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MÉTODOS CATEGÓRICOS EN TEORÍA DE MODELOS

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1 Acerca de este trabajo

1.1 Objetivos principales

El objetivo de este trabajo es exponer distintas herramientas de la teoría de categorías para proveer demostraciones de algunos resultados metamatemáticos clásicos en teoría de modelos. En primer lugar, se presenta una demostración del conocido teorema de completitud de Gödel para la lógica clásica (el cual ocupará un lugar preponderante) a través de métodos categóricos, ideada por André Joyal. Usando algunas de estas ideas se proporciona también demostraciones inéditas del teorema de Löwenheim-Skolem y del criterio conocido como test de Vaught para teorías completas.

La demostración aquí expuesta del teorema de completitud se basa en una serie de exposiciones llevadas a cabo por Joyal en 1978 en Montreal, en las que daba cuenta de cómo las técnicas categóricas usuales permitían formular una correcta interpretación del teorema y su demostración en un lenguaje categórico. La conexión entre la teoría de categorías y la lógica clásica había sido observada con anterioridad, y ya en la tesis doctoral de William Lawvere de 1963 se explica como la teoría de modelos podía beneficiarse con un enfoque funtorial de la semántica. Es con las ideas de Joyal que se logró unificar los acercamientos a distintos teoremas de completitud enunciados para lógicas diversas.

Se ha elegido para este trabajo la lógica clásica de primer orden por varios motivos. En primer lugar, el teorema de completitud para tal lógica, que proporciona el puente entre la semántica y la sintaxis, es un teorema clásico y célebre; en segundo lugar, la lógica clásica es el cimiento principal del desarrollo del edificio matemático, estando basadas en ella las más conocidas axiomatizaciones de la teoría de conjuntos y de otros desarrollos matemáticos. Por último, la lógica clásica ha sido caracterizada a través de ciertos resultados como la más fuerte (en un sentido preciso) entre aquellas que satisfacen determinadas propiedades de uso frecuente en teoría de modelos.

Siendo el teorema de completitud, el teorema de Löwenheim-Skolem y el test de Vaught resultados metamatemáticos, como todo aquel que verse sobre teoría de modelos, la práctica común es usar la propia teoría de conjuntos como el ámbito metamatemático adecuado para tratar el estudio de las teoría lógicas y sus modelos. Dentro de las axiomatizaciones más conocidas de la teoría de conjuntos, el sistema axiomático de Zermelo-Fraenkel sumado al Axioma de Elección es el más ampliamente usado, y es en este contexto que se inscriben dichos teoremas, debiendo sus enunciados ser entendidos como teoremas dentro de tal sistema. Por otro lado, nuestra intención ha sido realizar este trabajo teniendo en cuenta el rol que juega el propio Axioma de Elección, mostrando, por ejemplo, como el teorema de completitud se trata de un resultado no constructivo, en el sentido de que sólo los axiomas de Zermelo-Fraenkel no son suficientes para derivarlo. Ya Leon Henkin había demostrado en 1954 la equivalencia entre el teorema de completitud y el teorema que afirma la existencia de ideales primos en álgebras de Boole, que es de por sí un resultado no constructivo. Hemos decidido, pues, exponer la demostración de Joyal de manera de poder inferir claramente tal equivalencia, cuyo interés reside principalmente en el hecho de que el teorema del ideal primo es estrictamente más débil que el Axioma de Elección. Se obtiene también como corolario el teorema de compacidad para la lógica clásica de primer orden, así como el teorema de la existencia de modelos, ambos equivalentes también al teorema del ideal primo. Finalmente, haciendo uso de algunas construcciones efectuadas en la demostración del teorema de completitud, se exponen por último las demostraciones nuevas de los teoremas de Löwenheim-Skolem y del test de Vaught para teorías completas.

En cuanto a los fundamentos, adoptamos aquí la axiomatización conjuntista de la teoría de categorías, pero evitando hacer uso explícito de los universos de Grothendieck. Los pasajes en los que se mencionen categorías grandes se introducen de manera de articular el desarrollo del trabajo sin oscurecer el hilo del discurso con tecnicismos irrelevantes, pero el lector puede interpretar correctamente en su contexto las descripciones allí dadas sin necesidad de apelar a universos.

1.2 Descripción del trabajo

Salvo la presente introducción, este trabajo se desarrolla enteramente en idioma inglés. Luego de presentar en la sección 2 una breve introducción al enunciado y significación del teorema de completitud de Gödel, basada en su tesis doctoral original (1929), se muestra como es posible probar su equivalencia con el teorema del ideal primo una vez que se supone que éste es suficiente para deducir aquel, lo que ocupará las siguientes tres secciones.

La sección 3 está destinada a presentar en detalle los resultados propios de la teoría de categorías necesarios para comprender las ideas de Joyal. Se supone que el lector está ya familiarizado con las nociones categóricas básicas (categorías, funtores, transformaciones naturales, propiedades universales, adjunciones, límites, colímites), y se procede a desarrollar los lemas específicos requeridos para la demostración del teorema de completitud. Se repasan los resultados usuales asociados al funtor pullback, y se definen y desarrollan los conceptos de categorías regulares y booleanas, que jugarán un papel importante en la prueba. A continuación se describen las principales propiedades de los colímites filtrantes, en especial su relación con la propiedad de exactitud de la categoría de conjuntos. Se desarrolla seguidamente la noción de pseudofuntor y de bicolímite, y por último, se expone la noción de categoría fibrada, debida a Grothendieck, procediendo a demostrar los principales resultados asociados con sus construcciones usuales.

En la sección 4 se repasan conceptos específicos de la teoría de modelos, de los que se supone que el lector tiene ciertas nociones, y se procede luego a definir y desarrollar la interpretación categórica de las teorías. Esta parte esencial provee el vínculo entre la lógica clásica de primer orden y las categorías booleanas, y muestra cómo el estudio de estas categorías permite obtener resultados relacionados con la expresividad y el alcance demostrativo de las teorías lógicas.

La sección 5 está destinada a la exposición de la prueba de Joyal, usando los conceptos y desarrollos de las dos secciones anteriores. Se comienza por observar el carácter funtorial de la semántica y su uso en la caracterización categórica de la completitud, interpretando los modelos de determinadas teorías como funtores de ciertas categorías asociadas con valores en la categoría de conjuntos. Se exponen luego diversas construcciones ulteriores que permiten identificar una importante clase de modelos de las teorías lógicas, las cuales son utilizadas para llevar a cabo la demostración de la completitud.

Finalmente, la sección 6 contextualiza la prueba de la sección 5 al proveer una caracterización funtorial de todos los modelos de una determinada teoría. A diferencia de las otras secciones, se hace uso aquí del Axioma de Elección para poder demostrar correctamente estos resultados. Se explica también de qué manera puede usarse esta caracterización funtorial para derivar el teorema de Löwenheim-Skolem, lo que proporciona una prueba inédita de este resultado a través de métodos categóricos y justifica además el carácter no constructivo de las ideas que en esta sección se desarrollan. Por último, se expone una demostración también inédita del criterio conocido como test de Vaught, que permite enunciar condiciones suficientes para que una teoría de primer orden sea completa.

2 Introduction

2.1 Completeness

In his doctoral dissertation (see [3]), Kurt Gödel presented a proof of one of the most celebrated results in classical logic, the completeness theorem. This achievement generalized previous results by Paul Bernays (and present in the work of Hilbert and Ackermann) on the completeness of connective calculus to a wider important class of formulas containing quantifiers (those of first order classical logic), therefore establishing an important connection between semantic truth and syntactic provability. The theorem asserts that a formula which is valid in every model of classical logic is necessarily provable from its axioms using only certain specific rules of inference. Although the soundness of first order logic (i.e., the property that a provable formula is valid in every model) is easily verified by induction on the complexity of the formula in question, the converse result (that is, completeness) is not as direct. In fact, its proof heavily relies on the specific axiomatic context used to establish it, as there are non constructive aspects that are crucial for the result to hold.

By a model we mean a $\mathcal{S}et$-valued non trivial one, that is, a nonempty set interpreting the formulas of the theory in the usual way (see section 4), in which all logical axioms hold. By a first order theory we shall always refer to a theory within first order classical logic. Assuming we are working in Zermelo-Fraenkel set theory, ZF, some form of choice principle is needed to deduce completeness. Gödel's original result was intended for theories based on first order logic allowing a countable set of formulas (besides those of logic). A stronger version that we shall prove here holds, where there is no restriction on the cardinality of the set of formulas, namely:

Theorem 2.1.1. Completeness theorem*: Given a first order theory T, if a formula ϕ is valid in every model of T, then it is provable from the axioms of T (including first order logical axioms).*

It is easy to prove that the Completeness theorem implies the Boolean Prime Ideal theorem (BPI) in ZF, which is a principle derived from the Axiom of Choice but strictly weaker than it, although still independent of ZF (see for example [6]). To see this, we proceed by proving the following chain of implications deduced from completeness:

Theorem 2.1.2. Model existence theorem*: A first order theory is consistent if and only if it has a model.*

Proof. If the theory has a model and a contradiction could be derived from the theory, then $\perp$ would be, by soundness, valid in that model, which is absurd; therefore, the theory must be consistent. Conversely, suppose the theory is consistent. If it had no models, then $\perp$ would be trivially valid in every model, so by completeness it would be provable, which contradicts the fact that the theory is supposed to be consistent. Therefore, it must have a model. $\square$

As an immediate consequence we get to the important:

Theorem 2.1.3. Compactness theorem*: A first order theory consisting of an infinite number of formulas has a model if and only if every subtheory consisting of a finite number of such formulas has a model.*

Proof. By the Model existence theorem, the theory has a model if and only if it is consistent, which is true if and only if every subtheory with finite formulas is consistent. The result then follows through a second application of the Model existence theorem. $\square$

Finally, we get:

Theorem 2.1.4. *The Compactness theorem implies BPI.*

Proof. Let B be a Boolean algebra and let $\mathcal{L}$ be a language having a constant term for each element $a \in B$ (which we may identify) and a unary predicative variable I. Consider the theory given by the following formulas:

$$I(0), \neg I(1)$$

$$I(a) \vee I(\neg a)$$

for each $a \in B$, and

$$\bigwedge_{i=1}^{k} I(a_i) \to I\left(\bigvee_{i=1}^{k} a_i\right)$$

for all $a_1, ..., a_k \in B$. Then any finite set of formulas has a model, since they involve only a finite number of elements of B, which generate thus a finite subalgebra where BPI can be proved to hold (in ZF). By the Compactness theorem, the whole theory has a model, which yields in turn the prime ideal for B. □

The considerations above motivate the search of a proof of the Completeness theorem that uses only BPI, since this will therefore establish the equivalence of both theorems over ZF and will prove that the appeal to some choice principle is not superfluous. This is one of the virtues of the proof we shall expose in section 5, the main section of this work, for which only BPI is needed.

2.2 The axioms

We will now give a brief description of the axiom system and rules of inference used in [3]. Three primitive symbols are used ($\neg, \vee, \forall$) and others are defined as usual ($\wedge, \to, \leftrightarrow, \exists$). Individual variables are referred to by using small letters $x, y, z, ...$ (individual variables and constants may belong to different sorts), while capital letters $X, Y, Z, ...$ denote either sentential variables (if alone) or predicative variables (if followed by individual variables). The axioms include the four axioms of connective calculus, two axioms handling quantifiers and two axioms handling identity:

1) $X \vee X \to X$
2) $X \to X \vee Y$
3) $X \vee Y \to Y \vee X$
4) $(X \to Y) \to (Z \vee X \to Z \vee Y)$
5) $\forall x P(x) \to P(y)$
6) $\forall x(X \vee P(x)) \to X \vee \forall x P(x)$
7) $x = x$
8) $x = y \to (F(x) \to F(y))$

There are as well four rules of inference specified:

A) The inference scheme: from α and $\alpha \to \beta$ we can infer β
B) The substitution rule for sentential and predicative variables

C) From the formula $\phi(x)$ we can infer $\forall x \phi(x)$
D) Individual variables (either free or bounded) can be replaced by any other variables as long as renaming them does not change the reach of already existing quantifiers

Gödel then proceeds to prove the completeness theorem first for a special class of formulas and then deduces the general case. The proof we shall expose here follows different arguments, and it is based on methods in categorical logic that were developed a few decades after Gödel's first proof was published. These methods are based on the use of category theory to explore the concepts of model theory, and were initiated by Lawvere in his Phd. thesis [9], where he introduced the concept of functorial semantics. The idea was exploited mainly by Joyal, who inspired by Henkin's proof of the Completeness theorem, considered generalized models of theories that are not necessarily $\mathcal{S}et$-valued but instead have corresponding interpretations inside appropriate categories. As we shall see, it is possible to construct a category where first order logic can be interpreted, making use of certain functors from this category to $\mathcal{S}et$ to provide the usual semantics for the theory. We shall start in section 3 with some usual concepts and results that are known and will play an important role in the next sections. Section 4 will be devoted to the construction of a special category known as the syntactic category for first order theories, as well as a categorical model inside it, as exposed in [8]. Finally, section 5 will describe in detail the argument of Joyal's proof of Gödel's completeness theorem, and is based on a series of lectures he gave in 1978. Although these lectures were unpublished, the ideas were circulated amongstst his students and colleagues, and some related developments can be found, for example, in [8] and [13].

3 Categorical preliminaries

3.1 Pullbacks

Amongst the finite limits on a category, pullbacks occupy an important place. Recall that in a category $\mathcal{C}$, the pullback of an arrow $g : C \to B$ along an arrow $f : A \to B$ is a commutative square as shown below satisfying the following universal property: for every pair of arrows $j : Q \to A$, $k : Q \to C$ such that $gk = fj$, there exists a unique induced morphism $l : Q \to P$ such that $g'l = j$ and $f'l = k$:

$$\begin{tikzcd}
Q \arrow[dr, dashed, "l"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "j"'] & & \\
& P \arrow[r, "f'"] \arrow[d, "g'"'] & C \arrow[d, "g"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

The universal property above allows to define the greatest lower bound of two subobjects of an object X as the pullback of the corresponding monics; therefore, the poset $\mathcal{S}ub(X)$ has intersections ($\wedge$) provided the category has pullbacks.

Given a category $\mathcal{C}$, we can define for an object A in $\mathcal{C}$ the slice category $\mathcal{C}/A$ whose objects are arrows $f : C \to A$ and whose arrows are morphisms $g : C \to C'$ making commutative the triangle below:

$$
\begin{tikzcd}
C \arrow[rr, "g"] \arrow[dr, "f"'] & & C' \arrow[dl, "f'"] \\
& A &
\end{tikzcd}
$$

Pullbacks give rise to functors between slice categories in the following way. Given $f : A \to B$, define the pullback functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ acting on an object $g : C \to B$ by simply taking its pullback along f, while it acts on an arrow $i : C' \to C$ by assigning to it the unique induced morphism $f^*(C') \to f^*(C)$:

$$
\begin{tikzcd}
f^*(C') \arrow[rr, "f''"] \arrow[dr, dashed, "f^*(i)"] \arrow[dd, "h'"'] & & C' \arrow[dr, "i"] \arrow[dd, "h"] & \\
& f^*(C) \arrow[rr, "f'"] \arrow[dl, "g'"] & & C \arrow[dl, "g"] \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}
$$

(Note that in the diagram above all three squares are pullbacks).

A certain property of an arrow $g : C \to B$ is said to be stable under pullbacks (or under base change) if each time g has that property, for every $f : A \to B$ the corresponding arrow $g' : f^*(C) \to A$ in the pullback diagram also has the same property. Monomorphisms are stable under pullback, which leads to the following:

Definition 3.1.1. *The functor $f^{-1} : \mathcal{S}ub(B) \to \mathcal{S}ub(A)$ is the restriction $f^*|_{\mathcal{S}ub(B)}$ of the pullback functor.*

Remark 3.1.2. *Note that since monics are stable under pullbacks, the restriction of the domain to $\mathcal{S}ub(B)$ implies the restriction of the image to $\mathcal{S}ub(A)$.*

Pullbacks can be used to characterize monomorphisms, as in the following:

Lemma 3.1.3. *An arrow $f : A \to B$ is a monomorphism if and only if the diagram on the left is a pullback:*

$$
\begin{tikzcd}
A \arrow[r, "Id_A"] \arrow[d, "Id_A"'] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[dr, dashed, "\Delta"] \arrow[drr, bend left, "Id_A"] \arrow[ddr, bend right, "Id_A"'] & & \\
& P \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}
$$

Equivalently, f is a monomorphism is and only if the diagonal morphism Δ to the pullback P on the right is an isomorphism.

Proof. Clearly, the second assertion follows from the first one. To prove the first assertion, we just need to realize that if $m, n : C \to A$ satisfy $fm = fn$, then the diagram on the left is a pullback if and only if there is an induced morphism $l : C \to A$ such that $m = l = n$, i.e., if and only if f is monic. □

Another characterization of monomorphisms that uses pullbacks and is sometimes useful is the following:

Lemma 3.1.4. *An arrow $f : A \to B$ is a monomorphism if and only if the following square is a pullback:*

$$\begin{tikzcd} A \arrow[r, tail, "\Delta"] \arrow[d, "f"'] & A \times A \arrow[d, "f\times f"] \\ B \arrow[r, tail, "\Delta"] & B \times B \end{tikzcd}$$

Proof. Consider the kernel pair of f (i.e., the pullback of f along itself), $\pi_1, \pi_2 : R_f \rightrightarrows A$. Then we can see that the following square is a pullback:

$$\begin{tikzcd} R_f \arrow[r, tail, "(\pi_1,\pi_2)"] \arrow[d, "f\pi_1=f\pi_2"'] & A \times A \arrow[d, "f\times f"] \\ B \arrow[r, tail, "\Delta"] & B \times B \end{tikzcd}$$

Indeed, given a pair of morphisms $(g, h) : C \to A\times A$ and $p : C \to B$ satisfying $(f\times f)(g, h) = \Delta g$, we must have $fg = fh = p$, and hence there is a morphism $q : C \to R_f$, induced by the universal property of the pullback R_f (of f along itself), that satisfies $\pi_1 q = g$ and $\pi_2 q = h$. But these two equalities are equivalent to the equality $(\pi_1, \pi_2)q = (g, h)$, and since $f\pi_2 q = fh = p$, we see that q is the required induced morphism in the pullback above, and moreover, it is necessarily the unique such possible morphism.

Finally, to prove the lemma we just need to note that, according to lemma 3.1.3, f is a monomorphism if and only if $R_f = A$ and $\pi_1 = \pi_2 = Id_A$. But this implies precisely the statement we wanted to prove. □

One important fact about pullback functors is stated in the following:

Lemma 3.1.5. *If the category $\mathcal{C}$ has pullbacks, then the functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ has a left adjoint $\Sigma_f : \mathcal{C}/A \to \mathcal{C}/B$.*

Proof. In the special case where $B = 1$, the terminal object of $\mathcal{C}$, we have $\mathcal{C}/1 \cong \mathcal{C}$ and the pullback functor is just the functor $(-) \times A : \mathcal{C} \to \mathcal{C}/A$, sending each object C into the object $\pi_2 : C \times A \to A$ of $\mathcal{C}/A$. By the universal property of the product, it is easy to see that a left adjoint for this functor is given by the forgetful functor $\Sigma_f : \mathcal{C}/A \to \mathcal{C}$ which applies the object $g : C \to A$ in $\mathcal{C}/A$ into the object C. The general case follows now easily by noting that the arrow $f : A \to B$ is also an object (f) in the slice category $\mathcal{C}/B$ and we have $(\mathcal{C}/B)/(f) \cong \mathcal{C}/A$. Therefore, the corresponding pullback functor is $f^* = (-) \times (f) : \mathcal{C}/B \to (\mathcal{C}/B)/(f)$, which reduces to the previous case. □

3.2 Boolean categories

In the next sections we shall extend the notion of models using appropriate categories other than $\mathcal{S}et$, where first order theories can have an appropriate interpretation. This interpretation will be possible only if the category considered is complex enough to support the complexity of first order language. As we shall see, the right context to interpret first order theories is that of Boolean categories (see [13], ch. 1).

Definition 3.2.1. *A regular category is a category having the following three properties:*
1) It has all finite limits.
2) Every arrow $f : A \to B$ can be factored as $f : A \twoheadrightarrow C \rightarrowtail B$, where C, called the image of f, is the least subobject of B through which f can factor. The arrow $f : A \twoheadrightarrow C$ not factoring through any proper subobject of C is called a cover.
3) Images are stable under base change (i.e., pullbacks preserve covers).
A regular category is said to be Boolean if it also satisfies the following two conditions:
4) The poset $\mathcal{S}ub(X)$ of subobjects of a given object X has finite unions and these are stable under pullbacks.
5) Every subobject A in the poset $\mathcal{S}ub(X)$ has a complement, i.e., there exists a subobject B such that the intersection $A \wedge B$ is initial in $\mathcal{S}ub(X)$ and $A \vee B = X$ (in particular, $\mathcal{S}ub(X)$ is a Boolean algebra, and we denote $B = \neg A$).

Remark 3.2.2. *It follows from the definition that in a Boolean category complements are preserved by pullback functors f^{-1}, that is, $f^{-1}(\neg A) \cong \neg f^{-1}(A)$.*

Definition 3.2.3. *A functor between regular categories is regular provided it preserves finite limits and images factorizations. A regular functor between Boolean categories is called Boolean if it also preserves unions and complements.*

Boolean categories are specially adequate to interpret first order logic because all logical connectives have a definite meaning in it, due to the Boolean structure of the poset of subobjects of a given object. For example, since it has finite limits, the intersection $S \wedge S'$ of two subobjects of A is given by simply taking the pullback of the corresponding monics. Furthermore, as first observed by Lawvere, quantifiers have also a categorical interpretation in this context, which we mention in the following:

Lemma 3.2.4. *For every arrow $f : A \to B$ in a regular category $\mathcal{C}$, the functor $f^{-1} : \mathcal{S}ub(B) \to \mathcal{S}ub(A)$ has a left adjoint. Furthermore, if $\mathcal{C}$ is Boolean, then f^{-1} also has a right adjoint.*

Proof. Suppose $\mathcal{C}$ is regular. Now, $i_B : \mathcal{S}ub(B) \hookrightarrow \mathcal{C}/B$ has an image $Im(i_B)$, and the property of being the least subobject of its codomain through which i_B can factor says exactly that $Im : \mathcal{C}/B \to \mathcal{S}ub(B)$ is left adjoint to i_B.

$$
\begin{tikzcd}
\mathcal{C}/B \arrow[rr, "f^*"] \arrow[dd, bend left, "Im"] & & \mathcal{C}/A \arrow[ll, bend left, "\Sigma_f"] \\
& & \\
\mathcal{S}ub(B) \arrow[uu, hook, "i_B"] \arrow[rr, "f^{-1}"] & & \mathcal{S}ub(A) \arrow[uu, hook, "i_A"'] \arrow[ll, bend left, "\exists_f"]
\end{tikzcd}
$$

Therefore, we can take the left adjoint of f^{-1}, $\exists_f$, to be defined by the following composition:

$$
\begin{tikzcd}
\mathcal{S}ub(A) \arrow[r, hook, "i_A"] & \mathcal{C}/A \arrow[r, "\Sigma_f"] & \mathcal{C}/B \arrow[r, "Im"] & \mathcal{S}ub(B)
\end{tikzcd}
$$

where Σ_f is the left adjoint of $f^* : \mathcal{C}/B \to \mathcal{C}/A$.

Now suppose $\mathcal{C}$ is also Boolean. Then, the fact that each subobject has a complement defines an operation assigning to each subobject $A \rightarrowtail X$ its complement $\neg A$ in $\mathcal{S}ub(A)$. This amounts to having an idempotent contravariant endofunctor $\neg$ in $\mathcal{S}ub(A)$. Because of remark 3.2.2, we deduce that the functor $f^{-1} : \mathcal{S}ub(B) \to \mathcal{S}ub(A)$ also has a right adjoint $\forall_f$. Indeed, we can take $\forall_f = \neg\exists_f\neg$, which will necessarily be right adjoint to f^{-1} since $\neg$ is a (contravariant) isomorphism. $\square$

Note that any cover $A \twoheadrightarrow B$ is necessarily an epimorphism, for if f equalizes a pair of arrows of domain B, it would factor through their equalizer. Moreover, as the following proposition shows, covers are closed under composition:

Lemma 3.2.5. *In a regular category, the following holds:*
a) The composition of covers is a cover.
b) If $A \twoheadrightarrow 1, B \twoheadrightarrow 1$ are covers, then $A \times B \twoheadrightarrow 1$ is a cover.
c) If $A_i \twoheadrightarrow 1$ are covers for $i = 1, ..., n$, and $m \leq n$, then the canonical arrow $\prod_{i=1}^n A_i \twoheadrightarrow \prod_{i=1}^m A_i$ is a cover.

Proof. a) By definition, we have that an arrow $h : X \to Y$ is a cover if and only if $\exists_h(Y) = X$. Let $g : A \twoheadrightarrow B, f : B \twoheadrightarrow C$ be covers. Therefore, $\exists_g(B) = A$ and $\exists_f(C) = B$. But then $\exists_{fg}(C) = \exists_g\exists_f(C) = \exists_g(B) = A$, and fg is a cover.
b) It suffices to note that $A \times B \twoheadrightarrow 1$ is the composition of covers $A \times B \twoheadrightarrow A \twoheadrightarrow 1$, where $A \times B \twoheadrightarrow A$ is the pullback of the cover $B \twoheadrightarrow 1$ along $A \twoheadrightarrow 1$ (and is therefore a cover itself).
c) The given arrow is the pullback of $\prod_{i=m+1}^n A_i \twoheadrightarrow 1$ along $\prod_{i=1}^n A_i \twoheadrightarrow 1$, and is hence a cover, since the morphism $\prod_{i=m+1}^n A_i \twoheadrightarrow 1$ is a cover by b). $\square$

We should also mention that condition 2) of definition 3.2.1 has an equivalent form; instead of requiring that in the composition $f : A \twoheadrightarrow C \rightarrowtail B$ the first morphism be a cover, it can be required that it is a regular epimorphism, i.e., an epimorphism that occurs as a coequalizer. Indeed, as shown in [8], A1.3, we have:

Lemma 3.2.6. *In a regular category $\mathcal{C}$, the arrow $f : A \to B$ is a cover if and only if it is a regular epimorphism.*

Proof. Clearly, if f is a regular epimorphism, it is also a cover, because every factorization of it through a subobject of its codomain can be easily seen to be a coequalizer of the same pair of morphisms. Conversely, suppose f is a cover and let $a, b : R \rightrightarrows A$ be its kernel-pair (i.e., the pullback of f along itself). We shall prove that f is the coequalizer of a and b. For that purpose, let $c : A \to C$ be a morphism satisfying $ca = cb$ and consider the image factorization:

$$A \xrightarrow{d} D \xrightarrow{(g,h)} B \times C$$

of $(f, c) : A \to B \times C$. We shall prove that g is an isomorphism, so that $hg^{-1} : B \to C$ is a factorization of c through f (which will be clearly unique, since covers are epimorphisms). Because f and factors through g and f is a cover, g will be a cover as well, so it suffices to prove that it is monic. Suppose then that $k, l : E \to D$ are such that $gk = gl$. Form the pullback:

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "(m,n)"'] & E \arrow[d, "(k,l)"] \\ A \times A \arrow[r, "d\times d"] & D \times D \end{tikzcd}$$

We have $fm = gdm = gkp = glp = gdn = fn$, so m, n factor through a, b respectively by a morphism $q : P \to R$. Therefore, we have $hkp = hdm = cm = caq = cbq = cn = hdn = hlp$. Now, $d \times d$ is a cover, since it is the composite of the morphisms:

$$A \times A \xrightarrow{Id_A \times d} A \times D \xrightarrow{d \times Id_D} D \times D$$

both of which are pullbacks of d (and therefore covers). Hence, p is a cover, and in particular it is an epimorphism. This implies that $hk = hl$, and hence, that $(g, h)k = (g, h)l : E \to B \times C$. Since (g, h) is monic, we must have $k = l$, which completes the proof. □

Definition 3.2.7. *A finite coproduct of objects $\coprod_{i=1}^n A_i$ is said to be disjoint if the injections into the coproduct $A_i \rightarrowtail \coprod_{i=1}^n A_i$ are monomorphisms and their intersection is the initial subobject of $\coprod_{i=1}^n A_i$.*

The following lemma will be needed in section 5:

Lemma 3.2.8. *Regular (resp. Boolean) categories are stable under slicing; that is, if $\mathcal{C}$ is a regular (resp. Boolean) category and A is an object of C, then $\mathcal{C}/A$ is again regular (resp. Boolean). Moreover, if $\mathcal{C}$ has finite disjoint coproducts, then so does $\mathcal{C}/A$.*

Proof. $\mathcal{C}/A$ clearly has a terminal object. To see it has pullbacks if $\mathcal{C}$ has finite limits, note that for a pullback diagram we can take the pullback in $\mathcal{C}$ of the objects involved together with the evident arrow to A and verify at once that it is a pullback in $\mathcal{C}/A$. This shows $\mathcal{C}/A$ has finite limits. Now $\mathcal{C}/A$ will have images, unions and complements of subobjects respectively provided $\mathcal{C}$ does, for images, unions and complements in $\mathcal{C}/A$ are clearly those of $\mathcal{C}$ together with the obvious arrows to A. This implies, in turn, that covers and unions are stable under pullbacks if they already are in $\mathcal{C}$. Finally, to check that $\mathcal{C}/A$ has finite coproducts if $\mathcal{C}$ does, note that the universality of the coproduct in $\mathcal{C}$ of a set of objects from $\mathcal{C}/A$ endows it with an arrow to A, thus forming the coproduct in $\mathcal{C}/A$, which inherits then its disjointness. □

Definition 3.2.9. *A functor $F : \mathcal{C} \to \mathcal{D}$ is said to reflect isomorphisms when given an arrow $f : A \to B$ in $\mathcal{C}$, if the arrow $F(f) : F(A) \to F(B)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.*

Functors that reflects isomorphisms are called conservative. Basic is the following:

Lemma 3.2.10. *If $\mathcal{C}$ is a regular category and a regular functor $F : \mathcal{C} \to \mathcal{D}$ reflects all isomorphisms $F(f) : F(A) \to F(B)$ corresponding to monic arrows f, then F is conservative and faithful.*

Proof. Let us first prove that F is faithful. Given two arrows $f, g : A \to B$ such that $F(f) = F(g)$, take their equalizer $m : E \to A$, which is a monic. Since F preserves finite limits, $F(m)$ is the equalizer of $F(f)$ and $F(g)$, and therefore $F(E) \cong F(A)$. This implies that $E \cong A$ and then $f = g$.

Now suppose $f : A \to B$ is any arrow in $\mathcal{C}$ such that $F(f) : F(A) \to F(B)$ is an isomorphism. Consider the image factorization $f : A \twoheadrightarrow C \rightarrowtail B$; then the monic $F(C) \rightarrowtail F(B)$ must be a cover, and hence is an isomorphism. This means that C and B are isomorphic and therefore f is a cover. To prove it is an isomorphism, it remains to show that it is also monic. But if $a, b : C \to A$ are two morphisms such that $fa = fb$, then $F(f)F(a) = F(f)F(b)$, which implies that $F(a) = F(b)$ and therefore $a = b$. $\square$

Lemma 3.2.11. *For every cover $f : A \to B$ in a regular (resp. Boolean) category $\mathcal{C}$, the pullback functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ is conservative and regular (resp. Boolean).*

Proof. Suppose $\mathcal{C}$ is regular. Since pullback functors have left adjoint, they preserve limits. If f is a cover, then f^* clearly reflects isomorphisms that correspond to monic arrows in $\mathcal{C}/B$, and conservativity follows immediately from lemma 3.2.10. Finally, from 4) and 5) in definition 3.2.1 and remark 3.2.2, it follows that pullback functors must preserve unions and complements, which finishes the proof. $\square$

Finally, we state the following basic result that hold in regular categories:

Lemma 3.2.12. *In a regular category, if two subobjects $B_1 \rightarrowtail X, B_2 \rightarrowtail X$ are disjoint, then their union is as well their coproduct.*

Proof. This proof is mainly due to Joyal, and appears in [13], ch. 1. The lemma follows from a more general result stating that the following square in $\mathcal{S}ub(X)$ is a pushout in $\mathcal{C}$:

$$\begin{tikzcd}
B_1 \wedge B_2 \arrow[r, tail, "s"] \arrow[d, tail, "t"'] & B_2 \arrow[d, "j_2"] \\
B_1 \arrow[r, "j_1"] & B_1 \vee B_2
\end{tikzcd}$$

The special case where B_1, B_2 are disjoint makes this pushout a coproduct. Call $A = B_1 \vee B_2$ and suppose we are given the following commutative diagram:

$$\begin{tikzcd}
B_1 \wedge B_2 \arrow[r, tail, "s"] \arrow[d, tail, "t"'] & B_2 \arrow[d, "f_2"] \\
B_1 \arrow[r, "f_1"] & Z
\end{tikzcd}$$

Let $\Gamma_{f_1} : B_1 \to B_1 \times Z$ be the graph of $f_1 : B_1 \to Z$ and define Γ_{f_2} similarly. Consider the following compositions:

$$\begin{tikzcd} B_1 \arrow[r, tail, "\Gamma_{f_1}"] & B_1 \times Z \arrow[r, tail, "j_1 \times Id_Z"] & A \times Z \end{tikzcd}$$

$$\begin{tikzcd} B_2 \arrow[r, tail, "\Gamma_{f_2}"] & B_2 \times Z \arrow[r, tail, "j_2 \times Id_Z"] & A \times Z \end{tikzcd}$$

which being monomorphisms are isomorphic to their respective images $M_{f_1} = Im(j_1 \times Id_Z \circ \Gamma_{f_1}), M_{f_2} = Im(j_2 \times Id_Z \circ \Gamma_{f_2})$. Define now α to be the composition:

$$\begin{tikzcd} M_{f_1} \vee M_{f_2} \arrow[r, tail] & A \times Z \arrow[r, "\pi_1"] & A \end{tikzcd}$$

The idea is to prove that α is an isomorphism, since then, defining $g : A \to Z$ to be:

$$\begin{tikzcd} A \arrow[r, "\alpha^{-1}"] & M_{f_1} \vee M_{f_2} \arrow[r, tail] & A \times Z \arrow[r, "\pi_2"] & Z \end{tikzcd}$$

we clearly render the following diagram commutative:

$$\begin{tikzcd} B_1 \wedge B_2 \arrow[r, "s"] \arrow[d, tail, "t"'] & B_2 \arrow[d, tail, "j_2"] \arrow[ddr, bend left, "f_2"] & \\ B_1 \arrow[r, "j_1"] \arrow[drr, bend right, "f_1"'] & A \arrow[dr, dashed, "g"] & \\ & & Z \end{tikzcd}$$

and g would necessarily be the unique possible morphism that makes the above diagram commute. Now let us prove that $\alpha^{-1}(B_1) = M_{f_1}$ and $\alpha^{-1}(B_2) = M_{f_2}$, for which it suffices to prove the first case, since the second is similar. We have on one hand $\alpha^{-1}(B_1) = \pi_1^{-1}(B_1) \wedge (M_{f_1} \vee M_{f_2}) = (B_1 \times Z) \wedge (M_{f_1} \vee M_{f_2}) = ((B_1 \times Z) \wedge M_{f_1}) \vee ((B_1 \times Z) \wedge M_{f_2})$, and since $M_{f_1} \subseteq (B_1 \times Z)$, to prove that $\alpha^{-1}(B_1) = M_{f_1}$ it suffices to prove the inclusion $(B_1 \times Z) \wedge M_{f_2} \subseteq M_{f_1}$. For that purpose, consider the morphism $f_1 t = f_2 s : B_1 \wedge B_2 \to Z$ and let $\Gamma_f : (B_1 \wedge B_2) \to (B_1 \wedge B_2) \times Z$ be its graph. Then, in the diagram below:

$$
\begin{tikzcd}
B_1 \wedge B_2 \arrow[r, "\Gamma_f"] \arrow[d, tail, "t"] & (B_1 \wedge B_2) \times Z \arrow[r, "s \times Id_Z"] \arrow[d, tail, "t \times Id_Z"] & B_1 \times Z \arrow[d, tail, "j_1 \times Z"] \\
B_2 \arrow[r, tail, "\Gamma_{f_2}"] \arrow[rr, tail, bend right, "M_{f_2}"'] & B_2 \times Z \arrow[r, tail, "j_2 \times Id_Z"] & A \times Z
\end{tikzcd}
$$

both squares are pullbacks, making the whole diagram a pullback as well, which means that $(B_1 \times Z) \wedge M_{f_2} = B_1 \wedge B_2$. The sought inclusion can then be easily seen to hold noting that we have:

$$
\begin{tikzcd}
B_1 \wedge B_2 \arrow[r, tail, "s"] & B_1 \arrow[r, tail, "\Gamma_{f_1}"] \arrow[rr, tail, bend right, "M_{f_1}"'] & B_1 \times Z \arrow[r, tail, "j_1 \times Id_Z"] & A \times Z
\end{tikzcd}
$$

As a consequence, the following diagram is a pullback:

$$
\begin{tikzcd}
M_{f_i} \arrow[r, tail] \arrow[d, "r_i"'] & M_{f_1} \vee M_{f_2} \arrow[d, "\alpha"] \\
B_i \arrow[r, "j_1"] & A
\end{tikzcd}
$$

for $i = 1, 2$. But we can, moreover, affirm that the left arrow r_i is an isomorphism, since j_i factors through $A \times Z$ as the following diagram shows:

$$
\begin{tikzcd}
 & B_i \arrow[r, tail, "j_i"] & A \\
B_i \arrow[ur, "Id_{B_i}"] \arrow[r, tail, "\Gamma_{f_i}"] & B_i \times Z \arrow[u, "\pi_1"] \arrow[r, tail, "j_i \times Id_Z"] & A \times Z \arrow[u, "\pi_1"]
\end{tikzcd}
$$

We shall now deduce that α must be an isomorphism from the fact that each r_i is. Clearly, α is a cover, since $Im(\alpha) \supseteq B_1 \vee B_2 = A$. Thus we only need to be show that α is a monomorphism. Let $u, v : C \to M_{f_1} \vee M_{f_1}$ be arrows such that $\alpha u = \alpha v = h$; we need to prove that $Ker(u, v) = C$. Note that $C = h^{-1}(A) = h^{-1}(B_1 \vee B_2) = h^{-1}(B_1) \vee h^{-1}(B_2)$, and therefore it suffices to prove that $h^{-1}(B_i) \subseteq Ker(u, v)$, or equivalently, that $u|_{h^{-1}(B_i)} = v|_{h^{-1}(B_i)}$. But if we consider the following diagram on the left and pull it back along $B_i \rightarrowtail X$ to obtain the diagram on the right:

$$
\begin{tikzcd}
X \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[dr, "h"'] & M_{f_1} \vee M_{f_1} \arrow[d, "\alpha"] \\
& A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
h^{-1}(B_i) \arrow[r, bend left, "u|_{h^{-1}(B_i)}"] \arrow[r, bend right, "v|_{h^{-1}(B_i)}"'] \arrow[dr, "h"'] & M_{f_i} \arrow[d, "r_i"] \\
& B_i
\end{tikzcd}
$$

then the fact that r_i is an isomorphism implies that $u|_{h^{-1}(B_i)} = v|_{h^{-1}(B_i)}$. This finishes the proof. □

3.3 Filtered colimits

A special type of categories will prove to be useful for our purposes. Following [10] we give the following:

Definition 3.3.1. *A category $\mathcal{C}$ is called filtered if it satisfies the following two conditions:*
a) For every pair of objects A, B in $\mathcal{C}$ there is an object C such there are arrows $f : A \to C$ and $g : B \to C$.
b) For every pair of arrows $a, b : A \rightrightarrows B$ there is an arrow $c : B \to C$ such that $ca = cb$.

A colimit diagram $\Gamma : \mathcal{D} \to \mathcal{C}$ in the category $\mathcal{C}$ is called filtered if Γ is a filtered category. In section 5 we shall need the following result:

Lemma 3.3.2. *A filtered colimit of $\mathcal{S}et$-valued left exact functors is left exact.*

This is really a special case of the well known exactness property of $\mathcal{S}et$, by which all filtered colimits commute with finite limits. We present here a different proof of this particular case.

Proof. Suppose I is a filtered category, J is a finite category and $\{F_i/i \in I\}$ are $\mathcal{S}et$-valued left exact functors. We need to prove that $\varinjlim_{i\in I} \varprojlim_{j\in J} F_i(x_j) = \varprojlim_{j\in J} \varinjlim_{i\in I} F_i(x_j)$. We shall do so by proving that the colimit $\varinjlim_{i\in I} \varprojlim_{j\in J} F_i(x_j)$ satisfies the universal property of the limit $\varprojlim_{j\in J} \varinjlim_{i\in I} F_i(x_j)$. Let $x = \varinjlim_{i\in I} x_i$, $F = \varprojlim_{j\in J} F_j$, and consider the following diagram:

$$
\begin{tikzcd}
F_i(x_j) \arrow[r, "\theta_{x_j}"'] \arrow[dd, "F_i(g)"'] & F(x_j) \arrow[dd, bend left, "F(g)"] & & & & \\
& & F_i(x) \arrow[r, "\theta_x"] \arrow[ull, "F_i(f_j)"'] \arrow[dll, "F_i(f_{j'})"] & F(x) \arrow[ull, "F(f_j)"'] \arrow[dll, "F(f_{j'})"] & C \arrow[l, dashed, "m"'] \arrow[ulll, "a_j"'] \arrow[dlll, "a_{j'}"] & 1 \arrow[l, "c"'] \arrow[lll, bend left, "l"] \arrow[ulllll, bend right, "a'_j"'] \arrow[dlllll, bend left, "a'_{j'}"] \\
F_i(x_{j'}) \arrow[r, "\theta_{x_{j'}}"] & F(x_{j'}) & & & &
\end{tikzcd}
$$

We need to show that $F(x)$ is a limit diagram whose projections are the arrows $F(f_j)$ for $j \in J$. For that purpose, consider a cone $a_j : C \to F(x_j)$; we shall construct an induced morphism $m : C \to F(x)$ by defining the map m on each element $c \in C$. Take, thus, an element $c \in C$. Since J is finite and I is filtered, there exists some $i \in I$ such that all the elements $a_j(c)$ of the corresponding colimits have already representatives in some $F_i(x_j)$. Call a'_j the maps so defined, and let $\theta : F_i \to F$ be the natural transformation which is the injection into the colimit. Because $F_i(f_j) : F_i(x) \to F_i(x_j)$ is a universal cone and $a'_j : 1 \to F_i(x_j)$ can be supposed without loss of generality to be a cone, there is an induced morphism $l : 1 \to F_i(x)$ such that $F_i(f_j)l = a'_j$. Then we just define $m(c) = \theta_x l$. Indeed, we have, $F(f_j)m(c) = F(f_j)\theta_x l = \theta_{x_j} F_i(f_j) l = \theta_{x_j} a'_j = a_j(c)$, and similarly, $F(f_{j'})m(c) = a_{j'}(c)$. To prove that m is unique, suppose there is a morphism m' satisfying $F(f_j)m' = a_j$ and let $c \in C$ be such that $m(c) \neq m'(c)$. Find $i \in I$ such that $m'(c)$ factors as $\theta_x l'$. From diagram chasing we can infer that $\theta_{x_j} F_i(f_j) l' = \theta_{x_j} a'_j$. Since there must be some $j \in J$ such that $F_i(f_j)l' \neq F_i(f_j)l$, it follows that $F_i(f_j)l', F_i(f_j)l$ are in the same class in the colimit. Therefore, there is some $k \in I, \eta : F_i \to F_k$ for which $\eta_x l, \eta_x l'$ are two different morphisms from 1 into the limit $F_k(x)$ commuting with the limiting cone, contradicting universality of this limit. This finishes the proof. $\square$

Lemma 3.3.3. *Any colimit of $\mathcal{S}et$-valued left exact functors, each of which preserves covers and unions of subobjects, preserves as well covers and unions of subobjects.*

Proof. Suppose I is category and $\{F_i \ / \ i \in I\}$ are $\mathcal{S}et$-valued functors that preserve covers and unions of subobjects. Let us first prove that the colimit preserves covers, that is, given a cover $f : A \to B$, let us see that $\varinjlim_{i\in I} F_i(f) : \varinjlim_{i\in I} F_i(A) \to \varinjlim_{i\in I} F_i(B)$ is surjective. Take an element $c \in \varinjlim_{i\in I} F_i(B)$; it must have a representative $c' \in F_i(B)$ for some $i \in I$. Now, since colimits of functors are computed pointwise and $k_i : F_i \to \varinjlim_{i\in I} F_i$ are the injections into the colimit, the following diagram commutes:

$$\begin{tikzcd}
F_i(A) \arrow[r, "(k_i)_A"] \arrow[d, two heads, "F_i(f)"'] & \varinjlim_{i\in I} F_i(A) \arrow[d, dashed, two heads, "\varinjlim_{i\in I} F_i(f)"] \\
F_i(B) \arrow[r, "(k_i)_B"'] & \varinjlim_{i\in I} F_i(B)
\end{tikzcd}$$

and since $F_i(f) : F_i(A) \to F_i(B)$ is a cover, it is surjective and there is some $d \in F_i(A)$ such that $F_i(d) = c'$. Hence, $\varinjlim_{i\in I} F_i((k_i)_A(d)) = c$, which proves that $\varinjlim_{i\in I} F_i$ is regular.

Let us now see see that $\varinjlim_{i\in I} F_i$ preserves unions of subobjects. Given $i_A : A \rightarrowtail C$, $i_B : B \rightarrowtail C$ subobjects of C, because the F_i preserve finite limits, then $F_i(A \wedge B) = F_i(A) \wedge F_i(B)$, and hence the universal property of the pushout below (note that it is a pushout according to lemma 3.2.12) guarantees that the arrows $j_{AB} : F_i(A \vee B) = F_i(A) \vee F_i(B) \to \varinjlim_{i\in I} F_i(A) \vee \varinjlim_{i\in I} F_i(B)$ form a cocone diagram, which induces in turn the arrow m that makes the next diagram commute:

$$
\begin{tikzcd}
F_i(A \wedge B) = F_i(A) \wedge F_i(B) \arrow[r, tail, "j_B"] \arrow[d, tail, "j_A"'] & F_i(B) \arrow[d, tail, "j'_B"] \arrow[ddr, "(j'_{\varinjlim_{i\in I} F_i(A)}) \circ (k_i)_A"] & \\
F_i(A) \arrow[r, tail, "j'_A"] \arrow[drr, "(j'_{\varinjlim_{i\in I} F_i(B)}) \circ (k_i)_B"'] & F_i(A) \vee F_i(B) \arrow[dr, dashed, "j_{AB}"] & \\
& & \varinjlim_{i\in I} F_i(A) \vee \varinjlim_{i\in I} F_i(B)
\end{tikzcd}
$$

$$
\begin{tikzcd}
F_i(A \vee B) = F_i(A) \vee F_i(B) \arrow[r, "(k_i)_{A\vee B}"] \arrow[dr, "j_{AB}"'] & \varinjlim_{i\in I} F_i(A \vee B) \arrow[d, dashed, "m"] \\
& \varinjlim_{i\in I} F_i(A) \vee \varinjlim_{i\in I} F_i(B)
\end{tikzcd}
$$

It is now easy to see that m must be an isomorphism. Indeed, it is surjective, since if $c \in \varinjlim_{i\in I} F_i(A) \vee \varinjlim_{i\in I} F_i(B)$, supposing without loss of generality that $c \in \varinjlim_{i\in I} F_i(A)$, it has a representative c' in $F_i(A) \hookrightarrow F_i(A) \vee F_i(B) = F_i(A \vee B)$ for some $i \in I$, and hence $j_{AB}(c') = c$. Therefore, $c = m((k_i)_{A\vee B}(c'))$. To prove that m is injective we proceed similarly, noting that if $m(a) = m(b)$, then we can suppose without loss of generality that a', b' are the corresponding representatives in $F_i(A \vee B)$ for some $i \in I$. But since j_{AB} must map them into the same element in $\varinjlim_{i\in I} F_i(A) \vee \varinjlim_{i\in I} F_i(B)$ (which we can suppose to be, say, in $\varinjlim_{i\in I} F_i(A)$), then a', b' belong to the same class, i.e., $a = b$. $\square$

As a consequence of lemmas 3.3.2 and 3.3.3 we have now the following:

Corollary 3.3.4. *A filtered colimit of $\mathcal{S}et$-valued regular (resp. Boolean) functors is regular (resp. Boolean).*

Proof. Suppose I is a filtered category and $\{F_i \ / \ i \in I\}$ are $\mathcal{S}et$-valued regular (resp. Boolean) functors. By lemma 3.3.2, the colimit $\varinjlim_{i\in I} F_i$ preserves finite limits, and by lemma 3.3.3 it also preserves covers (when the functors are regular) and unions of subobjects (when the functors are Boolean). In this latter case, since the complement in $\mathcal{S}et$ of a subobject $s : A \to B$ is uniquely determined by the conditions $A \vee \neg A = B$ and $A \wedge \neg A = 0$, using that $\varinjlim_{i\in I} F_i$ preserves unions and intersections we can infer that $\varinjlim_{i\in I} F_i(\neg A) = \neg \varinjlim_{i\in I} F_i(A)$, which completes the proof. $\square$

One construction that we shall use later involves a filtered bicolimit of categories, so we recall here the definition and present an explicit construction.

We start, following [4], with the concept of pseudofunctor. This is a generalization of functors, except that it is only required to preserve composition up to isomorphism, and these isomorphisms have certain coherence properties, specified in the following:

Definition 3.3.5. *Given a category $\mathcal{D}$, a (normalized) pseudofunctor $F : \mathcal{D}^{op} \to \mathcal{C}at$ consists of the following:*
a) A function $\mathcal{O}b(F) : \mathcal{O}b(\mathcal{D}) \to \mathcal{O}b(\mathcal{C}at)$ (for convenience we shall refer to $F(D)$ for the category corresponding to the object D).
b) An application $\mathcal{A}r(F) : \mathcal{A}r(\mathcal{D}) \to \mathcal{A}r(\mathcal{C}at)$ which assigns to every arrow $f : C \to D$ in $\mathcal{D}$ a functor $f^* : F(D) \to F(C)$.
c) An application c defined in $\mathcal{A}r(\mathcal{D})^2$ which assigns to each pair (f, g) of arrows of $\mathcal{D}$ a natural isomorphism $c_{f,g} : g^*f^* \to (fg)^*$.
Furthermore, the following properties hold:
1) For every object C in $\mathcal{D}$ we have $(Id_C)^* = Id_{F(C)}$.
2) For every arrow $f : C \to D$ in $\mathcal{D}$, we have $c_{f,Id_C} = Id_{f^*}$ and $c_{Id_D,g} = Id_{f^*}$.
3) For a triple of composable arrows $f : C \to D$, $g : D \to E$ and $h : E \to G$, we have $c_{f,gh}(\xi) \circ c_{g,h}(f^*(\xi)) = c_{fg,h}(\xi) \circ h^*(c_{f,g}(\xi))$.

In the special case when $c_{f,g} = Id_{(fg)^*}$ the pseudofunctor reduces to a functor.
Following [1], we also have:

Definition 3.3.6. *Given a pseudofunctor $F : \mathcal{D}^{op} \to \mathcal{C}at$, a pseudococone with vertex at the category $\mathcal{X}$ consists of a family of functors $\{\phi_A : F(A) \to \mathcal{X} \ / \ A \in \mathcal{D}\}$ and a family of natural isomorphisms $\{\phi_u : \phi_B \circ u^* \to \phi_A \ / \ (u : A \to B) \in \mathcal{D}\}$ that satisfy the following conditions:*
a) $\phi_{Id_A} = Id_{\phi_A}$.
b) For $u : A \to B$ and $v : B \to C$, we have $\phi_{vu} = \phi_u \circ \phi_v Id_{v^*} \circ (Id_{\phi_C} \circ c^{-1}_{v,u})$:

$$
\begin{tikzcd}
 & F(C) \arrow[d, "v^*"] \arrow[dr, "\phi_A"] \arrow[dd, bend right=60, "u^*v^*"'] & \\
{} \arrow[r, phantom, "\overset{c^{-1}_{v,u}}{\Longrightarrow}"] & F(B) \arrow[r, "\phi_B"'] \arrow[d, "u^*"] \arrow[ur, phantom, "\Uparrow\phi_u"] \arrow[dr, phantom, "\Uparrow\phi_v"] & \mathcal{X} \\
 & F(A) \arrow[ur, "\phi_C"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F(C) \arrow[dd, "(vu)^*"'] \arrow[dr, "\phi_A"] & \\
{} \arrow[r, phantom, "\Uparrow\phi_{vu}"] & \mathcal{X} \\
F(A) \arrow[ur, "\phi_C"'] &
\end{tikzcd}
$$

Pseudococones allow us to consider a variation of the colimit notion that we shall call bicolimit, introduced in [14], Ex. VI 6.4.0, under the notation "$\underrightarrow{Lim}$" (with capital L). We have:

Definition 3.3.7. *Given a pseudofunctor $F : I^{op} \to \mathcal{C}at$, the bicolimit $\mathcal{C} = \lim_{\overrightarrow{i\in I}} F(i)$ is the universal pseudococone associated to F. In other words, it is a pseudococone $\phi : F \Rightarrow \mathcal{C}$ such that for every psedococone $\psi : F \Rightarrow \mathcal{D}$ there is a unique functor $\lambda : \mathcal{C} \to \mathcal{D}$ such that $\psi_i = \lambda\phi_i$ for every $i \in I$:*

$$
\begin{tikzcd}
 & \mathcal{C} \arrow[r, dashed, "\lambda"] & \mathcal{D} \\
F(i) \arrow[ur, "\phi_i"] \arrow[urr, "\psi_i"'] & &
\end{tikzcd}
$$

A construction of the bicolimit is described in [14] by means of categories of fractions applied to certain fibrations. This will prove to be useful in section 5.

3.4 Grothendieck fibrations

We now turn to Grothendieck construction of the bicolimit, as done in [14]. We need first some definitions.

Definition 3.4.1. *Let $\pi : \mathcal{C} \to \mathcal{D}$ be a functor, and $f : A \to B$ a morphism in $\mathcal{D}$. We say that a morphism $m : X \to Y$ in $\mathcal{C}$ satisfying $\pi(m) = f$ is cartesian if, for every morphism $m' : X' \to Y$ in $\mathcal{C}$ satisfying $\pi(m') = f$, there exists a unique morphism $p : X' \to X$ in $\mathcal{C}$ such that $\pi(p) = Id_A$ and $mp = m'$:*

$$\begin{tikzcd}
& X' \arrow[d, dashed, "p"'] \arrow[dr, "m'"] & \\
& X \arrow[r, "m"] & Y \\
{} \arrow[d, "\pi"'] & & \\
{} & A \arrow[r, "f"] & B
\end{tikzcd}$$

Definition 3.4.2. *A fibration is a functor $\pi : \mathcal{C} \to \mathcal{D}$ such that:*
a) For every morphism $f : A \to B$ in $\mathcal{D}$ and every Y in $\mathcal{C}$ satisfying $\pi(Y) = B$ there exists a cartesian morphism $m : X \to Y$ in $\mathcal{C}$ such that $\pi(m) = f$.
b) The composition of cartesian morphisms is cartesian.
In this case $\mathcal{C}$ is called fibered over $\mathcal{D}$.

Following [2], we have as well:

Definition 3.4.3. *Given a category $\mathcal{C}$, a set S of morphisms of $\mathcal{C}$ is said to satisfy a calculus of left fractions if the following holds:*
a) Morphisms of S are closed under composition.
b) If $f : B \to A$ and $g : C \to A$ are two morphisms and $g \in S$, there are morphisms $f' : D \to B$ and $g' : D \to C$ such that $fg' = gf'$ and $g' \in S$:

$$\begin{tikzcd}
& B \arrow[dr, "f"] & \\
D \arrow[ur, dashed, "g'"] \arrow[dr, dashed, "f'"'] & & A \\
& C \arrow[ur, "g"'] &
\end{tikzcd}$$

c) For every pair of morphisms $f, g : A \rightrightarrows B$ such that there is some $s \in S$ satisfying $sf = sg$, there exists some $t \in S$ satisfying $ft = gt$.

The terminology of definition 3.4.3 arises from the fact that the stated conditions provides another construction of the category of fractions. We recall here the definition:

Definition 3.4.4. *Given a category $\mathcal{C}$ and a set S of morphisms of $\mathcal{C}$, there exists a category $\mathcal{C}[S^{-1}]$, called the category of fractions, that satisfies the following universal property: there is a functor $j : \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that $j(s)$ is an isomorphism for every $s \in S$, and for every functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for every $s \in S$, there exists a unique functor $\overline{F} : \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that $\overline{F} j = F$:*

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, "j"] \arrow[dr, "F"'] & & \mathcal{C}[S^{-1}] \arrow[dl, dashed, "\overline{F}"] \\
& \mathcal{D} &
\end{tikzcd}
$$

We have now:

Theorem 3.4.5. *Let S be a set of morphisms of $\mathcal{C}$ that satisfies a calculus of left fractions. Then the category of fractions $\mathcal{C}[S^{-1}]$ can be constructed as follows:*
a) Objects are given by the objects of $\mathcal{C}$.
b) The hom-sets $[x, y]$ are given by the (filtered) colimits $\varinjlim_{(u \to x) \in S(x)} [u, y]$, where $S(x)$ is the category whose objects are morphisms $f : u \to x$ of S over x and whose morphisms between the objects $f : u \to x$ and $f' : u' \to x$ are represented by arrows $g : u \to u'$ such that $f'g = f$.

Proof. Given $x \in \mathcal{C}$, let us first check that $S(x)$ is cofiltered (i.e., that $S(x)^{op}$ is filtered). For objects A given by $f : u \to x$ and B given by $f' : u' \to x$, because of condition b) of definition 3.4.3 we can find $m : u'' \to u$ and $n : u'' \to u'$ such that $fm = f'n$. Therefore, for the object C given by $f'n : u'' \to x$ there exist morphisms $C \to A$ and $C \to B$ given by the arrows m and n, respectively, which proves that the first condition for filteredness in $S(x)^{op}$ is fulfilled. To prove the second condition, suppose there are two arrows $A \rightrightarrows B$ given by g, g'. Then, since $f'g = f'g'$, because of condition c) of definition 3.4.3 there exists $h : v \to u$ in S such that $gh = g'h$. Therefore, if D is the object given by $fh : v \to x$, the arrow $D \to A$ given by h equalizes those represented by g and g'.

To define the composition in $\mathcal{C}[S^{-1}]$, take representatives $f : u \to y$ whith $a : u \to x$ in S and $g : u' \to z$ with $b : u' \to y$ in S. Because of condition b) of definition 3.4.3, there are arrows $b' : v \to u$ and $f' : v \to u'$ such that $fb' = bf'$ and $b' \in S$. Then we define the composite $[g] \circ [f]$ as the class of the arrow $gf' : v \to z$, since $ab' : v \to x$ is in S. The conditions of definition 3.4.3 ensure that this composite is independent of the chosen v.

Finally, to verify the universal property of $\mathcal{C}[S^{-1}]$ note that we can assume without loss of generality that S contains all identity arrows, since if S' is the union of S and all these identities (which are inversible), it is easy to see that $\mathcal{C}[S'^{-1}]$ and $\mathcal{C}[S^{-1}]$ satisfy the same universal property. Define then the functor j by applying $x \in \mathcal{C}$ into $x \in \mathcal{C}[S^{-1}]$, and applying the arrow $f : x \to y$ into the class of $f : x \to y$ (which can be done since $Id_x : x \to x$ is in S). Then, an inverse for $j(s)$, where $s : x \to y$ is in S, is given by the representative of $Id_x : x \to x$. Also, it is easy to verify that the representative $f : z \to y$, where $s : z \to x$ is in S, is the composition $j(f) \circ (j(s))^{-1}$ in $\mathcal{C}[S^{-1}]$, from which we deduce that in order to have $\overline{F}j = F$ we must necessarily have $\overline{F}([f]) = F(f) \circ (F(s))^{-1}$. This defines the functor $\overline{F}$ and proves it is the unique such possible functor. □

The next lemma states sufficient conditions so that cartesian morphisms form a calculus of left fractions:

Lemma 3.4.6. *Let $\pi : \mathcal{C} \to \mathcal{D}$ be a fibration and suppose that $\mathcal{D}$ satisfies the following two conditions:*
1) For every pair of morphisms $f : B \to A$ and $g : C \to A$, there are morphisms $f' : D \to C$ and $g' : D \to B$ such that $fg' = gf'$:

$$\begin{tikzcd} & B \arrow[dr, "f"] & \\ D \arrow[ur, dashed, "g'"] \arrow[dr, dashed, "f'"'] & & A \\ & C \arrow[ur, "g"'] & \end{tikzcd}$$

2) c) For every pair of morphisms $f, g : A \rightrightarrows B$ such that there is some $s \in S$ satisfying $sf = sg$, there exists some $t \in S$ satisfying $ft = gt$.
Then the set S of cartesian morphisms of $\mathcal{C}$ satisfy a calculus of left fractions.

Proof. Since $\mathcal{C}$ is fibered over $\mathcal{D}$, condition a) of definition 3.4.3 is satisfied. To check condition b), suppose we have morphisms $f : B \to A$ and $g : C \to A$, which give rise to morphisms $\pi(f) : \pi(B) \to \pi(A)$ and $\pi(g) : \pi(C) \to \pi(A)$. By hypothesis there are arrows $f' : D \to \pi(C)$ and $g' : D \to \pi(B)$ such that $\pi(f)g' = \pi(g)f'$, and since $\pi : \mathcal{C} \to \mathcal{D}$ is a cofibration, there are cartesian morphisms $f'' : D' \to C$ and $g'' : D'' \to B$ such that $\pi(f'') = f'$ and $\pi(g'') = g'$. Therefore, there is a unique $p : D'' \to D'$ such that $gg''p = ff''$, which means that the following square commutes:

$$\begin{tikzcd} & B \arrow[dr, "f"] & \\ D \arrow[ur, dashed, "f''"] \arrow[dr, dashed, "g''p"'] & & A \\ & C \arrow[ur, "g"'] & \end{tikzcd}$$

as required.

Finally, to verify condition c), suppose we have morphisms $f, g : A \rightrightarrows B$ and a cartesian s satisfying $sf = sg$. Then $\pi(s)$ coequalizes $\pi(f)$ and $\pi(g)$. By hypothesis, there is some h in $\mathcal{D}$ such that $\pi(f)h = \pi(g)h$. If h' in $\mathcal{C}$ is a cartesian morphism such that $\pi(h') = h$, then $sfh' = sgh'$, which implies $fh' = gh'$, since s is cartesian. This completes the proof. □

Corollary 3.4.7. *If $\pi : \mathcal{C} \to \mathcal{D}$ is a fibration and $\mathcal{D}$ is cofiltered, the set of cartesian morphisms in $\mathcal{C}$ satisfies a calculus of left fractions.*

Proof. It is an immediate consequence of the previous lemma. □

A natural type of fibration that often arises is described in the following:

Definition 3.4.8. Grothendieck construction: *Let I be a category and $F : I^{op} \to \mathcal{C}at$ a pseudofunctor. We define the category Γ_F as follows:*
a) The objects of Γ_F are given by pairs (x, i), where $i \in I$ and $x \in F(i)$.
b) The morphisms $(x, i) \to (y, j)$ are given by pairs (f, ϕ), where $f : i \to j$ is an arrow of I and $\phi : x \to f^*(y)$ is a morphism in $F(i)$. Moreover, the composite $(g, \psi) \circ (f, \phi)$ of two objects $(f, \phi) : (x, i) \to (y, j)$ and $(g, \psi) : (y, j) \to (z, k)$ is defined as $(gf, (c_{g,f})_z \circ f^*(\psi) \circ \phi)$.

Remark 3.4.9. *If we define the functor $\pi : \Gamma_F \to I$ by setting $\pi((x, i)) = i$ and $\pi((f, \phi)) = f$, it can be seen (see [4], Ex. VI 8 for details) that Γ_F becomes a fibered category over I, called the fibration associated to the pseudofunctor F. Furthermore, the cartesian morphisms are precisely those morphisms (f, ϕ) for which ϕ is an isomorphism. Indeed, a morphism $(f, \phi) : (x, i) \to (y, j)$*

is cartesian if and only if for every morphism of the form $(f, \psi) : (x', i) \to (y, j)$ there exists a unique morphism of the form $(Id_i, \eta) : (x', i) \to (x, i)$ such that $(f, \psi) = (f, \phi) \circ (Id_i, \eta) = (f, \phi\eta)$. But this is equivalent to stating that, in $F(i)$, $\phi : x \to f^*(y)$ is such that for every $\psi : x' \to f^*(y)$ there exists a unique $\eta : x \to x$ satisfying $\phi\eta = \psi$, which clearly implies that ϕ is an isomorphism. Finally, Grothendieck also states in [4] that for an arbitrary fibration over I and a choice of cartesian morphisms, a pseudofunctor F can be defined in such a way that its associated fibration is precisely the original one.

Since, given an arbitrary fibration, the existence of the pseudofunctor described in the previous remark makes an essential appeal to the Axiom of Choice, we shall not use it in our proof in section 5. Instead, we shall work directly with the fibration in question.

The connection between Grothendieck's construction of categories of fractions and universal pseudococones is contained in the following:

Theorem 3.4.10. *Let I be category, $F : I^{op} \to \mathcal{C}at$ a pseudofunctor and $\pi : \Gamma_F \to I$ the fibration associated to F. Then, if S is the set of cartesian morphisms in Γ_F, the category of fractions $\Gamma_F[S^{-1}]$ is the universal pseudococone associated to F.*

Proof. The pseudococone is defined through the functors $jg_i : F(i) \to \Gamma_F[S^{-1}]$, where j is the functor defined in 3.4.5 and $g_i(x) = (x, i), g_i(l) = (Id_i, l)$, and through the natural isomorphisms $h_f = Id_j g_f$, where $g_f : g_i F(f) \Rightarrow g_j$ is defined for each arrow $f : i \to j$ in I by $(g_f)_x = (f, Id_{F(f)(x)})$.

For a pseudococone $\eta : F \Rightarrow \mathcal{D}$, there is an induced morphism $\theta : \Gamma_F \to \mathcal{D}$ defined as $\theta((x, i)) = \eta_i(x)$ and $\theta((f, \phi)) = \eta_j(\phi)(\eta_f)_x$. This morphism θ satisfies $\theta g_i = \eta_i$, and furthermore, applies cartesian morphisms in Γ_F into invertible morphisms in $\mathcal{D}$. Therefore the universal property of $\Gamma_F[S^{-1}]$ yields the desired result. □

Remark 3.4.11. *The usual notion of colimits in $\mathcal{C}at$ (the universal cocone, as given, for instance, in [2]) do not coincide in general with the bicolimit of categories (the universal pseudococone associated to a functor F, as in definition 3.3.7). In the filtered case both notions coincide ([14], Ex VI 6.8), and the standard construction of [2] of the universal cocone yields a category which is equivalent to the category constructed in theorem 3.4.5, although they are not strictly isomorphic.*

The following lemma provides evidence of the fact that all finitary constructions in the categories of a filtered diagram that are preserved by the transition functors are inherited by the bicolimit:

Lemma 3.4.12. *Let I be a cofiltered category and $F : I^{op} \to \mathcal{C}at$ a pseudofunctor such that each $F(i)$ is a regular (resp. Boolean) category and each transition functor $f^* : F(j) \to F(i)$ (for $f : i \to j$ in I) is regular (resp. Boolean). Then, if S is the set of cartesian morphisms in the fibered category Γ_F over I, the category of fractions $\Gamma_F[S^{-1}]$ is also regular (resp. Boolean). Moreover, for each i in I, the injection into the bicolimit $I_i : F(i) \to \Gamma_F[S^{-1}]$ is a regular (resp. Boolean) functor.*

Proof. In what follows, we we shall use the construction given in the proof of theorem 3.4.5, as well as the filteredness of I^{op} to find specific arrows needed for the constructions.

Using the fact that for, f in I, each functor f^* is regular (resp. Boolean), it can now be verified that the following holds: the product of two objects (C, i) and (D, j) is given by the object $(u^*(C) \times v^*(D), k)$, where $u : k \to i$, $v : k \to j$ are arrows of I. The corresponding projections are given by the classes of the arrows on the right:

$$(u^*(C)\times v^*(D),k) \xleftarrow{(Id_k, Id_{u^*(C)\times v^*(D)})} (u^*(C)\times v^*(D),k) \xrightarrow{\quad (u,\pi_1) \quad} (C,i)$$

$$(u^*(C)\times v^*(D),k) \xleftarrow{(Id_k, Id_{u^*(C)\times v^*(D)})} (u^*(C)\times v^*(D),k) \xrightarrow{\quad (v,\pi_2) \quad} (D,j)$$

The equalizer of two morphisms represented by the arrows on the right:

$$(C,i) \xleftarrow{(v,\psi)} (X,k) \xrightarrow{(u,\phi)} (D,j)$$

$$(C,i) \xleftarrow{(v',\psi')} (X',k') \xrightarrow{(u',\phi')} (D,j)$$

where $(v,\psi),(v',\psi')$ are the corresponding cartesian morphisms, i.e., where ψ,ψ' are isomorphisms, is represented by the following morphism on the right:

$$(E,l) \xleftarrow{(Id_l, Id_E)} (E,l) \xrightarrow{(xu,(c_{x,u})_C g)} (C,i)$$

where (Id_l, Id_E) is cartesian, in which $u : k \to i$, $v : k \to j$, $u' : k' \to i$, $v' : k' \to j$ are arrows of I, $x : l \to k$ and $y : l \to k'$ are such that $x^*u^*(C) = y^*u'^*(C)$, $x^*v^*(D) = y^*v'^*(D)$ and $g : E \to x^*u^*(C)$ is the equalizer of the arrows $x^*(\phi\psi^{-1}), y^*(\phi'\psi'^{-1}) : x^*u^*(C) \to y^*u'^*(D)$ in $\mathcal{C}_l$. The image of a morphism represented by the arrow on the right:

$$(C,i) \xleftarrow{(v,\psi)} (X,k) \xrightarrow{(u,\phi)} (D,j)$$

where (v,ψ) is the corresponding cartesian morphism, i.e., where ψ is an isomorphism, is given by the subobject represented by the following right morphism:

$$(M,k) \xleftarrow{(Id_k, Id_{(M,k)})} (M,k) \xrightarrow{\quad (v,m) \quad} (v^*(D),k)$$

where $(Id_k, Id_{(M,k)}) : (M,k) \to (M,k)$ is cartesian, in which $u : k \to i$, $v : k \to j$ are arrows of I and $m : M \rightarrowtail v^*(D)$ is the subobject corresponding to the image of $\phi\psi^{-1} : u^*(C) \to v^*(D)$ in $\mathcal{C}_k$. In particular, the class of (u,ϕ) is a cover if and only if $\phi\psi^{-1} : u^*(C) \to v^*(D)$ is a cover in $\mathcal{C}_k$, from which it can be proved, using the fact that each $\mathcal{C}_i$ is regular, that covers are stable under pullbacks. The union of two subobjects represented by arrows on the right:

$$(C,i) \xleftarrow{(v,\psi)} (X,k) \xrightarrow{(u,\phi)} (D,j)$$

$$(C,i) \xleftarrow{(v',\psi')} (X',k') \xrightarrow{(u',\phi')} (D,j)$$

where $(v,\psi),(v',\psi')$ are the corresponding cartesian morphisms, i.e., where ψ,ψ' are isomorphisms, is given by the subobject represented by the following right morphism:

$$(U,l) \xleftarrow{(Id_l, Id_{(U,l)})} (U,l) \xrightarrow{(yv',(c_{y,v'})_D g)} (D,j)$$

where $(Id_l, Id_{(U,l)})$ is cartesian, in which $u : k \to i$, $v : k \to j$, $u' : k' \to i'$, $v' : k' \to j$ are arrows of I, $x : l \to k$ and $y : l \to k'$ are such that $x^*v^*(D) = y^*v'^*(D)$ and $g : U \rightarrowtail y^*v'^*(D)$ is the subobject corresponding to the union of the subobjects $x^*(\phi\psi^{-1}) : x^*u^*(C) \rightarrowtail y^*v'^*(D)$ and $y^*(\phi'\psi'^{-1}) : y^*u'^*(C) \rightarrowtail y^*v'^*(D)$ in $\mathcal{C}_l$. In case each $\mathcal{C}_i$ is also Boolean, it can be verified from the definitions above that unions are stable under pullbacks too. On the other hand, the complement of a subobject represented by right arrow:

$$(C, i) \xleftarrow{(v,\psi)} (X, k) \xrightarrow{(u,\phi)} (D, j)$$

where (v, ψ) is cartesian, i.e., where ψ is an isomorphism), is given by the subobject represented by the following morphism on the right:

$$(V, k) \xleftarrow{(Id_k, Id_{(V,k)})} (V, k) \xrightarrow{(s,v)} (D, j)$$

where $(Id_k, Id_{(V,k)})$ is cartesian, in which $u : k \to i$, $v : k \to j$ are arrows of I and $s : V \to v^*(D)$ is the complement of the subobject $\phi\psi^{-1} : u^*(C) \rightarrowtail v^*(D)$ in $\mathcal{C}_k$. Finally, it is straightforward to check that with these specifications the functors $I_i : \mathcal{C}_i \to \Gamma_F[S^{-1}]$ are regular (resp. Boolean). □

Theorem 3.4.10 and lemma 3.4.12 require the existence of a pseudofunctor $F : I^{op} \to \mathcal{C}at$. In many contexts in which the Axiom of Choice is not used, this pseudofunctor is not available, and therefore a new approach must be taken. This leads to the following:

Definition 3.4.13. *Let $\pi : \mathcal{C} \to I$ be a fibration such that each fiber F_i over i is a regular category. We say that the set S of cartesian morphisms is locally regular if for every arrow $f : i \to j$ in I, the following conditions hold:*
a) For every finite limit diagram $k_l : L \to C_l$ in F_j and every choice of cartesian morphisms $g_l : C'_l \to C_l$, $g_L : L' \to L$ satisfying $\pi(g_l) = \pi(g_L) = f$, the unique induced arrows $k'_l : L' \to C'_l$ in F_i form a limit diagram there.
b) For every cover $p : C \to D$ in F_j and every choice of cartesian morphisms $g_C : C' \to C$, $g_D : D' \to D$ satisfying $\pi(g_C) = \pi(g_D) = f$, the unique induced arrow $p' : C' \to D'$ is a cover in F_i.
In case the fibers F_i are also Boolean categories, we say that S is locally Boolean if, in addition:
c) For every pair of subobjects $s : C_1 \rightarrowtail D$, $t : C_2 \rightarrowtail D$ in F_j and every choice of cartesian morphisms $g_i : C'_i \to C_i$, $g_D : D' \to D$, $g : U \to C_1 \vee C_2$ satisfying $\pi(g_i) = \pi(g_D) = \pi(g) = f$, then U is the union, in F_i, of the unique induced subobjects $s' : C'_1 \rightarrowtail D'$, $t' : C'_2 \rightarrowtail D'$.
d) For every subobject $s : C \rightarrowtail D$ in F_j and every choice of cartesian morphisms $g_C : C' \to C$, $g_D : D' \to D$, $g : V \to \neg C$ satisfying $\pi(g_C) = \pi(g_D) = \pi(g) = f$, then V is the complement, in F_i, of the unique induced subobject $s' : C' \rightarrowtail D'$.

The definition above allows to consider the preservation of regularity (resp. Booleanness) in a local sense, in a context where we may not have a global choice of cartesian morphisms that define a functor between fibers. This leads to the following reformulation of lemma 3.4.12 that does not appeal to the existence of a pseudofunctor:

Lemma 3.4.14. *Let $\pi : \mathcal{C} \to I$ be a fibration such that $\mathcal{C}$ and the fibers F_i are regular (resp. Boolean) categories, I is a cofiltered category with a terminal object and the set S of cartesian morphisms in $\mathcal{C}$ is locally regular (resp. locally Boolean). Then, the category $\mathcal{C}_1 = \mathcal{C}[S^{-1}]$ is regular (resp. Boolean). Furthermore, there exists a regular (resp. Boolean) functor $I_1 : \mathcal{C} \to \mathcal{C}_1$.*

Proof. In what follows, we we shall use the filteredness of I^{op} to find specific arrows of it that will be used in the constructions. Being these finitary constructions, we can choose some fixed cartesian morphisms for each one of the objects involved (a finite number of them), in case such a morphism is not determined by the context. In such a case, the notation $(-)^*$ will indicate a specific choice of a cartesian morphism over the corresponding arrow $(-)$ of I. We shall use as well the construction of the category of fractions used in the proof of theorem 3.4.5.

Using the local regularity (resp. local Booleanness) of S, it can now be verified that the following holds: the product of two objects C and D is given by the object $u^*(C)\times v^*(D)$, where $u : k \to i$, $v : k \to j$ are arrows of I and $u^*(C)\times v^*(D)$ is the product in F_k. The corresponding projections are given by the classes of the morphisms corresponding to the composition of the arrows on the right (the first arrows on the left are cartesian morphisms):

$$\begin{tikzcd} {} \arrow[d, "\pi"'] & u^*(C)\times v^*(D) & u^*(C)\times v^*(D) \arrow[l] \arrow[r, "\pi_1"] & u^*(C) \arrow[r] & C \\ {} & k & k \arrow[l, "Id_k"'] \arrow[r, "Id_k"'] & k \arrow[r, "u"'] & i \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[d, "\pi"'] & u^*(C)\times v^*(D) & u^*(C)\times v^*(D) \arrow[l] \arrow[r, "\pi_2"] & v^*(D) \arrow[r] & D \\ {} & k & k \arrow[l, "Id_k"'] \arrow[r, "Id_k"'] & k \arrow[r, "v"'] & j \end{tikzcd}$$

To get the equalizer of two morphisms between C and D represented by the arrows on the right:

$$\begin{tikzcd} {} \arrow[d, "\pi"'] & C & u^*(C) \arrow[l] \arrow[r, "a"] & D \\ {} & i & k \arrow[l, "u"'] \arrow[r, "v"'] & j \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[d, "\pi"'] & C & u'^*(C) \arrow[l] \arrow[r, "a'"] & D \\ {} & i & k' \arrow[l, "u'"'] \arrow[r, "v'"'] & j \end{tikzcd}$$

we first find arrows $x : l \to k$ and $y : l \to k'$ in I such that $x^*u^*(C) = y^*u'^*(C)$ and $x^*v^*(D) = y^*v'^*(D)$. Then the equalizer is given by the morphism represented by the composition of the two right arrows in the following diagram:

$$\begin{tikzcd} {} \arrow[d, "\pi"'] & E & E \arrow[l] \arrow[r, "h"] & x^*u^*(C) \arrow[r] & C \\ {} & l & l \arrow[l, "Id_l"'] \arrow[r, "Id_l"'] & l \arrow[r, "ux"'] & i \end{tikzcd}$$

where $h : E \to x^*u^*(C)$ is the equalizer of the arrows $x^*(\phi), y^*(\psi) : x^*u^*(C) \to x^*v^*(D)$ in F_l, and $\phi : u^*(C) \to v^*(D)$, $\psi : u'^*(C) \to v'^*(D)$ are the unique arrows induced by the corresponding cartesian morphisms $v^*(D) \to D$ and $v'^*(D) \to D$.

The image of a morphism $C \to D$ represented by the arrow on the right:

$$\begin{tikzcd} \arrow[dd, "\pi"'] & C & u^*(C) \arrow[l] \arrow[r, "a"] & D \\ & & & \\ & i & k \arrow[l, "u"] \arrow[r, "v"'] & j \end{tikzcd}$$

is given by the subobject represented by the composition of the two right arrows:

$$\begin{tikzcd} \arrow[dd, "\pi"'] & M & M \arrow[l] \arrow[r, tail, "m"] & v^*(D) \arrow[r] & D \\ & & & & \\ & k & k \arrow[l, "Id_k"] \arrow[r, "Id_k"'] & k \arrow[r, "v"'] & j \end{tikzcd}$$

where $u : k \to i$, $v : k \to j$ are arrows of I and $m : M \rightarrowtail v^*(D)$ is the subobject corresponding to the image in F_k of $\phi : u^*(C) \to v^*(D)$, the arrow induced by the cartesian morphism $v^*(D) \to D$.

In particular, such a morphism is a cover if and only if $\phi : u^*(C) \to v^*(D)$ is a cover in F_k, from which it can be proved, using the fact that F_k is regular, that covers are stable under pullbacks.

To get the union of two subobjects $C \rightarrowtail D$ and $C' \rightarrowtail D$ represented by the arrows on the right:

$$\begin{tikzcd} \arrow[dd, "\pi"'] & C & u^*(C) \arrow[l] \arrow[r, "a"] & D \\ & & & \\ & i & k \arrow[l, "u"] \arrow[r, "v"'] & j \\ \arrow[dd, "\pi"'] & C' & u^*(C) \arrow[l] \arrow[r, "a'"] & D \\ & & & \\ & i' & k' \arrow[l, "u'"] \arrow[r, "v'"'] & j \end{tikzcd}$$

we first find arrows $x : l \to k$ and $y : l \to k'$ in I such that $x^*v^*(D) = y^*v'^*(D)$. Then the union is given by the subobject represented by the composition of the two right arrows:

$$\begin{tikzcd} \arrow[dd, "\pi"'] & U & U \arrow[l] \arrow[r, tail, "h"] & x^*v^*(D) \arrow[r] & D \\ & & & & \\ & l & l \arrow[l, "Id_l"] \arrow[r, "Id_l"'] & l \arrow[r, "vx"'] & j \end{tikzcd}$$

where $h : U \rightarrowtail x^*v^*(D)$ is the subobject corresponding to the union of the subobjects $x^*(\phi) : x^*u^*(C) \rightarrowtail x^*v^*(D)$ and $y^*(\psi) : y^*u'^*(C) \rightarrowtail y^*v'^*(D)$ in F_l, and where $\phi : u^*(C) \to v^*(D)$, $\psi : u'^*(C) \to v'^*(D)$ are the unique arrows induced by the corresponding cartesian morphisms $v^*(D) \to D$ and $v'^*(D) \to D$.

In case $\mathcal{C}$ is also Boolean, it can be verified from the definitions above that unions are stable under pullbacks too. On the other hand, the complement of a subobject represented by the arrow on the right:

$$\begin{tikzcd} & C & u^*(C) \arrow[l] \arrow[r, "a"] & D \\ {} \arrow[from=1-1, to=2-1, "\pi"'] & i & k \arrow[l, "u"] \arrow[r, "v"'] & j \end{tikzcd}$$

is given by the subobject represented by the composition of the following right arrows:

$$\begin{tikzcd} & V & V \arrow[l] \arrow[r, tail, "s"] & v^*(D) \arrow[r] & D \\ {} \arrow[from=1-1, to=2-1, "\pi"'] & k & k \arrow[l, "Id_k"] \arrow[r, "Id_k"'] & k \arrow[r, "v"'] & j \end{tikzcd}$$

where $s : V \to v^*(D)$ is the complement, in F_k, of the subobject $\phi : u^*(C) \rightarrowtail v^*(D)$, the arrow induced by the cartesian morphism $v^*(D) \to D$.

Finally, the functor $I_1 : \mathcal{C} \to \mathcal{C}_1$ can be defined by applying the object C in $\mathcal{C}$ into the object C in $\mathcal{C}_1$, and applying each morphism $f : C \to D$ in $\mathcal{C}$ into the morphism represented by the arrow on the right:

$$\begin{tikzcd} & C & C \arrow[l] \arrow[r, "f"] & D \\ {} \arrow[from=1-1, to=2-1, "\pi"'] & 1 & 1 \arrow[l] \arrow[r] & 1 \end{tikzcd}$$

A straightforward verification, using the constructions above, shows now that the functor so defined is regular (resp. Boolean). □

A natural type of fibration that often arises involves a category described in the following:

Definition 3.4.15. *Let $\mathcal{C}$ be a category with pullbacks and I a subcategory. The category $\mathcal{A}_I$ of arrows over I is defined as follows:*
a) Objects are given by arrows $u : c \to i$ where i is in I.
b) The arrows between two objects $[u : c \to i]$ and $[v : d \to j]$ are given by commutative squares:

$$\begin{tikzcd} c \arrow[r, "g"] \arrow[d, "u"'] & d \arrow[d, "v"] \\ i \arrow[r, "f"'] & j \end{tikzcd}$$

where f is an arrow of I. Composition is defined in the obvious way.

There is a functor $\partial_1 : \mathcal{A}_I \to I$ which assigns to each arrow its codomain, and to each commutative square the arrow of I at the base. The fiber over i is given by the slice category $\mathcal{C}/i$. It is easy to check that ∂_1 is a fibration and that the cartesian morphisms correspond precisely to those commutative squares in $\mathcal{C}$ that are pullbacks. If a choice of pullbacks is not assumed, and hence no pseudofunctor F can be defined, this fibration may be used instead of the fibration associated to F, which allows to use lemma 3.4.14. This will be done in section 5.

4 Categorical models

4.1 Theories and models

For the sake of completeness we recall here the main notions from model theory. Following [5], we have:

Definition 4.1.1. *A (multityped) structure A consists of the following:*
a) A family of sets $\{A_i \ / \ i \in I\}$, each of which is called the sort of its corresponding elements.
b) A family of elements of the sets A_i called constants, denoted by constant symbols c (each of a given sort).
c) A set of n-ary relations, i.e., subsets of $A_{i_1} \times ... \times A_{i_n}$, denoted by relation symbols R.
d) A set of n-ary operations, i.e., functions from $A_{i_1} \times ... \times A_{i_n}$ to $A_{i_{n+1}}$, denoted by function symbols f.

Definition 4.1.2. *The signature Σ of a structure A consists of the set of all constants of A, the set of all n-ary relations symbols and the set of all n-ary function symbols, for $n \in \mathbf{N}$.*

Given a signature Σ, we can build the language of a first order theory through the definition of terms and formulas:

Definition 4.1.3. *A term of a language over a signature Σ is defined to be one of the following elements: a variable x or a constant symbol c of given sorts, or a function symbol $f(x_1, ..., x_n)$ (symbolizing the value of f at $(x_1, ..., x_n)$).*
A formula of the language is a string of terms and logical symbols ($\forall, \exists, \vee, \wedge, \neg$) built according to the following recursive clauses:
a) Atomic formulas. These have either the form $R(x_1, ..., x_n)$ for a relation symbol R and variables x_i of adequate sorts, or $s = t$ where s, t are terms of the same sort.
b) Formulas built from connectives. If ϕ, ψ are formulas, then $\neg\phi, \phi \vee \psi, \phi \wedge \psi$ are formulas.
c) Quantified formulas. If $\phi(x)$ is a formula where x is a free variable of a given sort, then $\forall x\phi(x), \exists x\phi(x)$ are formulas.

Remark 4.1.4. *The recursive clauses above define the class of all formulas within a language. At the metamathematical level, a proposition is said to hold for the class of all formulas if there are inductively verified with respect to the complexity of the formula; that is, if it holds for all formulas of type a), and whenever they hold for formulas of a certain type, they hold as well for combinations of these with logical symbols that define the next level of complexity.*

A first order theory consists of the class of all formulas derived through the use of the rules of inference from both the logical axioms and a certain set of non logical axioms.
Finally, we have:

Definition 4.1.5. *Given a first order theory T over a signature Σ, a (Set-valued) interpretation of T is given by a so called Σ-structure $M(\Sigma)$, that has a constant symbol $M(c)$ for every constant term c in T, a function symbol $M(f)$ for every function f and a relation symbol $M(R)$ for every predicative variable R.*

A sentence is a formula with no free variables. Therefore, interpretations assign to sentences of a language corresponding sentences in terms of relationships within the structure. More concretely:

Definition 4.1.6. *We say that the interpretation $M(T)$ is a (Set-valued) model of T if $M(T)$ satisfies all formulas of T in the sense defined recursively as follows:*

a)If ϕ is the atomic sentence $s = t$, then $M(T)$ satisfies ϕ if s, t are interpreted by the same element in $M(T)$.
b)If $\phi = \psi \vee \eta$, then $M(T)$ satisfies ϕ if it satisfies ψ or if it satisfies η. Similarly, $\phi = \psi \wedge \eta$ is satisfied if both ψ and η are satisfied, and $\phi = \neg\psi$ is satisfied if $M(T)$ does not satisfy ψ.
c)If $\phi = \forall x \psi(x)$, $M(T)$ satisfies ϕ if it satisfies all sentences of the form $\psi(a)$ where $a \in M(T)$ ranges over all elements of the same sort as x. Similarly, if $\phi = \exists x \psi(x)$, $M(T)$ satisfies ϕ if there exists some $a \in M(T)$ of the same sort as x such that it satisfies $\psi(a)$.

In order to introduce the notion of submodel we give the following:

Definition 4.1.7. *A substructure N of a (multityped) structure M is a structure whose sorts and corresponding set of constants are subsets of those of M, and whose functions and relations are the restriction of those of M to elements of N.*

We have now:

Definition 4.1.8. *A submodel N of a given model M of certain theory is a substructure of M that is itself a model of the theory and satisfies the following condition: for every formula $\phi(x_1, ..., x_n)$ and all n-tuples $(a_1, ..., a_n)$ in N where each a_i is of the same sort as x_i, then $\phi(a_1, ..., a_n)$ is true in M if and only if it is true in N.*

These definitions will be referred to later.

4.2 Interpretation of theories in a categorical setting

Following [8], D1.2, given a Boolean category $\mathcal{C}$, for each signature Σ of a first order language we can associate the so called Σ-structure within $\mathcal{C}$ in a way that generalizes the $\mathcal{S}et$-valued interpretations to all Boolean categories. More precisely, for each sort A of variables in Σ there is a corresponding object $M(A)$; for each function symbol f there is a morphism $M(f) : M(A_1, ..., A_n) = M(A_1) \times ... \times M(A_n) \to M(B)$ and for each n-ary predicative variable R there is a subobject $M(R) \rightarrowtail M(A_1, ..., A_n)$, where A_i are the sorts corresponding to the individual variables related to R (which will specify, by definition, the type of f and R). The Σ-structure will serve as a setup for interpreting all formulas of the language considered. Due to the need of distinguishing the free variables of the formula for the purpose of a correct interpretation, we shall adopt the notation $(\overline{x}, \phi)$ to represent a formula ϕ whose free variables occur within $\overline{x} = x_1...x_n$. We now define the interpretation of such formulas by induction on their complexity:

Definition 4.2.1. *Given a term s of a first order theory, its interpretation within the Boolean category $\mathcal{C}$ is a morphism of $\mathcal{C}$ defined in the following way:*

1) If s is a variable, it is necessarily some x_i, and then the corresponding morphism is $[[\overline{x}, x_i]] = \pi_i : M(A_1, ..., A_n) \to M(A_i)$, the i-th product projection.
2) If s is a function symbol $f(x_1, ..., x_n)$, its interpretation is $M(f) : M(A_1, ..., A_n) \to M(B)$.

The interpretation in $\mathcal{C}$ of the formula $(\overline{x}, \phi)$, where $\overline{x} = x_1...x_n$ and x_i is a variable of sort A_i, is defined as a subobject $[[\overline{x}, \phi]] \rightarrowtail M(A_1, ..., A_n)$ in the following way:

3) If ϕ is the formula $R(t_1, ..., t_n)$, where R is a n-ary predicative variable of type $B_1, ..., B_n$, then $[[\overline{x}, \phi]]$ is given by the pullback:

$$\begin{tikzcd}
{[[\overline{x},\phi]]} \arrow[r] \arrow[d, tail] & M(R) \arrow[d, tail] \\
M(A_1,...,A_n) \arrow[r, "{([[\overline{x},t_1]],...,[[\overline{x},t_n]])}"] & M(B_1,...,B_n)
\end{tikzcd}$$

4) If ϕ is the formula $s = t$ where s, t are variables of sort B, then $[[\overline{x}, \phi]]$ is the equalizer of the arrows:

$$\begin{tikzcd}
M(A_1,...,A_n) \arrow[r, shift left, "{[[\overline{x},s]]}"] \arrow[r, shift right, "{[[\overline{x},t]]}"'] & M(B)
\end{tikzcd}$$

Equivalently, $[[\overline{x}, \phi]]$ is the pullback of the diagonal $M(B) \rightarrowtail M(B) \times M(B)$ along the morphism $([[\overline{x}, s]], [[\overline{x}, t]])$.
5) If ϕ is the formula $\eta \vee \psi$, then $[[\overline{x}, \phi]]$ is the union $[[\overline{x}, \eta]] \vee [[\overline{x}, \psi]]$ in $\mathcal{S}ub(M(A_1, ..., A_n))$. Similarly, if ϕ is the formula $\neg\psi$, the corresponding subobject is $\neg[[\overline{x}, \psi]]$.
6) The formulas $\top$ and $\bot$ are interpreted respectively as $M(A_1, ..., A_n)$ and the initial subobject in $\mathcal{S}ub(M(A_1, ..., A_n))$.
7) If ϕ is the formula $\exists y(\psi)$, then $[[\overline{x}, \phi]]$ is the image of the composite:

$$[[\overline{x}y, \psi]] \rightarrowtail M(A_1, ..., A_n, B) \xrightarrow{\pi} M(A_1, ..., A_n)$$

where π is the projection to the first n coordinates.

For the corresponding notion of substructure we have the following:

Definition 4.2.2. *A substructure Σ' of a (multityped) structure Σ consists of a set of monic arrows $h_A : M_{\Sigma'}(A) \to M_\Sigma(A)$ indexed by the sorts of Σ such that:*
i) for each function symbol f the following square commutes:

$$\begin{tikzcd}
M_{\Sigma'}(A_1,...,A_n) \arrow[r, "M_{\Sigma'}(f)"] \arrow[d, tail, "h_{A_1}\times...\times h_{A_n}"'] & M_{\Sigma'}(B) \arrow[d, tail, "h_B"] \\
M_\Sigma(A_1,...,A_n) \arrow[r, "M_\Sigma(f)"] & M_\Sigma(B)
\end{tikzcd}$$

ii) for each relation symbol R there is a commutative square:

$$\begin{tikzcd}
M_{\Sigma'}(R) \arrow[r, tail] \arrow[d, tail] & M_{\Sigma'}(A_1,...,A_n) \arrow[d, tail, "h_{A_1}\times...\times h_{A_n}"] \\
M_\Sigma(R) \arrow[r, tail] & M_\Sigma(A_1,...,A_n)
\end{tikzcd}$$

The reader should be able to verify that in the particular case where $\mathcal{C} = \mathcal{S}et$, the definitions above coincide with the $\mathcal{S}et$-valued one as in 4.1.5, 4.1.6 and 4.1.7. They provide a translation of the language over Σ into a subcategory of $\mathcal{C}$, interpreting notions that are purely syntactical in nature through the use of categorical concepts. Furthermore, as we shall see, there is a tight relation between the categorical properties of $\mathcal{C}$ and the properties of theories defined over that language. For that purpose it is useful to derive a concept of validity, inside $\mathcal{C}$, for subobjects that interpret certain formulas.

Definition 4.2.3. *The formula $\forall \overline{x}(\phi(\overline{x}) \to \psi(\overline{x}))$ is said to be satisfied by the Σ-structure M if the corresponding subobjects for ϕ, ψ satisfy $[[\overline{x}, \phi]] \leq [[\overline{x}, \psi]]$ in $\mathcal{S}ub(M(A_1, ..., A_n))$. More generally, the formula $\phi(\overline{x})$ is said to be satisfied by M if M satisfies $\forall \overline{x}(\top \to \phi(\overline{x}))$, (or, equivalently, if $[[\overline{x}, \phi]] = [[\overline{x}, \top]]$). In this case $\phi(\overline{x})$ is said to have full extension.*

Definition 4.2.4. *Given a theory T over a language Σ interpretable in $\mathcal{C}$, we say that the Σ-structure M is a model for T if M satisfies all the axioms of T (including first order logical axioms).*

Note that this reveals the functorial nature of semantics: if $F : \mathcal{C} \to \mathcal{S}et$ is a Boolean functor (i.e., a functor that preserves finite limits, covers, unions and complements), then such a functor applies any model of T in $\mathcal{C}$ into a model of the theory in the sense of definition 4.1.6. This provides a new insight to the study of the relationship between semantics and syntax, since we can specify models of a theory by specifying Boolean functors to $\mathcal{S}et$ from a Boolean category that contains a (categorical) model of the theory. Joyal's proof of the Completeness theorem can now be seen to have two instances:

a) Constructing a canonical categorical model of a theory inside a convenient Boolean category, such that a formula is satisfied by that model if and only if it is provable in the theory.

b) Constructing a special family of Boolean functors from that category to $\mathcal{S}et$ which jointly reflect the satisfiability of the formulas, providing thus the link between semantics and the syntactical properties of the theory.

4.3 The syntactic category of a theory

Given a first order theory T, we shall define, as done in [8], D1.4, its syntatic category $\mathcal{C}_T$ and a categorical model M_T inside in such a way that a formula in T will be provable if and only if its interpretation in $\mathcal{C}_T$ is satisfied by the model M_T. This already gives a hint regarding what the objects of $\mathcal{C}_T$ should be. We will say that two formulas $(\overline{x}, \phi), (\overline{y}, \psi)$ are equivalent if the formula $\phi(\overline{x}) \leftrightarrow \psi(\overline{y})$ is provable in T through the use of logical and non-logical axioms as well as the rules of inference specified in section 2. We therefore take the objects of $\mathcal{C}_T$ to be the provable-equivalence classes of formulas $(\overline{x}, \phi)$ (note that the set of free variables specified may be empty $\{\}$). To describe the morphisms, consider two objects $[\overline{x}, \phi], [\overline{y}, \psi]$, and assume, without loss of generality, that their set of variables $\overline{x} = x_1, ..., x_n, \overline{y} = y_1, ..., y_m$ are disjoint. Consider now a formula θ that satisfies the following conditions:

a) Its free variables are amongstst $\overline{xy}$.
b) The following formulas are provable in T:

$$\forall \overline{xy}(\theta(\overline{x}, \overline{y}) \to \phi(\overline{x}) \wedge \psi(\overline{y}))$$

$$\forall \overline{xy}(\phi(\overline{x}) \rightarrow \exists \overline{y}(\theta(\overline{x},\overline{y})))$$
$$\forall \overline{xyz}(\theta(\overline{x},\overline{y}) \wedge \theta(\overline{x},\overline{z}) \rightarrow (\overline{y} = \overline{z}))$$

Define now the morphisms between $[\overline{x},\phi]$ and $[\overline{y},\psi]$ to be the provable-equivalence class of all those formulas of T that satisfy conditions a) and b) above (note that the formula $\theta(\overline{x},\overline{z})$ denotes the formula $\theta(\overline{xy})$ after variables occurring in $\overline{y}$ have been replaced for those occurring in $\overline{z}$; in particular, these two sets are implicitely assumed to contain the same number of variables. Also, $\exists\overline{y}$ stands for $\exists y_1 ... \exists y_m$, and similarly, $\overline{y} = \overline{z}$ stands for $y_1 = z_1, ..., y_m = z_m$).

The idea behind this definition is to allow only those morphisms that are exactly needed for our purposes. More precisely, the first formula in condition b) restricts the interpretation $[[\theta(\overline{x},\overline{y})]]$ in any model to be a subobject of $[[\phi(\overline{x}) \wedge \psi(\overline{y})]]$, while the last two formulas imply, if the category has finite limits, that it will be the graph of a morphism from $[[\phi(\overline{x})]]$ to $[[\psi(\overline{y})]]$. Because of the particular construction of the model M_T, this says exactly that the class $[\theta(\overline{x},\overline{y})]$ is a morphism from $[\phi(\overline{x})]$ to $[\psi(\overline{y})]$.

The composite of two morphisms:

$$[\overline{x},\phi] \xrightarrow{[\overline{xy},\theta]} [\overline{y},\psi] \xrightarrow{[\overline{yz},\delta]} [\overline{z},\eta]$$

is defined to be the class $[\overline{xz}, \exists\overline{y}(\theta \wedge \delta)]$. It can be verified that this definition does not depend on the choice of representatives θ, δ and that this morphism so defined satisfies conditions a) and b) above. It can also be verified that composition of morphisms is associative. Finally, the identity morphism on an object $[\overline{x},\phi]$ can be defined to be arrow:

$$[\overline{x},\phi] \xrightarrow{[\overline{xy},\phi(\overline{x})\wedge(\overline{x}=\overline{y})]} [\overline{y},\phi]$$

where $\overline{x} = x_1, ..., x_n$ and $\overline{y} = y_1, ..., y_n$ have both the same number of variables and the representative formula of the codomain class has been obtained by replacing each x_i by y_i in the representative formula of the domain (in what follows we shall assume this is the case provided the morphism between them contains the subformula $\overline{x} = \overline{y}$). Again, it is easily checked that this morphism satisfies condition a) and b) and that it is the unity for composition. Also, note that these definitions do not depend on the choices of representatives in each class.

This makes $\mathcal{C}_T$ a small category. But we can actually prove a much stronger result, namely: $[[\overline{x},\phi]] \leq [[\overline{x},\psi]]$

Theorem 4.3.1. *If T is a first order theory, then $\mathcal{C}_T$ is a Boolean category.*

Proof. We proceed to verify conditions 1)-5) in definition 3.2.1. To prove $\mathcal{C}_T$ has finite limits it suffices to prove it has binary products and equalizers. As the product of two objects $[\overline{x},\phi]$, $[\overline{y},\psi]$ (where $\overline{x}$ and $\overline{y}$ are assumed to be disjoint) we can take the class $[\overline{xy}, \phi \wedge \psi]$ together with the projections indicated below:

$$\begin{tikzcd}
 & {[\overline{z},\chi]} \arrow[dl, "{[\overline{zx'},\theta]}"'] \arrow[d, dashed, "{[\overline{zxy},\theta\wedge\delta]}"] \arrow[dr, "{[\overline{zy'},\delta]}"] & \\
{[\overline{x'},\phi]} & {[\overline{xy},\phi\wedge\psi]} \arrow[l, "{[\overline{x'}\overline{xy},\phi\wedge\psi\wedge(\overline{x'}=\overline{x})]}"'] \arrow[r, "{[\overline{xy}\overline{y'},\phi\wedge\psi\wedge(\overline{y'}=\overline{y})]}"] & {[\overline{y'},\psi]}
\end{tikzcd}$$

Given morphisms $[\overline{zx}, \theta]$ and $[\overline{zy}, \delta]$, the induced morphism into the product is given by the class $[\overline{zxy}, \theta \wedge \phi]$, since it can be easily verified that this is the only morphism that makes the diagram commute.

For the equalizer of a parallel pair of morphisms $[\overline{xy}, \theta], [\overline{xy}, \delta]$, we take:

$$\begin{tikzcd}
{[\overline{x'}, \exists \overline{y}(\theta(\overline{x'}, \overline{y}) \wedge \delta(\overline{x'}, \overline{y}))]} \arrow[rr, "{[\overline{x'}\overline{x}, \exists\overline{y}(\theta\wedge\delta\wedge(\overline{x'}=\overline{x}))]}"] & & {[\overline{x}, \phi]} \arrow[r, bend left, "{[\overline{xy},\theta]}"] \arrow[r, bend right, "{[\overline{xy},\delta]}"'] & {[\overline{y}, \psi]} \\
{[\overline{z}, \chi]} \arrow[u, dashed, "{[\overline{z}\overline{x'},\eta]}"] \arrow[urr, "{[\overline{zx},\eta]}"'] & & &
\end{tikzcd}$$

and the universal property is satisfied with the indicated induced morphism. This proves that $\mathcal{C}_T$ has finite limits. Note as well that there is an initial object given by $[[\{\}, \perp]]$, and a terminal object given by $[[\{\}, \top]]$.

To prove point 2) of definition 3.2.1, given a morphism $[\overline{xy}, \theta] : [\overline{x}, \phi] \to [\overline{y}, \psi]$ we take its image as the subobject $[\overline{y}, \exists \overline{x}(\theta)] \rightarrowtail [\overline{y}, \psi]$. In particular, $[\overline{xy}, \theta]$ is a cover if and only if the formula $\forall \overline{y}(\psi(\overline{y}) \to \exists \overline{x}(\theta(\overline{x}, \overline{y})))$ is provable in T. Then, from the construction of limits above, it can be verified straightforwardly that covers are stable under pullbacks, proving point 3) of the definition. To prove condition 4), take two subobjects $[\overline{x}, \psi], [\overline{x}, \eta]$ from $[\overline{x}, \phi]$ and define their union to be $[\overline{x}, \psi \vee \eta]$. Finally, point 5) is easily verified by considering a subobject $[\overline{x}, \psi] \rightarrowtail [\overline{x}, \phi]$ and taking its complement to be $[\overline{x}, \neg\psi \wedge \phi]$. This concludes the proof. $\square$

Our goal is to relate syntactical provability in T with semantic validity in the categorical model M_T. One aspect of this relation is given by the following lemma, which highlights the syntactical aspects of the properties of $\mathcal{C}_T$:

Lemma 4.3.2. *1) A morphism $[\overline{xy}, \theta] : [\overline{x}, \phi] \to [\overline{y}, \psi]$ is an isomorphism if and only if $[\overline{xy}, \theta] : [\overline{y}, \psi] \to [\overline{x}, \phi]$ is a valid morphism in $\mathcal{C}_T$ (i.e., it satisfies conditions a) and b) of the definition of morphism).*
2) A morphism $[\overline{xy}, \theta] : [\overline{x}, \phi] \to [\overline{y}, \psi]$ is a monomorphism if and only if the formula $\forall \overline{xyz}(\theta(\overline{x}, \overline{y}) \wedge \theta(\overline{z}, \overline{y}) \to \overline{x} = \overline{z})$ is provable in T.
3) Every subobject of $[\overline{x}, \phi]$ is isomorphic to one of the form:

$$[\overline{x}, \psi] \overset{[\psi\wedge(\overline{x}=\overline{y})]}{\rightarrowtail} [\overline{y}, \phi]$$

where ψ is such that the formula $\forall \overline{x}(\psi(\overline{x}) \to \phi(\overline{x}))$ is provable in T. Moreover, any two subobjects $[\overline{x}, \psi], [\overline{x}, \eta]$ in $\mathcal{S}ub([\overline{y}, \phi])$ satisfy $[\overline{x}, \psi] \leq [\overline{x}, \eta]$ if and only if the formula $\forall \overline{x}(\psi(\overline{x}) \to \eta(\overline{x}))$ is provable in T.

Proof. To prove 1), suppose $[\overline{xy}, \theta]$ is a valid morphism from $[\overline{y}, \psi]$ to $[\overline{y}, \phi]$. Then it can be easily checked that $[\overline{xy}, \theta]$ itself is an inverse for $[\overline{xy}, \theta]$. Conversely, if $[\overline{xy}, \theta] : [\overline{x}, \phi] \to [\overline{y}, \psi]$ has an inverse $[\overline{xy}, \delta]$ (which is a valid morphism), then it can be verified that θ and δ are necessarily provable-equivalent in T, from which the result follows.

To prove 2), construct the kernel pair of $[\overline{xy}, \theta] : [\overline{x}, \phi] \to [\overline{y}, \psi]$, which, using the construction of products and equalizers given in the proof of theorem 4.3.1, can be verified to be the class $[\overline{xz}, \exists\overline{y}(\theta(\overline{x}, \overline{y}) \wedge \theta(\overline{z}, \overline{y}))]$. Then, as can be easily checked, the provability of the stated formula is equivalent, by 1), to the fact that the diagonal morphism from $[\overline{x}, \phi]$ to this kernel pair is an isomorphism, which is in turn equivalent, by lemma 3.1.3, to the fact that $[\overline{xy}, \theta]$ is a monomorphism.

Finally, suppose we have a monomorphism $[\overline{xy}, \theta] : [\overline{y}, \psi] \to [\overline{x}, \phi]$. By 1), the morphism $[\overline{xy}, \theta] : [\overline{y}, \psi] \to [\overline{x}, \exists\overline{y}(\theta(\overline{x}, \overline{y}))]$ is an isomorphism. Then, composing its inverse with the original monomorphism we have a subobject of the stated form, where $\psi(\overline{x})$ is the formula $\exists\overline{y}(\theta(\overline{x}, \overline{y}))$. Now, two subobjects $[\overline{y}, \psi], [\overline{y}, \eta]$ of $[\overline{x}, \phi]$ satisfy $[\overline{y}, \psi] \leq [\overline{y}, \eta]$ if and only if there exists a monomorphism $[\overline{y}, \psi] \rightarrowtail [\overline{y}, \eta]$, which by the previous argument must have the form $[\psi' \wedge (\overline{x} = \overline{y})] : [\overline{x}, \psi'] \rightarrowtail [\overline{y}, \eta]$ for some ψ'. But then, since ψ and ψ' must be provable equivalent, this is a valid morphism if and only if the formula $\forall\overline{x}(\psi(\overline{x}) \to \eta(\overline{x}))$ is provable in T. This completes the proof of 3). □

To construct the desired model M_T in the syntactic category of T, note that there is a natural Σ-structure assigning to the sort A the formula $[x, \top]$ where x is a variable of sort A, and to the predicative variable R over variables $x_1, ..., x_n$ of sorts $A_1, ..., A_n$ respectively, the subobject $[x_1, ..., x_n, R(x_1, ..., x_n)] \rightarrowtail [x_1, ..., x_n, \top]$. We have now finally gotten to the important relationship between syntactic provability and semantic validity in M_T:

Theorem 4.3.3. *The formula $\forall\overline{x}(\phi(\overline{x}) \to \psi(\overline{x}))$ is satisfied by the model M_T if and only if it is provable in T. Consequently, a formula $\eta(\overline{x})$ has full extension in M_T if and only if it is provable in T.*

Proof. By definition, the stated formula is satisfied by M_T if and only if the corresponding subobjects in the interpretation satisfy $[[\overline{x}, \phi]] \leq [[\overline{x}, \psi]]$. By the construction of M_T, a straightforward induction on the complexity of ϕ (see remark 4.1.4) proves that the interpretation $[[\overline{x}, \phi]]$ is the subobject $[\overline{x}, \phi] \rightarrowtail [\overline{x}, \top]$. For example, the base of the induction corresponds to the verification of this property for atomic formulas. If $[\overline{x}, \phi]$ is the formula $[R(x_1, ..., x_n)]$ (which in the described interpretation has a sort corresponding to $[x_1, ..., x_n, \top]$), the interpretation $[[\overline{x}, \phi]]$ is by definition the pullback of $[R(x_1, ..., x_n)] \rightarrowtail [x_1, ..., x_n, \top]$ along $[x_1, ..., x_n, \top]$, that is, it is precisely the subobject $[R(x_1, ..., x_n)] \rightarrowtail [x_1, ..., x_n, \top]$. If $[\overline{x}, \phi]$ is the atomic formula $x = x'$, the sort of the variables x, x' correspond to $[y, \top]$ and hence, by definition, the interpretation $[[\overline{x}, \phi]]$ is the equalizer of $[x, x], [x', x'] : [xx', \top] \rightarrowtail [y, \top]$, that is, the subobject $[xx', x = x'] \rightarrowtail [xx', \top]$. Similarly, the rest of the cases of the induction process can be carried out.
Therefore, the assertion $[[\overline{x}, \phi]] \leq [[\overline{x}, \psi]]$ is equivalent to the fact that the two subobjects $[\overline{x}, \phi], [\overline{x}, \psi]$ of $[\overline{x}, \top]$ satisfy $[\overline{x}, \phi] \leq [\overline{x}, \psi]$, which, by lemma 4.3.2 3), is in turn equivalent to the fact that $\forall\overline{x}(\phi(\overline{x}) \to \psi(\overline{x}))$ is provable in T. □

Theorem 4.3.3 says in a way that the model M_T reflects all syntactical relations in the theory T; therefore, the analysis of categorical properties of M_T will reveal facts about provability in T. This is the start of Joyal's proof of the Completeness theorem, which we shall expose in section 5.
To conclude this section, we state some results concerning the identification of models of a first order theory with $\mathcal{S}et$-valued Boolean functors:

Theorem 4.3.4. *Every Boolean functor $F : \mathcal{C}_T \to \mathcal{S}et$ determines a $\mathcal{S}et$-valued model M of T whose domain is given by the image of F. Conversely, for every $\mathcal{S}et$-valued model M of T there is a Boolean functor $F : \mathcal{C}_T \to \mathcal{S}et$ whose image is the domain of M.*

Proof. The first assertion follows after a straightforward induction on the complexity of the formulas of T. To prove the converse result, suppose we have a $\mathcal{S}et$-valued model M, and define F on objects by assigning to the object $[\overline{x}, \phi]$ the extension of the formula $\phi(\overline{x})$ within M, and to the arrow $[\overline{xy}, \theta]$, the function whose graph corresponds to the extension of the formula $\theta(\overline{x}, \overline{y})$ (note that, by soundness, these definitions do not depend on the choice of representatives $\phi(\overline{x}), \theta(\overline{x}, \overline{y})$). It is immediate to verify that this defines a functor, and from the construction of finite limits, images, unions and complements specified in the proof of theorem 4.3.1, it follows that this functor is Boolean. □

With the above identification, natural transformations between two Boolean functors representing models of the same theory correspond, as we shall soon see, to the notion of submodel embedding, defined as follows:

Definition 4.3.5. *A submodel of a model M of T is a substructure N such that for every formula ϕ the following square is a pullback:*

$$\begin{tikzcd}
{[[\overline{x}, \phi]]_N} \arrow[r, tail] \arrow[d] & M_{\Sigma'}(A_1, ..., A_n) \arrow[d, "h_{A_1} \times ... \times h_{A_n}"] \\
{[[\overline{x}, \phi]]_M} \arrow[r, tail] & M_{\Sigma}(A_1, ..., A_n)
\end{tikzcd}$$

It is immediate to verify that this definition generalizes definition 4.1.8, which is a particular case for $\mathcal{S}et$-valued models.

Lemma 4.3.6. *If there exists a natural transformation $\eta : N \Rightarrow M$ between two Boolean functors $N, M : \mathcal{C}_T \to \mathcal{S}et$ which give models of T, then N is a submodel of M. Conversely, for every submodel there is a corresponding natural transformation between the involved functors.*

Proof. The last sentence follows easily from the definition of submodel. To prove the first sentence, note that, by definition, if Σ', Σ are the signatures corresponding to N, M respectively, we have arrows $\eta_A : M_{\Sigma'}(A) \to M_{\Sigma}(A)$ for each sort A, as well as commutative squares of the form:

$$\begin{tikzcd}
{[[\overline{x}, \phi]]_N} \arrow[r, tail] \arrow[d] & M_{\Sigma'}(A_1, ..., A_n) \arrow[d, "\eta_{A_1} \times ... \times \eta_{A_n}"] \\
{[[\overline{x}, \phi]]_M} \arrow[r, tail] & M_{\Sigma}(A_1, ..., A_n)
\end{tikzcd}$$

for every formula ϕ, since $[[\overline{x}, \top]] \cong M(A_1, ..., A_n)$. Moreover, the commutativity of the square above for all formulas ϕ imply that it is necessarily a pullback square. Indeed, consider two complementary formulas ϕ and $\neg\phi$, together with the pullbacks P, Q along $\eta_{A_1} \times ... \times \eta_{A_n}$ of their respective interpretations in M:

$$\begin{tikzcd}
P \arrow[r, tail] \arrow[d] & M_{\Sigma'}(A_1, ..., A_n) \arrow[d, "\eta_{A_1} \times ... \times \eta_{A_n}"] & Q \arrow[l, tail] \arrow[d] \\
{[[\overline{x}, \phi]]_M} \arrow[r, tail] & M_{\Sigma}(A_1, ..., A_n) & {[[\overline{x}, \neg\phi]]_M} \arrow[l, tail]
\end{tikzcd}$$

Then P, Q are subobjects of $M_{\Sigma'}(A_1, ..., A_n)$ which must be disjoint (since $[[\overline{x}, \phi]]_M$ and $[[\overline{x}, \neg\phi]]_M$ are disjoint) and must contain $[[\overline{x}, \phi]]_N$ and $[[\overline{x}, \neg\phi]]_N$ respectively (because of the universal property of pullbacks). It follows then that $P = [[\overline{x}, \phi]]_N$ and $Q = [[\overline{x}, \neg\phi]]_N$, as asserted.

It only rests to prove that each η_A is a monomorphism. But since the square above is a pullback for atomic formulas (in particular, for the equality $x = y$, where x, y are variables of sort A), it follows that the square below is a pullback:

$$\begin{tikzcd}
M_{\Sigma'}(A) \arrow[r, tail, "\Delta"] \arrow[d, "\eta_A"'] & M_{\Sigma'}(A) \times M_{\Sigma'}(A) \arrow[d, "\eta_A \times \eta_A"] \\
M_{\Sigma}(A) \arrow[r, tail, "\Delta"] & M_{\Sigma}(A) \times M_{\Sigma}(A)
\end{tikzcd}$$

Hence, a simple application of lemma 3.1.4 proves that each η_A is monic, which completes the proof. $\square$

Remark 4.3.7. *Lemma 4.3.6 can be generalized in a precise way by showing that the bijection between natural transformations and submodel embeddings is functorial. As a consequence, it can be shown that the category of $\mathcal{S}et$-valued models of a first order theory, where the morphisms are given by submodel embeddings, is equivalent to the full subcategory of Boolean functors in $\mathcal{S}et^{\mathcal{C}_T}$.*

5 Joyal's proof

5.1 Idea of the proof

In section 3 we have established a correspondence between provability in a first order theory T and satisfiability in the canonical model M_T of its syntactic category. As mentioned in section 2, this allows to analyze usual $\mathcal{S}et$-valued models of the theory by means of Boolean functors $F : \mathcal{C}_T \to \mathcal{S}et$.

We are interested now in functors that provide models, but such that the converse process can take place, that is, from properties of $\mathcal{S}et$-valued models we would like to decide satisfiability conditions in M_T, which would be linked directly to syntactic provability. One of the properties that makes such a reverse process possible is conservativity of those functors. Semantically, this means that satisfiability in a $\mathcal{S}et$-valued model implies satisfiability in M_T. Of course, this is almost never the case, but instead we can ask whether a certain class of functors is jointly conservative (i.e., whether for a subobject $A \rightarrowtail B$, $F(A) \cong F(B)$ for every F in the class implies $A \cong B$). In fact, jointly conservativity is exactly the concept that will lead to completeness:

Theorem 5.1.1. *A first order theory T satisfies the Completeness theorem if and only if there exists a family of jointly conservative Boolean functors from $\mathcal{C}_T$ to $\mathcal{S}et$.*

Proof. If a formula is valid in every $\mathcal{S}et$-valued model, its interpretation has full extension in every one of them. Therefore, its interpretation in M_T will have full extension (that is, the formula will be provable in T) if and only if there exists a family of Boolean functors that is jointly conservative. $\square$

Thus, the whole proof reduces to find the appropriate class of jointly conservative Boolean functors. Should representable functors be Boolean, then such a class would provide an answer, since representable functors are jointly conservative by Yoneda's lemma. However, they fail to satisfy all conditions needed for a functor to be Boolean, and we must look for a somewhat different aproach.

It should also be noted that since we are planning to carry out the proof within ZF, it is not possible to mention the large category $\mathcal{S}et$, since it requires either an axiomatic system suitable for handling proper classes, such as von Neumann-Bernays-Gödel NBG system, or some extra assumption regarding the existence of inaccessible cardinals, which support the construction of appropriate Grothendieck universes. It is known, however, that such an assumption is not provable within ZF (see, for example, [7]), and therefore we shall be compelled to work in a convenient small subcategory of $\mathcal{S}et$, big enough to support all Boolean functors of a jointly conservative class.

5.2 The finite coproduct completion

The first step of the proof shall be to find an appropriate immersion of $\mathcal{C}_T$ into a Boolean category that has finite disjoint coproducts, since these will be needed for the construction of Boolean functors. Note that although $\mathcal{C}_T$ has finite limits, it is not true that it has even finite coproducts, which means we shall need to enlarge it to admit them. This is done in [13], ch. 4, by considering the category of sheaves $\neg\neg\mathcal{C}_T$ for the double negation topology, which is known to be Boolean (see, for instance, [11]). However, we shall follow a much simpler and elementary approach, which is to consider the completion of $\mathcal{C}_T$ by finite coproducts, as explained in [8], A1.4. This alternative has the essential advantage of avoiding large topoi and fits better for our purposes.

Lemma 5.2.1. *If $\mathcal{C}$ is a regular (resp. Boolean) category, there exists an immersion J of $\mathcal{C}$ into a regular (resp. Boolean) category $P(\mathcal{C})$ that also has finite disjoint coproducts. Moreover, $P(\mathcal{C})$ satisfies the following universal property: for every regular (resp. Boolean) functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is regular (resp. Boolean) and has finite disjoint coproducts, there exists a regular (resp. Boolean) functor $\overline{F} : P(\mathcal{C}) \to \mathcal{D}$ satisfying $\overline{F}J = F$, such that for any other functor $G : P(\mathcal{C}) \to \mathcal{D}$ satisfying $GJ = F$, there exists a unique natural isomorphism $\phi : G \Rightarrow \overline{F}$ such that $\phi Id_J = Id_F$:*

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "J"] \arrow[dr, "F"'] & & P(\mathcal{C}) \arrow[dl, "\overline{F}"] \arrow[dl, bend left=60, "G"] \arrow[dl, phantom, shift left=5, "\overset{\phi}{\Leftarrow}"] \\
& \mathcal{D} &
\end{tikzcd}$$

Proof. Consider the category $P(\mathcal{C})$ whose objects are n-tuples $(A_1, ..., A_n)$ of objects of $\mathcal{C}$. Define a morphism between $(A_1, ..., A_n)$ and $(B_1, ..., B_m)$ to be specified by the following:
i) A m-fold decomposition of each A_i, that is, a set of m pairwise disjoint subobjects $A_{i_1}, ..., A_{i_m}$ of A_i such that $\bigvee_{j=1}^{m} A_{i_j} = A_i$ (we allow some of the subobjects to be 0, the initial object of $\mathcal{C}$).
ii) A set of $\mathcal{C}$-morphisms $f_{ij} : A_{i_j} \to B_j$ for each $i = 1, ..., n, j = 1, ..., m$.
Given morphisms $F : (A_1, ..., A_n) \to (B_1, ..., B_m)$ associated to arrows $f_{ij} : A_{i_j} \to B_j$ and $G : (B_1, ..., B_m) \to (C_1, ..., C_p)$ associated to arrows $g_{jk} : B_{j_k} \to C_k$, define the composite $GF : (A_1, ..., A_n) \to (C_1, ..., C_p)$ to be the morphism given by the arrows $h_{ik} = (\exists_{g_{1k} f_{i1}}, ..., \exists_{g_{mk} f_{im}}) : \bigvee_{j=1}^{n}(f_{ij}^{-1}(B_{j_k}) \wedge A_{i_j}) \to C_k$, where $f_{ij}^{-1} : \mathcal{S}ub(B_j) \to \mathcal{S}ub(A_i)$ is the usual restriction of the pullback functor. Note that this definition uses here the fact asserted in lemma 3.2.12 that the union of disjoint subobjects is their coproduct, since the morphism $(\exists_{g_{1k} f_{i1}}, ..., \exists_{g_{mk} f_{im}})$ is well defined only if $\bigvee_{j=1}^{n}(f_{ij}^{-1}(B_{j_k}) \wedge A_{i_j})$ is a coproduct. Also, the fact that composition of arrows is associative is similar to the fact that matrix multiplication is, given the resemblance of $\bigvee_{j=1}^{n}(f_{ij}^{-1}(B_{j_k}) \wedge A_{i_j})$ with the (k, i)-th entry $\sum_{j=1}^{n} b_{j_k} a_{i_j}$ of a matrix product. Finally, define the identity morphism $Id_{(A_1,...,A_n)}$ to be given by trivial decompositions of each A_i and the arrows $f_{ii} = Id_{A_i}$, f_{ij} initial for $i \neq j$. With these specifications, $P(\mathcal{C})$ becomes a category that contains $\mathcal{C}$ as the full subcategory of all 1-tuples.

Suppose now that $\mathcal{C}$ is regular (resp. Boolean), and let us see that $P(\mathcal{C})$ is necessarily regular (resp. Boolean). It is a straightforward verification that the following works: the product of two objects $(A_1, ..., A_n)$ and $(B_1, ..., B_m)$ is given by the nm-tuple whose entries are the products $A_i \times B_j$, with the obvious projections. Their coproduct is just given by the $(n+m)$-tuple $(A_1, ..., A_n, B_1, ..., B_m)$; moreover, it can be checked that coproducts are disjoint. The equalizer of a pair of morphisms $M, N : (A_1, ..., A_n) \to (B_1, ..., B_m)$ associated with arrows f_{ij}, g_{ij} respectively, is given by the morphism $(C_1, ..., C_n) \to (A_1, ..., A_n)$, where $C_i = \bigvee_{j=1}^{m} E_{ij}$ and the associated arrows $e : E_{ij} \to A_j$ are the equalizers of the arrows $f_{ij}, g_{ij} : A_{i_j} \to B_j$. The image of a morphism $M : (A_1, ..., A_n) \to (B_1, ..., B_m)$ associated to the arrows $f_{ij} : A_{i_j} \to B_j$ is given by the subobject $(M_1, ..., M_m) \rightarrowtail (B_1, ..., B_m)$, where $M_j = \bigvee_{i=1}^{m} Im(f_{ij})$ is a subobject of B_j. This explicit expression for the image provides as well a criterion for a morphism to be a cover, and it follows from the regularity (resp. Booleanness) of $\mathcal{C}$ that covers must be stable under pullbacks.

In case that $\mathcal{C}$ is Boolean, then since $P(\mathcal{C})$ has images and coproducts, it follows that it also has finite unions for subobjects and that these are stable under pullbacks. On the other hand, given a subobject $M : (A_1, ..., A_n) \rightarrowtail (B_1, ..., B_m)$ we can form its complement as the subobject $(D_1, ..., D_m) \rightarrowtail (B_1, ..., B_m)$, where $D_j = \bigvee_{i=1}^{n} \neg A_{i_j}$ is a subobject $f_j : D_j \rightarrowtail B_j$ and the monomorphism is given by the trivial decomposition on each D_j, while the associated arrows are given by $h_{ii} = f_i$ and h_{ij} initial for $i \neq j$.

Finally, by construction, for every regular (resp. Boolean) functor $F : \mathcal{C} \to \mathcal{D}$ we can consider the functor $\overline{F}$ defined in the evident manner, i.e., applying each n-tuple $(A_1, ..., A_n)$ into the coproduct $\coprod_{i=1}^{n} F(A_i)$ in $\mathcal{D}$, and each morphism into the corresponding induced morphism between coproducts. A straightforward calculation shows it must be regular (resp. Boolean). Moreover, if there are two functors $\overline{F}, G$ with the stated properties that extend F, then they can only differ in the choice of the coproducts above, and hence there are canonical isomorphisms $\phi_C : G(C) \to \overline{F}(C)$, induced by the universal property of the coproduct. Given a morphism $f : C \to D$, the arrows $\phi_D \overline{F}(f)$ and $G(f)\phi_C$ would be two induced morphisms between the coproducts $\overline{F}(C)$ and $G(D)$; therefore, they must coincide, and then the isomorphisms ϕ_C define a natural isomorphism $\phi : G \Rightarrow \overline{F}$, as stated. Finally, it is clear that $\phi Id_J = Id_F$. $\square$

5.3 Making the terminal object projective

Once we have a (Boolean) immersion of $\mathcal{C}_T$ into a Boolean category with finite disjoint coproducts (by using the construction in the proof of lemma 5.2.1), we are interested in finding Boolean functors to (a small subcategory of) $\mathcal{S}et$. As we have already seen, in general representable functors are not even regular, mainly because they do not preserve covers. In fact, for a representable functor $[A, -]$ to preserve covers it is necessary that A be projective with respect to covers (i.e., to be cover-projective), since covers in $\mathcal{S}et$ are precisely the surjections. For a regular category, we can find an equivalent condition:

Lemma 5.3.1. *The representable functor* $[A, -]$ *preserves covers (i.e.,* A *is cover-projective) if and only if every cover* $p : X \twoheadrightarrow A$ *has a section.*

Proof. If $[A, -]$ is cover-projective, given a cover $p : X \twoheadrightarrow A$ we can take the factorization of $Id_A : A \to A$ through X, which provides a section for p. Conversely, suppose that every cover over A has a section. Given a morphism $f : A \to Y$ and a cover $p : X \twoheadrightarrow Y$, form the pullback P of p along f:

$$
\begin{tikzcd}
P \arrow[r, "f'"] \arrow[d, two heads, "p'"'] & X \arrow[d, two heads, "p"] \\
A \arrow[r, "f"] \arrow[u, bend left, "s"] & Y
\end{tikzcd}
$$

Then p' must be a cover over A, and if $s : A \to P$ is a section, we get the factorization $f = pf's$. $\square$

Lemma 5.3.1 motivates the search of those objects for which every cover on them has a section.

Our next step will be enlarging our category $P(\mathcal{C})$ appropriately so that the terminal object becomes projective. *This idea can be interpreted as the categorical version of Henkin's proof of the Completeness theorem, where the process of making the terminal object projective corresponds to the process of adding constants to the language of the theory.*

Lemma 5.3.2. *If* $\mathcal{C}$ *is a regular (resp. Boolean) category and* $f : A \twoheadrightarrow B$ *is a cover in* $P(\mathcal{C})$*, then the pullback functor* f^* *is a conservative regular (resp. Boolean) functor that preserves finite coproducts.*

Proof. Because of lemma 3.2.11, we just need to prove that f^* preserves coproducts. But because injections into the coproduct are monomorphisms whose intersection is initial, the coproduct is their union and is thus preserved by pullbacks. $\square$

Let $\mathcal{C}_0 = P(\mathcal{C})$; our goal is to make the terminal object cover-projective. We shall do so by constructing succesive categories $\{\mathcal{C}_n / n \in \mathbf{N}\}$, each one embedded in the next, such that the terminal object of $\mathcal{C}_n$ is cover-projective for all covers that are images of covers in $\mathcal{C}_{n-1}$.

First, let us describe an embedding $I_1 : \mathcal{C}_0 \to \mathcal{C}_1$ that has this property for all covers in $\mathcal{C}_0$. Let Γ be the indexing set of all such covers $\{A_i \twoheadrightarrow 1 / i \in \Gamma\}$. Consider, for each finite $F \subseteq \Gamma$, the set of covers $\{A_i \twoheadrightarrow 1 / i \in F\}$ together with the canonical projections $\pi_{FG} : \prod_{i \in G} A_i \twoheadrightarrow \prod_{i \in F} A_i$ for $F \subseteq G$. Define the category I whose objects are all finite products of objects $P_F = \prod_{i \in F} A_i$, $F \subseteq \Gamma$, and whose arrows are given by the corresponding canonical (induced) morphisms π_{FG} between such products. Note that even if the products are not canonical, the morphisms are nevertheless canonical (even those morphisms between isomorphic products). Then I is clearly filtered, since between any two objects there is at most one morphism, and for a pair of objects P_F, P_G we can consider the object $P_{F\cup G}$ along with the corresponding pullback functors.

Consider now in $\mathcal{C}_0$ the category $\mathcal{A}_I$ of arrows over I, with the corresponding fibration $\pi : \mathcal{A}_I \to I$ (see definition 3.4.15 and the following comments). If we denote by S the set of cartesian morphisms, we have the following:

Lemma 5.3.3. *The category $\mathcal{C}_1 = \mathcal{A}_I[S^{-1}]$ is a regular (resp. Boolean) category that has finite disjoint coproducts, and there is a conservative regular (resp. Boolean) functor $I_1 : \mathcal{C}_0 \to \mathcal{C}_1$ that preserve coproducts. Furthermore, if $A' \rightarrowtail A$ is a proper subobject in $\mathcal{C}_0$ and $A \twoheadrightarrow 1$ is a cover, the corresponding cover $I_1(A) \twoheadrightarrow 1_{\mathcal{C}_1}$ in $\mathcal{C}_1$ has a section not factoring through $I_1(A')$.*

Proof. Since the same argument in the proof of lemma 5.3.2 proves that S is locally regular (resp. locally Boolean), then, by lemma 3.4.14, $\mathcal{C}_1$ is a regular (resp. Boolean) category and the functor $I_1 : \mathcal{C}_0 \to \mathcal{C}_1$ is regular (resp. Boolean); moreover, since finite coproducts are the unions of the corresponding injections, I_1 must preserve them as well. Let us prove it is conservative. If a morphism $f : A \to B$ has an inverse f^{-1} in $\mathcal{C}_1$, it must necessarily be of the form represented by the right square:

$$\begin{tikzcd} B \arrow[d] & B \times C \arrow[l, "\pi_1"'] \arrow[r] \arrow[d, "\pi_2"] & A \arrow[d] \\ 1 & C \arrow[l] \arrow[r] & 1 \end{tikzcd}$$

But there are isomorphisms $[A \to 1] \cong [\pi_2 : A \times C \to C]$ and $[B \to 1] \cong [\pi_2 : B \times C \to C]$, and thus f^{-1} corresponds to the morphism represented by the right square:

$$\begin{tikzcd} B \times C \arrow[d, "\pi_2"'] & B \times C \arrow[l] \arrow[r, "g"] \arrow[d, "\pi_2"] & A \times C \arrow[d, "\pi_2"] \\ C & C \arrow[l, "Id_C"] \arrow[r] & C \end{tikzcd}$$

for some $g : B \times C \to A \times C$. Regarding this as a morphism over C in $\mathcal{C}/C$, we see that it is an inverse of the morphism $f \times Id_C : A \times C \to B \times C$ over C. This latter morphism can be written as $\pi^*_{FG}(f)$, where $F = \emptyset$ and $G = C$, and hence we conclude that the arrow π_{FG} maps f into an invertible morphism in $\mathcal{C}/C$. Therefore, the same argument used in the proof of lemma 5.3.2 shows that "local" conservativity holds and f has already an inverse in $\mathcal{C}_0$.

Finally, given a proper subobject $A' \rightarrowtail A$ in $\mathcal{C}_0$, then the dual of the morphism represented by the commutative square on the left:

$$\begin{tikzcd} 1 \arrow[d] & A \arrow[l] \arrow[r, "Id_A"] \arrow[d, "Id_A"] & A \arrow[d] \\ 1 & A \arrow[l] \arrow[r] & 1 \end{tikzcd}$$

provides a section for $I_1(A) \twoheadrightarrow 1_{\mathcal{C}_1}$ not factoring through $I_1(A')$. Indeed, there are isomorphisms $[A \to 1] \cong [\pi_2 : A \times A \to A]$ and $[1 \to 1] \cong [Id_A : A \to A]$, and the section above corresponds to the morphism represented by the following right square:

$$\begin{tikzcd}
A \arrow[d, "Id_A"'] & A \arrow[l, "Id_A"'] \arrow[r, "\Delta"] \arrow[d, "Id_A"] & A \times A \arrow[d, "\pi_2"] \\
A & A \arrow[l, "Id_A"] \arrow[r, "Id_A"'] & A
\end{tikzcd}$$

Hence, if the section factored through $I_1(A')$, the diagonal Δ would factor through $A' \times A$, which is impossible. This completes the proof. $\square$

Remark 5.3.4. *In the presence of the axiom of choice, the construction above can be performed with the aid of theorem 3.4.10 and lemma 3.4.12. Indeed, it suffices to define a pseudofunctor $F : I^{op} \to \mathcal{C}at$ such that the category $\mathcal{C}_1$ is its bicolimit, constructed as the category of fractions of the fibration associated to F. This issue is therefore reduced to the problem of finding a pseudofunctor F whose associated fibration is precisely the fibration corresponding to the category of arrows over I. This can be done as follows: define F on an object P_F as the slice category $\mathcal{C}/P_F$; for each identity arrow Id_{P_F} define F as the corresponding identity $Id_{\mathcal{C}/P_F}$, and for each arrow π_{FG} of I select a fixed pullback functor π^*_{FG} and set it as the value of F on such an arrow. Then, if for arrows f, g of I we define $c_{f,g}$ as the corresponding induced natural isomorphism between pullbacks, F becomes a pseudofunctor, as can be easily checked. Finally, the fibration corresponding to the category of arrows over I can be interpreted as the fibration associated to F, since each commutative square can be completely defined by specifying the arrow at its base and the morphism from the upper left corner to the selected pullback along that arrow (see [4], Ex. VI 8).*

Note that the procedure described above allows to have a section for each cover $I_1(A_i) \twoheadrightarrow 1_{\mathcal{C}_1}$, where $A_i \twoheadrightarrow 1$ is in the set of covers of $\mathcal{C}$. The next step to take is to extend this property to every cover $A \twoheadrightarrow 1_{\mathcal{C}_1}$ in $\mathcal{C}_1$. But the process to follow now is clear: repeating the whole construction above for $\mathcal{C}_1$ instead of $\mathcal{C}$ we get an embedding $I_2 : \mathcal{C}_1 \to \mathcal{C}_2$ preserving all the structure, such that in the new category $\mathcal{C}_2$ every cover $I_2(A) \twoheadrightarrow 1_{\mathcal{C}_2}$ has a section for each A in the set of covers $A \twoheadrightarrow 1_{\mathcal{C}_1}$ of $\mathcal{C}_1$. Iterating this construction, we can obtain a sequence of conservative regular (resp. Boolean) embeddings preserving coproducts, $I_n : \mathcal{C}_{n-1} \to \mathcal{C}_n$. This amounts to having a functor $F : \omega \to \mathcal{C}at$, or, formally dualizing, a functor $F : \omega^{op} \to \mathcal{C}at$, and if T is the set of cartesian morphisms in the fibration Γ_F associated to F, we can consider the category $\mathcal{C}_\omega = \Gamma_F[T^{-1}]$. It is now easy to verify that $\mathcal{C}_\omega$ is the category we need, as shown in the following:

Theorem 5.3.5. *There is a regular (resp. Boolean) conservative functor $I_0 : \mathcal{C}_0 \to \mathcal{C}_\omega$, where $\mathcal{C}_\omega$ is a regular (resp. Boolean) category with finite disjoint coproducts such that every cover over the terminal object 1_ω has a section. Moreover, for every proper subobject $f : S \rightarrowtail A$ such that $A \twoheadrightarrow 1_\omega$ is a cover, there exists a section $s : 1_\omega \to A$ not factoring through S.*

Proof. The fact that $\mathcal{C}_\omega$ is a regular (resp. Boolean) category and the induced functors $\mathcal{C}_n \to \mathcal{C}_\omega$ are regular (resp. Boolean) follows from lemma 3.4.12, and it is not difficult to see that these induced functors are conservative and preserve finite coproducts. Finally, given a proper subobject $A' \rightarrowtail A$ in $\mathcal{C}_1$, where $A \twoheadrightarrow 1$ is a cover, there exists some $n \in \mathbf{N}$ such that f has a representative that lies in some $\mathcal{C}_n$, and in $\mathcal{C}_{n+1}$ we can find a representative of a section $1_{\mathcal{C}_\omega} \to A$ not factoring through A'. $\square$

5.4 Ultra-representability and $\mathcal{S}et$-valued models

Our work so far has been developed in a parallel way for regular and Boolean categories. In order to get the desired jointly conservative family of functors that will prove the Completeness theorem, we have to apply theorem 5.3.5 to the category $\mathcal{C}_0 = \mathcal{C}_T$. This latter category will have different properties according to which type of theory is under consideration. In particular, for first order classical logic we shall now make explicit use of the fact that $\mathcal{C}_T$ has complements. It is at this point where our work leaves the main course, since it is possible to develop completeness theorems for different types of logic by considering different theories T. Also, the consideration of first order theories will need the Axiom of Choice (or a weaker choice principle) as we shall see, while for weaker types of logics this is not needed.

We have obtained a conservative Boolean functor $\mathcal{C}_T \to \mathcal{C}_\omega$, where $\mathcal{C}_\omega$ has finite coproducts and satisfies the property that $1 (= 1_\omega)$ is cover-projective. Actually, we can get a better result:

Lemma 5.4.1. *Every subobject of* 1 *in* $\mathcal{C}_\omega$ *is cover-projective.*

Proof. Let $S \rightarrowtail 1$ be a subobject and S' its complement in $\mathcal{S}ub(1)$. If $\pi : A \twoheadrightarrow S$ is a cover, then we also have a cover $\pi \coprod Id_{S'} : A \coprod S' \twoheadrightarrow S \coprod S' \cong 1$. Since 1 is projective, this cover splits, and because coproducts are disjoint, its section must map S into A, which yields a section for π. $\square$

Consider now the (small) category $\mathcal{C}_\omega$ and define in ZF the set of all its hom-sets. Consider all possible finite limits and coproducts as well as colimits indexed over diagrams in $\mathcal{S}ub(1)$ (note that because limits and colimits in $\mathcal{S}et$ can be constructively described, there is a canonical choice for each one of them). Call the resulting set $\mathcal{P}$ and consider the full subcategory of $\mathcal{S}et$, $\mathcal{S}$, whose objects are elements of $\mathcal{P}$. Clearly, every finite limit, coproduct and colimit (of the type specified) of representable functors, has image in $\mathcal{S}$. Thus, from now on these functors will be always understood to be constructed in this way, to avoid any mention to the large category $\mathcal{S}et$ (which is replaced by $\mathcal{S}$) or multiple choices between colimits. Note that it is essential for this to be carried out that finite limits in $\mathcal{C}_\omega$ be determined effectively and not just up to isomorphism. This in turn can be possible due to the special care taken when defining $\mathcal{C}_\omega$ through the use of lemma 3.4.14 instead of lemma 3.4.12, avoiding therefore choices of the corresponding pullback functors. Furthermore, the reader may verify that all constructions made upon $\mathcal{C}_T$ to obtain $\mathcal{C}_\omega$ were done in a constructive way (particularly the choice of a canonical filtered bicolimit in $\mathcal{C}at$), from which we can be sure to be able to formalize the whole argument entirely within ZF+BPI.

So far the construction of $\mathcal{C}_\omega$ makes the representable functors $[s, -]$ for subobjects $s \rightarrowtail 1$ have all properties of a Boolean functor except preservation of unions. To get this latter property we introduce the following:

Definition 5.4.2. *A functor* $h : \mathcal{C}_\omega \to \mathcal{S}$ *is said to be ultra-representable if there is* $C \in \mathcal{C}_\omega$ *such that it can be expressed as a (filtered) colimit* $h = \varinjlim_{A \in \Phi} [A, -]$ *for some ultrafilter* Φ *in* $\mathcal{S}ub(C)$.

Note that we can use BPI to ensure the existence of such ultrafilters, and then these functors are well defined. We have now:

Theorem 5.4.3. *If* Φ *is an ultrafilter in* $\mathcal{S}ub(S)$ *for a subobject* $S \rightarrowtail 1$*, the ultra-representable functor* $h_\Phi : \mathcal{C}_\omega \to \mathcal{S}$ *is Boolean.*

Proof. According to lemma 3.3.2, filtered colimits of representable functors (which are left exact) preserve finite limits in $\mathcal{S}et$. Hence, h will necessarily preserve finite limits. As for images, note that since the injections into the colimit form a jointly regular epimorphic family and this property is preserved by h, it is mapped onto a jointly epic family in $\mathcal{S}et$ (where covers are surjections), and hence h will preserve covers provided each representable functor $[A, -]$ does, which is clearly the case because each subobject A is cover-projective by lemma 5.4.1. Therefore, we just have to prove that h preserves unions, i.e., given B, C subobjects of D, we need to prove that $h(B \vee C) = h(B) \vee h(C)$. Clearly, $h(B) \vee h(C) \leq h(B \vee C)$, since h preserves finite limits, and therefore monomorphisms. To prove the converse inequality, note that, according to the usual construction of filtered colimits in $\mathcal{S}et$ (see [10]), we have $h(X) = \coprod_{A \in \Phi} [A, X]/\sim$, where $\sim$ is the equivalence relation which identifies $f : U \to X$ with $g : V \to X$ if and only if there exists some $W \in \Phi$ such that the following square commutes:

$$\begin{tikzcd}
U \arrow[r, "f"] & X \\
W \arrow[u, tail] \arrow[r, tail] & V \arrow[u, "g"']
\end{tikzcd}$$

Take (some representative of) an arrow $f : U \to B \vee C$, for some $U \in \Phi$. Since unions are stable under pullback, we have $U = f^{-1}(B) \vee f^{-1}(C)$. Now, since Φ is an ultrafilter, either $f^{-1}(B)$ or $f^{-1}(C)$ is in Φ; suppose without loss of generality $f^{-1}(B) \in \Phi$. Then the following pullback:

$$\begin{tikzcd}
B \arrow[r, tail, "s"] & B \vee C \\
f^{-1}(B) \arrow[u, "f'"] \arrow[r, tail] & U \arrow[u, "f"']
\end{tikzcd}$$

shows that f and sf' are in the same class in $h(B \vee C)$. We may therefore assign to each $f \in h(B \vee C)$ either an arrow $f' \in h(B)$ or an arrow $f'' \in h(C)$, from which we conclude that $h(B \vee C) \leq h(B) \vee h(C)$. Hence, we must have $h(B \vee C) = h(B) \vee h(C)$ in $\mathcal{S}$. □

The final step in Joyal's proof is given by the following:

Theorem 5.4.4. *The family of ultra-representable functors $h_\Phi : \mathcal{C}_\omega \to \mathcal{S}$, where Φ ranges over all ultrafilters in $\mathcal{S}ub(1)$, is jointly conservative.*

Proof. Suppose we have a proper subobject $t : A \rightarrowtail X$. We shall prove that there exists some ultra-representable functor h such that $h(A) \rightarrowtail h(X)$ is a proper subobject in $\mathcal{S}$. Since $X \coprod 1 \twoheadrightarrow 1$ is a cover, by theorem 5.3.5 there exists a section $u : 1 \to X \coprod 1$ not factoring through the proper subobject $A \coprod 1 \rightarrowtail X \coprod 1$. Pulling back along the first injection, we obtain a subobject $S \rightarrowtail 1$ and an arrow $s : S \to X$ not factoring through A. Form the further pullback P along s:

$$\begin{tikzcd}
 & A \arrow[r, tail, "t"] & X \\
W \arrow[ur, "f"] & P \arrow[u] \arrow[r, tail] & S \arrow[u, "s"'] \\
Q \arrow[u] \arrow[ur, dashed] \arrow[urr] & &
\end{tikzcd}$$

Clearly, $P \rightarrowtail S$ will be then a proper subobject. Consider the ideal $\mathcal{I}$ in $\mathcal{S}ub(1)$ of all subobjects of P. Since $S \notin \mathcal{I}$, we deduce from BPI that there must be some ultrafilter Φ in $\mathcal{S}ub(1)$ containing S and disjoint from $\mathcal{I}$. Then define $h = \lim_{\overrightarrow{U\in\Phi}}[U, -]$; it is now easy to check that the class of s in $h(X)$ is an element not belonging to $h(A)$. Indeed, if s were in the same class that tf for some $f : W \to A$, then there would exist some $Q \in \Phi$ making commutative the diagram above. This would, hence, induce a monomorphism $Q \rightarrowtail P$, contradicting the fact that $Q \notin \mathcal{I}$. $\square$

Theorem 5.4.4 finishes the study of completeness for first order theories, according to the result stated in theorem 5.1.1.

6 A characterization of models

6.1 Boolean models and the Löwenheim-Skolem theorem

The proof of the Completeness theorem as presented in section 5 only requires to verify validity in those models defined by means of ultra-representable functors associated to ultrafilters in $\mathcal{S}ub(1)$. Assuming the Axiom of Choice, and if the signature of the language of a given theory has cardinality κ, it is not difficult to see that the objects and morphisms of the category $\mathcal{C}_T$ both have cardinality κ. Furthermore, an analysis of the construction of lemmas 5.3.3 and 5.3.5 shows that the same holds for the category $\mathcal{C}_\omega$, which implies that the mentioned ultra-representable functors $h : \mathcal{C}_\omega \to \mathcal{S}$ provide models of cardinality at most κ. Thus, we have actually proved the following:

Theorem 6.1.1. *If a formula of a first order theory is valid in all models of cardinality not greater than that of the signature, then the formula is provable within the theory.*

Note that according to the well known generalized version of the Löwenheim-Skolem theorem, a theory with an infinite model has models of all infinite cardinalities. This implies that the theorem above provides a stronger notion of completeness, since it follows that in order to deduce the provability of a formula it is enough to verify its satisfiability in a (small) set of models rather than in the (large) class of all models. In fact, assuming the Axiom of Choice, we can actually provide a detailed description of all Boolean models of a given theory, of which those used in the proof of the Completeness theorem form a relatively small subclass.

We start with the following:

Lemma 6.1.2. *Let $\mathcal{C}$ be a regular (resp. Boolean) category and let A be an object with full support (i.e. such that $f : A \twoheadrightarrow 1$ is a cover); let $F : \mathcal{C} \to \mathcal{D}$ be a regular (resp. Boolean) functor and let s be a section of $F(A) \twoheadrightarrow F(1)$ in the regular (resp. Boolean) category $\mathcal{D}$. Then, for each fixed pullback functor f^*, there is a regular (resp. Boolean) functor $\overline{F} : \mathcal{C}/A \to \mathcal{D}$ satisfying $\overline{F}(\Delta) = s$ and $\overline{F}f^* = F$, such that for every functor $G : \mathcal{C}/A \to \mathcal{D}$ and natural isomorphism $\psi : Gf^* \Rightarrow F$, there exists a unique natural isomorphism $\phi : G \Rightarrow \overline{F}$ such that $\phi Id_{f^*} = \psi$:*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "f^*"] \arrow[dr, "F"'] & & \mathcal{C}/A \arrow[dl, "\overline{F}"'] \arrow[dl, bend left=60, "G"] \arrow[dl, phantom, bend left=30, "\Leftarrow\phi"] \\ & \mathcal{D} & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C} \arrow[rr, "f^*"] \arrow[dr, "F"'] & \arrow[d, phantom, "\Leftarrow\psi"] & \mathcal{C}/A \arrow[dl, "G"] \\ & \mathcal{D} & \end{tikzcd}$$

Proof. The square of the left diagram is a pullback in $\mathcal{C}$, which implies that the square on the right is a pullback in $\mathcal{C}/A$:

$$
\begin{tikzcd}
C \arrow[dr, dashed, "\pi_1 g"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "h"'] & & \\
& X \arrow[r, "(Id_X \times f)\circ\Delta_X"] \arrow[d, "f"] & X \times A \arrow[d, "f \times Id_A"] \\
& A \arrow[r, "\Delta"'] & A \times A
\end{tikzcd}
\qquad
\begin{tikzcd}
& [p : C \to A] \arrow[d, dashed, "\pi_1 g"] \arrow[dr, bend left, "g"] \arrow[dd, bend right, "h"'] & \\
& [f : X \to A] \arrow[r, "(Id_X \times f)\circ\Delta_X"] \arrow[d, "f"] & [\pi_2 : X \times A \to A] \arrow[d, "f \times Id_A"] \\
& [Id_A : A \to A] \arrow[r, "\Delta"'] & [\pi_2 : A \times A \to A]
\end{tikzcd}
$$

This leads to define $\overline{F}$ on $[f : X \to A]$ as the pullback of $F(f) : F(X) \to F(A)$ along $s : 1 \to F(A)$, in order to preserve the pullback above. More precisely, since f^*A is the arrow $\pi_2 : X \times A \to A$, to get $\overline{F}f^* = F$ we just need to select, amongst all pullbacks of $\pi_2 : F(X) \times F(A) \to F(A)$ along $s : 1 \to F(A)$, precisely the arrow $F(X) \to 1$, while the choice of the rest of the pullbacks is arbitrary. As for arrows, we define $\overline{F}$ in the obvious way using the induced arrows between pullback diagrams. It can now be shown that the functor so defined enjoys all the required properties. Moreover, if there is another extension of F, G, with the stated properties, then there are canonical isomorphisms $\phi_C : G(C) \to \overline{F}(C)$, induced by the universal property of the pullback. Given a morphism $f : C \to D$, the arrows $\overline{F}(f)\phi_C$ and $\phi_D G(f)$ would be two induced morphisms between the pullbacks $G(C)$ and $\overline{F}(D)$; therefore, they must coincide, and then the isomorphisms ϕ_C define a natural isomorphism $\phi : G \Rightarrow \overline{F}$, as stated. Finally, it is clear that $\phi Id_{f^*} = \psi$. $\square$

As an application of the preceding lemma we have the following:

Lemma 6.1.3. *For any Boolean functor $F : \mathcal{C}_T \to \mathcal{S}et$ there exists a Boolean functor $\overline{F} : \mathcal{C}_\omega \to \mathcal{S}et$ such that $\overline{F}I = F$, where I is the composition $I_0 J : \mathcal{C}_T \to \mathcal{C}_\omega$ and I_0, J are the embeddings defined in lemmas 5.3.5 and 5.2.1 respectively:*

$$
\begin{tikzcd}
\mathcal{C}_T \arrow[rr, "I"] \arrow[dr, "F"'] & & \mathcal{C}_\omega \arrow[dl, "\overline{F}"] \\
& \mathcal{S} &
\end{tikzcd}
$$

Proof. Because of lemma 5.2.1, we can choose a functor $F_0 : \mathcal{C}_0 = P(\mathcal{C}) \to \mathcal{S}$ such that $F_0 J = F$. We shall use the axiom of choice to define succesive functors $F_i : \mathcal{C}_i \to \mathcal{S}$ that form a pseudococone (see definition 3.3.6). Then, the universal property of the bicolimit $\mathcal{C}_\omega$ will give the desired functor $\overline{F}$, according to remark 5.3.4 and theorem 3.4.10. In fact, this is the idea that will be used to get F_1 from F_0 (we only show here this case since the others are similar).

Consider then the set of all finite products of covers $\{t_{P_F} : P_F = \prod_{i \in F} A_i \twoheadrightarrow 1 \ / \ F \subseteq \Gamma, finite\}$ in $\mathcal{C}_0$ (as in the considerations preceding lemma 5.3.3), which are mapped by F_0 into corresponding surjections $F_0(t_{P_F}) : F_0(P_F) \twoheadrightarrow F_0(1) = 1$. Choose a section s_{P_F} for each one of the surjections. Let π^*_{FG} be the pullback corresponding to π_{FG} through the pseudofunctor considered in remark 5.3.4. By lemma 6.1.2, F_0 provides Boolean functors $H_P : \mathcal{C}_0/P \to \mathcal{S}$ such that $H_P t^*_P = F$, where each t^*_P is the pullback selected by the pseudofunctor.

We shall now prove that the functors H_P form a pseudococone diagram in $\mathcal{C}at$. To do this we need to define natural isomorphisms $\phi_{FG} : H_{P_G} \pi^*_{FG} \Rightarrow H_{P_F}$, where $F \subseteq G$, for each morphism π^*_{FG}. Now, note that we have a natural isomorphism given by $Id_{H_{P_G}} c_{t_{P_F}, \pi_{FG}} : H_{P_G} \pi^*_{FG} t^*_{P_F} \Rightarrow H_{P_G} t^*_{PG} = F_0$, and since H_{P_F} satisfies the universal property stated in lemma 6.1.2 with respect to the triangular diagram below with vertices $\mathcal{C}, \mathcal{C}/P_F, \mathcal{S}$, there is a unique natural isomorphism $\phi_{FG} : H_{P_G} \pi^*_{FG} \Rightarrow H_{P_F}$ such that $\phi_{FG} Id_{t^*_{P_F}} = Id_{H_{P_G}} c_{t_{P_F}, \pi_{FG}}$.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{C}_0 \arrow[r, "t^*_{P_F}"'] \arrow[rr, bend left=30, "t^*_{P_G}"] \arrow[rrr, bend left=40, "t^*_{P_T}"] \arrow[d, "F_0"'] \arrow[rr, phantom, shift left=4, "\Uparrow c_{t_{P_F}, \pi_{FG}}"] & \mathcal{C}_0/P_F \arrow[r, "\pi^*_{FG}"'] \arrow[dl, "H_{P_F}"] \arrow[rr, phantom, shift left=6, "\Uparrow c_{t_{P_G}, \pi_{GT}}"] \arrow[r, phantom, shift right=6, "\Leftarrow \phi_{FG}"] & \mathcal{C}_0/P_G \arrow[r, "\pi^*_{GT}"'] \arrow[dll, "H_{P_G}"] \arrow[r, phantom, shift right=6, "\Leftarrow \phi_{GT}"] & \mathcal{C}_0/P_T \arrow[dlll, "H_{P_T}"] \\
\mathcal{S} & & &
\end{tikzcd}$$

If for $F \subseteq G \subseteq T$ we define similarly ϕ_{GT} and ϕ_{FT}, we just need to verify that with these natural isomorphisms the diagram becomes a pseudococone, which reduces in turn to verify condition b) of definition 3.3.6. But this is again a consequence of the universal property of lemma 6.1.2, since the natural isomorphism $\phi_{FG} \circ \phi_{GT} Id_{\pi^*_{FG}} \circ Id_{H_{P_T}} c^{-1}_{\pi_{FG}, \pi_{GT}} : H_{P_T} \pi^*_{FT} \Rightarrow H_{P_F}$ satisfies:

$$(\phi_{FG} \circ \phi_{GT} Id_{\pi^*_{FG}} \circ Id_{H_{P_T}} c^{-1}_{\pi_{FG}, \pi_{GT}}) Id_{t^*_{P_F}} = \phi_{FG} Id_{t^*_{P_F}} \circ \phi_{GT} Id_{\pi^*_{FG}} Id_{t^*_{P_F}} \circ Id_{H_{P_T}} c^{-1}_{\pi_{FG}, \pi_{GT}} Id_{t^*_{P_F}}$$

$$= Id_{H_{P_T}} Id_{\pi^*_{GT}} c_{t_{P_F}, \pi_{FG}} \circ Id_{H_{P_T}} c_{t_{P_G}, \pi_{GT}} \circ Id_{H_{P_T}} c^{-1}_{\pi_{FG}, \pi_{GT}} Id_{t^*_{P_F}}$$

$$= Id_{H_{P_T}} (Id_{\pi^*_{GT}} c_{t_{P_F}, \pi_{FG}} \circ c_{t_{P_G}, \pi_{GT}} \circ c^{-1}_{\pi_{FG}, \pi_{GT}} Id_{t^*_{P_F}}) = Id_{H_{P_T}} c_{t_{P_F}, \pi_{FT}}$$

according to property 3) of definition 3.3.5. Then, because of the uniqueness of ϕ_{FT}, we must have $\phi_{FT} = \phi_{FG} \circ \phi_{GT} Id_{\pi^*_{FG}} \circ Id_{H_{P_T}} c^{-1}_{\pi_{FG}, \pi_{GT}}$, as we wanted to prove. Therefore, the diagram is a pseudococone and the required functor $F_1 : \mathcal{C}_1 \to \mathcal{S}et$ is induced.

Finally, because the functors H_{P_F} in the pseudococone diagram are Boolean, the construction of the filtered colimit in $\mathcal{C}at$ shows, after a straightforward verification, that F_1 preserves finite limits, covers, unions and disjoint coproducts; in particular, it is Boolean, which finishes the proof. □

We finally get to the important characterization of Boolean models of a first order theory. This achievement is a reformulation, due mainly to Joyal, of existing results on limit ultrapowers (see [13]):

Theorem 6.1.4. *Every model of a first order theory T is given by (the image of) a functor $M \circ I : \mathcal{C}_T \to \mathcal{S}et$, where $I : \mathcal{C}_T \to \mathcal{C}_\omega$ is the immersion defined in lemma 6.1.3 and $M : \mathcal{C}_\omega \to \mathcal{S}et$ is a filtered colimit of ultra-representable functors. Moreover, if the ultra-representable functors correspond to ultrafilters in $\mathcal{S}ub(S)$ for subobjects $S \rightarrowtail 1$, every such composition $M \circ I$ defines a model of T.*

Proof. According to theorem 5.4.3, if Φ is an ultrafilter in $\mathcal{S}ub(S)$, for a subobject $S \rightarrowtail 1$, every ultra-representable functor $h_\Phi : \mathcal{C}_\omega \to \mathcal{S}et$ is Boolean, and hence, by corollary 3.3.4, every filtered colimit of these functors is as well Boolean. This proves that the composition $F \circ I$ is always Boolean and thus defines a model of the theory.

Supose now that $M' : \mathcal{C}_T \to \mathcal{S}et$ is a Boolean functor corresponding to a model M_T of T. By lemma 6.1.3 there exists a Boolean functor $M : \mathcal{C}_\omega \to \mathcal{S}et$ such that $MI = M'$. Define the category $\mathcal{X}$ whose objects are pairs (A, ξ) where A is an object of $\mathcal{C}_\omega$ and $\xi \in M(A)$, and whose morphisms $(A, \xi) \to (B, \eta)$ are given by those arrows $f : A \to B$ in $\mathcal{C}_\omega$ such that $\eta = M(f)(\xi)$. Because $\mathcal{C}_\omega$ has finite limits and M preserves them, $\mathcal{X}$ will have finite limits. Therefore, its dual $\mathcal{W} = \mathcal{X}^{op}$ is a (small) filtered category. We shall now define a functor H from $\mathcal{W}$ to the category of ultra-representable functors. For each object $(A, \xi) \in \mathcal{W}$, let $\Phi(A, \xi)$ be the set of all subobjects $C \rightarrowtail A$ such that $\xi \in M(C)$. Because M is Boolean, it is easy to check that $\Phi(A, \xi)$ is an ultrafilter in $\mathcal{S}ub(A)$. Define then $H((A, \xi)) = h_{\Phi(A,\xi)}$, the ultra-representable functor corresponding to $\Phi(A, \xi)$. Given an arrow $f : (A, \xi) \to (B, \eta)$, the mapping $C \mapsto f^{-1}(C)$, defined for each $C \in \Phi(B, \eta)$, determines a natural transformation $H(f^{op}) : h_{\Phi(B,\eta)} \to h_{\Phi(A,\xi)}$ in the following way: for each representative $a : C \to X$ in $h_{\Phi(B,\eta)}(X)$ we let $H(f)_X([a]) = [af']$, where $f' : f^{-1}(C) \to C$ is the arrow arising from the pullback of $C \rightarrowtail B$ along f. This application is well defined and makes H a functor.

We shall prove that $\varinjlim_{(A,\xi)\in\mathcal{W}} h_{\Phi(A,\xi)} \cong M$. To define an isomorphism $K : \varinjlim_{(A,\xi)\in\mathcal{W}} h_{\Phi(A,\xi)} \to M$ it suffices to define natural transformations $\psi_{(A,\xi)} : h_{\Phi(A,\xi)} \to M$ which will induce the required morphism. This can be done by setting $(\psi_{(A,\xi)})_X([f]) = M(f)(\xi)$, where $f : C \to X$ and $C \in \Phi(A, \xi)$. It can be easily checked that the definition does not depend on the representative of the class $[f]$; also, note that $C \in \Phi(A, \xi)$ implies that $\xi \in M(C)$, and therefore $M(f)(\xi) \in M(X)$ and $(\psi_{(A,\xi)})_X$ is well defined.

Let us first prove that K is a monomorphism. For this it is enough to verify that each $\psi_{(A,\xi)} : h_{\Phi(A,\xi)} \to M$ is monic, for which it suffices in turn to check that each $(\psi_{(A,\xi)})_X$ is injective. So suppose that we have arrows $f : C \to X, g : C' \to X$ such that $(\psi_{(A,\xi)})_X([f]) = (\psi_{(A,\xi)})_X([g])$. Then, $M(f)(\xi) = M(g)(\xi)$. Take the intersection $C \wedge C'$ in $\mathcal{S}ub(A)$, which gives monics $a : C \wedge C' \rightarrowtail C$ and $b : C \wedge C' \rightarrowtail C'$. Consider the equalizer E of fa and gb. Since M preserves equalizers and $\xi \in M(C \wedge C')$, then $\xi \in M(E)$, that is, $E \in \Phi(A, \xi)$. Therefore, f and g belong to the same class, i.e., $[f] = [g]$.

Finally, let us prove that K is an epimorphism, for which it suffices to show that each $K(D) : \varinjlim_{(A,\xi)\in\mathcal{W}} h_{\Phi(A,\xi)}(D) \to M(D)$ is surjective. Given an element $\chi \in M(D)$, consider the ultrafilter $\Phi(D, \chi)$, that contains D. Since $(\psi_{(D,\chi)})_D : h_{\Phi(D,\chi)}(D) \to M(D)$ satisfies $(\psi_{(D,\chi)})_D([Id_D]) = M(Id_D)(\chi) = \chi$, we conclude that the family $\{(\psi_{(A,\xi)})_D/(A, \xi) \in \mathcal{W}\}$ is jointly epic, from which we can deduce that $K(D)$ is necessarily surjective. $\square$

Corollary 6.1.5. Löwenheim-Skolem theorem: *Every model M of a first order theory T with countable signature has a submodel for T which is at most countable.*

Proof. Suppose that T is a first order theory with countable signature that has a model $M' : \mathcal{C}_T \to \mathcal{S}$. By theorem 6.1.4, M' has an extension $M : \mathcal{C}_\omega \to \mathcal{S}$ that is a filtered colimit of ultra-representable functors. Furthermore, since M is not trivial, we see from the proof of that theorem that $(1, *)$ is an object in $\mathcal{W}$, and hence the corresponding ultrafilter $\Phi(1, *)$ in $\mathcal{S}ub(1)$ (defined as $\{S \in \mathcal{S}ub(1) \ / \ M(S) \neq \emptyset\}$) gives an ultra-representable functor $h_{\Phi(1,*)}$ that belongs to the colimit diagram for M. But then, because of lemma 4.3.6, it is not difficult to see that $h_{\Phi(1,*)}I$ is a submodel for T which is at most countable, which finishes the proof. □

Remark 6.1.6. *The use of the Axiom of Choice throughout this section is not entirely avoidable, in the sense that some form of choice is needed to deduce theorem 6.1.4. For suppose we could prove in ZF that M' has an extension $M : \mathcal{C}_\omega \to \mathcal{S}$; then, the Löwenheim-Skolem theorem would be derivable in ZF, while it is known to be unprovable there (see [12] for references). As a consequence, the existence of the Boolean extension M must as well be unprovable in ZF.*

A stronger form of the Löwenheim-Skolem theorem (the downward form) can also be proven introducing a slight modification into the proof above:

Theorem 6.1.7. *For every model M of a first order theory T of cardinality κ, every cardinal $\mu \leq \kappa$ at least equal to the cardinality of the signature of T, and every subset $S \subseteq M$ of cardinality $\lambda \leq \mu$, there exists a submodel of M of cardinality μ that contains S.*

Proof. Consider the theory T' whose language has μ many constants and whose axioms are those of T plus a set of axioms expressing that the constants are pairwise different. Then M can be turned into a model of T' by interpreting the constants as elements of a subset S' such that $S \subseteq S' \subseteq M$. Hence, T' is consistent. Let $F : \mathcal{C}_{T'} \to \mathcal{S}$ be the corresponding Boolean functor for this new model, and $F' : \mathcal{C}_\omega \to \mathcal{S}$ its corresponding extension. Then, if $\Phi(1, *)$ is the ultrafilter in $\mathcal{S}ub(1)$ defined as $\{S \in \mathcal{S}ub(1) \ / \ F'(S) \neq \emptyset\}$), it can be seen that $h_{\Phi(1,*)}I$ is the required submodel. □

Remark 6.1.8. *Note that the previous proof does not make reference to the usual Skolem functions, which are commonly used in the proofs of this theorem.*

6.2 Complete theories and Vaught's test

Theorem 6.1.4 characterizes Boolean models by giving an explicit description of the functors associated to them. Ultra-representable functors in $\mathcal{S}ub(1)$ are therefore a relatively small subclass of these, but as seen in section 5, they suffice nevertheless to deduce the Completeness theorem. Another application of this set of models is related to a result in model theory known as Vaught's test, which provides a sufficient condition for a theory to be complete. Recall that a theory T is said to be complete if, for every sentence ϕ (i.e., every formula with no free variables), either ϕ or $\neg\phi$ is provable in T. We say that T is κ-categorical for some cardinal κ if any two models of T of cardinality κ are isomorphic. We have now:

Theorem 6.2.1. Vaught's test: *Let T be a first order theory that has no finite models and is κ-categorical for some cardinal κ at least equal to the cardinality of the signature of T. Then T is complete.*

Proof. Consider the theory T' defined by adding κ many constant symbols to the language of T, whose axioms are those of T plus a set of axioms expressing that the constants are pairwise different. Then, the model of T of cardinality κ is also a model for T', which is therefore consistent; moreover, since T is κ-categorical, then so is T'. Consider the models of T' given by ultra-representable functors in $\mathcal{C}_\omega$, $h_\Phi I : \mathcal{C}_{T'} \to \mathcal{S}$ for ultrafilters Φ in $\mathcal{S}ub(1)$. Since these models are non-trivial, they contain κ many distinct individuals, and since they have cardinality at most κ, it follows that their cardinality is exactly κ; moreover, because T' is κ-categorical, all these models are isomorphic. Let $[\{\}, \phi]$ be the object in $\mathcal{C}_{T'}$ corresponding to the sentence ϕ; then $[\{\}, \phi]$ is a subobject of $[\{\}, \top] = 1_{\mathcal{C}_{T'}}$. Let $s = I([\{\}, \phi])$ be the corresponding subobject in $\mathcal{C}_\omega$. Then, if $s = 0$, it follows by the conservativity of I that $[\{\}, \phi] = 0_{\mathcal{C}_{T'}} = [\{\}, \perp]$, and hence $\neg\phi$ is provable in T'. If, on the other hand, $s \neq 0$, then s belongs to some ultrafilter Ψ in $\mathcal{S}ub(1)$, and therefore $h_\Psi(s) = 1$. But since all models $h_\Phi I$ are isomorphic, it follows from lemma 4.3.6 that for every Φ there is a natural isomorphism $h_\Psi I \to h_\Phi I$, and hence for every Φ we have $h_\Phi(s) = 1$. This implies that s belongs to the intersection of all ultrafilters in $\mathcal{S}ub(1)$, and consequently, $s = 1$. Since I is conservative, it follows then that $[\{\}, \phi] = [\{\}, \top]$, and hence that T' proves ϕ. We have thus established that T' is complete.

To see that T must also be complete, let ϕ be a sentence in the language of T. Suppose without loss of generality that T' proves ϕ, and note that a deduction of ϕ in T' must use only a finite number of the extra added axioms. Therefore, if T'' is the theory whose axioms are those of T plus a countable set of axioms expressing that some countable set of constant symbols are pairwise distinct, then T'' is already complete. Since T has no finite models, all models given by the functors $h_\Phi I : \mathcal{C}_T \to \mathcal{S}$ have cardinality greater or equal than $\aleph_0$, and hence they are also models of T'' for some adequate interpretation of the constants. Because T'' proves ϕ, it must be valid in all these models, which implies that $h_\Phi I([\{\}, \phi]) = 1$ for all ultrafilters Φ. But then an argument similar to the one used before implies that $[\{\}, \phi] = [\{\}, \top]$ in $\mathcal{C}_T$, which finishes the proof. $\square$

The proof above should also convince the reader of the flexibility of categorical methods in proving metamathematical results, making Joyal's characterization of Boolean models an important line of approach to the study of model theory.

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