hence, since $b \leq c$ and *(ii)*, we have $b = c \ominus (c \ominus b) \leq c \ominus a$. Moreover $c \ominus a = ((c \ominus a) \ominus b) \oplus b$, then $(c \ominus a) \oplus a = (((c \ominus a) \ominus b) \oplus b) \oplus a$, hence $c = (((c \ominus a) \ominus b) \oplus a) \oplus b$ and, since $c = (c \ominus b) \oplus b$, we obtain $(c \ominus b) \oplus b = (((c \ominus a) \ominus b) \oplus a) \oplus b$.

Therefore, by the cancellative property, $(c \ominus b) = ((c \ominus a) \ominus b) \oplus a$ and thus $(c \ominus a) \ominus b = (c \ominus b) \ominus a$. □

Lemma 1.4.8 Let E be an effect algebra. If $a \oplus b$ is defined, then $(a \oplus b)' = a' \ominus b = b' \ominus a$.

Proof. By Lemma 1.4.3 (iv) if $a \oplus b$ is defined, then $b \oplus (a \oplus b)'$ is defined and $a = (b \oplus (a \oplus b)')'$. Thus $a' = ((b \oplus (a \oplus b)')')' = b \oplus (a \oplus b)'$.
From definition of $\ominus$, $(a \oplus b)' = a' \ominus b$. The rest follows by symmetry. □

Let E_1, E_2 be effect algebras. A mapping $\phi : E_1 \to E_2$ is called a *homomorphism of effect algebras* iff

- $\phi(1) = 1$
- The existence of $a \oplus b$ implies the existence of $\phi(a) \oplus \phi(b)$ and

  $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$

Remark: Let $a \in E_1$, then $\phi(a') = (\phi(a))'$ in E_2.

Lemma 1.4.9 Let E_1, E_2 be effect algebras and let $\phi : E_1 \to E_2$ be a homomorphism of effect algebras.

(i) If $a, b \in E_1$ and $a \leq b$, then $\phi(a) \leq \phi(b)$.

(ii) If $a, b \in E_1$ and $a \leq b$, then $\phi(b \ominus a) = \phi(b) \ominus \phi(a)$.

Proof.
(i) $a \leq b \Rightarrow \exists c \in E_1$ such that $b = a \oplus c$. Then $\phi(a) \oplus \phi(c)$ is defined in E_2 and $\phi(b) = \phi(a) \oplus \phi(c)$. Thus $\phi(a) \leq \phi(b)$.
(ii) By *(i)* $a \leq b \Rightarrow \phi(a) \leq \phi(b)$, and thus $\phi(b) \ominus \phi(a)$ is defined. From $b = (b \ominus a) \oplus a$, we have $\phi(b) = \phi(b \ominus a) \oplus \phi(a)$ and thus $\phi(b \ominus a) = \phi(b) \ominus \phi(a)$.□

A homomorphism $\phi : E_1 \to E_2$ is *full* iff whenever $\phi(a) \perp \phi(b)$ and $\phi(a) \oplus \phi(b) \in \phi(E_1)$, then there are $a_1, b_1 \in E_1$ such that $\phi(a) = \phi(a_1)$, $\phi(b) = \phi(b_1)$ and $a_1 \perp b_1$.

A homomorphism $\phi : E_1 \to E_2$ is an *isomorphism* iff ϕ is bijective and full.

Note that even if E_1 and E_2 are lattice ordered, a homomorphism of effect algebras need not to preserve joins and meets.

1.5 MV-effect algebras [7]

Definition 1.5.1 An MV-effect algebra is a lattice ordered effect algebra M in which, for all $a, b \in M$, $(a \vee b) \ominus a = b \ominus (a \wedge b)$.

Example 1.5.2 The examples 1.4.1 and 1.4.2 are MV-effect algebras (see examples 1.4.4 and 1.4.6).

Proposition 1.5.3 Let M be an MV-effect algebra and let $a, b, c \in M$.

(i) If $a \leq c$ and $b \leq c$, then $c \ominus (a \vee b) = (c \ominus a) \wedge (c \ominus b)$.

In particular, if $a \perp b$, then $(a \oplus b) \ominus (a \vee b) = a \wedge b$.

(ii) If $c \leq a$ and $c \leq b$, then $(a \wedge b) \ominus c = (a \ominus c) \wedge (b \ominus c)$.

(iii) $((a \vee b) \ominus a) \wedge ((a \vee b) \ominus b) = 0$.

(iv) If $c \leq a$ and $c \leq b$, then $(a \ominus c) \vee (b \ominus c) = (a \vee b) \ominus c$.

(v) If $a \leq c$ and $b \leq c$ then $c \ominus (a \wedge b) = (c \ominus a) \vee (c \ominus b)$.

In particular, if we put $c = a \vee b$,

$$(a \vee b) \ominus (a \wedge b) = ((a \vee b) \ominus a) \vee ((a \vee b) \ominus b).$$

Proof.
(i)
From the inequalities $a \leq a \vee b \leq c$ and $b \leq a \vee b \leq c$ and Lemma 1.4.7 $(viii)$ we have $c \ominus (a \vee b) \leq c \ominus a$ and $c \ominus (a \vee b) \leq c \ominus b$. For any other $w \in M$ with $w \leq c \ominus a$ and $w \leq c \ominus b$, by Lemma 1.4.7 (i), (ii) and $(viii)$, $a = c \ominus (c \ominus a) \leq c \ominus w$ and $b = c \ominus (c \ominus b) \leq c \ominus w$, therefore $a \vee b \leq c \ominus w \leq c$, and so $w = c \ominus (c \ominus w) \leq c \ominus (a \vee b)$, which implies that $c \ominus (a \vee b)$ is the greatest lower bound of the set $\{c \ominus a, c \ominus b\}$, which concludes the proof of (i).

(ii)

$c \leq a$ and $c \leq b$ imply $c \leq a \wedge b \leq a$ and $c \leq a \wedge b \leq b$ then,

by Lemma 1.4.7 (iii), $(a \wedge b) \ominus c \leq a \ominus c$ and $(a \wedge b) \ominus c \leq b \ominus c$.

If $w \in M$ is such that $w \leq a \ominus c$ and $w \leq b \ominus c$ then, since $(a \ominus c) \oplus c$ is defined and Lemma 1.4.5 (v), $w \oplus c$ is defined and, by Lemma 1.4.5 (vii), $w \oplus c \leq (a \ominus c) \oplus c = a$ and $w \oplus c \leq (b \ominus c) \oplus c = b$.

Therefore $c \leq w \oplus c \leq a \wedge b$ and thus, by Lemma 1.4.7 (iii) and Remark page 39, $w = (w \oplus c) \ominus c \leq (a \wedge b) \ominus c$. Hence $(a \wedge b) \ominus c$ is the greatest lower bound of $\{a \ominus c, b \ominus c\}$.

(iii)

In (i) put $c = a \vee b$ and Lemma 1.4.7 (v).

(iv)

From $c \leq a \leq a \vee b$ and $c \leq b \leq a \vee b$ we get, by Lemma 1.4.7 (iii), $a \ominus c \leq (a \vee b) \ominus c$ and $b \ominus c \leq (a \vee b) \ominus c$. Let $w \in M$ be such that $a \ominus c \leq w$ and $b \ominus c \leq w$ then $a \ominus c \leq w \wedge ((a \vee b) \ominus c) \leq (a \vee b) \ominus c$ and thus

$((a \vee b) \ominus c) \ominus (w \wedge ((a \vee b) \ominus c)) \leq ((a \vee b) \ominus c) \ominus (a \ominus c) = (a \vee b) \ominus a$ by Lemma 1.4.7 (vi); similarly, $((a \vee b) \ominus c) \ominus (w \wedge ((a \vee b) \ominus c)) \leq (a \vee b) \ominus b$.

Therefore $((a \vee b) \ominus c) \ominus (w \wedge ((a \vee b) \ominus c)) \leq ((a \vee b) \ominus a) \wedge ((a \vee b) \ominus b) = 0$ by (iii). Hence $((a \vee b) \ominus c) \ominus (w \wedge ((a \vee b) \ominus c)) = 0$, then

$(((a \vee b) \ominus c) \ominus (w \wedge ((a \vee b) \ominus c))) \oplus (w \wedge ((a \vee b) \ominus c)) = 0 \oplus (w \wedge ((a \vee b) \ominus c))$ and thus $(a \vee b) \ominus c = w \wedge ((a \vee b) \ominus c) \leq w$.

(v)

From the inequalities $a \wedge b \leq a \leq c$ and $a \wedge b \leq b \leq c$ it follows, by Lemma 1.4.7 $(viii)$, that $c \ominus a \leq c \ominus (a \wedge b)$ and $c \ominus b \leq c \ominus (a \wedge b)$. For $w \in M$ with $c \ominus a \leq w$ and $c \ominus b \leq w$, then $c \ominus a = (c \ominus a) \wedge c \leq w \wedge c \leq c$, which gives $c \ominus (w \wedge c) \leq c \ominus (c \ominus a) = a$ (by Lemma 1.4.7 (ii)), and similarly $c \ominus (w \wedge c) \leq b$, therefore, $c \ominus (w \wedge c) \leq a \wedge b$. Then, since $a \wedge b \leq c$, we obtain $c \ominus (a \wedge b) \leq c \ominus (c \ominus (w \wedge c)) = w \wedge c \leq w$ (by Lemma 1.4.7 $(viii)$ and (ii)), which implies that $c \ominus (a \wedge b)$ is the least upper bound of the set $\{c \ominus a, c \ominus b\}$.

□

Proposition 1.5.4 Let M be an MV-effect algebra and let $a, b, c \in M$.

(i) If $a \oplus b$ and $a \oplus c$ are defined then $a \oplus (b \wedge c) = (a \oplus b) \wedge (a \oplus c)$.

(ii) If $a \oplus b$ and $a \oplus c$ are defined then $a \oplus (b \vee c) = (a \oplus b) \vee (a \oplus c)$.

Proof.
(i)
By Proposition 1.5.3 *(ii)* $((a \oplus b) \wedge (a \oplus c)) \ominus a = ((a \oplus b) \ominus a) \wedge ((a \oplus c) \ominus a) = b \wedge c$, therefore $(((a \oplus b) \wedge (a \oplus c)) \ominus a) \oplus a = (b \wedge c) \oplus a$ and thus
$(a \oplus b) \wedge (a \oplus c) = (b \wedge c) \oplus a$.
(ii)
By Proposition 1.5.3 *(iv)* $((a \oplus b) \vee (a \oplus c)) \ominus a = ((a \oplus b) \ominus a) \vee ((a \oplus c) \ominus a) = b \vee c$, whence $(((a \oplus b) \vee (a \oplus c)) \ominus a) \oplus a = (b \vee c) \oplus a$, and thus
$(a \oplus b) \vee (a \oplus c) = (b \vee c) \oplus a$ $\square$

Lemma 1.5.5 (De Morgan's Identities) Let M be an MV-effect algebra and let a, b in M. Then

(i) $(a \vee b)' = a' \wedge b'$ and

(ii) $(a \wedge b)' = a' \vee b'$.

Proof.
(i) By definition an MV-effect algebra is a lattice, by Lemma 1.4.5 (i) $a \leq b$ if and only if $b' \leq a'$ and by Lemma 1.4.3 (i) $a'' = a$. Thus, since $a' \wedge b' \leq a'$ we have $a \leq (a' \wedge b')'$. Similarly $b \leq (a' \wedge b')'$.
Suppose $a \leq e$ and $b \leq e$ then $e' \leq a'$ and $e' \leq b'$ therefore $e' = e' \wedge e' \leq a' \wedge b'$ and thus $(a' \wedge b')' \leq e$, but this means $a \vee b = (a' \wedge b')'$. Hence $(a \vee b)' = (a' \wedge b')'' = a' \wedge b'$ which completes the proof of (i).
(ii) If we simultaneously replace a by a' and b by b' in (i), we obtain
$(a' \vee b')' = a'' \wedge b'' = a \wedge b$ and then $a' \vee b' = (a \wedge b)'$. $\square$

In section 4 is given the definition of *MV-algebras* and it is proved that there is a natural, one-to-one correspondence between MV-effect algebras and MV-algebras given by the following rules. Let $(M, \oplus, 0, 1)$ be an MV-effect algebra. Let $\boxplus$ be a total operation given by $x \boxplus y = x \oplus (x' \wedge y)$. Then $(M, \boxplus, ', 0)$ is an MV-algebra.
Similarly, let $(M, \boxplus, \neg, 0)$ be an MV-algebra. Restrict the operation $\boxplus$ to the pairs (x, y) satisfying $x \leq y'$ and call the new partial operation $\oplus$. Then

$(M, \oplus, 0, 1)$ is an MV-effect algebra.

Proposition 1.5.6 On each MV-effect algebra E the natural order determines a bounded distributive lattice structure.

Proof. En el apéndice se muestra que en la mencionada correspondencia entre MV-álgebras y MV-effect álgebras, el orden en una MV-effect álgebra $(M, \oplus, 0, 1)$ coincide con el orden de su respectiva MV-álgebra $(M, \boxplus, \neg, 0)$ y como por la Proposición 4.1.6, $(M, \boxplus, \neg, 0)$ es un reticulado acotado y distributivo, entonces $(M, \oplus, 0, 1)$ también lo es. $\square$

Proposition 1.5.7 Let E be an MV-effect algebra. Then there exist the Boolean algebra R-generated by E.

Proof. By Proposition 1.5.6 E is a bounded distributive lattice, and by Lemma 1.3.7 and Theorem 1.3.13 there exist the Boolean algebra R-generated by E. $\square$

2 The function ϕ_M [13]

Let M be an MV-effect algebra (and thus M is a bounded distributive lattice), and let $B(M)$ be the Boolean algebra R-generated by M.
For every element x of $B(M)$, there exists a finite chain $x_1 \leq \ldots\ldots \leq x_n$ in M such that $x = x_1 + \ldots\ldots + x_n$ (lemma 1.3.6). We then say than $\{x_i\}_{i=1}^n$ is a M-chain representation of x. It is easy to see that every element of $B(M)$ has a M-chain representation of even length (if $x_1 \leq \ldots\ldots \leq x_n$ is a M-chain representation of odd length, then $0 \leq x_1 \leq \ldots\ldots \leq x_n$ is a M-chain representation of even length).

Theorem 2.0.8 ***(Main result).*** Let M be an MV-effect algebra.
The mapping $\phi_M : B(M) \to M$ given by $\phi_M(x) = \bigoplus_{i=1}^n (x_{2i} \ominus x_{2i-1})$, where $\{x_i\}_{i=1}^{2n}$ is a M-chain representation of x, is a surjective homomorphism of effect algebras.

We have divided the proof into a secuence of lemmas. We use the notation of Lemmas 1.1.16 and 1.1.17.

Lemma 2.0.9 Let L be a finite sublattice of an MV-effect algebra M. Let C be a maximal chain of L, let $a \in J(L)$ and let $x \in C$, $\pi_C(a) \succ_L x$. Then $\pi_C(a) \ominus x = a \ominus (a \wedge m(a))$.

Proof. Since M is a distributive lattice, L is distributive. By Lemma 1.1.16, we have $a \vee x = \pi_C(a)$ and $a \wedge x = a \wedge m(a)$. Since M is an MV-effect algebra $\pi_C(a) \ominus x = (a \vee x) \ominus x = a \ominus (a \wedge x) = a \ominus (a \wedge m(a))$. $\square$

Lemma 2.0.10 Let L be a finite sublattice of an MV-effect algebra M. Let C_1, C_2 be a maximal chains of L. There exists a bijection $b : C_1 \to C_2$ such that, for all $x_1, x_2 \in C_1$ with $x_2 \succ_L x_1$, $x_2 \ominus x_1 = b(x_2) \ominus y$, where $y \in C_2$ and $b(x_2) \succ_L y$.

Proof. Since M is distributive, L is distributive. Let us put $b(x) = \pi_{C_2}(\pi_{C_1}^{-1}(x))$. By Lemma 1.1.17 (iii), b is a bijection. Write $a = \pi_{C_1}^{-1}(x_2)$. By Corollary 2.0.9, $\pi_{C_1}(a) \ominus x_1 = x_2 \ominus x_1 = a \ominus (a \wedge m(a))$. Similarly, by Lemma 2.0.9, $b(x_2) \ominus y = \pi_{C_2}(a) \ominus y = a \ominus (a \wedge m(a))$. Thus $x_2 \ominus x_1 = b(x_2) \ominus y$. $\square$

Lemma 2.0.11 Let L be a finite 0,1-sublattice of an MV-effect algebra M. The mapping $\psi_L : 2^{J(L)} \to M$ given by

$$\psi_L(X) = \bigoplus_{a \in X} a \ominus (a \wedge m(a))$$

is a homomorphism of effect algebras and, for all $x \in L$, $\psi_L(r(x)) = x$, (note that the sum $\bigoplus$ is finite).

Proof. By definition $\psi_L(\emptyset) = 0$. Let $x \in L$ and write $L_x = \{y \in L : y \leq x\}$ (L_x is a lattice). Note that $r(x) = J(L_x)$. Let $C = \{0 = x_0, x_1, \ldots, x_n = x\}$ with $x_{i+1} \succ_L x_i$ be a maximal chain of L_x. We claim that the sum

$$\bigoplus_{i=1}^{n} x_i \ominus x_{i-1}$$

exists in M and equals x.

We proceed by induction on n. If $n = 1$, $\bigoplus_{i=1}^{n} x_i \ominus x_{i-1} = x_1 \ominus x_0 = x_1 \ominus 0$ and this is defined and equals x_1.

Let $C = \{0 = x_0, x_1, \ldots, x_n, x_{n+1} = x\}$ with $x_{i+1} \succ_L x_i$ be a maximal chain of L_x, then $\{0 = x_0, x_1, \ldots, x_n\}$ is a maximal chain of L_{x_n} with $x_{i+1} \succ_L x_i$. Then by the induction hypothesis $\bigoplus_{i=1}^{n} x_i \ominus x_{i-1}$ exists in M and equals x_n. Thus $\bigoplus_{i=1}^{n+1} x_i \ominus x_{i-1} = (\bigoplus_{i=1}^{n} x_i \ominus x_{i-1}) \oplus (x_{n+1} \ominus x_n) = x_n \oplus (x_{n+1} \ominus x_n)$ and (by definition of $\ominus$ page 39 and $x_n \leq x_{n+1}$) this is defined and equals x_{n+1}, so the claim is proved.

By Corollary 2.0.9 (replacing a by $\pi_C^{-1}(x_i)$ and x by x_{i-1}) we have

$$x_i \ominus x_{i-1} = \pi_C^{-1}(x_i) \ominus (\pi_C^{-1}(x_i) \wedge m(\pi_C^{-1}(x_i))).$$

Since π_C is a bijection , we have $r(x) = \left\{\pi_C^{-1}(x_i) : i \in \{1, \ldots, n\}\right\}$, hence $\psi_L(r(x))$ exists and equals x. As a consequence, $\psi_L(2^{J(L)})) = \psi_L(r(1)) = 1$. The additivity of ψ_L is trivial. □

Since, for every finite lattice L, $r(L)$ R-generates $2^{J(L)}$ (Lemma 1.3.5), the injective mapping $r : L \to 2^{J(L)}$ uniquely extends to an isomorphism of Boolean algebras $\hat{r} : B(L) \to 2^{J(L)}$ (by Corollary 1.3.18).

Lemma 2.0.12 Let L be a finite 0,1-sublattice of an MV-effect algebra M. Let ψ_L, $\hat{r}$ be the mapping given above. Then $\psi_L \circ \hat{r}$ is a homomorphism of effect agebras satisfying

$$\psi_L \circ \hat{r}(x_1 + x_2 + \ldots + x_{2n}) = \bigoplus_{i=1}^{n} (x_{2i} \ominus x_{2i-1})$$

for every chain $x_1 \leq \ldots \leq x_{2n}$ of L.

Proof. Evidently, $\psi_L \circ \hat{r} : B(L) \to M$ is a homomorphism of effect algebras. Let $x_1 \leq \ldots \leq x_{2n}$ be a chain in L. Then

$$\begin{aligned}\psi_L(\hat{r}(x_1 + x_2 + \ldots + x_{2n})) &= \psi_L(\hat{r}(x_1) + \hat{r}(x_2) + \ldots + \hat{r}(x_{2n})) = \\ &= \psi_L(r(x_1) + r(x_2) + \ldots + r(x_{2n})).\end{aligned}$$

Since r is a lattice homomorphism , $r(x_1) \leq \ldots \leq r(x_{2n})$. Thus, in the Boolean algebra $2^{J(L)}$ we obtain (by Lemma 1.3.3 (v) and examples 1.4.2 and 1.4.6)

$$r(x_1) + \ldots + r(x_{2n}) = \bigoplus_{i=1}^{n} (r(x_{2i}) \ominus r(x_{2i-1})).$$

Finally, by Lemma 2.0.11 and since ψ_L is a homomorphism of Effect algebras

$$\begin{aligned}\psi_L(r(x_1)+r(x_2)+\ldots+r(x_{2n})) &= \bigoplus_{i=1}^{n}\psi_L(r(x_{2i}))\ominus\psi_L(r(x_{2i-1}))= \\ &= \bigoplus_{i=1}^{n}(x_{2i}\ominus x_{2i-1})= \\ &= \phi_L(x_1+x_2+\ldots+x_{2n}).\end{aligned}$$

□

Proof of the main result. Let $x_1 \leq \ldots \leq x_{2n}$, $y_1 \leq \ldots \leq y_{2m}$ be two chains of M. Let L be the 0,1-sublattice of M generated by $\{x_1,\ldots,x_{2n},y_1,\ldots,y_{2m}\}$. Then $B(L)$ is a Boolean subalgebra of $B(M)$, $\{x_1,\ldots,x_{2n},y_1,\ldots,y_{2m}\}\subseteq B(L)$ and, by Lemma 2.0.12, $\phi_L : B(L)\rightarrow M$ is a homomorphism of effect algebras. Let us prove that ϕ_M is well defined. Suppose that $x_1+\ldots+x_{2n}=y_1+\ldots+y_{2m}$. By Lemma 2.0.12, $\bigoplus_{i=1}^{n}(x_{2i}\ominus x_{2i-1})=\bigoplus_{i=1}^{m}(y_{2i}\ominus y_{2i-1})$, hence ϕ_M is well defined on $B(L)$ and hence on the whole set M. Moreover, ϕ_L is just the restriction of ϕ_M to $B(L)$.
Suppose now that $x=x_1+\ldots+x_{2n}\perp y_1+\ldots+y_{2m}=y$. Again by Lemma 2.0.12, $\phi_L(x)\perp\phi_L(y)$ and $\phi_L(x\oplus y)=\phi_L(x)\oplus\phi_L(y)$. Obviously, $\phi_M(1)=1$.
For the proof of surjectivity, it suffices to observe that, for all $x\in M$,
$\phi_M(x)=x$. □

Example 2.0.13 Let $x\in B([0,1])$ and let $\{x_i\}_{i=1}^{2n}$ be a M-chain representation of x of even length (see examples 1.5.2 and 1.3.14).
Then,
$x=x_1+x_2+\ldots+x_{2n-1}+x_{2n}=(x_1]+(x_2]+\ldots+(x_{2n-1}]+(x_{2n}]=$
$=(x_1,x_2]\,\dot\cup\ldots\ldots\dot\cup\,(x_{2n-1},x_{2n}]$,
and
$\phi_M(x)=\phi_M(x_1+\ldots\ldots+x_{2n})=(x_2\ominus x_1)\oplus\ldots\ldots\oplus(x_{2n}\ominus x_{2n-1})=$
$=(x_2-x_1)+\ldots\ldots+(x_{2n}-x_{2n-1})=$ the "length" of x.

3 From MV-effect algebras to MV-pairs

3.1 Effect algebra congruence [12]

A binary relation $a \sim b$ defined for arbitrary elements a, b of a non-empty set A is an *equivalence relation* in A iff it is reflexive, symmetric and transitive, i.e., for arbitrary elements $a, b, c \in A$:

$a \sim a$,

if $a \sim b$ then $b \sim a$,

if $a \sim b$ and $b \sim c$ then $a \sim c$.

Let E be an effect algebra. A relation $\sim$ on E is a *weak congruence* iff the following conditions are satisfied.
$(C1)$ $\sim$ is an equivalence relation.
$(C2)$ If $a_1 \sim a_2$, $b_1 \sim b_2$ and $a_1 \oplus b_1$, $a_2 \oplus b_2$ exist, then $a_1 \oplus b_1 \sim a_2 \oplus b_2$.

We denote the class in $E/\sim$ of a element a of E by $|a|$
(i.e. $|a| = \{b \in E / a \sim b\}$).

$|a| \oplus |b|$ is defined on $E/\sim$ iff there are $a_1, b_1 \in E$ such that $a_1 \sim a$, $b_1 \sim b$ and $a_1 \oplus b_1$ exist. In this case we define $|a| \oplus |b| := |a_1 \oplus b_1|$.

If E is an effect algebra and $\sim$ is a weak congruence on E, the quotient $E/\sim$ need not to be a partial abelian monoid, since the associativity condition may fail (c.f. [11]). This fact motivates the study of sufficient conditions for a weak congruence to preserve associtivity. The following condition was considered in [4].

$(C5)$ If $a \sim b \oplus c$, then there are b_1, c_1 such that $b_1 \sim b$, $c_1 \sim c$, $b_1 \oplus c_1$ exists and $a = b_1 \oplus c_1$.

Lemma 3.1.1 Let P be a partial monoid and let $\sim$ be a weak congruence satisfying $(C5)$. Then, the quotient $P/\sim$ is again a partial abelian monoid.

Proof.
$|a| \oplus |b|$ is well defined by (C2).
Associativity:
Suppose $|a| \oplus |b|$ and $(|a| \oplus |b|) \oplus |c|$ are defined.
$|a| \oplus |b|$ is defined $\Rightarrow \exists a_1, b_1$ such that $a_1 \sim a$, $b_1 \sim b$ and $a_1 \oplus b_1$ is defined.
Then $|a| \oplus |b| := |a_1 \oplus b_1|$.
$(|a| \oplus |b|) \oplus |c| = (|a_1 \oplus b_1|) \oplus |c|$ is defined $\Rightarrow \exists d, c_1$ such that
$c_1 \sim c$, $a_1 \oplus b_1 \sim d$ and $d \oplus c_1$ is defined.
Then $(|a| \oplus |b|) \oplus |c| = |d \oplus c_1|$.
By (C5) $\exists a_2 \sim a_1, b_2 \sim b_1$ such that $a_2 \oplus b_2$ is defined and $d = a_2 \oplus b_2$.
Thus $d \oplus c_1 = (a_2 \oplus b_2) \oplus c_1$. Since P is a partial monoid, $b_2 \oplus c_1$ and $a_2 \oplus (b_2 \oplus c_1)$ are defined, and $d \oplus c_1 = a_2 \oplus (b_2 \oplus c_1)$.
Thus $(|a| \oplus |b|) \oplus |c| = |d \oplus c_1| = |a_2 \oplus (b_2 \oplus c_1)| = |a_2| \oplus (|b_2| \oplus |c_1|) =$
$= |a| \oplus (|b| \oplus |c|)$. □

Let E be an effect algebra, the (C1) (C2) (C5) properties of $\sim$ does not guarantee that the $'$ operation is preserved by $\sim$. The operation $'$ is preserved by $\sim$ if condition
(C6) If $a \sim b$ then $a' \sim b'$ is satisfied.
A relation on an effect algrbra satisfying (C1) (C2) (C5) (C6) is called an *effect algebra congruence.*

Lemma 3.1.2 Let $(E, \oplus, 0, 1)$ be an effect algebra and let $\sim$ be an effect algebra congruence, then

(i) $(E/\sim, \oplus, |0|, |1|)$ is an effect algebra.

(ii) The mapping $a \to |a|$ is a full morphism of effect algebras.

Proof.
(i)
$(E1)$ If $|a| \oplus |b|$ is defined, then $\exists a_1 \sim a, b_1 \sim b$ such that $a_1 \oplus b_1$ exist. Since E is a effect algebra $b_1 \oplus a_1$ is defined and $a_1 \oplus b_1 = b_1 \oplus a_1$. Thus $|b| \oplus |a|$ is defined and $|a| \oplus |b| = |a_1 \oplus b_1| = |b_1 \oplus a_1| = |b| \oplus |a|$.
$(E2)$ Lemma 3.1.1

$(E3)$ We will show that $|a|' = |a'|$. Let $a \in E$. Since $a \oplus a'$ is defined, then $|a| \oplus |a'|$ is defined and $|a| \oplus |a'| = |a \oplus a'| = |1|$.
Unicity:
If $|a| \oplus |b| = |1|$, then $\exists a_1 \sim a, b_1 \sim b$ such that $a_1 \oplus b_1$ is defined and $|1| = |a| \oplus |b| = |a_1 \oplus b_1|$, thus $a_1 \oplus b_1 \sim 1$. By (C5) $\exists a_2 \sim a_1, b_2 \sim b_1$ such that $a_2 \oplus b_2$ is defined and $a_2 \oplus b_2 = 1$, then (since E is an effect algebra) $b_2 = a_2'$. Now $a_2 \sim a_1 \sim a \Rightarrow a_2 \sim a \Rightarrow a_2' \sim a'$ (by $(C6)$). Therefore $a' \sim a_2' = b_2 \sim b_1 \sim b \Rightarrow b \sim a' \Rightarrow |b| = |a'|$.
$(E4)$ If $|a| \oplus |1|$ is defined, then $\exists a_1 \sim a, b \sim 1$ such that $a_1 \oplus b$ is defined.
By Lemma 1.4.3 (iv) $b' = (b \oplus a_1)' \oplus a_1$.
On the other hand by (C6) $b \sim 1 \Rightarrow b' \sim 1' = 0$. Thus $0 \sim (b \oplus a_1)' \oplus a_1$.
By (C5) $\exists u \sim (b \oplus a_1)', v \sim a_1$ such that $u \oplus v$ is defined and $0 = u \oplus v$.
By Lemma 1.4.3 (vi) $v = 0$. Therefore $0 = v \sim a_1 \sim a \Rightarrow 0 \sim a \Rightarrow |a| = |0|$.
(ii)
It follows from definition of $\oplus$ on $E/\sim$. □

Lemma 3.1.3 Let E be an effect algrbra and let $\sim$ be an effect algebra congruence. For all $x, y \in E$, the following are equivalent.

(a) $|x| \leq |y|$.

(b) There is $x_1 \sim x$ such that $x_1 \leq y$.

(c) There is $y_1 \sim y$ such that $x \leq y_1$.

Proof.
$(b \Rightarrow a)$
$x_1 \leq y \Rightarrow \exists a \in E$ such that $x_1 \oplus a$ is defined and
$y = x_1 \oplus a \Rightarrow |y| = |x_1| \oplus |a| \Rightarrow |x| = |x_1| \leq |y|$.
$(c \Rightarrow a)$
Similar to $b \Rightarrow a$.
$(a \Rightarrow b)$
$|x| \leq |y| \Rightarrow \exists u \in E$ such that $|x| \oplus |u|$ is defined, and $|x| \oplus |u| = |y|$.
Then $\exists x_0, u_0 \in E$ such that $x_0 \sim x$, $u_0 \sim u$, $x_0 \oplus u_0$ exists and $x_0 \oplus u_0 \sim y$.
By the (C5) property, there are x_1, u_1 such that $x_1 \sim x_0$, $u_1 \sim u_0$, $x_1 \oplus u_1$ exists, and $x_1 \oplus u_1 = y$. This proves $a \Rightarrow b$.

$(a \Rightarrow c)$
By Lemma 3.1.2, Lemma 1.4.5 (i) and (C6) property, $|y'| \leq |x'|$. As $a \Rightarrow b$ there is $z \sim y'$ such that $z \leq x'$ and this is equivalent with $x \leq z'$. By the (C6) property, $z \sim y'$ iff $z' \sim y$ and we can put $y_1 = z'$. $\square$

3.2 MV-pairs [12]

Let B be a Boolean algebra. Let G be a subgroup of $Aut(B)$. For $a, b \in B$ we write $a \sim_G b$ (or $a \sim b$) iff there exists $f \in G$ such that $f(a) = b$. Obviously $\sim_G$ is a equivalence relation. We write $|a|_G$ (or $|a|$) for the equivalence class of an element a of B.
Also we denote $B_\sim = B_{\sim_G} = \{|a|_G : a \in B\}$.
For all $a, b \in B$ we write:
$L(a, b) = \{a \wedge f(b) : f \in G\}$
$L^+(a, b) = \{g(a) \wedge f(b) : f, g \in G\}$
$max(L(a, b)) = \{m \in L(a, b) : \forall x \in L(a, b) \text{ con } x \geq m \Rightarrow x = m\}$
(the set of all maximal elements of $L(a, b)$)
$max(L^+(a, b)) = \{m \in L^+(a, b) : \forall x \in L^+(a, b) \text{ con } x \geq m \Rightarrow x = m\}$
(the set of all maximal elements of $L^+(a, b)$)

Definition 3.2.1 Let B be a Boolean algebra and let G be a subgroup of $Aut(B)$. We say that (B,G) is an $MV-pair$ iff the following two conditions are satisfied:

(MVP1) For all $a, b \in B, f \in G$ such that $a \leq b$ and $f(a) \leq b$, there is $h \in G$ such that $h(a) = f(a)$ and $h(b) = b$.

(MVP2) For all $a, b \in B$ and $x \in L(a, b)$ there exist $m \in max(L(a, b))$ with $m \geq x$.

Example 3.2.2 For every finite Boolean algebra B, $(B; Aut(B))$ is an $MV-pair$.

Example 3.2.3 Let B be a Boolean algebra with three atoms a_1, a_2, a_3. The mapping f given by

x	0	a_1	a_2	a_3	a_1^c	a_2^c	a_3^c	1
f(x)	0	a_2	a_3	a_1	a_2^c	a_3^c	a_1^c	1

is an automorphism of B and $G = \{id, f, f^2\}$ is a subgroup of $Aut(B)$. However, (B, G) is not an $MV - pair$. Indeed, we have $a_1 \leq a_3^c$ and $f(a_1) = a_2 \leq a_3^c$, but there is no $h \in G$ such that $h(a_1) = f(a_1)$ and $h(a_3^c) = a_3^c$.

Example 3.2.4 Let $2^{\mathbb{Z}}$ be the Boolean algebra of all subsets of $\mathbb{Z}$. Then $(2^{\mathbb{Z}}, Aut(2^{\mathbb{Z}}))$ is not an $MV - pair$. Indeed, let $f \in Aut(2^{\mathbb{Z}})$ be the automorphism of $2^{\mathbb{Z}}$ associated with the permutation $f(n) = n + 1$. Let $A = B = \mathbb{N}$. We see that $f(A) = A \setminus \{0\}$, $A \subseteq B$ and $f(A) \subseteq B$. However, there is no $h \in Aut(2^{\mathbb{Z}})$ such that $h(A) = f(A)$ and $h(B) = B$, simply because $A = B$ implies that $h(A) = h(B)$, but $f(A) \neq B$.

Lemma 3.2.5 Let B be a Boolean algebra, let G be a subgroup of $Aut(B)$. Then the following conditions are eqivalent:

(i) $MVP2$

(ii) For all $a, b \in B$ there exist $m \in max(L(a, b))$ with $m \geq a \wedge b$.

Proof.
$(i) \Rightarrow (ii)$ is clear.
$(ii) \Rightarrow (i)$ Let $a, b \in B$ and $f \in G$. If $g \in G$ we have $a \wedge g(b) =$
$= a \wedge g(f^{-1}(f(b))) = (g \circ f^{-1})(f(b))$. Therefore $L(a, b) \subseteq L(a, f(b))$. If $g \in G$ we have $a \wedge g(f(b)) = a \wedge (g \circ f)(b)$. Therefore $L(a, f(b)) \subseteq L(a, b)$. Thus $L(a, f(b)) = L(a, b)$ and $max(L(a, f(b))) = max(L(a, b))$.
Now, let $x \in L(a, b)$, then $x = a \wedge f(b)$ for some $f \in G$. From (ii) there exist $m \in max(L(a, f(b)))$ with $m \geq a \wedge f(b)$.
Thus $m \geq x = a \wedge f(b)$ with $m \in max(L(a, f(b))) = max(L(a, b))$. $\square$

Lemma 3.2.6 Let B be a Boolean algebra, let G be a subgroup of $Aut(B)$. Then the following condition are equivalent.

(a) (MVP1).

(b) For all $a, b \in B, f \in G$ such that $a \leq b$ and $a \leq f(b)$, there is $h \in G$ such that $h(b) = f(b)$ and $h(a) = a$.

(c) For all $a, b \in B, f \in G$ such that $a \wedge b = 0$ and $a \wedge f(b) = 0$, there is $h \in G$ such that $h(b) = f(b)$ and $h(a) = a$.

Proof.
$(a) \Rightarrow (b)$: Replace a with b^c and b with a^c and apply the fact that f is an automorphism.
$(b) \Rightarrow (c)$: Replace b with b^c.
$(c) \Rightarrow (a)$: Replace b with a and a with b^c. □

3.3 From MV-effect algebras to MV-pairs [12]

Notation: In what follows, we will deal with an MV-effect algebra M and a Boolean algebra $B(M)$ such that M is a 0,1-sublattice of $B(M)$. In this particular situation, a small notational problem arises: both M and $B(M)$ are MV-effect algebras, but the $\oplus, \ominus$ and $'$ operations on $B(M)$ and M differ. To avoid confusion, we denote the partial operation of dijoint join (the $\oplus$ of Boolean algebras) on a Boolean algebra by $\dot{\vee}$. The partial difference of comparable elements and the complement in a Boolean algebra are denoted by $\setminus$ and ^{c} respectively.

The next Theorem is prved in [12] and in Guillermo Herrmann's Licentiate Dissertation.

Theorem 3.3.1 *[12]* Let (B, G) be an MV-pair, then $(B_{\sim}, \oplus, 0, 1)$ es una *MV-effect algebra*, where $0 = |0| = \{0\}$, $1 = |1| = \{1\}$ and $|a| \oplus |b|$ is defined iff there are $a_1 \sim a$, $b_1 \sim b$ such that $a_1 \wedge b_1 = 0$ and in this case $|a| \oplus |b| := |a_1 \dot{\vee} b_1|$. Furthermore $|a|' = |\neg a|$ and $|a| \wedge |b| = |a \wedge f(b)|$ with $a \wedge f(b) \in max(L^+(a, b))$.

Remark 3.3.2 $|a| \wedge |b| = max(L^+(a, b))$ where the $=$ is a set equality.

The last Theorem prove that for every MV-pair (B, G) there is an MV-effect algebra $\mathcal{A}(B, G)$ arising from it. The next Theorem prove that for every MV-effect algebra M there is a MV-pair (B, G) such that $\mathcal{A}(B, G) \cong M$. Let M

be an MV-effect algebra. Let S be a subset of $B(M)$ (the Boolean algebra R-generated by M). We say that a mapping $f : S \to B(M)$ is $\phi_M - preserving$ iff for all $x \in S$, $\phi_M(x) = \phi_M(f(x))$ or, in other words, ϕ_M restricted to S equals $\phi_M \circ f$. Let $G(M)$ be the set of all ϕ_M-preserving automorphisms of $B(M)$. It is easy to see that $G(M)$ is a subgroup of $Aut(B)$.

Theorem 3.3.3 Let M be an MV-effect algebra. Let $G(M)$ be the set of all ϕ_M-preserving automorphisms of $B(M)$. Then $(B(M), G(M))$ is an MV-pair and $\mathcal{A}(B(M), G(M))$ is isomorphic to M.

As in Section 2, we have divided the proof into a sequence of lemmas. In this section, M is an MV-effect algebra and $G(M)$ is the subgroup of Aut(B(M)) described in Theorem 3.3.3.

Lemma 3.3.4 Let $c, d \in M$, $d \leq c$. There is a ϕ_M-preserving isomorphism

$$\psi : B([0, c \ominus d]_M) \to [0, c \setminus d]_{B(M)} .$$

Proof. Consider the mapping $\psi_0 : [0, c \ominus d]_M \to [0, c \setminus d]_{B(M)}$, given by $\psi_0(x) = (x \oplus d) \setminus d$. Note that $d \leq c' \oplus d \leq x'$ (since that $x \leq c \ominus d = (c' \oplus d)'$) and thus $x \oplus d$ is defined. We see that $\psi_0(0) = 0$ and $\psi_0(c \ominus d) = c \setminus d$.
ψ_0 preserves joins and meets:
By Proposition 1.5.4 (ii) and 1.2.7 (i) $\psi_0(x \vee y) = ((x \vee y) \oplus d) \setminus d =$
$= ((x \oplus d) \vee (y \oplus d)) \setminus d = ((x \oplus d) \setminus d) \vee ((y \oplus d) \setminus d) = \psi_0(x) \vee \psi_0(y)$.
By Proposition 1.5.4 (i) and 1.2.7 (ii) $\psi_0(x \wedge y) = ((x \wedge y) \oplus d) \setminus d =$
$= ((x \oplus d) \wedge (y \oplus d)) \setminus d = ((x \oplus d) \setminus d) \wedge ((y \oplus d) \setminus d) = \psi_0(x) \wedge \psi_0(y)$.
From Lemma 1.2.7 (iii) and Lemma 1.4.3 (v) ψ_0 is injective, hence ψ_0 is a $\{0, 1\}$-lattice embedding of $[0, c \ominus d]_M$ into $[0, c \setminus d]_{B(M)}$.
We shall prove that the range of ψ_0 R-generates the Boolean algebra $[0, c \setminus d]_{B(M)}$. ψ_0 then uniquely extends to an isomorphism (by Corollary 1.3.18). $\psi : B([0, c \ominus d]_M) \to [0, c \setminus d]_{B(M)}$.
Let $x \in [0, c \setminus d]_{B(M)}$. Let $\{x_i\}_{i=1}^{2n}$ be an M-chain representation of x. For all $1 \leq i \leq n$, $x_{2i} \setminus x_{2i-1} \leq c \setminus d$ (since, by Lemma 1.2.7 (iv), $x_{2i} \setminus x_{2i-1} \leq x_{2i} \leq c \setminus d$). Then, by Lemma 1.2.7 (v)

$$x_{2i} \setminus x_{2i-1} = ((x_{2i} \vee d) \wedge c) \setminus ((x_{2i-1} \vee d) \wedge c).$$

For all $1 \leq j \leq 2n$, $(x_j \vee d) \wedge c \in [d, c]$. By Lemma 1.3.3 (v) $x = x_1 + \ldots + x_{2n} =$
$= (x_{2n} \backslash x_{2n-1}) \dot{\vee} \ldots \dot{\vee} (x_2 \backslash x_1) = (y_{2n} \backslash y_{2n-1}) \dot{\vee} \ldots \dot{\vee} (y_2 \backslash y_1) = y_1 + \ldots + y_{2n}$ (where $y_j = (x_j \vee d) \wedge c$). Therefore x has a M-chain representation $\{y_j\}_{j=1}^{2n} \subseteq [d, c]_M$. Since for all $1 \leq i \leq n$, $d \leq y_{2i-1} \leq y_{2i} \leq c$ then, by Lemma 1.2.7 (vii),

$$y_{2i} \setminus y_{2i-1} = (y_{2i} \setminus d) \setminus (y_{2i-1} \setminus d)$$

and $\{y_i \setminus d\}_{i=1}^{2n}$ is a chain representation of x. It remain to observe that, for all $1 \leq i \leq 2n$,

$$y_i \setminus d = ((y_i \ominus d) \oplus d) \setminus d = \psi_0(y_i \ominus d)$$

and that $y_i \ominus d \in [0, c \ominus d]_M$ (since $d \leq y_i \leq c$ and Lemma 1.4.7 (iii)). Thus, every element of $[0, c \setminus d]_{B(M)}$ has a $\psi_0([0, c \ominus d]_M)$-chain representation.
Let us prove that ψ is a ϕ_M-preserving mapping. Let $z \in B([0, c \ominus d]_M)$, let $\{z_i\}_{i=1}^{2n}$ be a $[0, c \ominus d]_M$-chain representation of z. Then, by Lemma 1.3.3 (v) and since ψ is a homomorphism of lattices and ϕ_M is a homomorphism of effect algebras

$$\begin{gathered} \phi_M(\psi(z)) = \phi_M(\psi(\dot{\vee}_{i=1}^{n}(z_{2i} \setminus z_{2i-1}))) = \\ = \phi_M(\dot{\vee}_{i=1}^{n}\psi(z_{2i} \setminus z_{2i-1})) = \bigoplus_{i=1}^{n} \phi_M(\psi(z_{2i} \setminus z_{2i-1})) \end{gathered}$$

and for all $1 \leq i \leq n$ (by Lemma 1.2.7 (vii), Lemma 1.4.9 (ii), Lemma 1.4.7 (vii), and since $\forall x \in M \quad \phi_M(x) = x$)

$$\begin{gathered} \phi_M(\psi(z_{2i} \setminus z_{2i-1})) = \phi_M(\psi(z_{2i}) \setminus (\psi z_{2i-1})) = \\ = \phi_M(((z_{2i} \oplus d) \setminus d) \setminus ((z_{2i-1} \oplus d) \setminus d)) = \\ = \phi_M((z_{2i} \oplus d) \setminus (z_{2i-1} \oplus d)) = \phi_M(z_{2i} \oplus d) \ominus \phi_M(z_{2i-1} \oplus d) = \\ = (z_{2i} \oplus d) \ominus (z_{2i-1} \oplus d) = z_{2i} \ominus z_{2i-1} = \phi_M(z_{2i} \setminus z_{2i-1}), \end{gathered}$$

so we obtain

$$\phi_M(\psi(z)) = \bigoplus_{i=1}^{n} \phi_M(\psi(z_{2i} \setminus z_{2i-1})) = \bigoplus_{i=1}^{n} \phi_M(z_{2i} \setminus z_{2i-1}) = \phi_M(z).$$

$\square$

Corollary 3.3.5 Let $c_1, d_1, c_2, d_2 \in M$ be such that $c_1 \geq d_1, c_2 \geq d_2$ and $c_1 \ominus d_1 = c_2 \ominus d_2$. There is a ϕ_M-preserving isomorphism
$\psi : [0, c_1 \setminus d_1]_{B(M)} \to [0, c_2 \setminus d_2]_{B(M)}$.

Proof.
By Lemma 3.3.4 there are ϕ_M-preserving isomorphisms
$\psi_1 : B([0, c_1 \ominus d_1]_M) \to [0, c_1 \setminus d_1]_{B(M)}$ and
$\psi_2 : B([0, c_2 \ominus d_2]_M) \to [0, c_2 \setminus d_2]_{B(M)}$.
Since $c_1 \ominus d_1 = c_2 \ominus d_2$ we can take
$\psi = \psi_2 \circ \psi_1^{-1} : [0, c_1 \setminus d_1]_{B(M)} \to [0, c_2 \setminus d_2]_{B(M)}$, and ψ is a ϕ_M-preserving isomorphism (since $G(M)$ is a subgroup of $Aut(M)$). $\square$

Lemma 3.3.6 For every $a \in B(M)$, there is a ϕ_M-preserving isomorphism of Boolean algebras $\psi : B([0, \phi_M(a)]_M) \to [0, a]_{B(M)}$.

Proof. Let $\{a_i\}_{i=1}^{2n}$ be an M-chain representation of a. Then $\{a_{2i} \setminus a_{2i-1}\}_{i=1}^{2n}$ is a decomposition of unit in the Boolean algebra $[0, a]_{B(M)}$ (see Lemma 1.3.3 (v) and Lemma 1.2.7 $(viii)$) and $\phi_M(a) = \bigoplus_{i=1}^{n}(a_{2i} \ominus a_{2i-1})$. For $j \in \{0, \ldots, n\}$, write $b_j = \bigoplus_{i=1}^{j}(a_{2i} \ominus a_{2i-1})$. Then $\{b_j\}_{j=0}^{n}$ is a finite chain in $[0, \phi_M(a)]_M$ with $b_0 = 0$ and $b_n = \phi_M(a)$. Thus $\{b_j \setminus b_{j-1}\}_{i=1}^{n}$ is a decomposition of unit in the Boolean algebra $B([0, \phi_M(a)]_M)$. For every $x \in B([0, \phi_M(a)]_M)$, $x = \dot{\bigvee}_{j=1}^{n} x \wedge (b_j \setminus b_{j-1})$. Since, for all j, $b_j \ominus b_{j-1} = a_{2j} \ominus a_{2j-1}$, Corollary 3.3.5 implies that, for all $1 \leq j \leq n$, there is a ϕ_M-preserving isomorphism
$\psi_j : [0, b_j \setminus b_{j-1}]_{B(M)} \to [0, a_{2j} \setminus a_{2j-1}]_{B(M)}$.
Define $\psi : B([0, \phi_M(a)]_M) \to [0, a]_{B(M)}$, $\psi(x) = \dot{\bigvee}_{j=1}^{n} \psi_j(x \wedge (b_j \setminus b_{j-1}))$.
$\psi(x)$ is a homomorphism of Boolean algebras:
$\psi(0) = \dot{\bigvee}_{j=1}^{n} \psi_j(0 \wedge (b_j \setminus b_{j-1})) = \dot{\bigvee}_{j=1}^{n} \psi_j(0) = \dot{\bigvee}_{j=1}^{n} 0 = 0$.
$\psi(\phi_M(a)) = \dot{\bigvee}_{j=1}^{n} \psi_j(\phi_M(a) \wedge (b_j \setminus b_{j-1})) = \dot{\bigvee}_{j=1}^{n} \psi_j(b_j \setminus b_{j-1}) =$
$= \dot{\bigvee}_{j=1}^{n}(a_{2j} \setminus a_{2j-1}) = a_n + \ldots + a_1 = a$.
$\psi(x \vee y) = \dot{\bigvee}_{j=1}^{n} \psi_j((x \vee y) \wedge (b_j \setminus b_{j-1})) =$
$= \dot{\bigvee}_{j=1}^{n} \psi_j((x \wedge (b_j \setminus b_{j-1})) \vee (y \wedge (b_j \setminus b_{j-1}))) =$
$= \dot{\bigvee}_{j=1}^{n} \psi_j(x \wedge (b_j \setminus b_{j-1})) \vee \psi_j(y \wedge (b_j \setminus b_{j-1})) =$
$= (\dot{\bigvee}_{j=1}^{n} \psi_j(x \wedge (b_j \setminus b_{j-1}))) \vee (\dot{\bigvee}_{j=1}^{n} \psi_j(y \wedge (b_j \setminus b_{j-1}))) = \psi(x) \vee \psi(y)$.
$\psi(x \wedge y) = \dot{\bigvee}_{j=1}^{n} \psi_j((x \wedge y) \wedge (b_j \setminus b_{j-1})) =$

$= \dot{\bigvee}_{j=1}^{n}\psi_j((x \wedge (b_j \setminus b_{j-1})) \wedge (y \wedge (b_j \setminus b_{j-1}))) =$
$= \dot{\bigvee}_{j=1}^{n}(\psi_j(x \wedge (b_j \setminus b_{j-1})) \wedge \psi_j(y \wedge (b_j \setminus b_{j-1}))) =$
$= (\dot{\bigvee}_{j=1}^{n}\psi_j((x \wedge (b_j \setminus b_{j-1})))) \wedge (\dot{\bigvee}_{j=1}^{n}\psi_j((y \wedge (b_j \setminus b_{j-1})))) = \psi(x) \wedge \psi(y)$ (by Lemma 1.2.7 (ix)).
Thus $\psi(x)$ is a homomorphism of Boolean algebras (see Remark page 21).
$\psi(x)$ is injective:
$\psi(x) = \psi(y) \Rightarrow \dot{\bigvee}_{j=1}^{n}\psi_j(x \wedge (b_j \setminus b_{j-1})) = \dot{\bigvee}_{j=1}^{n}\psi_j(y \wedge (b_j \setminus b_{j-1}))$. Then, by Lemma 1.2.7 (ix), $\psi_j(x \wedge (b_j \setminus b_{j-1})) = \psi_j(y \wedge (b_j \setminus b_{j-1}))$ and thus , since ψ_j an isomorphism, $x \wedge (b_j \setminus b_{j-1}) = y \wedge (b_j \setminus b_{j-1}) \quad 1 \le j \le n$. Therefore $x = \dot{\bigvee}_{j=1}^{n}x \wedge (b_j \setminus b_{j-1}) = \dot{\bigvee}_{j=1}^{n}y \wedge (b_j \setminus b_{j-1}) = y$.
$\psi(x)$ is surjective:
Let $y \in [0,a]_{B(M)}$, then $y = \dot{\bigvee}_{j=1}^{n}y \wedge (a_{2j} \setminus a_{2j-1})$. Since for all $1 \le j \le n$ $y \wedge (a_{2j} \setminus a_{2j-1}) \le a_{2j} \setminus a_{2j-1}$ and $\psi_j : [0, b_j \setminus b_{j-1}]_{B(M)} \to [0, a_{2j} \setminus a_{2j-1}]_{B(M)}$ an isomorphism, there exist $x_j \in [0, b_j \setminus b_{j-1}]_{B(M)}$ such that $\psi_j(x_j) =$
$= y \wedge (a_{2j} \setminus a_{2j-1})$. Let $x = \dot{\bigvee}_{j=1}^{n}x_j$. Then $x \in B([0,\phi_M(a)]_M)$ and
$\psi(x) = \dot{\bigvee}_{j=1}^{n}\psi_j(x \wedge (b_j \setminus b_{j-1})) = \dot{\bigvee}_{j=1}^{n}\psi_j((\dot{\bigvee}_{k=1}^{n}x_k) \wedge (b_j \setminus b_{j-1})) =$
$= \dot{\bigvee}_{j=1}^{n}\psi_j(x_j \wedge (b_j \setminus b_{j-1})) = \dot{\bigvee}_{j=1}^{n}\psi_j(x_j) = \dot{\bigvee}_{j=1}^{n}y \wedge (a_{2j} \setminus a_{2j-1}) = y$.
ψ is ϕ_M-preserving:
Let $x \in B([0,\phi_M(a)]_M)$, then $\phi_M(\psi(x)) = \phi_M(\dot{\bigvee}_{j=1}^{n}\psi_j(x \wedge (b_j \setminus b_{j-1}))) =$
$= \bigoplus_{j=1}^{n}(\phi_M(\psi_j(x \wedge (b_j \setminus b_{j-1})))) =$ (since ψ_j is ϕ_M-preserving)
$= \bigoplus_{j=1}^{n}(\phi_M(x \wedge (b_j \setminus b_{j-1}))) = \phi_M(\dot{\bigvee}_{j=1}^{n}x \wedge (b_j \setminus b_{j-1})) = \phi_M(x)$. □

Corollary 3.3.7 Let $a, b \in B(M)$ be such that $\phi_M(a) = \phi_M(b)$. Then there is a ϕ_M-preserving isomorphism $\psi : [0,a]_{B(M)} \to [0,b]_{B(M)}$.

Proof. Use Lemma 3.3.6 twice. □

Lemma 3.3.8 Let $u, v \in B(M)$, $u \wedge v = 0$ and $\phi_M(u) = \phi_M(v)$. Then there is a ϕ_M-preserving automorphism f of $B(M)$ such that $f(u) = v$, $f(v) = u$ and for all $x \le (u \dot{\vee} v)^c$, $f(x) = x$.

Proof. By Corollary 3.3.7, there is an isomorphism $\psi : [0,u]_{B(M)} \to [0,v]_{B(M)}$. Let $f : B(M) \to B(M)$ be a mapping given by

$$f(x) = \psi^{-1}(x \wedge v)\dot{\vee}\psi(x \wedge u)\dot{\vee}(x \wedge (u\dot{\vee}v)^c).$$

It is easy to check that, for all $x \in B(M)$, $f(f(x)) = x$. Thus f is a bijection. Moreover, we see that $f(0) = 0$, $f(1) = 1$ and for all $x, y \in B(M)$
$f(x \vee y) = \psi^{-1}((x \vee y) \wedge v)\dot{\vee}\psi((x \vee y) \wedge u)\dot{\vee}((x \vee y) \wedge (u\dot{\vee}v)^c) =$
$= \psi^{-1}((x \wedge v) \vee (y \wedge v))\dot{\vee}\psi((x \wedge u) \vee (y \wedge u))\dot{\vee}((x \wedge (u\dot{\vee}v)^c) \vee (y \wedge (u\dot{\vee}v)^c)) =$
$= (\psi^{-1}(x \wedge v)\dot{\vee}\psi(x \wedge u)\dot{\vee}(x \wedge (u\dot{\vee}v)^c)) \vee (\psi^{-1}(y \wedge v)\dot{\vee}\psi(y \wedge u)\dot{\vee}(y \wedge (u\dot{\vee}v)^c)) =$
$= f(x) \vee f(y)$.
and
$f(x^c) = \psi^{-1}(x^c \wedge v)\dot{\vee}\psi(x^c \wedge u)\dot{\vee}(x^c \wedge (u\dot{\vee}v)^c) =$
$= \psi^{-1}(v \setminus (x \wedge v))\dot{\vee}\psi(u \setminus (x \wedge u))\dot{\vee}(x^c \wedge (u\dot{\vee}v)^c) =$
$= (u \setminus \psi^{-1}(x \wedge v))\dot{\vee}(v \setminus \psi(x \wedge u))\dot{\vee}(x^c \wedge (u\dot{\vee}v)^c) =$
$= (\psi^{-1}(x \wedge v)\dot{\vee}\psi(x \wedge u)\dot{\vee}(x \wedge (u\dot{\vee}v)^c))^c = (f(x))^c$.
The latter equality follows by elementary Boolean calculus.
Since f preserves $0, 1, \vee$ and ^{c}, it is a homomorphism of Boolean algebras. □

Lemma 3.3.9 Let $u, v \in B(M)$, $\phi_M(u) = \phi_M(v)$. Then there is a ϕ_M-preserving automorphism f of $B(M)$ such that $f(u) = v$, $f(v) = u$ and for all $x \leq (u\dot{\vee}v)^c$, $f(x) = x$.

Proof. Put $u_0 = u \setminus u \wedge v$ and $v_0 = u \setminus u \wedge v$ then
$\phi_M(u_0) \oplus \phi_M(u \wedge v) = \phi_M(u) = \phi_M(v) = \phi_M(v_0) \oplus \phi_M(u \wedge v)$and thus $\phi_M(u_0) = \phi_M(v_0)$. Since $u_0 \wedge v_0 = 0$, by Lemma 3.3.8, there is $f \in G(M)$ such that $f(u_0) = v_0$, $f(v_0) = u_0$ and for all $x \in B(M)$ such that $x \leq (u_0\dot{\vee}v_0)^c$ we have $f(x) = x$. Since $u \wedge v \leq (u_0\dot{\vee}v_0)^c$, $f(u \wedge v) = u \wedge v$. Therefore,

$$f(u) = f(u_0\dot{\vee}u \wedge v) = f(u_0)\dot{\vee}(u \wedge v) = v_0\dot{\vee}(u \wedge v) = v$$

and similarly, $f(v) = u$.
Let $x \leq (u \vee v)^c$. Since $x \leq (u_0 \vee v_0)^c$, $f(x) = x$. □

Corollary 3.3.10 For all $u, v \in B(M)$, $u \sim_{G(M)} v$ iff $\phi_M(u) = \phi_M(v)$.

Proof. One implication follows by the definition of $G(M)$, the other one follows by Lemma 3.3.9. □

Corollary 3.3.11 For all $u \in B(M)$, $u \sim_{G(M)} \phi_M(u)$.

Proof. Put $v = \phi_M(u)$ in Corollary 3.3.10. $\square$

Proof of Theorem 3.3.3.
$(MVP1)$: Let $a, b \in B(M)$, $f \in G(M)$ be such that $a \le b$, $a \le f(b)$. Let $u = b \setminus (b \wedge f(b)), v = f(b) \setminus (b \wedge f(b))$. We have

$$\begin{gathered}\phi_M(u) = \phi_M(b \setminus (b \wedge f(b))) = \phi_M(b) \ominus \phi_M(b \wedge f(b)) = \\ = \phi_M(f(b)) \ominus \phi_M(b \wedge f(b)) = \phi_M(f(b) \setminus b \wedge f(b)) = \phi_M(v).\end{gathered}$$

By Lemma 3.3.8, there is a ϕ_M-preserving automorphism h of $B(M)$ with $h(u) = v$. Moreover, since $a \wedge u = a \wedge v = 0$ we have $h(a) = a$ ($a \wedge u = 0$ and $a \wedge v = 0$ imply $a \wedge (u \vee v) = 0$ and then, by Lemma 1.3.3 (i), $a \le (u \vee v)^c$). Similarly, since $(b \wedge f(b)) \wedge u = (b \wedge f(b)) \wedge v = 0$ we have $h(b \wedge f(b)) = b \wedge f(b)$. This implies that

$$h(b) = h((b \wedge f(b)) \dot{\vee} u) = h((b \wedge f(b))) \dot{\vee} h(u) = (b \wedge f(b)) \dot{\vee} u = f(b).$$

Thus, there is $h \in G(M)$ such that $h(a) = a$ and $h(b) = f(b)$. By Lemma 3.2.6, this implies $(MVP1)$.

$(MVP2)$: Let $a \wedge f(b)$ be an element of $L(a, b)$. By Corollary 3.3.11, there is $f_1 \in G(M)$ such that $f_1(a) = \phi_M(a)$. Since f_1 is ϕ_M-preserving,
$\phi_M(f_1(a \wedge f(b))) = \phi_M(a \wedge f(b))$. By Corollary 3.3.11, there is $g \in G(M)$ such that $g(f_1(a \wedge f(b))) = \phi_M(a \wedge f(b))$.
Since

$$f_1(a \wedge f(b)) \le f_1(a) = \phi_M(a)$$

and

$$g(f_1(a \wedge f(b))) = \phi_M(a \wedge f(b)) \le \phi_M(a),$$

$(MVP1)$ implies that there is $h \in G(M)$ such that $h(f_1(a \wedge f(b))) =$
$= \phi_M(a \wedge f(b))$ and $h(\phi_M(a)) = \phi_M(a)$.
Put $y = a \wedge f_1^{-1}(h^{-1}(\phi_M(f(b))))$. We shall prove that $y \ge a \wedge f(b)$ and that y is a maximal element of $L(a, b)$.
Indeed, we have

$$h(f_1(a)) = h(\phi_M(a)) = \phi_M(a),$$

therefore

$$\begin{aligned} h(f_1(y)) &= h(f_1(a \wedge f_1^{-1}(h^{-1}(\phi_M(f(b)))))) = \\ &= h(f_1(a)) \wedge h(f_1(f_1^{-1}(h^{-1}(\phi_M(f(b)))))) = \\ &= \phi_M(a) \wedge \phi_M(f(b)) = \phi_M(a) \wedge \phi_M(b) \end{aligned}$$

and

$$h(f_1(a \wedge f(b))) = \phi_M(a \wedge f(b)) \leq \phi_M(a) \wedge \phi_M(f(b)) = h(f_1(y)).$$

Since both h and f_1 are automorphisms of $B(M)$, the latter inequality cleary implies that $a \wedge f(b) \leq y$. Moreover, since h and f_1 are ϕ_M-preserving and ϕ_M restricted to M is the identity mapping, we obtain

$$\phi_M(y) = \phi_M(h(f_1(y))) = \phi_M(\phi_M(a) \wedge \phi_M(b)) = \phi_M(a) \wedge \phi_M(b).$$

Let us prove that y is maximal in $L(a,b)$. Suppose that $z \in L(a,b)$, $z \geq y$. Since $z = a \wedge f_2(b)$ for some $f_2 \in G(M)$, we see that

$$\phi_M(z) = \phi_M(a \wedge f_2(b)) \leq \phi_M(a) \wedge \phi_M(f_2(b)) = \phi_M(a) \wedge \phi_M(b) = \phi_M(y).$$

This implies that $\phi_M(z) = \phi_M(y)$. As $\phi_M(z \setminus y) = \phi_M(z) \ominus \phi_M(y) = 0$ and ϕ_M is faithful (i.e. $\phi_M(x) = 0 \Rightarrow x = 0$), $z \setminus y = 0$ and thus (since $y \leq z$) $z = y$.

Let us prove that $\mathcal{A}(B(M), G(M))$ is isomorphic to M.
The isomorphism $\psi : \mathcal{A}(B(M), G(M)) \to M$ is given by

$$\psi(|a|_{G(M)}) = \phi_M(a).$$

By Corollary 3.3.10, ψ is well-defined and injective. Since, for all $a \in M$, $\psi(|a|_{G(M)}) = a$, ψ is surjecective. Obviously, $\psi(|1|_{G(M)}) = 1$. Let $|a|_{G(M)}$, $|b|_{G(M)} \in \mathcal{A}(B(M), G(M))$ be such that $|a|_{G(M)} \perp |b|_{G(M)}$. We may always select the elements $a, b \in B(M)$ so that $a\dot{\vee}b$ exists, that means, $a \wedge b = 0$. Since ϕ_M is a morphism of effect algebras, $\phi_M(a) \oplus \phi_M(b)$ exists in M and we may compute

$$\begin{aligned}\psi(|a|_{G(M)} \oplus |b|_{G(M)}) &= \psi(|a\dot{\vee}b|_{G(M)}) = \phi_M(a\dot{\vee}b) \\ &= \phi_M(a) \oplus \phi_M(b) = \psi(|a|_{G(M)}) \oplus \psi(|b|_{G(M)}),\end{aligned}$$

hence ψ is a morphism of effect algebras. It remains to prove that ψ is a full morphism. Suppose that $\psi(|a|_{G(M)}) \oplus \psi(|b|_{G(M)})$ exists in M. Consider the elements $\phi_M(a)$ and $(\phi_M(a) \oplus \phi_M(b)) \setminus \phi_M(a)$. We see that

$$\phi_M(a) \wedge ((\phi_M(a) \oplus \phi_M(b)) \setminus \phi_M(a)) = 0,$$

that means, $\phi_M(a)\dot{\vee}((\phi_M(a) \oplus \phi_M(b)) \setminus \phi_M(a))$ exists in $B(M)$. This implies that $\phi_M(a) \oplus ((\phi_M(a) \oplus \phi_M(b)) \setminus \phi_M(a))$ exists in $\mathcal{A}(B(M), G(M))$. Finally,

$$\psi(|\phi_M(a)|_{G(M)}) = \phi_M(\phi_M(a)) = \phi_M(a) = \psi(|a|_{G(M)})$$

and
$\psi((\phi_M(a) \oplus \phi_M(b)) \setminus \phi_M(a)) = \phi_M((\phi_M(a) \oplus \phi_M(b)) \setminus \phi_M(a)) =$
$= \phi_M((\phi_M(a) \oplus \phi_M(b))) \ominus \phi_M(\phi_M(a)) = (\phi_M(a) \oplus \phi_M(b)) \ominus \phi_M(a) =$
$\phi_M(b) = \psi(|b|_{G(M)})$. □

Example 3.3.12 Let M be the MV-effect algebra $[0,1]$ (or M as in example 1.5.2). Then (see examples 1.5.2, 1.3.14 y 2.0.13) if $a \in B(M)$,
$a = (x_1, x_2] \,\dot{\cup} \ldots\ldots \dot{\cup}\, (x_{2n-1}, x_{2n}]$,
and
$\phi_M(a) = (x_2 - x_1) + \ldots\ldots + (x_{2n} - x_{2n-1})$ = the "length" of x.

Therefore $|a| = |b| \Leftrightarrow$ the "length" of a = the "length" of b,

and $\psi : B(M)/_{G(M)} \to [0,1]\,, \quad \psi(|a|)$ = the "length" of a,
is an isomorphism of effect-algebras.
Also
$|a|' = |a^c| = \{x \in B(M) :$ the "length" of $x = 1 -$ (the "length" of $a)\}$, and

$|a| \oplus |b|$ is defined $\Leftrightarrow$ the "length" of $a \leq 1-$ (the "length" of b) $\Leftrightarrow$
$\Leftrightarrow \exists a_1 \sim a$ y $b_1 \sim b$ such that $a_1 \cap b_1 = \emptyset$,

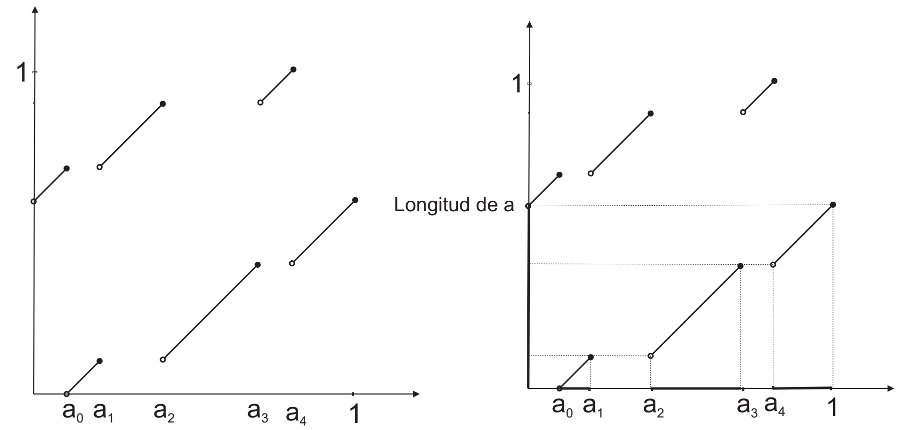

Figure 2:

and, in this case, $|a| \oplus |b| = |a_1 \dot{\cup} b_1| = \{x \in B(M) :$ the "length" of $x =$ (the "length" of a) $+$ (the "length" of b) $\}$

For example,
Let $a = (a_0, a_1] \dot{\cup} (a_2, a_3] \dot{\cup} (a_4, 1]$ and $f : [0,1] \rightarrow [0,1]$ as in Figure 2 to the left, and let $\tilde{f} : B(M) \rightarrow B(M)$, $\tilde{f}(y) = f(y)$ (the "image of y" by f).
Then , $\tilde{f} \in Aut(M)$, $\tilde{f}$ is ϕ_M-preserving and
$\tilde{f}(a) = (0, \text{the "length" of } a]$ (Figure 2 to the right).

4 Correspondence between MV-algebras and MV-effect algebras

4.1 MV-algebras [2]

An *MV-algebra* is an algebra $\langle A, \oplus, \neg, 0 \rangle$ with a binary operation $\oplus$, a unary operation $\neg$ and a constan 0 satisfying the following equations:

MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

MV2) $x \oplus y = y \oplus x$

MV3) $x \oplus 0 = x$

MV4) $\neg\neg x = x$

MV5) $x \oplus \neg 0 = \neg 0$

MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

Following tradition, we denote an MV-algebra $\langle A, \oplus, \neg, 0\rangle$ by its universe A.

On each MV-algebra A we define the constant 1 and the operations $\odot$ and $\ominus$ as follows:

$$1 := \neg 0$$

$$x \odot y := \neg(\neg x \oplus \neg y)$$

$$x \ominus y := x \odot \neg y \quad (\; = \neg(\neg x \oplus y) \;)$$

The following identities are inmediate consequences of MV4):

MV7) $\neg 1 = 0$

MV8) $x \oplus y = \neg(\neg x \odot \neg y)$

Axioms MV5) and MV6) can now be written as:

MV5') $x \oplus 1 = 1$, and

MV6') $(x \ominus y) \oplus y = (y \ominus x) \oplus x$.

Setting $y = \neg 0$ in MV6) we obtain:

MV9) $x \oplus \neg x = 1$.

Lemma 4.1.1 Let A be an MV-algebra and $x, y \in A$. Then the following conditions are equivalent:

(i) $\neg x \oplus y = 1$;

(ii) $x \odot \neg y = 0$;

(iii) $y = x \oplus (x \ominus y)$;

(iv) there is an element $z \in A$ such that $x \oplus z = y$.

Proof. $(i) \Rightarrow (ii)$ By MV4) and MV7). $(ii) \Rightarrow (iii)$ Inmediate from MV3) and MV6'). $(iii) \Rightarrow (iv)$ Take $z = y \ominus x$. $(iv) \Rightarrow (i)$ By MV9), $\neg x \oplus x \oplus z = 1$. □

Let A be an MV-algebra. For any two element x and y of A let us agree to write

$$x \leq y$$

iff x and y satisfy the above equivalent conditions $(i) - (iv)$. It follows that $\leq$ is a partial order, called the *natural order* of A. Indeed, reflexivity is equivalent to MV9), antisymetry follows from conditions (ii) and (iii), and transitivity follows from condition (iv).

Lemma 4.1.2 Let A be an MV-algebra. For each $a \in A$, $\neg a$ is the unique solution x of the simultaneous equations:

$$\begin{cases} a \oplus x = 1 \\ a \odot x = 0 \end{cases}$$

Proof. By Lemma 4.1.1, these two equations amount to writing
$\neg a \leq x \leq \neg a$. □

Lemma 4.1.3 In every MV-algebra A the natural order $\leq$ has the following properties:

(i) $x \leq y$ if and only if $\neg y \leq \neg x$;

(ii) If $x \leq y$ then for each $z \in A, x \oplus z \leq y \oplus z$ and $x \odot z \leq y \odot z$;

(iii) $x \odot y \leq z$ iff $x \leq \neg y \oplus z$;

(iv) $y \leq x \oplus y$;

(v) $x \odot y \leq y$.

(vi) $0 \leq x \quad \forall x \in A$

(vii) $x \leq 1 \quad \forall x \in A$

Proof. (i) This follows from Lemma 4.1.1 (i), since $\neg x \oplus y = \neg\neg y \oplus \neg x$. (ii) The monotonicity of $\oplus$ is an easy consequence of Lemma 4.1.1 (iv); using (i), one inmediately proves the monotonicity of $\odot$. (iii) It is sufficient to note that $x \odot y \leq z$ is equivalent to $1 = \neg(x \odot y) \oplus z = \neg x \oplus \neg y \oplus z$. (iv) It is inmediate from definition of $\leq$, Lemma 4.1.1 (iv). (v) By (iv) $\neg y \leq \neg x \oplus \neg y$, then by (i) $\neg(\neg x \oplus \neg y) \leq y$ and thus $x \odot y \leq y$. (vi) It is inmediate from definition of $\leq$, Lemma 4.1.1 (iv) and MV3). (vii) It is inmediate from definition of $\leq$, Lemma 4.1.1 (iv) and MV5') $\square$

Proposition 4.1.4 On each MV-algebra A the natural order determines a bounded lattice structure. Specifically, the join $x \vee y$ and the meet $x \wedge y$ of the elements x and y are given by

$$x \vee y = (x \odot \neg y) \oplus y = (x \ominus y) \oplus y = \neg(\neg x \oplus y) \oplus y, \tag{3}$$

$$x \wedge y = \neg(\neg x \vee \neg y) = x \odot (\neg x \oplus y). \tag{4}$$

Proof. To prove 3, by MV6'), MV9) and Lemma 4.1.3 (ii), $x \leq (x \ominus y) \oplus y$ and $y \leq (x \ominus y) \oplus y$. Suppose $x \leq z$ and $y \leq z$. By (i) and (iii) in Lemma 4.1.1, $\neg x \oplus z = 1$ and $z = (z \ominus y) \oplus y$. Then by MV6') we can write
$\neg((x \ominus y) \oplus y) \oplus z = (\neg(x \ominus y) \ominus y) \oplus y \oplus (z \ominus y) =$
$= (y \ominus \neg(x \ominus y)) \oplus \neg(x \ominus y) \oplus (z \ominus y) =$
$= (y \ominus \neg(x \ominus y)) \oplus \neg x \oplus y \oplus (z \ominus y) =$
$= (y \ominus \neg(x \ominus y)) \oplus \neg x \oplus z = 1$.
It follows that $((x \ominus y) \oplus y) \leq z$, which completes the proof of (3). We now inmediately obtain (4) as a consequence of (3) together with Lemma 4.1.3 (i). Also A is a bounded lattice by Lemma 4.1.3 (vi) and (vii). $\square$

Proposition 4.1.5 The following equations hold in every MV-algebra:

(i) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,

(ii) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$.

(iii) $\neg(x \wedge y) = \neg x \vee \neg y$

(iv) $\neg(x \vee y) = \neg x \wedge \neg y$

Proof. By MV6') and Lemma 4.1.3 (*ii*), $x \odot y \leq x \odot (y \vee z)$ and $x \odot z \leq x \odot (y \vee z)$. Suppose $x \odot y \leq t$ and $x \odot z \leq t$. Then by 4.1.3 (*iii*), $y \leq \neg x \oplus t$ and $z \leq \neg x \oplus t$, whence $y \vee z \leq \neg x \oplus t$. One more application of Lemma 4.1.3 (*iii*) yield $(y \vee z) \odot x \leq t$, which completes the proof of (*i*). It is now easy to see that (*ii*) is a consequence of (*i*), using Lemma 4.1.3 (*i*), together with MV4) and MV8). (*iii*) It follows that Proposition 4.1.4 (4) and MV4). (*iv*) By 4.1.4 (4) $\neg x \wedge \neg y = \neg(\neg\neg x \vee \neg\neg y) = \neg(x \vee y)$ by MV4). □

Proposition 4.1.6 Let A be an MV-algebra. Then A with the natural order is a bounded distributive lattice.

Proof. By Proposition 4.1.4, A is a bounded lattice. Now

$a \wedge (b \vee c) = (a \oplus \neg(b \vee c)) \odot (b \vee c)$ By Proposition 4.1.4 (4)
$= (a \oplus (\neg b \wedge \neg c)) \odot (b \vee c)$ By Proposition 4.1.5 (*iv*)
$= ((a \oplus \neg b) \wedge (a \oplus \neg c)) \odot (b \vee c)$ By Proposition 4.1.5 (*ii*)
$= (((a \oplus \neg b) \wedge (a \oplus \neg c)) \odot b) \vee (((a \oplus \neg b) \wedge (a \oplus \neg c)) \odot c)$
By Proposition 4.1.5 (*i*)
$= (((a \oplus \neg c) \oplus \neg(a \oplus \neg b)) \odot (a \oplus \neg b) \odot b)$
$\vee (((a \oplus \neg b) \oplus \neg(a \oplus \neg c)) \odot (a \oplus \neg c) \odot c)$ By Proposition 4.1.4 (4)
$= (((a \oplus \neg c) \oplus \neg(a \oplus \neg b)) \odot (a \wedge b)) \vee (((a \oplus \neg b) \oplus \neg(a \oplus \neg c)) \odot (a \wedge c))$
By Proposition 4.1.4 (4)
$\leq (a \wedge b) \vee (a \wedge c)$. By Lemma 4.1.3 (*v*)

On the other hand, $a \wedge b \leq a \wedge (b \vee c)$, $a \wedge c \leq a \wedge (b \vee c)$ imply
$(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$, and therefore
$(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$.

Replacing a by $b \vee a$ in $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ we obtain

$$
\begin{aligned}
(b \vee a) \wedge (b \vee c) &= ((b \vee a) \wedge b) \vee ((b \vee a) \wedge c) = \\
&= b \vee ((b \vee a) \wedge c) \quad (\text{since } (b \vee a) \wedge b = b) \\
&= b \vee ((b \wedge c) \vee (a \wedge c)) = (b \vee (b \wedge c)) \vee (a \wedge c) = \\
&= b \vee (a \wedge c) \quad (\text{since } b \vee (b \wedge c) = b).
\end{aligned}
$$

□

4.2 Correspondence between MV-algebras and MV-effect algebras

Proposition 4.2.1 Let $(M, \boxplus, \neg, 0)$ be an MV-algebra. Restrict the operation $\boxplus$ to the pairs (x, y) satisfying $x \leq \neg y$ and call the new partial operation $\oplus$. Then $M^{\mathcal{P}} = (M, \oplus, 0, 1)$ is an MV-effect algebra (where $1 = \neg 0$).

Proof.

$M^{\mathcal{P}} = (M, \oplus, 0, 1)$ **is an effect algebra :**

E_1

If $x \oplus y$ is defined then $x \leq \neg y$, hence (by Lemma 4.1.3 (i) and MV4) $y \leq \neg x$. Therefore $y \oplus x$ is defined and (by MV2) $x \oplus y = x \boxplus y = y \boxplus x = y \oplus x$.

E_2

Let a, b, c in M such that $b \oplus c$ and $a \oplus (b \oplus c)$ are defined (i.e. $b \leq \neg c$ and $a \leq \neg(b \oplus c)$). By lemma 4.1.3 (iv) $b \leq b \boxplus c = b \oplus c$. By hypothesis $a \leq \neg(b \oplus c)$, then $(b \oplus c) \leq \neg a$ and thus $b \leq (b \oplus c) \leq \neg a$. Therefore $b \leq \neg a$ and then $a \oplus b$ is defined.

By hypothesis $a \leq \neg(b \oplus c)$, then by Lemma 4.1.3 (ii) and since $a \oplus b$ is defined, $a \oplus b = a \boxplus b \leq \neg(b \oplus c) \boxplus b = \neg(b \boxplus c) \boxplus b = \neg(b \boxplus \neg(\neg c)) \boxplus b = \neg(\neg b \boxplus \neg c) \boxplus \neg c$ (by MV6).

On the other hand, by hypothesis $b \leq \neg c$, then by Lemma 4.1.1 (i) $1 = \neg c \boxplus \neg b$. Therefore $a \oplus b \leq \neg(\neg b \boxplus \neg c) \boxplus \neg c = \neg 1 \boxplus \neg c = 0 \boxplus \neg c = \neg c$ and thus $(a \oplus b) \oplus c$ is defined and $(a \oplus b) \oplus c = (a \boxplus b) \boxplus c = a \boxplus (b \boxplus c) = a \oplus (b \oplus c)$.

E_3

Let $x \in M$, then there exist a unique $\neg x$ in M such that $x \boxplus \neg x = 1$ (see MV9 and Lemma 4.1.2). Also, since $x \leq x$ and $\neg\neg x = x$, $x \oplus \neg x$ is defined and $x \oplus \neg x = x \boxplus \neg x = 1$.

Remark. x' in $M^{\mathcal{P}}$ is $\neg x$ in M.

E_4
If $x \oplus 1$ is defined then $x \leq \neg 1$ and thus $x \leq 0$. By Lemma 4.1.3 (vi) $x = 0$.

The natural order of the MV-algebra M and the natural order of the effect-algebra $M^{\mathcal{P}}$ are the same:
In other words $a \leq b$ in M iff $a \leq b$ in $M^{\mathcal{P}}$.
Let $a \leq b$ in M. then $\exists z \in M$ such that $a \boxplus z = b$ (Lemma 4.1.1 (iv)).
Since $z \wedge \neg a \leq \neg a$ then $a \leq \neg(z \wedge \neg a)$. Thus $a \oplus (z \wedge \neg a)$ is defined and

$$\begin{aligned} a \oplus (z \wedge \neg a) = a \boxplus (z \wedge \neg a) &= (a \boxplus z) \wedge (a \boxplus \neg a) \text{ (by Proposition 4.1.5 (ii))} \\ &= (a \boxplus z) \wedge 1 = a \boxplus z = b. \end{aligned}$$

Therefore $a \leq b$ in $M^{\mathcal{P}}$.
Now assume that $a \leq b$ in $M^{\mathcal{P}}$, then $\exists z \in M^{\mathcal{P}}$ $(z \in M)$ such that $a \oplus z$ is defined and $b = a \oplus z = a \boxplus z$.
Therefore $a \leq b$ in M.

$M^{\mathcal{P}}$ is a bounded distributive lattice:
The MV-algebra M is a bounded distributive lattice (Proposition 4.1.6) and since the natural order of the MV-algebra M and the natural order of the effect-algebra $M^{\mathcal{P}}$ are the same, then $M^{\mathcal{P}}$ is a bounded distributive lattice.

If $a \leq b$ in $M^{\mathcal{P}}$, then $b \ominus a$ in $M^{\mathcal{P}}$ is $b \boxminus a$ in M:
$a \leq \neg b \boxplus a \quad \Rightarrow \quad \neg(\neg b \boxplus a) \leq \neg a$, i.e. $b \boxminus a \leq \neg a$.
Thus $a \oplus (b \boxminus a)$ is defined in $M^{\mathcal{P}}$.
Also, $a \leq b$ in $M^{\mathcal{P}} \quad \Rightarrow \quad a \leq b$ in $M \quad \Rightarrow \quad$ (Lemma 4.1.1 (iii)) $a \boxplus (b \boxminus a) = b$.
Therefore, if $a \leq b$, $a \oplus (b \boxminus a) = a \boxplus (b \boxminus a) = b$, i.e. $a \leq b \quad \Rightarrow \quad b \ominus a = b \boxminus a$.

The lattice ordered effect algebra $M^{\mathcal{P}}$ satisfies the ecuation
$(a \vee b) \ominus a = b \ominus (a \wedge b)$:

$$\begin{aligned} (a \vee b) \ominus a = (a \vee b) \boxminus a = \neg(\neg(a \vee b) \boxplus a) &= \neg((\neg a \wedge \neg b) \boxplus a) = \\ &= \neg((\neg a \boxplus a) \wedge (\neg b \boxplus a)) \text{ (by Proposition 4.1.5 (ii))} \\ &= \neg(1 \wedge (\neg b \boxplus a)) = \neg(\neg b \boxplus a) = (b \boxminus a). \end{aligned}$$

$$\begin{aligned} b \ominus (a \wedge b) = b \boxminus (a \wedge b) = \neg(\neg b \boxplus (a \wedge b)) &= \neg((\neg b \boxplus a) \wedge (\neg b \boxplus b)) = \\ &= \neg(\neg b \boxplus a) \wedge 1) = \neg(\neg b \boxplus a) = (b \boxminus a). \end{aligned}$$

Therefore $(a \vee b) \ominus a = b \ominus (a \wedge b)$.

This completes the proof of Proposition 4.2.1. $\square$

Proposition 4.2.2 Let $M = (M, \oplus, 0, 1)$ be an MV-effect algebra. Let $\boxplus$ be a total operation given by $x \boxplus y = x \oplus (x' \wedge y)$. Then $M^{\mathcal{T}} = (M, \boxplus, {}', 0)$ is an MV-algebra.

Proof.
MV2)

$$\begin{aligned} x \boxplus y = x \oplus (x' \wedge y) &= (x' \ominus (x' \wedge y))' = \quad &&\text{(Lemma 1.4.8)} \\ &= ((x' \vee y) \ominus y)' = &&\text{(since } M \text{ an MV-effect algebra)} \\ &= y \oplus (x' \vee y)' = y \oplus (x \wedge y') = y \boxplus x &&\text{(by De Morgan's Identities).} \end{aligned}$$

MV1)
We will need the next results. We define a total binary operation on M given by $a \boxminus b = a \ominus (a \wedge b)$.

Lemma 4.2.3 Let M an MV-effect algebra and $a, b, c \in M$, then

(i) If $b \leq a$ then $a \boxminus b = a \ominus b$.

(ii) $a \boxminus (b \wedge c) = (a \boxminus b) \vee (a \boxminus c)$.

(iii) $(a \boxminus b) \boxminus c = a \boxminus (b \boxminus c)$.

(iv) $a \boxplus b = (a' \boxminus b)' = (b' \boxminus a)'$.

Proof.
(i) $a \boxminus b = a \ominus (a \wedge b) = a \ominus b$ (since $b \leq a$).
(ii) $(a \boxminus b) \vee (a \boxminus c) = (a \ominus (a \wedge b)) \vee (a \ominus (a \wedge c)) =$
$= a \ominus ((a \wedge b) \wedge (a \wedge c)) =$ (by Lemma 1.5.3 (v))
$= a \ominus (a \wedge (b \wedge c)) = a \boxminus (b \wedge c)$.
(iii) $(a \boxminus b) \boxminus c = (a \boxminus b) \ominus ((a \boxminus b) \wedge c) = (a \ominus (a \wedge b)) \ominus ((a \ominus (a \wedge b)) \wedge c) =$
$= (a \ominus ((a \ominus (a \wedge b)) \wedge c)) \ominus (a \wedge b) =$ (by Lemma 1.4.7 (ix))
$= (a \boxminus ((a \boxminus (a \wedge b)) \wedge c)) \ominus (a \wedge b) =$ (by (i))
$= ((a \boxminus (a \boxminus (a \wedge b))) \vee (a \boxminus c)) \ominus (a \wedge b) =$ (by (ii))
$= ((a \ominus (a \ominus (a \wedge b))) \vee (a \boxminus c)) \ominus (a \wedge b) =$ (by (i))

$= ((a \wedge b) \vee (a \boxminus c)) \ominus (a \wedge b) =$ (by Lemma 1.4.7 (ii))
$= (a \boxminus c) \ominus ((a \wedge b) \wedge (a \boxminus c)) =$ (since M an MV-effect algebra)
$= (a \boxminus c) \ominus ((a \boxminus c) \wedge b) =$ (since, by Lemma 1.4.7 (i), $a \boxminus c \leq a$)
$= (a \boxminus c) \boxminus b$.
(iv) $a \boxplus b = a \oplus (a' \wedge b) = (a' \ominus (a' \wedge b))' =$ (by Lemma 1.4.8)
$= (a' \boxminus b)'$. The rest follows by symmetry and $MV2$. □

Now, we will prove **MV1)**.
By Lemma 4.2.3 (iv), $(a \boxplus b) \boxplus c = ((a \boxplus b)' \boxminus c)' = (((b' \boxminus a)')' \boxminus c)' =$
$= ((b' \boxminus a) \boxminus c)' = ((b' \boxminus c) \boxminus a)' =$ (by Lemma 4.2.3 (iii))
$= ((b \boxplus c)' \boxminus a)' = ((((b \boxplus c)')' \boxplus a)')' = (b \boxplus c) \boxplus a = a \boxplus (b \boxplus c)$ (by $MV2$).

MV3)
$x \boxplus 0 = x \oplus (x' \wedge 0) = x \oplus 0 = x$. (Lemma 1.4.3 (iii))

MV4)
By Lemma 1.4.3 (i) $x'' = x$.

MV5)
$x \boxplus 0' = x \oplus (x' \wedge 0') = x \oplus (x' \wedge 1) = x \oplus x' = 1 = 0'$
(by E_3 and Lemma 1.4.3 (ii)).

MV6)
By MV2) $(x' \boxplus y)' \boxplus y = y \boxplus (y \boxplus x')' = y \oplus [y' \wedge (y \oplus (y' \wedge x'))'] =$
$= y \oplus (y \oplus (y' \wedge x'))'$ (since $(y \oplus (y' \wedge x'))' \leq y'$)
$= y \oplus (y' \ominus (y' \wedge x')) = (y' \ominus (y' \ominus (y' \wedge x')))' = ((y' \wedge x'))'$ (by Lemma 1.4.7 (ii))
$= y \vee x$.
Thus $(x' \boxplus y)' \boxplus y = y \vee x$ and by symmetry $(y' \boxplus x)' \boxplus x = x \vee y$.
Therefore $(x' \boxplus y)' \boxplus y = (y' \boxplus x)' \boxplus x$. □

The natural order of the MV-effect algebra M and the natural order of the MV- algebra $M^{\mathcal{T}}$ are the same:
If $a \leq b$ in M $\Rightarrow$ $\exists z \in M$ such that $a \oplus z$ is defined and $a \oplus z = b$.
$a \oplus z$ is defined $\Rightarrow a \leq z' \Rightarrow z \leq a' \Rightarrow a' \wedge z = z$.

Thus $a \boxplus z = a \oplus (a' \wedge z) = a \oplus z = b$, and then $a \leq b$ in $M^{\mathcal{T}}$.

$a \leq b$ in $M^{\mathcal{T}} \Rightarrow \exists z \in M$ such that $a \boxplus z = b$ that is $a \oplus (a' \wedge z) = b$.
Therefore $a \leq b$ in M.

Proposition 4.2.4

(i) Let $(M, \oplus, 0, 1)$ be an MV-effect algebra, then $(M^{\mathcal{T}})^{\mathcal{P}} = M$,

(ii) Let $(M, \boxplus, \neg, 0)$ be an MV algebra, then $(M^{\mathcal{P}})^{\mathcal{T}} = M$.

Proof.
(i)
Let a, b in $(M^{\mathcal{T}})^{\mathcal{P}}$ such that $a \oplus_{(M^{\mathcal{T}})^{\mathcal{P}}} b$ is defined, i.e. $a \leq b'$ in $(M^{\mathcal{T}})^{\mathcal{P}}$.
Now $a \leq b'$ in $(M^{\mathcal{T}})^{\mathcal{P}} \;\Leftrightarrow\; a \leq b'$ in $M^{\mathcal{T}} \;\Leftrightarrow\; a \leq b'$ in M.
Then $b \leq a'$ in M and thus $a' \wedge b = b$.
Therefore $a \oplus_{(M^{\mathcal{T}})^{\mathcal{P}}} b = a \boxplus_{M^{\mathcal{T}}} b = a \oplus_M (a' \wedge b) = a \oplus_M b$.
(ii)

$$\begin{aligned} a \boxplus_{(M^{\mathcal{P}})^{\mathcal{T}}} b &= a \oplus_{M^{\mathcal{P}}} (a' \wedge b) = \\ &= a \boxplus_M (a' \wedge b) \text{ (since } a \leq (a' \wedge b)' \text{ in } M^{\mathcal{P}} \;\Rightarrow\; a \leq (a' \wedge b)' \text{ in } M) \\ &= (a \boxplus_M a') \wedge (a \boxplus_M b) \quad \text{(Proposition 4.1.5 (ii))} \\ &= 1 \wedge (a \boxplus_M b) = \\ &= a \boxplus_M b. \end{aligned}$$

□

5 Appendix

Let M an MV-algebra, we call *radical* of M ($Rad(M)$) the intersection of all maximal ideals of M. An element a in M is said to be *infinitely small* or *infinitesimal* if and only if $a \neq 0$ and $na \leq \neg a$ for each integer $n \geq 0$ (where na is $a \boxplus \ldots \boxplus a$ n-times). The set of all infinitesimals in M will be denoted by $Infinit(M)$. An MV-algebra M is said to be *semisimple* if and only if $Rad(M) = \{0\}$.

Remark 5.0.5 It is proved in [2, Proposition 3.6.4] that
$Rad(M) = \{0\} \cup Infinit(M)$.

Example 5.0.6 Let $C = [0, 1]$, then it is easy to see that $C = (C, \boxplus, \neg, 0)$ is an MV-algebra where for all x, y in C $x \boxplus y = min(x + y,\ 1)$ and $\neg x = 1 - x$. It is called the standard MV-algebra. Also the natural order on the MV-algebra C is the usual order of numbers of C and C is semisimple since $Infinit(M) = \emptyset$.

Remark 5.0.7 Let $C = [0, 1]$ as example 5.0.6 and $C^{\mathcal{P}}$ as Proposition 4.2.1. Then $C^{\mathcal{P}} = (C, \oplus, 0, 1)$ where $a \oplus b$ is defined if and only if $a \leq 1 - b$ and, in this case, $a \oplus b = a + b$. Also $a' = 1 - a$ and $a \ominus b$ is defined if and only if $b \leq a$ and, in this case, $a \ominus b = a - b$.

Example 5.0.8 It is proved in [2] Proposition 3.6.1 (page 72) that an MV-algebra M is semisimple if and only if M is a subdirect product of subalgebras of the standard MV-algebra $[0, 1]$, that is, there is an injective homomorphism of MV-algebras $h : M \to \prod_{i \in I} C_i$ such that for each $j \in I$, C_j is subalgebra of $[0, 1]$ and $p^j \circ h : M \to C_j$ is a homomorphism onto C_j, where p^j is the j^{th} *projection.*
We identify M with the subalgebra (and the sublattice) $h(M) \subseteq \prod_{i \in I} C_i$ and $M^{\mathcal{P}}$ with $h(M)^{\mathcal{P}} \subseteq \prod_{i \in I} C_i$. Thus, we can think of the elements $x \in M^{\mathcal{P}}$ as elements $(x^l)_{l \in I}$ with $x^l \in C^l$ $l \in I$ and if $(x^l)_{l \in I}, (y^l)_{l \in I} \in M^{\mathcal{P}}$ we have that $(x^l)_{l \in I} \oplus (y^l)_{l \in I}$ is defined in $M^{\mathcal{P}}$ if and only if $(x^l)_{l \in I} \leq 1 - (y^l)_{l \in I} = (1 - y^l)_{l \in I}$ (i.e. for all $l \in I$ $\ x^l \leq 1 - y^l$) and, in this case, $(x^l)_{l \in I} \oplus (y^l)_{l \in I} = (x^l + y^l)_{l \in I}$. Also $(x^l)_{l \in I} \ominus (y^l)_{l \in I}$ is defined in $M^{\mathcal{P}}$ if and only if $(x^l)_{l \in I} \geq (y^l)_{l \in I}$ (i.e. for all $l \in I$ $\ x^l \geq y^l$) and, in this case, $(x^l)_{l \in I} \ominus (y^l)_{l \in I} = (x^l - y^l)_{l \in I}$.

Example 5.0.9 Let M be a semisimple MV-algebra, then (see example 5.0.8) $M^{\mathcal{P}} \subseteq \prod_{i \in I} C_i$. It is easy to see that the map $f : \prod_{i \in I} C_i \to \prod_{i \in I} B(C_i)$ defined by $f((x^l)_{l \in I}) = ((0, x^l])_{l \in I}$ (see example 1.3.14 and Lemma 1.3.15) is an order isomorphism onto the sublattices $f(M^{\mathcal{P}})$ of $\prod_{i \in I} B(C_i)$. Then ([10] II.4 Corollary 8) $B(M^{\mathcal{P}}) \cong B(f(M^{\mathcal{P}})) \cong B$ where B is the subalgebra of $\prod_{i \in I} B(C_i)$ R-generate by $f(M^{\mathcal{P}})$. Thus, we can think of the elements $x \in B(M^{\mathcal{P}})$ as elements in $\prod_{i \in I} B(C_i)$. Furthermore, let $x \in B(M^{\mathcal{P}})$, then $x = x_1 + \ldots + x_{2n}$ with $x_1, \ldots, x_{2n} \in M^{\mathcal{P}}$ and $x_1 \leq \ldots \leq x_{2n}$ and $\phi_{M^{\mathcal{P}}}(x) = \bigoplus_{i=1}^{n} (x_{2i} \ominus x_{2i-1}) = \bigoplus_{i=1}^{n} (x^l_{2i} - x^l_{2i-1})_{l \in I} = \left[\sum_{i=1}^{n} (x^l_{2i} - x^l_{2i-1})\right]_{l \in I}$. Thus if we think in $B(C_i)$ the element $x^i_1 + \ldots + x^i_{2n}$ as $(x^i_1, x^i_2] \dot\cup \ldots \dot\cup \left(x^i_{2n-1}, x^i_{2n}\right]$ (see example 1.3.14) then $\phi_{M^{\mathcal{P}}}(x)$ is in each coordinate i the "length" of $(x^i_1, x^i_2] \dot\cup \ldots \ldots \dot\cup \left(x^i_{2n-1}, x^i_{2n}\right]$.

5.1 Vetterlein's Boolean ambiguity algebras

A Boolean algebra $B = (B, \wedge, \vee, {}^c, 0, 1)$ with a countable dense subset is called *separable*. Furthermore, for $g \in Aut(B)$ and $a \in B$, we denote by $g|_a$ the restriction of g to the interval $[0, a]$. We define $a \to b = \neg a \vee b$ for $a, b \in B$. We furthermore write $a \perp b$ if there are no non-zero $a_0 \leq a$ and $b_0 \leq b$ such that $a_0 \sim b_0$. Let B be a separable Boolean algebra, and G be a group of automorphisms of B. Then we call the pair (B, G) a **Boolean ambiguity algebra**. Given (B, G), we introduce the following notions:

i We call G *compact* if for all non-zero $a \in B$, every set of pairwise disjoint elements of the form $g(a)$, where $g \in G$, is finite.

ii Let B be a $\sigma-$*complete* Boolean algebra. We call G *full* if for any two partitions of unity $(a_i)_{i \leq \lambda}$ and $(b_i)_{i \leq \lambda}$, where $\lambda \leq \omega$, and a system $g_i \in G$, $i \leq \lambda$, such that $g(b_i) = a_i$, the automorphism g defined by $g|_{a_i} = g_i|_{a_i}$, belong to G as well.

iii We call G *f-full* if for any two partitions of unity $(a_i)_{i<l}$ and $(b_i)_{i<l}$, where $l < \omega$, and a system $g_i \in G$, $i < l$, such that $g(b_i) = a_i$, the automorphism g defined by $g|_{a_i} = g_i|_{a_i}$ for each $i < l$, belong to G as well.

iv We say that G has the *decomposition property*, or (DP) for short, if for any $a, b \in B$, there are $c \leq a$ and $d \leq b$ such that $c \sim d$ and $a \setminus c \perp b \setminus d$.

A Boolean ambiguity algebra (B, G) will be called **complete** if B is $\sigma-$complete and G is compact and full, and it is called **normal** if G is compact f-full and fulfils (DP).

Remark 5.1.1 As a matter of fact, all the results concerning normal ambiguity algebras stated in this paper do not depend on the separability of the corresponding Boolean algebras. Hence this condition can be eliminated from the definition of normal ambiguity algebras.

Let (B, G) be a complete Boolean ambiguity algebra or a normal Boolean ambiguity algebra then it is proved in [17, Propositions 2.8 and 4.5] that $(B_{\sim_G}, \leq, 0, 1)$ is a lattice with smallest element $0 = |0| = \{0\}$, greatest element

$1 = |1| = \{1\}$ and $|a| \leq |b|$ if and only if there exist $a_1 \sim a$ and $b_1 \sim b$ such that $a_1 \leq b_1$. Moreover, for any $a, b \in B$, there is a $b_1 \sim b$ such that $|a \wedge b_1| = |a| \wedge |b|$ and $|a \vee b_1| = |a| \vee |b|$. Furthermore if $a, b \in B$, then $\{|a_1 \wedge b_1| : \ a_1 \sim a, b_1 \sim b\}$ has a minimal element, and $\{|a_1 \to b_1| : a_1 \sim a, b_1 \sim b\}$ has a maximal element. Let (B, G) be a complete Boolean ambiguity algebra or a normal Boolean ambiguity algebra. We define:

$|a| \odot |b| = \bigwedge \{|a_1 \wedge b_1| : a_1 \sim a, \ b_1 \sim b\}, \qquad \neg |a| = |a| \to 0 = |a^c|,$

$|a| \to |b| = \bigvee \{|a_1 \to b_1| : a_1 \sim a, \ b_1 \sim b\}, \qquad |a| \oplus |b| = \neg(\neg |a| \odot \neg |b|).$

Proposition 5.1.2 *[17, Propositions 2.12 and 4.7]* Let (B, G) be a complete Boolean ambiguity algebra or a normal Boolean ambiguity algebra. Let $a, b \in B$ such that $|a| \wedge |b| = |a \wedge b|$. Then $|a| \odot |b^c| = |a \wedge b^c|$ and $|a| \to |b| = |a \to b|$.

Theorem 5.1.3 *[17, Theorems 2.14 and 4.8]* Let (B, G) be a complete Boolean ambiguity algebra or a normal Boolean ambiguity algebra. Then $(B_\sim, \oplus, \neg, 0)$ is an MV-algebra.

5.2 Normal Boolean ambiguity algebras and MV-pairs

Let us start this section by showing that if (B, G) is a Normal Boolean ambiguity algebra then (B, G) is an MV-pair. We need first to prove the following lemmas:

Lemma 5.2.1 Let (B, G) be a Boolean ambiguity algebra with G compact, let $f \in G$ and let $x, a, b \in B$ such that $a \wedge b = 0$, $x \leq a$ and $f^n(x) \leq b$ for all $n \in \mathbb{N}$. Then $x = 0$.

Proof. Since $x \leq a$, $a \wedge b = 0$ and $f^n(x) \leq b$ for all $n \in \mathbb{N}$ we have $x \wedge f^n(x) = 0$ for all $n \in \mathbb{N}$, then $f^i(x) \wedge f^j(x) = 0$ for all $i \neq j \ \ i, j \in \mathbb{N}$. Since G is compact $\{f^n(x) : n \in \mathbb{N}\}$ is finite, i.e. $\{f^n(x) : n \in \mathbb{N}\} = \big\{f^1(x), f^2(x), \ldots, f^k(x)\big\}$. Let $f^{k+1}(x)$, then $\exists j$, $1 \leq j \leq k$ such that $f^{k+1}(x) = f^j(x)$, therefore $f^{-j}(f^{k+1}(x)) = f^{-j}(f^j(x))$, that is $f^{k+1-j}(x) = x$ (note that $k + 1 - j > 0$) then $x = 0$ since $f^{k+1-j}(x) \leq b$, $x \leq a$ and $a \wedge b = 0$. $\square$

Lemma 5.2.2 Let (B, G) be a Boolean ambiguity algebra and let $a, b \in B$, then the following conditions are equivalent:

(i) $a \perp b$

(ii) for all $h \in G \quad h(a) \wedge b = 0$

Proof.
$(i) \Rightarrow (ii)$: If $h(a) \wedge b \neq 0$, let $b_0 = h(a) \wedge b$ and let $a_0 = h^{-1}(b_0) = a \wedge h^{-1}(b)$. Then $0 \neq a_0 \leq a$, $0 \neq b_0 \leq b$ and $a_0 \sim b_0$ which contradicts $a \perp b$.
$(ii) \Rightarrow (i)$: Let $a_0 \leq a$, $b_0 \leq b$ and $a_0 \sim b_0$. Then there exist $h \in G$ with $b_0 = h(a_0)$, therefore $b_0 = b_0 \wedge b = h(a_0) \wedge b \leq h(a) \wedge b = 0$ and thus $b_0 = 0$ and $a_0 = 0$. □

Remark 5.2.3 Let (B, G) be a complete Boolean ambiguity algebra or a normal Boolean ambiguity algebra. Then it is proved in [17] Lemmas 2.3 and 4.4 that if $a, b \in B$ are such that $a \sim b$ and $a \leq b$, then $a = b$.

Lemma 5.2.4 Let (B, G) be a complete Boolean ambiguity algebra or a normal Boolean ambiguity algebra and let $a, b, b' \in B$ such that $b \sim b'$ and $|a \wedge b'| = |a| \wedge |b|$. Then $a \wedge b' \in max(L^+(a, b))$.

Proof.
Let $f, g \in G$ such that $a \wedge b' \leq g(a) \wedge f(b)$. Then we have:

$|g(a) \wedge f(b)| \leq |a|$ and $|g(a) \wedge f(b)| \leq |b|$, thus $|g(a) \wedge f(b)| \leq |a| \wedge |b|$.

$a \wedge b' \leq g(a) \wedge f(b)$ imply $|a \wedge b'| \leq |g(a) \wedge f(b)|$.

Therefore $|a| \wedge |b| = |a \wedge b'| \leq |g(a) \wedge f(b)| \leq |a| \wedge |b|$ and then
$|a \wedge b'| = |g(a) \wedge f(b)|$.
Since $a \wedge b' \leq g(a) \wedge f(b)$ and $a \wedge b' \sim g(a) \wedge f(b)$ then, by Remark 5.2.3, $a \wedge b' = g(a) \wedge f(b)$ and thus $a \wedge b' \in max(L^+(a, b))$. □

Proposition 5.2.5 Let (B, G) be a normal Boolean ambiguity algebra, then (B, G) is an MV-pair.

Proof.

MVP1

Let (B, G) be a normal Boolean ambiguity algebra, $a, b \in B$ and $f \in G$ such that $a \leq b$ and $f(a) \leq b$.

- If $a = b$ then $f(a) = f(b) \leq b$ and then, by Remark 5.2.3, $f(a) = f(b) = b = a$. Therefore $h = id$ satisfy the requirement.

- If $f(a) = b$ then $a \sim b$ and $a \leq b$. As above we have $a = b$. Therefore $f(a) = b = a$ and again $h = id$ satisfy the requirement.

- If $a < b$ and $f(a) < b$ it is proved in [17, Lemma 4.3] that $\exists h \in G$ such that $h(b) = b$ and $h|_a = f|_a$. In particular $h(b) = b$ and $h(a) = f(a)$.

MVP2 Let (B, G) be a complete Boolean ambiguity algebra. From Lemma 3.2.5 it suffices to prove that for all $a, b \in B$ there exist $m \in max(L(a, b))$ such that $m \geq a \wedge b$. Let $a, b \in B$. Since (B, G) is a normal Boolean ambiguity algebra, we can apply (DP) property to $a \setminus b$ and $b \setminus a$ and we obtain that there are

$$c \leq a \setminus b,\ d \leq b \setminus a \text{ and } \ g \in G \text{ with } g(d) = c \text{ and } (a \setminus b) \setminus c \perp (b \setminus a) \setminus d \quad (5)$$

(note that $c \wedge d = 0$). Since G is f-full the automorphism $\tilde{g}$ defined by $\tilde{g}|_d = g|_d$, $\tilde{g}|_c = g^{-1}|_c$ and $\tilde{g}|_{(c \vee d)^c} = id|_{(c \vee d)^c}$ is in G. Let $b' = \tilde{g}(b)$. It is easy to see that $b' = (b \setminus d) \dot{\vee} c$, $a \wedge b' = (a \wedge b) \dot{\vee} c$, $b' \setminus a = (b \setminus a) \setminus d$ and $a \setminus b' = (a \setminus b) \setminus c$,

$$\text{and thus, } \ a \wedge b \leq a \wedge b' \text{ and, from (5), } \ b' \setminus a \perp a \setminus b'. \quad (6)$$

We claim that for all $b'' \sim b$ there exist an $h \in G$ such that $h(a \wedge b'') \leq a \wedge b'$. Indeed, since $b' \sim b''$, it is proved in [17] Lemma 4.3 (ii) that there exist an $h \in G$ such that

$$h(b'' \setminus b') = b' \setminus b'', \quad h(b' \setminus b'') = b'' \setminus b' \quad \text{and} \quad h|_{b' \wedge b''} = id|_{b' \wedge b''}. \quad (7)$$

By (6) and Lemma 5.2.2 $h(a \setminus b') \wedge (b' \setminus a) = 0$ that is $h(a \wedge b'^c) \wedge b' \wedge a^c = 0$, and we also obtain $h(a \wedge b'^c \wedge b'') \wedge b' \wedge a^c \wedge b''^c = 0$. Since, from (7), $h(a \wedge b'^c \wedge b'') \leq h(b'' \setminus b') = b' \setminus b'' = ((b' \setminus b'') \wedge a) \dot{\vee} ((b' \setminus b'') \wedge a^c) = (b' \wedge b''^c \wedge a) \dot{\vee} (b' \wedge b''^c \wedge a^c)$ we have that $h(a \wedge b'^c \wedge b'') \leq b' \wedge a \wedge b''^c \leq b' \wedge a$.
On the other hand, by (7), $h(a \wedge b' \wedge b'') = id(a \wedge b' \wedge b'') = a \wedge b' \wedge b'' \leq a \wedge b'$. Therefore $h(a \wedge b'') = h(a \wedge b'' \wedge b') \dot{\vee} h(a \wedge b'' \wedge b'^c) \leq a \wedge b' \vee a \wedge b' = a \wedge b'$ and the claim is proved.

We will prove that $|a \wedge b'| = |a| \wedge |b|$. It is clear that $|a \wedge b'| \leq |a|$ and $|a \wedge b'| \leq |b|$. Let $x \in B$ such that $|x| \leq |a|$ and $|x| \leq |b|$, then there are $f_1, f_2 \in G$ such that $f_1(x) \leq a$ and $x \leq f_2(b')$ (and thus $f_1(x) \leq f_1(f_2(b'))$). Therefore $f_1(x) \leq a \wedge f_1(f_2(b'))$. From the claim, there is an $h \in G$ such that $h(a \wedge f_1(f_2(b'))) \leq a \wedge b'$, that is $a \wedge f_1(f_2(b')) \leq h^{-1}(a \wedge b')$. Then $f_1(x) \leq h^{-1}(a \wedge b')$, and thus $(h \circ f_1)(x) \leq a \wedge b'$ that is $|x| \leq |a \wedge b'|$. Therefore $|a \wedge b'| = |a| \wedge |b|$.

Finally, from Lemma 5.2.4, $a \wedge b' \in max(L^+(a, b))$ and, by (6), $a \wedge b \leq a \wedge b'$.□

Summing up, we have:
Let (B, G) be a normal Boolean ambiguity algebra then,

- **(I)** From Theorem 5.1.3, $(B_\sim, \boxplus, \neg, 0)$ is an MV-algebra. We call it $\mathcal{V}(B, G)$.
- **(II)** From Proposition 5.2.5 (B, G) is an MV-pair and then, from Theorem 3.3.1, $M = (B_\sim, \oplus, 0, 1)$ is an MV-effect algebra . Therefore from Proposition 4.2.2, $M^{\mathcal{I}} = (B_\sim, \hat{\boxplus}, \hat{\neg}, 0)$ is an MV-algebra. We call it $\mathcal{J}(B, G)$.

Proposition 5.2.6 Let (B, G) be a normal Boolean ambiguity algebra and let the MV-algebras $\mathcal{V}(B, G)$ and $\mathcal{J}(B, G)$ as **(I)** and **(II)**.
Then $\mathcal{V}(B, G) = \mathcal{J}(B, G)$.

Proof. In $\mathcal{V}(B, G)$ and $\mathcal{J}(B, G)$ we have $0 = |0| = \{0\}$. Let $|a| \in B_\sim$, in $\mathcal{V}(B, G)$ $\neg |a| = |a^c|$ and in $\mathcal{J}(B, G)$ $\hat{\neg} |a| = |a|' = |a^c|$. Thus $\hat{\neg} = \neg$ on $B_\sim$. So we only need to show that $\boxplus = \hat{\boxplus}$ on $B_\sim$.

$|a| \boxplus |b| = \neg(\neg |a| \odot \neg |b|) = \neg(|a^c| \odot |b^c|) = \neg(|a^c \wedge f(b)^c|)$ where, by Proposition 5.1.2, $f(b)$ is such that $|a^c \wedge f(b)| = |a^c| \wedge |b|$ and then, from Lemma 5.2.4 $a^c \wedge f(b) \in max(L^+(a^c, b))$. Then we have
$|a| \boxplus |b| = \neg(|a^c \wedge f(b)^c|) = |(a^c \wedge f(b)^c)^c| = |a \vee f(b)|$ with
$a^c \wedge f(b) \in max(L^+(a^c, b))$

On the other hand
$|a| \hat{\boxplus} |b| = |a| \oplus (|a|' \wedge |b|) = |a| \oplus (|a^c| \wedge |b|) = |a| \oplus |a^c \wedge g(b)|$ with
$a^c \wedge g(b) \in max(L^+(a^c, b))$. From Remark 3.3.2 $|a^c \wedge g(b)| = |a^c \wedge f(b)|$,
and thus $|a| \hat{\boxplus} |b| = |a| \oplus |a^c \wedge g(b)| = |a| \oplus |a^c \wedge f(b)| =$

$= |a \dot{\vee} a^c \wedge f(b)| = |a \vee f(b)|$. □

Let us proceed with this section by showing that the MV-algebras obtained in Proposition 5.2.6 are semisimple.

Let (B, G) be an MV-pair and let $M = (B_\sim, \oplus, 0, 1)$ be the MV-effect algebra as in Theorem 3.3.1. Let $|a| \in M$, we write $n\,|a|$ $(n \in \mathbb{N})$ for $|a| \oplus \ldots \oplus |a|$ (n times) provided $|a| \oplus \ldots \oplus |a|$ (n times) is defined. Then:

Lemma 5.2.7 Let M and a as above, then $2\,|a|$, $3\,|a|$, $\ldots$, $n\,|a|$, are defined in M if and only if there are $f_1, \ldots, f_n \in G$ such that $f_i(a) \wedge f_j(a) = 0$ for all $i \neq j$ $i, j = 1, \ldots, n$. In this case $n\,|a| = |f_1(a) \dot{\vee} \ldots \dot{\vee} f_n(a)|$.

Proof. We use induction on n. If $n = 2$, from Theorem 3.3.1 $|a| \oplus |a|$ is defined if and only if there are $f_1, f_2 \in G$ such that $f_1(a) \wedge f_2(a) = 0$ and in this case $|a| \oplus |a| = |f_1(a) \dot{\vee} f_2(a)|$. Suppose that $2\,|a|$, $3\,|a|$, $\ldots$, $n\,|a|$, $(n+1)\,|a|$ are defined in M. By the induction hypothesis, there are $g_1, \ldots, g_n \in G$ such that $g_i(a) \wedge g_j(a) = 0$ for all $i \neq j$ $i, j = 1, \ldots, n$ and $n\,|a| = |g_1(a) \dot{\vee} \ldots \dot{\vee} g_n(a)|$. Therefore $(n+1)\,|a| = |g_1(a) \dot{\vee} \ldots \dot{\vee} g_n(a)| \oplus |a|$ and this is defined if and only if there are $h_1, h_2 \in G$ such that $h_1(g_1(a) \dot{\vee} \ldots \dot{\vee} g_n(a)) \wedge h_2(a) = 0$, that is, if and only if $h_1(g_i(a)) \wedge h_1(g_j(a)) = 0$ $\forall i \neq j$ $i, j = 1, \ldots, n$ and $h_1(g_i(a)) \wedge h_2(a) = 0$ $i = 1, \ldots, n$ and $(n+1)\,|a| = |h_1(g_j(a)) \dot{\vee} \ldots \;\ldots \dot{\vee} h_1(g_j(a)) \dot{\vee} h_2(a)|$. The induction is complete if we call $f_i = h_1 \circ g_i$ $i = 1, \ldots, n$ and $f_{n+1} = h_2$. □

Lemma 5.2.8 Let (B, G) be a normal Boolean ambiguity algebra, let $M = (B_\sim, \oplus, 0, 1)$ be the MV-effect algebra as in **(II)** and let $0 \neq |a| \in M$. Then there exist $n \in \mathbb{N}$ such that $2\,|a|$, $3\,|a|$, $\ldots$, $(n-1)\,|a|$, are defined in M and $n\,|a|$ is not defined in M.

Proof. If $m\,|a|$ is defined for all $m \in \mathbb{N}$ then, from Lemma 5.2.7, there are $f_1, f_2, \ldots \in G$ such that $f_i(a) \wedge f_j(a) = 0$ for all $i \neq j$ $i, j \in \mathbb{N}$ wich is a contradiction since $a \neq 0$ and G is compact. □

Corollary 5.2.9 Let (B, G) be a normal Boolean ambiguity algebra and let $M = (B_\sim, \oplus, 0, 1)$ be the MV-effect algebra as in **(II)**. Then for all $a \in B$, $a \neq 0$, there exist $n \in \mathbb{N}$ such that $m\,|a| \leq |a|'$ $m = 1, \ldots, n-1$ and $n\,|a| \not\leq |a|'$ in M.

Proof. Let E be an effect algebra and let $x, y \in E$, it is easy to see that $x \oplus y$ is defined if and only if $y \leq x'$. Therefore $|a| \oplus |a|$ is defined in M if and only if $|a| \leq |a|'$, $(|a| \oplus |a|) \oplus |a|$ is defined if and only if $|a| \oplus |a| \leq |a|'$, ..., in general $m\,|a|$ is defined if and only if $(m-1)\,|a| \leq |a|'$. Therefore the proof follows from Lemma 5.2.8. $\square$

Proposition 5.2.10 Let (B, G) be a normal Boolean ambiguity algebra and let $\mathcal{V}(B, G)$ as in **(I)**. Then $\mathcal{V}(B, G)$ is a semisimple MV-algebra.

Proof. Let $0 \neq a \in B$. Let $M = (B_\sim, \oplus, 0, 1)$ be the MV-effect algebra as in **(II)**, then from Corollary 5.2.9, there exist $n \in \mathbb{N}$ such that $n\,|a| \nleq |a|'$ in M. Since $|a|' = \hat{\neg}\,|a|$ in $\mathcal{J}$ and, from Proposition 4.2.2, the order in M and $M^{\mathcal{T}} = \mathcal{J}(B, G)$ are the same we obtain that, in $\mathcal{J}(B, G)$, $n\,|a| = |a|\,\hat{\boxplus} \ldots \hat{\boxplus}\,|a| = |a| \oplus \ldots \oplus |a| \nleq \hat{\neg}\,|a|$.
By Proposition 5.2.6 the MV-algebras $\mathcal{J}(B, G) = (B_\sim, \hat{\boxplus}, \hat{\neg}, 0)$ and $\mathcal{V}(B, G) = (B_\sim, \boxplus, \neg, 0)$ are equals and thus we have that for all $0 \neq |a| \in \mathcal{V}(B, G)$ there exist $n \in \mathbb{N}$ such that $n\,|a| \nleq \hat{\neg}\,|a|$ in $\mathcal{V}$ that is $Infinit(\mathcal{V}(B, G)) = \emptyset$. Therefore, from Remark 5.0.5, $Rad(\mathcal{V}(B, G)) = 0$ and $\mathcal{V}(B, G)$ is semisimple. $\square$

Finally we will see that if we build on a semisimple MV-algebra and obtain, through Proposition 4.2.2 and Theorem 3.3.3, an MV-pair, the latter is a normal Boolean ambiguity algebra.

To prove that $G(M^{\mathcal{P}})$ is compact we need the following results:

Let C_i be a subalgebra of the standard MV-algebra $[0, 1]$ as example 5.0.6 and let $B(C_i)$ be the Boolean algebra R-generate by C_i. Let $a \in B(C_i)$,
$a = a_1 + a_2 + \ldots + a_{2n-1} + a_{2n}$ with $a_1, \ldots, a_{2n} \in C_i$ $a_1 \leq \ldots \leq a_{2n}$ then (see example 1.3.14) we can represent a as $(a_1, a_2]\,\dot{\cup} \ldots \dot{\cup}\,(a_{2n-1}, a_{2n}]$. We denote $lenght(a)$ for $(a_2 - a_1) + \ldots + (a_{2n} - a_{2n-1})$.

Remark 5.2.11 *Let C_i and $B(C_i)$ as above, and $a = (a_1, a_2]\,\dot{\cup} \ldots \dot{\cup}\,(a_{2n-1}, a_{2n}]$ and $b = (b_1, b_2]\,\dot{\cup} \ldots \dot{\cup}\,(b_{2m-1}, b_{2m}]$ in $B(C_i)$, then $a \wedge b = 0$ if and only if $(a_{2r-1}, a_{2r}] \cap (b_{2s-1}, b_{2s}] = \emptyset$ for all $0 \leq r \leq n$ and $0 \leq s \leq m$.*

Lemma 5.2.12 Let C_i and $B(C_i)$ as above and let $\{a_r\}_{r \in \mathbb{N}}$ be a secuence of pairwise disjoint elements in $B(C_i)$ with the same lenght $l = lenght(a_1) = lenght(a_2) = \ldots$. Then $l = 0$ (and thus $a_1 = \emptyset, a_2 = \emptyset, \ldots$).

Proof.
It follows from Remark 5.2.11, that
$a_1 \wedge a_2 = 0$ imply that $l \leq \frac{1}{2}$.
In the same form, if $\{a_1, a_2, a_3\}$ are pairwise disjoint, then $l \leq \frac{1}{3}$,
$\vdots$
if $\{a_1, a_2, \ldots, a_n\}$ are pairwise disjoint, then $l \leq \frac{1}{n}$,
and thus $l = 0$. $\square$

Corollary 5.2.13 Let C_i and $B(C_i)$ as above and let $\{a_j\}_{j\in J}$ be a secuence of pairwise disjoint elements in $B(C_i)$ with the same lenght $l = lenght(a_j)$ for all $j \in J$ and $l > 0$. Then J is finite.

Proposition 5.2.14 Let M be a semisimple MV-algebra, let $M^{\mathcal{P}}$ as in Proposition 4.2.2 and let $(B(M^{\mathcal{P}}), G(M^{\mathcal{P}}))$ be the MV-pair as Theorem 3.3.3 . Then $(B(M^{\mathcal{P}}), G(M^{\mathcal{P}}))$ is a normal Boolean ambiguity algebra.

Proof.
$G(M^{\mathcal{P}})$ **is compact**:
From examples 5.0.8 and 5.0.9 we have that $M^{\mathcal{P}} \subseteq \prod_{i\in I} C_i$ and $B(M^{\mathcal{P}}) \subset \prod_{i\in I} B(C_i)$ where, for all $i \in I$, C_i is a subalgebra of MV-algebra $[0,1]$. Let $x \in B(M^{\mathcal{P}})$, $x = x_1 + \ldots + x_{2n}$ with $x_1, \ldots, x_{2n} \in M^{\mathcal{P}}$ and $x_1 \leq \ldots \leq x_{2n}$. Note that if $f \in G(M^{\mathcal{P}})$ then $\phi_{M^{\mathcal{P}}}(f(x)) = \phi_{M^{\mathcal{P}}}(x)$ and thus, if $f(x) = y_1 + \ldots + y_{2m}$ with $y_1, \ldots, y_{2m} \in M^{\mathcal{P}}$ and $y_1 \leq \ldots \leq y_{2m}$, we have that (see example 5.0.9) for all $i \in I$ $(y_2^i - y_1^i) + \ldots + (y_{2m}^i - y_{2m-1}^i) = (x_2^i - x_1^i) + \ldots + (x_{2n}^i - y_{2n-1}^i) > 0$ that is for all $i \in I$ $(x_1^i, x_2^i] \dot\cup \ldots \dot\cup \left(x_{2n-1}^i, x_{2n}^i\right]$ and $(y_1^i, y_2^i] \dot\cup \ldots \dot\cup \left(y_{2m-1}^i, y_{2m}^i\right]$ have the same length in $B(C_i)$, that is, for all $i \in I$ $lenght(x^i) = lenght(f(x)^i)$ in $B(C_i)$.

Now, let $x \in B(M^{\mathcal{P}})$, $x \neq 0$ and let $\{f_\alpha(x)\}_{\alpha\in A}$ be a set of pairwise disjoint elements with $f_\alpha \in G(M^{\mathcal{P}})$ for all $\alpha \in A$. Since $x \neq 0$ then, from Theorem 3.3.3, $\phi_{M^{\mathcal{P}}}(x) \neq 0$ in $\prod_{i\in I} C_i$ and thus $\exists j \in I$ such that $p^j(\phi_{M^{\mathcal{P}}}(x)) = (\phi_{M^{\mathcal{P}}}(x))^j \neq 0$ in C_j, that is, $\sum_{i=1}^n (x_{2i}^j - x_{2i-1}^j) > 0$ that is $lenght(x^j) > 0$ in $B(C_j)$. On the other hand (since, $f_\alpha(x) \in B(M^{\mathcal{P}}) \subset \prod_{i\in I} B(C_i)$ for all $\alpha \in A$) $\{f_\alpha\}_{\alpha\in A}$ are pairwise disjoint if and only if for all $i \in I$ $\{(f_\alpha)^i\}_{\alpha\in A}$ are pairwise disjoint in $B(C_i)$. In particular $\{(f_\alpha)^j\}_{\alpha\in A}$ are pairwise disjoint in $B(C_j)$ and, from above, $0 < lenght(x^j) = lenght((f_\alpha(x))^j)$ in $B(C_j)$ for all $\alpha \in A$. Therefore from Corollary 5.2.13 A is finite.

$G(M^{\mathcal{P}})$ **is f-full**:
Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be two partitions of unity of $B(M^{\mathcal{P}})$, let $g_1, \ldots, g_n$ in $G(M^{\mathcal{P}})$ such that $g_k(a_k) = b_k$ $k = 1, \ldots, n$ and let g defined by $g|_{a_i} = g_i|_{a_i}$. Then $\phi_{M^{\mathcal{P}}}(g(x)) = \phi_{M^{\mathcal{P}}}(g(x \wedge a_1 \dot{\vee} \ldots \dot{\vee} x \wedge a_n)) =$
$= \phi_{M^{\mathcal{P}}}(g_1(x \wedge a_1) \dot{\vee} \ldots \dot{\vee} g_n(x \wedge a_n))$ and, since $\phi_{M^{\mathcal{P}}}$ is a homomorphism of effect algebras and the sum operation in $B(M^{\mathcal{P}})$ is $\dot{\vee}$, we have that
$\phi_{M^{\mathcal{P}}}(g(x)) = \bigoplus_{k=1}^n \phi_{M^{\mathcal{P}}}(g_k(x \wedge a_k)) = \bigoplus_{k=1}^n \phi_{M^{\mathcal{P}}}(x \wedge a_k)$ (since g_k is $\phi_M - preserving$ $k = 1, \ldots, n$). Therefore $\phi_{M^{\mathcal{P}}}(g(x)) = \bigoplus_{k=1}^n \phi_{M^{\mathcal{P}}}(x \wedge a_k) =$
$= \phi_{M^{\mathcal{P}}}(x \wedge a_1 \dot{\vee} \ldots \dot{\vee} x \wedge a_n) = \phi_{M^{\mathcal{P}}}(x)$ and thus $g \in G(M^{\mathcal{P}})$ and $G(M^{\mathcal{P}})$ is f-full.

$(B(M^{\mathcal{P}}), G(M^{\mathcal{P}}))$ **fulfils (DP):**
It is proved in [12] Lemma 4.4 that for every $a \in B(M)$, there is a ϕ_M-preserving isomorphism of Boolean algebras
$\psi : B([0, \phi_M(a)]_M) \to [0, a]_{B(M)}$. Let $a, b \in B(M^{\mathcal{P}})$ and let
$t = \phi_{M^{\mathcal{P}}}(a) \wedge \phi_{M^{\mathcal{P}}}(b)$, then there are two $\phi_{M^{\mathcal{P}}}$-preserving isomorphisms of Boolean algebras $h_1 : B([0, \phi_{M^{\mathcal{P}}}(a)]_{M^{\mathcal{P}}}) \to [0, a]_{B(M^{\mathcal{P}})}$ and
$h_2 : B([0, \phi_{M^{\mathcal{P}}}(b)]_{M^{\mathcal{P}}}) \to [0, b]_{B(M^{\mathcal{P}})}$ (note that $t \in [0, \phi_{M^{\mathcal{P}}}(a)]_{M^{\mathcal{P}}}$ and
$t \in [0, \phi_{M^{\mathcal{P}}}(b)]_{M^{\mathcal{P}}}$), let $c = h_1(t)$ and $d = h_2(t)$ then $0 \le c \le a$ and $0 \le d \le b$ and $c \sim d$ since h_1 and h_2 are $\phi_{M^{\mathcal{P}}}$-preserving and Theorem 3.3.3. Let $r, s \in B(M^{\mathcal{P}})$ be such that $0 \le r \le a \setminus c$, $0 \le s \le b \setminus d$ and $r \sim s$ (and thus by Theorem 3.3.3, $\phi_{M^{\mathcal{P}}}(r) = \phi_{M^{\mathcal{P}}}(s)$) then $\phi_{M^{\mathcal{P}}}(0) \le \phi_{M^{\mathcal{P}}}(r) \le \phi_{M^{\mathcal{P}}}(a \setminus c)$ and $\phi_{M^{\mathcal{P}}}(0) \le \phi_{M^{\mathcal{P}}}(s) \le \phi_{M^{\mathcal{P}}}(b \setminus d)$. Now, in the effect algebra $B(M^{\mathcal{P}})$ the partial difference is defined if and only if $x \le y$ and it is $x \setminus y$ and, since $\phi_{M^{\mathcal{P}}}$ is a homomorphism of effect algebras, we obtain $0 \le \phi_{M^{\mathcal{P}}}(r) \le \phi_{M^{\mathcal{P}}}(a) \ominus \phi_{M^{\mathcal{P}}}(c)$ and $0 \le \phi_{M^{\mathcal{P}}}(s) \le \phi_{M^{\mathcal{P}}}(b) \ominus \phi_{M^{\mathcal{P}}}(d)$ in $M^{\mathcal{P}}$. Therefore , since $\phi_{M^{\mathcal{P}}}(r) = \phi_{M^{\mathcal{P}}}(s)$, $\quad 0 \le \phi_{M^{\mathcal{P}}}(r) \le (\phi_{M^{\mathcal{P}}}(a) \ominus \phi_{M^{\mathcal{P}}}(c)) \wedge (\phi_{M^{\mathcal{P}}}(b) \ominus \phi_{M^{\mathcal{P}}}(d))$.
On the other hand $\phi_{M^{\mathcal{P}}}(a) \ominus \phi_{M^{\mathcal{P}}}(c) = \phi_{M^{\mathcal{P}}}(a) \ominus \phi_{M^{\mathcal{P}}}(h_1(t)) =$
$= \phi_{M^{\mathcal{P}}}(a) \ominus \phi_{M^{\mathcal{P}}}(t) = \phi_{M^{\mathcal{P}}}(a) \ominus t = \phi_{M^{\mathcal{P}}}(a) \ominus (\phi_{M^{\mathcal{P}}}(a) \wedge \phi_{M^{\mathcal{P}}}(b)) =$
$= (\phi^i_{M^{\mathcal{P}}}(a) - \phi^i_{M^{\mathcal{P}}}(a) \wedge \phi^i_{M^{\mathcal{P}}}(b))_{i \in I}$ and

$$\phi^i_{M^{\mathcal{P}}}(a) - \phi^i_{M^{\mathcal{P}}}(a) \wedge \phi^i_{M^{\mathcal{P}}}(b) = \begin{cases} 0 & \text{if } \phi^i_{M^{\mathcal{P}}}(a) \leq \phi^i_{M^{\mathcal{P}}}(b) \\ \phi^i_{M^{\mathcal{P}}}(a) - \phi^i_{M^{\mathcal{P}}}(b) & \text{if } \phi^i_{M^{\mathcal{P}}}(a) > \phi^i_{M^{\mathcal{P}}}(b) \end{cases}$$

Similarly $\phi_{M^{\mathcal{P}}}(b) \ominus \phi_{M^{\mathcal{P}}}(d) = (\phi^i_{M^{\mathcal{P}}}(b) - \phi^i_{M^{\mathcal{P}}}(a) \wedge \phi^i_{M^{\mathcal{P}}}(b))_{i\in I}$ and

$$\phi^i_{M^{\mathcal{P}}}(b) - \phi^i_{M^{\mathcal{P}}}(a) \wedge \phi^i_{M^{\mathcal{P}}}(b) = \begin{cases} 0 & \text{if } \phi^i_{M^{\mathcal{P}}}(b) \geq \phi^i_{M^{\mathcal{P}}}(b) \\ \phi^i_{M^{\mathcal{P}}}(b) - \phi^i_{M^{\mathcal{P}}}(a) & \text{if } \phi^i_{M^{\mathcal{P}}}(a) < \phi^i_{M^{\mathcal{P}}}(b) \end{cases}$$

Thus $(\phi_{M^{\mathcal{P}}}(a) \ominus \phi_{M^{\mathcal{P}}}(c)) \wedge (\phi_{M^{\mathcal{P}}}(b) \ominus \phi_{M^{\mathcal{P}}}(d)) = 0$ in $M^{\mathcal{P}}$ and then $\phi_{M^{\mathcal{P}}}(r) = 0$. Therefore $r = 0$ and $s = 0$ and thus $(B(M^{\mathcal{P}}), G(M^{\mathcal{P}}))$ fulfils (DP).

5.3 Complete Boolean ambiguity algebras and MV-pairs

Let us start this section by showing that if (B, G) is a Complete Boolean ambiguity algebra then (B, G) is an MV-pair. However, we need first to prove the following lemma:

Lemma 5.3.1 Let (B, G) be a complete Boolean ambiguity algebra, let $a, b \in B$ and let $g \in G$ with $g(b) \leq a$. Then there is an automorphism $\bar{g} \in G$ such that $\bar{g}(b) \leq a$ and $\bar{g}(a \wedge b) = a \wedge b$.

Proof.

It is clear if $a \wedge b = 0$. Suppose that $a \wedge b \neq 0$.

To define $\bar{g}$ let us build an appropriate partition of the unit, appropriate automorphisms, and based on the fact that G is full. The proof is quite simple when the Boolean algebra is atomic; in general, the idea is the same but the operations are more cumbersome.

Let us start by defining certain elements b_j in B so that in the event that the Boolean algebra should be atomic, then b_j would be the set of atoms x in set $b \setminus a$ such that $g^i(x) \in a \wedge b \;\; i = 1, \ldots, j-1$ and $g^j(x) \in a \setminus b$. See Figure 3.

Let $b_1 = b \wedge a^c \wedge g^{-1}(a \setminus b)$, $\qquad b_2 = b \wedge a^c \wedge b_1^c \wedge g^{-2}(a \setminus b)$, $\qquad \ldots$

$\ldots$, $\qquad b_j = b \wedge a^c \wedge b_1^c \wedge \ldots \wedge b_{j-1}^c \wedge g^{-j}(a \setminus b)$, $\qquad \ldots$

We have divided the proof into a sequence of remarks.

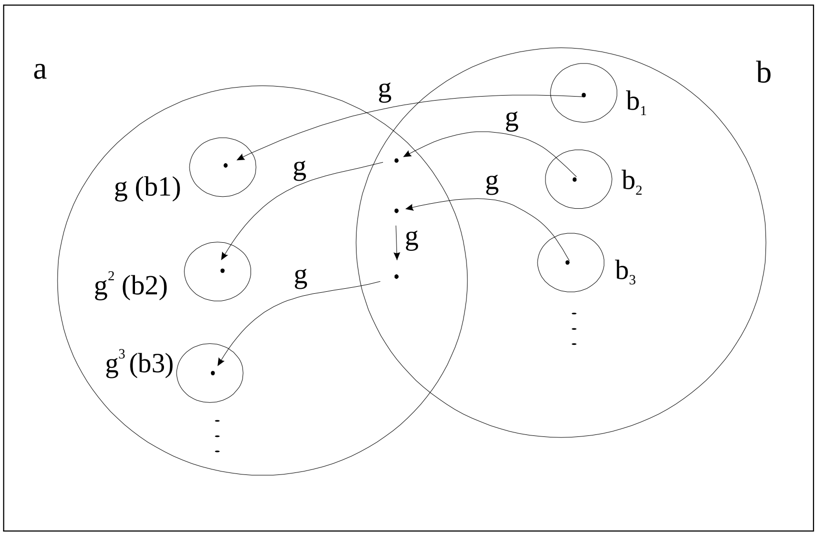

Figure 3:

R1) If $i \neq j$ it is clear from definition, that $b_i \wedge b_j = 0$.

R2) $b \setminus a = \bigvee_1^\infty b_i$.

$$\text{Let } r = (b \backslash a) \backslash (\bigvee_1^\infty b_i). \text{ We note that } g(r) \leq g(b) \leq a \text{ and, for all } i \in \mathbb{N}\; r \wedge b_i = 0. \tag{8}$$

We claim that for all $i \in \mathbb{N}$, $g^i(r) \leq a \wedge b$. We use induction on i.

Let $i = 1$ and $t = g(r) \wedge (a \backslash b)$. Then $g^{-1}(t) = r \wedge g^{-1}(a \backslash b) \leq (b \backslash a) \wedge g^{-1}(a \backslash b) = b \wedge a^c \wedge g^{-1}(a \backslash b) = b_1$ and $g^{-1}(t) \leq r$. Therefore $g^{-1}(t) \leq r \wedge b_1 = 0$ (by (8)) and thus $g^{-1}(t) = 0$ and $t = 0$. Finally $0 = t = g(r) \wedge (a \setminus b)$ imply $g(r) \leq (a \setminus b)^c = a^c \vee b$ and since by (8) $g(r) \leq a$, we obtain $g(r) = g(r) \wedge a \leq (a^c \vee b) \wedge a$, and then $g(r) \leq a \wedge b$.

Induction hypothesis: $g^k(r) \leq a \wedge b, \;\; k = 1, 2, \ldots, i$. Let $t = g^{i+1}(r) \wedge (a \setminus b)$, then
$g^{-1}(t) = g^i(r) \wedge g^{-1}(a \setminus b) \leq g^i(r) \leq a \wedge b$,

$$\begin{array}{lll} \vdots & \vdots & \vdots \\ g^{-i}(t) = g^1(r) \wedge g^{-i}(a \setminus b) \leq g^1(r) \leq a \wedge b. \end{array}$$

Thus, for $0 \leq k \leq i-1$, we have $g^{-i+k}(t) \leq a \wedge b$.

If we take infimum with $g^{k+1}(b \wedge a^c \wedge b_1^c \wedge \ldots \wedge b_k^c) \wedge (a \setminus b)$ in both side to the last inequality we obtain $g^{-i+k}(t) \wedge g^{k+1}(b \wedge a^c \wedge b_1^c \wedge \ldots \wedge b_k^c) \wedge (a \setminus b) = 0$ (since that $(a \setminus b) \wedge a \wedge b = 0$) and then
$g^{-k-1}(g^{-i+k}(t) \wedge g^{k+1}(b \wedge a^c \wedge b_1^c \wedge \ldots \wedge b_k^c) \wedge (a \setminus b)) = g^{-k-1}(0)$ that is
$g^{-i-1}(t) \wedge \underbrace{(b \wedge a^c \wedge b_1^c \wedge \ldots \wedge b_k^c) \wedge g^{-k-1}(a \setminus b)}_{b_{k+1}} = 0$ for $0 \leq k \leq i-1$.

Therefore $g^{-i-1}(t) \wedge b_1 = 0$, and thus $g^{-i-1}(t) \leq b_1^c$,
$g^{-i-1}(t) \wedge b_2 = 0$, and thus $g^{-i-1}(t) \leq b_2^c$,
$\vdots \qquad \vdots \qquad \vdots$
$g^{-i-1}(t) \wedge b_i = 0$, and thus $g^{-i-1}(t) \leq b_i^c$.

On the other hand $g^{-i-1}(t) = g^{-i-1}(g^{i+1}(r) \wedge (a \setminus b)) = r \wedge g^{-i-1}(a \setminus b)$.
Thus $g^{-i-1}(t) \leq r \leq b \setminus a = b \wedge a^c$ and $g^{-i-1}(t) \leq g^{-i-1}(a \setminus b)$.
Therefore $g^{-i-1}(t) \leq b \wedge a^c \wedge b_1^c \wedge b_2^c \ldots \wedge b_i^c \wedge g^{-i-1}(a \setminus b) = b_{i+1}$.

Then, we have: $g^{-i-1}(t) \leq b_{i+1}$ and $g^{-i-1}(t) \leq r$. Therefore $g^{-i-1}(t) \leq b_{i+1} \wedge r = 0$ (by (8)) and thus $g^{-i-1}(t) = 0$ and $t = 0$.
Since $g(b) \leq a$ and, by induction hypothesis, $g^i(r) \leq a \wedge b \leq b$, we have $g^{i+1}(r) \leq a$. But $0 = t = g^{i+1}(r) \wedge (a \setminus b)$ and $g^{i+1}(r) \leq a$ imply (as before, in case $i = 1$) $g^{i+1}(r) \leq a \wedge b$. Thus the induction is complete and we have proved that $g^i(r) \leq a \wedge b \; \forall i \in \mathbb{N}$.

Finally, we have $r \leq b \setminus a$ and $g^i(r) \leq a \wedge b \; \forall i \in \mathbb{N}$, then from Lemma 5.2.1 $r = 0$.
Since $r = (b \setminus a) \setminus (\bigvee_1^\infty b_i)$ and $\bigvee_1^\infty b_i \leq b \setminus a$, we obtain $b \setminus a = \bigvee_1^\infty b_i$.

We intend to prove that if $i \neq j$ then $g^i(b_i) \wedge g^j(b_j) = 0$, but we need first to prove the following observation.
R3) Let $r \neq 0$ and $r \leq b_i$ for some i $\in \mathbb{N}$, then
$g(r) \wedge (a \wedge b) \neq 0$,
$g[g(r) \wedge (a \wedge b)] \wedge (a \wedge b) = g^2(r) \wedge g(a \wedge b) \wedge (a \wedge b) \neq 0$,

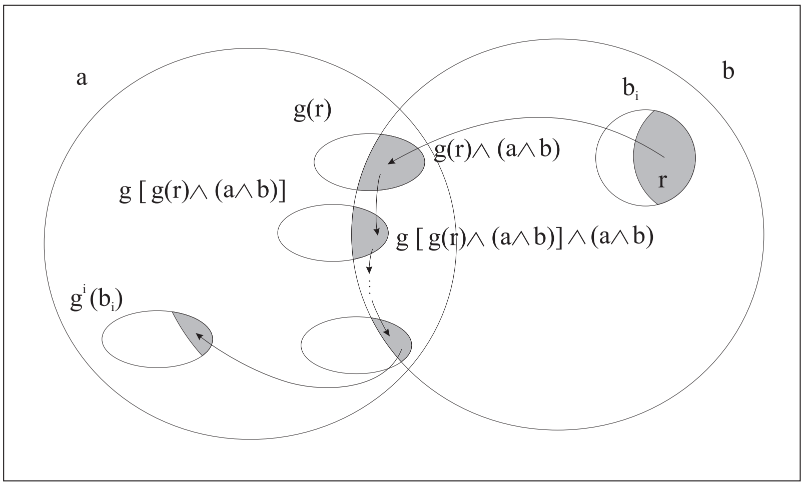

Figure 4:

$g\,\{g[g(r) \wedge (a \wedge b)] \wedge (a \wedge b)\} \wedge (a \wedge b) = g^3(r) \wedge g^2(a \wedge b) \wedge g(a \wedge b) \wedge (a \wedge b) \neq 0,$
$\vdots$
$g^{i-1}(r) \wedge g^{i-2}(a \wedge b) \wedge \ldots \wedge g(a \wedge b) \wedge (a \wedge b) \neq 0$ (see Figure 4).

Suppose that $g^l(r) \wedge g^{l-1}(a \wedge b) \wedge \ldots \wedge g(a \wedge b) \wedge (a \wedge b) = 0$ for some $1 \leq l \leq i-1$. Let $k = min\,\{l : g^l(r) \wedge g^{l-1}(a \wedge b) \wedge \ldots \wedge g(a \wedge b) \wedge (a \wedge b) = 0,$
$1 \leq l \leq i-1\}.$

Now, $0 = g^k(r) \wedge g^{k-1}(a \wedge b) \wedge \ldots \wedge g(a \wedge b) \wedge (a \wedge b) =$
$= g[g^{k-1}(r) \wedge g^{k-2}(a \wedge b) \wedge \ldots \wedge g(a \wedge b) \wedge (a \wedge b)] \wedge (a \wedge b).$
We call $m = g^{k-1}(r) \wedge g^{k-2}(a \wedge b) \wedge \ldots \wedge g(a \wedge b) \wedge (a \wedge b).$

We have that $g(m) \leq a$ (since $m \leq b$ and then $g(m) \leq g(b) \leq a$) and $g(m) \wedge (a \wedge b) = 0$. Then
$g(m) \leq a \setminus b$. Also $m \neq 0$ (since $k - 1 < k$). (9)

Let $z = g^{-k+1}(m) = r \wedge g^{-1}(a \wedge b) \ldots \wedge g^{-k+2}(a \wedge b) \wedge g^{-k+1}(a \wedge b) \leq r \leq b_i.$
Therefore $z \leq b_i = b \wedge a^c \wedge b_1^c \wedge \ldots \wedge b_{k-1}^c \wedge \ldots \wedge b_{i-1}^c \wedge g^{-i}(a \setminus b) \leq$

$\leq b\wedge a^c\wedge b_1^c\wedge\ldots\wedge b_{k-1}^c$. On the other hand $g^k(z)=g(m)\leq a\setminus b$ (by (9)), and thus $z\leq g^{-k}(a\setminus b)$. Therefore $z\leq b\wedge a^c\wedge b_1^c\wedge\ldots\wedge b_{k-1}^c\wedge g^{-k}(a\setminus b)=b_k$. Then $z\leq b_k$ and $z\leq b_i$ and thus $z\leq b_k\wedge b_i=0$ (since $k\neq j$) which is a contradiction since (by (9)) $m\neq 0$. Therefore $g^k(r)\wedge g^{k-1}(a\wedge b)\ldots\wedge g(a\wedge b)\wedge(a\wedge b)\neq 0$ for all $k=1,2,\ldots,i-1$.

R4) Now we will prove that if $i\neq j$ then $g^i(b_i)\wedge g^j(b_j)=0$. Suppose that $g^i(b_i)\wedge g^j(b_j)\neq 0$ with $i\neq j$ and suppose $i>j$ (the other case is similar). Let $0\neq t=g^i(b_i)\wedge g^j(b_j)$. Since $t\leq g^i(b_i)$ we have $0\neq g^{-i}(t)\leq b_i$. Let $0\neq r=g^{-i}(t)$.

From R3) $g^k(r)\wedge g^{k-1}(a\wedge b)\ldots\wedge g(a\wedge b)\wedge(a\wedge b)\neq 0$ for all $k=1,2,\ldots,i-1$.

Let $0\neq m_{i-1}=g^{i-1}(r)\wedge g^{i-2}(a\wedge b)\ldots\wedge g(a\wedge b)\wedge(a\wedge b)\leq a\wedge b$,

$$0\neq m_{i-2}=g^{-1}(m_{i-1})=g^{i-2}(r)\wedge g^{i-3}(a\wedge b)\ldots\wedge(a\wedge b)\wedge g^{-1}(a\wedge b)\leq a\wedge b,$$
$$0\neq m_{i-3}=g^{-1}(m_{i-2})=g^{i-3}(r)\wedge\ldots\wedge(a\wedge b)\wedge g^{-1}(a\wedge b)\wedge g^{-2}(a\wedge b)\leq a\wedge b,$$
$$\vdots$$
$$0\neq m_1=g^{-1}(m_2)=g(r)\wedge(a\wedge b)\wedge g^{-1}(a\wedge b)\wedge\ldots\wedge g^{-i+2}(a\wedge b)\leq a\wedge b,$$

$$0\neq s=g^{-1}(m_1)=r\wedge g^{-1}(a\wedge b)\wedge\ldots\wedge g^{-i+1}(a\wedge b)\leq r\quad\text{see Figure 5.}\quad (10)$$

Note that by construction $g^{i-j}(s)=m_{i-j}\leq a\wedge b$ and by (10) $g^i(s)\leq g^i(r)=$ $=g^i(g^{-i}(t))=t=g^i(b_i)\wedge g^j(b_j)\leq g^j(b_j)$ and thus $g^{i-j}(s)\leq b_j\leq b\setminus a$. Therefore $g^{i-j}(s)\leq(a\wedge b)\wedge(b\setminus a)=0$ and then $g^{i-j}(s)=0$ and $s=0$ wich contradict (10).

Therefore $g^i(b_i)\wedge g^j(b_j)=0$ for all $i\neq j$.

Thus we have proved that $b_1,b_2,b_3\ldots,g(b_1),g^2(b_2),g^3(b_3),\ldots,((\dot{\bigvee}_{i=1}^{\infty}b_i)\dot{\vee}(\dot{\bigvee}_{i=1}^{\infty}g^i(b_i)))^c$ is a partition of unity (note that $a\wedge b\leq((\dot{\bigvee}_{i=1}^{\infty}b_i)\dot{\vee}(\dot{\bigvee}_{i=1}^{\infty}g^i(b_i)))^c$).

Since (B,G) is a complete Boolean ambiguity algebra, the automorphism $\bar{g}$ defined by $\bar{g}|_{b_i}=g^i|_{b_i}$, $\bar{g}|_{g^i(b_i)}=g^{-i}|_{g^i(b_i)}$, and $\bar{g}|_{((\dot{\bigvee}_{i=1}^{\infty}b_i)\dot{\vee}(\dot{\bigvee}_{i=1}^{\infty}g^i(b_i)))^c}=id|_{((\dot{\bigvee}_{i=1}^{\infty}b_i)\dot{\vee}(\dot{\bigvee}_{i=1}^{\infty}g^i(b_i)))^c}$ belong to G as well.

Also $\bar{g}(a\wedge b)=id(a\wedge b)=a\wedge b$ and $\bar{g}(b)=\bar{g}((\dot{\bigvee}_{i=1}^{\infty}b_i)\dot{\vee}(a\wedge b))=$

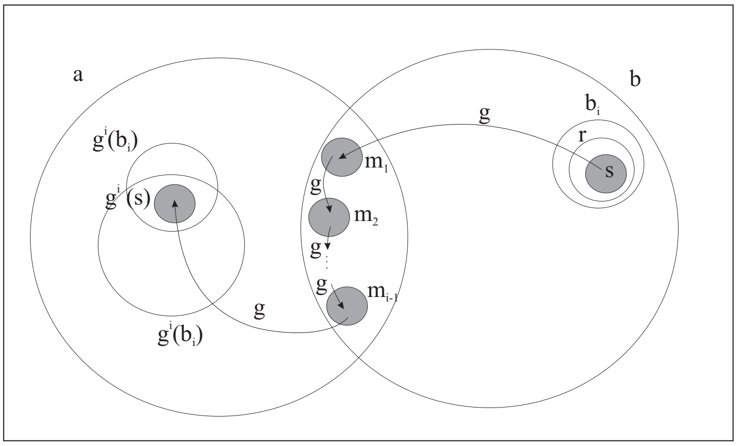

Figure 5:

$= (\dot{\bigvee}_{i=1}^{\infty} g^i(b_i))\dot{\vee} id(a \wedge b) \leq (\dot{\bigvee}_{i=1}^{\infty}(a \setminus b))\dot{\vee}(a \wedge b) = (a \setminus b)\dot{\vee}(a \wedge b) = a.$ $\square$

Proposition 5.3.2 Let (B, G) be a complete Boolean ambiguity algebra, then (B, G) is an MV-pair.

Proof.

MVP1 As Proposition 5.2.5 (we only must to use [17] Lemma 2.4 instead of [17] Lemma 4.3).

MVP2

Let (B, G) be a complete Boolean ambiguity algebra. From Lemma 3.2.5 it suffices to prove that for all $a, b \in B$ there exist $m \in max(L(a, b))$ such that $m \geq a \wedge b$.
Vetterlein, in [17] Section 2, make use of parts of theory developed by Kawada in [14]. Lemma 16 [14] and Lemma 2.7 [17], It shows that there is a pair $e, f \in B$ of disjoint elements wich are invariant under G and $g_1, g_2, g_3 \in G$ such that $g_1(a \wedge e) \leq b \wedge e$, $g_2(b \wedge f) \leq a \wedge f$ and $g_3(a \wedge (e \vee f)^c) = b \wedge (e \vee f)^c$. From Lemma 5.3.1 there is $\bar{g_2} \in G$ such that $\bar{g_2}(b \wedge f) \leq a \wedge f$ and $\bar{g_2}((b \wedge f) \wedge (a \wedge f)) = (b \wedge f) \wedge (a \wedge f) = a \wedge b \wedge f$.

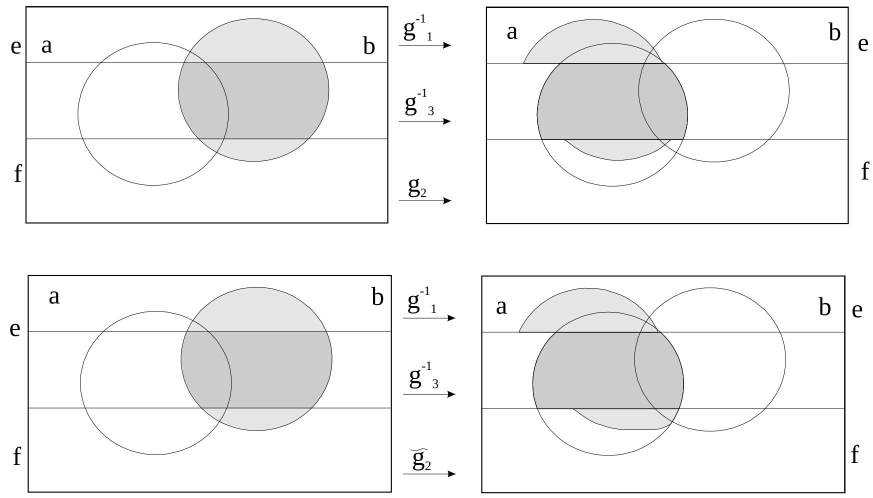

Figure 6:

Therefore we have

$$a \wedge e \leq g_1^{-1}(b \wedge e),\ a \wedge (e \vee f)^c = g_3^{-1}(b \wedge (e \vee f)^c), \tag{11}$$

$$\bar{g_2}(b \wedge f) \leq a \wedge f \tag{12}$$

$$\text{and } \bar{g_2}((b \wedge f) \wedge (a \wedge f)) = (b \wedge f) \wedge (a \wedge f) = a \wedge b \wedge f \tag{13}$$

(see Figure 6).

Since (B, G) is full and the elements e and f are invariant under G, the automorphism g defined by

$$g|_e = g_1^{-1}|_e, \quad g|_{(e \vee f)^c} = g_3^{-1}|_{(e \vee f)^c} \quad \text{and} \quad g|_f = \bar{g_2}|_f \quad \text{is in } G. \tag{14}$$

We call $b' = g(b)$. From (11), (12), (13) and (14) we have:

$$a \wedge b \wedge e \leq a \wedge e \leq g_1^{-1}(b \wedge e) = g(b \wedge e) =$$

$$= g(b) \wedge g(e) = g(b) \wedge e = b' \wedge e, \tag{15}$$

$$a \wedge b \wedge (e \vee f)^c \leq a \wedge (e \vee f)^c = g_3^{-1}(b \wedge (e \vee f)^c) = g(b \wedge (e \vee f)^c) =$$

$$= g(b) \wedge g((e \vee f)^c) = g(b) \wedge (e \vee f)^c = b' \wedge (e \vee f)^c \text{ ,and}$$

$$a \wedge b \wedge f = \bar{g_2}(a \wedge b \wedge f) \leq \bar{g_2}(b \wedge f) = g(b \wedge f) = g(b) \wedge g(f) = \tag{16}$$

$$= g(b) \wedge f = b' \wedge f,$$

Therefore $(a \wedge b \wedge e) \vee (a \wedge b \wedge (e \vee f)^c) \vee (a \wedge b \wedge f) \leq (b' \wedge e) \vee (b' \wedge (e \vee f)^c) \vee (b' \wedge f)$ that is $a \wedge b \leq b'$ and then $a \wedge b \leq a \wedge b'$.

We will prove that $|a \wedge b'| = |a| \wedge |b|$.
It is clear thah $|a \wedge b'| \leq |a|$ and $|a \wedge b'| \leq |b|$. Let $x \in B$ such that $|x| \leq |a|$ and $|x| \leq |b|$, then $\exists f_1, f_2 \in G$ such that $f_1(x) \leq a$ and $f_2(x) \leq b$. We have:

$f_1(x \wedge e) = f_1(x) \wedge f_1(e) = f_1(x) \wedge e \leq a \wedge e \leq b' \wedge e$ (by(15)) and thus

$$f_1(x \wedge e) \leq a \wedge b' \wedge e. \tag{17}$$

Let $f_3 = g \circ f_2$. Note that $e^c = (e \vee f)^c \vee f$ (since $e \wedge f = 0$) and e and f are invariant under G, then from (11),(12) and (14) we have that
$f_3(x \wedge e^c) = f_3(x) \wedge f_3(e^c) = f_3(x) \wedge e^c = g(f_2(x)) \wedge e^c = g(f_2(x)) \wedge ((e \vee f)^c \vee f) \leq g(b) \wedge ((e \vee f)^c \vee f) = (g(b) \wedge (e \vee f)^c) \vee (g(b) \wedge f) = g(b \wedge (e \vee f)^c) \vee g(b \wedge f) = g_3^{-1}(b \wedge (e \vee f)^c) \vee \bar{g_2}(b \wedge f) = (a \wedge (e \vee f)^c) \vee \bar{g_2}(b \wedge f) \leq (a \wedge (e \vee f)^c) \vee a \wedge f = a \wedge ((e \vee f)^c \vee f) = a \wedge e^c$. Furthermore $f_3(x \wedge e^c) \leq f_3(x) = g(f_2(x)) \leq g(b) = b'$. Therefore

$$f_3(x \wedge e^c) \leq a \wedge b' \wedge e^c. \tag{18}$$

Since G is full and e and f are invariant under G, the automorphism h defined by $h|_e = f_1|_e$ and $h|_{e^c} = f_3|_{e^c}$ is in G. Then by (17) and (18)
$h(x) = h((x \wedge e) \vee (x \wedge e^c)) = h(x \wedge e) \vee h(x \wedge e^c) = f_1(x \wedge e) \vee f_3(x \wedge e^c) \leq$
$\leq (a \wedge b' \wedge e) \vee (a \wedge b' \wedge e^c) = a \wedge b'$.
Therefore $|x| \leq |a \wedge b'|$ and then $|a \wedge b'| = |a| \wedge |b|$.

Finally, from Lemma 5.2.4, we have that $a \wedge b' \in max(L^+(a, b))$. $\square$

As previous section, let (B, G) be a Complete Boolean ambiguity algebra then,

(I) From Theorem 5.1.3, $(B_\sim, \boxplus, \neg, 0)$ is an MV-algebra. We call it $\mathcal{V}(B, G)$.

(II) From Proposition 5.3.2 (B, G) is an MV-pair and then, from Theorem 3.3.1, $M = (B_\sim, \oplus, 0, 1)$ is an MV-effect algebra . Therefore from Proposition 4.2.2, $M^{\mathcal{T}} = (B_\sim, \hat{\boxplus}, \hat{\neg}, 0)$ is an MV-algebra. We call it $\mathcal{J}(B, G)$.

Proposition 5.3.3 Let (B, G) be a normal Boolean ambiguity algebra and let the MV-algebras $\mathcal{V}(B, G)$ and $\mathcal{J}(B, G)$ as **(I)** and **(II)**.
Then $\mathcal{V}(B, G) = \mathcal{J}(B, G)$ and It are semisimple.

Proof. It is proved in exactly the same form that Proposition 5.2.6, Lemma 5.2.8, Corollary 5.2.9 and Proposition 5.2.10. $\square$

We will see now that if we build on a semisimple MV-algebra and obtain, through Proposition 4.2.2 and Theorem 3.3.3, an MV-pair it does not necessarily constitute a Complete Boolean ambiguity algebra.

Lemma 5.3.4 Let $C = [0, 1]$ the semisimple MV-algebra as examlpe 5.0.6 and $C^{\mathcal{P}}$ as example 5.0.7. Let $B(C^{\mathcal{P}})$ be the Boolean algebra R-generated by $C^{\mathcal{P}}$ then $B(C^{\mathcal{P}})$ is not $\sigma-$complete.

Proof. ([10] II.4 Lemma 25) Let $0 < x_1 < x_2 < \ldots < x_n < \ldots < 1$ (for example $x_n = \frac{n}{n+1}$, $n \in \mathbb{N}$) and let $a_n = x_1 + x_2 + \ldots + x_{2n}$, $n \in \mathbb{N}$. We claim that $\bigvee \{a_n, n \in \mathbb{N}\}$ does not exist. Indeed, let a be an upper bound for $\{a_n, n \in \mathbb{N}\}$. By example 1.3.14 we can represent a_n as $(x_1, x_2] \cup (x_3, x_4] \cup \ldots \cup (x_{2n-1}, x_{2n}]$ and a as $(a_1, a_2] \cup (a_3, a_4] \cup \ldots \cup (a_{2m-1}, a_{2m}]$ with $0 \leq a_1 < a_2 < \ldots < a_{2m} < 1$. Since a contains each a_n, there must exist an n and $j < m$ such that both $(x_{2n-1}, x_{2n}]$ and $(x_{2n+1}, x_{2n+2}]$ are contained in $(a_{2j-1}, a_{2j}]$. Therefore, the interval $(x_{2n}, x_{2n+1}]$ can be deleted from a, and it will still contain all the a_i, that is, $a + x_{2n+1} + x_{2n+2}$ is an upper bound for $\{a_n, n \in \mathbb{N}\}$ and
$a + x_{2n+1} + x_{2n+2} < a$. We conclude that $\{a_n, n \in \mathbb{N}\}$ does not have a least upper bound. $\square$

Corollary 5.3.5 Let $C^{\mathcal{P}}$ as above and let $(B(C^{\mathcal{P}}), G(C^{\mathcal{P}}))$ be the MV-pair as

Theorem 3.3.3, then $(B(C^{\mathcal{P}}), G(C^{\mathcal{P}}))$ is not a complete Boolean ambiguity algebra.

Proof. Lemma 5.3.4. □

5.4 Final remark

We have proved (Propositions 5.2.5 and 5.2.10) that if (B, G) is a normal Boolean ambiguity algebra, then (B, G) is an MV-pair and there is a semisimple MV-algebra $M^{\mathcal{T}} = (B_{\sim}, \hat{\boxplus}, \hat{\neg}, 0)$ arising from it. Following [6], we denote it $B_{\sim_G}$. Furthermore if M is a semisimple MV-algebra then as shown in Propositions 5.2.14, the pair $(B(M^{\mathcal{P}}), G(M^{\mathcal{P}}))$ is a normal Boolean ambiguity algebra (and thus an MV-pair). Following again [6], we denote it $(B(M), G(M))$.
We want to show that these constructions are functorial. The followings definitions and results are taken from [6].

Let (B_1, G_1) and (B_2, G_2) be MV-pairs, we say that ψ is a morphism of MV-pairs iff

(i) $\psi : B_1 \to B_2$ is a morphism of Boolean algebras.

(ii) For all $x, y \in B_1$, $x \sim_{G_1} y$ implies $\psi(x) \sim_{G_2} \psi(y)$.

(iii) For all $x, y \in B_1$ and $f_2 \in G_2$ there exists $f_1 \in G_1$ such that

$$|\psi(x) \wedge f_2(\psi(y))|_{G_2} \leq |\psi(x \wedge f_1(y))|_{G_2}.$$

The class of MV-pairs equipped with morphisms of MV-pairs forms a category $\mathcal{P}$.

It is proved that if M_1 and M_2 are MV-algebras then the map $\psi_M : B_{1\sim_{G_1}} \to B_{2\sim_{G_2}}$ given by $\psi_M(|x|_{G_1}) = |\psi(x)|_{G_2}$ is a morphism of MV-algebras. Moreover the map $\Delta : \mathcal{P} \to \mathcal{M}$ (where $\mathcal{M}$ is the category of MV-algebras) given by $\Delta((B, G)) = B_{\sim_G}$ and $\Delta(\psi) = \psi_M$ is a functor.
On the other hand using the fact [10] that all morphisms of bounded distributive lattices $\varphi : M_1 \to M_2$ uniquely extends to a homomorphism of Boolean

algebras $\varphi_B : B(M_1) \to B(M_2)$ (where $B(M_1)$ and $B(M_2)$ are the Boolean algebras R-generates by M_1 and M_2) it is proved in [6] that φ_B is a morphism between the MV-pairs $(B(M_1), G(M_1))$ and $(B(M_2), G(M_2))$, and the map $\nabla : \mathcal{M} \to \mathcal{P}$ given by $\nabla(M) = (B(M), G(M))$ and $\nabla(\varphi) = \varphi_B$ is a faithful functor.

Note that if M is an MV-algebra then $\Delta(\nabla(M)) = B(M)_{\sim_{G(M)}}$. Therefore from Theorem 3.3.3 $\Delta(\nabla(M)) = B(M)_{\sim_{G(M)}} \cong M$ and the map $\eta_M : M \to B(M)_{\sim_{G(M)}}$ defined by $\eta_M(x) = |x|_{G(M)}$ is an isomorphism of MV-algebras.

Furthermore it is proved in [6] that if $\psi : M_1 \longrightarrow M_2$ is a morphism of MV-algebras then the diagram

$$\begin{tikzcd} M_1 \arrow[r, "\eta_{M_1}"] \arrow[d, "\psi"'] & \Delta(\nabla(M_1)) \arrow[d, "\Delta(\nabla(\psi))"] \\ M_2 \arrow[r, "\eta_{M_2}"] & \Delta(\nabla(M_2)) \end{tikzcd}$$

commutes. Therefore $\eta : 1_{MV} \approx \Delta\nabla$ is a natural equivalence, where 1_{MV} is the identity functor on $\mathcal{M}$.

Let $\mathcal{N}$ denote the full subcategory of $\mathcal{P}$ whose objects are the normal Boolean ambiguity algebras, and let $\mathcal{S}$ denote the full subcategory of $\mathcal{M}$ whose objects are the semisimple MV-algebras. Propositions 5.2.10 and 5.2.14 show that we can consider the restrictions of the functors Δ and ∇ to $\mathcal{N}$ and $\mathcal{S}$. Formally:

> Let $(B, G), (B_1, G_1)$ and (B_2, G_2) be normal Boolean ambiguity algebras and let $\psi : (B_1, G_1) \longrightarrow (B_2, G_2)$ be a morphism of MV-pairs. We call $\tilde{\Delta}$ to the map $\tilde{\Delta} : \mathcal{N} \to \mathcal{S}$ given by $\tilde{\Delta}((B, G)) = B_{\sim_G}$ and $\tilde{\Delta}(\psi) = \psi_M$.
>
> Let M, M_1 and M_2 be semisimple MV-algebras and let $\varphi : M_1 \to M_2$ be a morpfism of MV-algebras. We call $\tilde{\nabla}$ to the map $\tilde{\nabla} : \mathcal{S} \to \mathcal{N}$ given by $\tilde{\nabla(M)} = (B(M), G(M))$ and $\tilde{\nabla}(\varphi) = \varphi_B$.

We immediately obtain that $\tilde{\Delta} : \mathcal{N} \to \mathcal{S}$ is a functor, $\tilde{\nabla} : \mathcal{S} \to \mathcal{N}$ is a faithful functor and $\eta : 1_S \approx \tilde{\Delta}\tilde{\nabla}$ is a natural equivalence, where 1_S is the identity functor on $\mathcal{S}$.

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