UNIVERSIDAD DE BUENOS AIRES
Facultad de Ciencias Exactas y Naturales
Departamento de Matemática

Soluciones conjuntistas a la ecuación de Yang-Baxter, invariantes de nudos y cohomología.

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

Juliana García Galofre

Director de tesis: Marco A. Farinati
Consejero de estudios: Marco A. Farinati

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Resumen

En el Capítulo 2 definimos una biálgebra B cuya homología y cohomología coinciden con las de biquandle definidas en [CJKS] y otras generalizaciones de cohomología del caso quandle o rack (por ejemplo la definida en [CES2]). La estructura algebraica encontrada permite demostrar con transparencia la existencia de un producto asociativo en la cohomología de biquandles. Este producto era conocido para el caso rack (con una demostración topológica, por lo que nuestra construcción provee de una prueba completamente algebraica e independiente) pero era desconocido en el caso general de biquandles. También esta estructura algebraica descubierta permite mostrar la existencia de morfismos de comparación con cohomología de Hochschild que, eventualmente, podrán proveer de ejemplos de cálculo de cociclos, que (en grado dos para nudos, y en grado tres para superficies) pueden ser utilizados para calcular invariantes. Más aún, explicitamos un morfismo de comparación que se factoriza por un complejo que, como bimódulo, es la extensión de escalares de un álgebra de Nichols.

En [AG] se define un 2-cociclo de quandle como una aplicación $\beta : X \times X \to H$ donde $(X, \star)$ es un quandle y H es un grupo (no necesariamente abeliano) tal que

$$\beta(x_1, x_2)\beta(x_1 \star x_2, x_3) = \beta(x_1, x_3)\beta(x_1 \star x_3, x_2 \star x_3)$$

y $\beta(x, x) = 1$.

En el Capítulo 3 generalizamos esa definición para biquandles (X, σ) adaptando las ecuaciones existentes y agregando una equación más:

Una función $f : X \times X \to H$ es un *2-cociclo trenzado no conmutativo* si

- $f\Big(x_1, x_2\Big) f\Big(\sigma^2(x_1, x_2), x_3\Big) = f\Big(x_1, \sigma^1(x_2, x_3)\Big) f\Big(\sigma^2(x_1, \sigma^1(x_2, x_3)), \sigma^2(x_2, x_3)\Big)$, y
- $f\Big(\sigma^1(x_1, x_2), \sigma^1(\sigma^2(x_1, x_2), x_3)\Big) = f\Big(x_2, x_3\Big)$

$\forall x_1, x_2, x_3 \in X$.

Definimos un grupo, U_{nc}, y un 2-cociclo no conmutativo universal, π, tales que para todo grupo H y $f : X \times X \to H$ 2-cociclo no conmutativo, existe un único morfismo de grupos $\overline{f} : U_{nc} \to H$ tal que $f = \overline{f} \circ \pi$. Mostramos que U_{nc} es funtorial. Definimos una asignación de pesos a cada cruce en un nudo/link y, probando que cierto producto es invariante por movimientos de Reidemeister obtuvimos un nuevo invariante de nudos/links que generaliza el invariante obtenido en [CEGS].

Para cada grupo U_{nc} definimos cocientes U_{nc}^{γ} y mostramos que estos, si bien son en general mucho más chicos que U_{nc}, guardan la misma información que el primero con respecto al cálculo de invariantes. Hemos calculado U_{nc} y U_{nc}^{γ} para ciertos ejemplos de biquandles pequeños. Para poder trabajar con ejemplos de cardinal mayor a tres utilizamos GAP (System for Computational Discrete Algebra). Esto último nos permitió

colorear links con biquandles (no provenientes de quanldles) de mayor cardinal y así distinguir nudos-links concretos (e.g.: el trebol de su imagen especular, la no trivialidad del link Whitehead, etc). Es decir, encontramos ejemplos que muestran que nuestro invariante generaliza estrictamente el definido en [CEGS]. Estos ejemplos ya se dan con biquandles de tamaño muy chico (cardinal 3) y permiten distinguir sensiblemente nudos distintos (e.g.: link Borromeo de tres “no nudos” separados , link de Whitehead de dos “no nudos”, trebol de su imagen especular).

Palabras clave: invariante de knot-links, cohomología de Yang-Baxter.

Quandle, biquandle, rack, biálgebra, álgebra de Hopf, algebra trenzada.

Set theoretic solutions of the Yang-Baxter equation, knot invariants and cohomology.

Abstract

In Chapter 2, we define a bialgebra B such that its homology and cohomology are the same as the biquandle ones defined in [CJKS] and other genalizations of cohomology of the quandle-rack case (for example defined in [CES2]).This algebraic structure enable us to show an associative product in biquandle cohomology. This product was known for the rack case (with topological proof) but unknown in biquandle case. This algebraic structure also allows to define comparison morphisms with other cohomology theories that could eventually provide cocycle examples (of degree two for knots and degree three for surfaces) for computing invariants. Furthermore, we give an explicit comparison morphism that factorizes by a complex that, as bimodule, is the scalar extension of a Nichols algebra.

In [AG] a quandle 2-cocycle is defined as a map $\beta : X \times X \to H$ where $(X, \star)$ is a quandle and H is a group (not necessarily abelian) such that

$$\beta(x_1, x_2)\beta(x_1 \star x_2, x_3) = \beta(x_1, x_3)\beta(x_1 \star x_3, x_2 \star x_3)$$

and $\beta(x, x) = 1$.

In Chapter 3 we generalized this definition to biquandles (X, σ):

A function $f : X \times X \to H$ is a *non commutative braided 2-cocycle* if verifies both

- $f\Big(x_1, x_2\Big)f\Big(\sigma^2(x_1, x_2), x_3\Big) = f\Big(x_1, \sigma^1(x_2, x_3)\Big)f\Big(\sigma^2(x_1, \sigma^1(x_2, x_3)), \sigma^2(x_2, x_3)\Big)$, and
- $f\Big(\sigma^1(x_1, x_2), \sigma^1(\sigma^2(x_1, x_2), x_3)\Big) = f\Big(x_2, x_3\Big)$

$\forall x_1, x_2, x_3 \in X$.

We define a group, U_{nc} and a universal non commutative 2-cocycle π such that for every group H and $f : X \times X \to H$ a non commutative 2-cocycle, exist a unique group morphism $\overline{f} : U_{nc} \to H$ such that $f = \overline{f} \circ \pi$. We show that U_{nc} is functorial. Define an assignment of a weight to each crossing in a knot-link. A certain product of these weights is invariant under Reidemester moves, then a new invariant for knot-links is obtained generalising the one obtained in [CEGS]. For each group U_{nc} we defined quotients U_{nc}^{γ} which keep the same data when computing the invariant and have smaller cardinal. We calculated U_{nc} and U_{nc}^{γ} for certain biquandles of small cardinality. To be able to work with more examples we worked with GAP (System for Computational Discrete Algebra). Creating programs we were able to color links with bigger biquandles (not coming from quandles) and found examples that show our invariant generalizes the one defined in [CEGS]. This examples are achived using biquandles of cardinality three and distinguish knots-links (e.g.: Borromean link from three separeted unknots, Whitehead link from two unknots, trefoil knot and its mirror). .

Key words: Knot-links invariants, Yang-Baxter cohomology.

Quandle, biquandle, rack, bialgebra, Hopf algebra, braided algebra.

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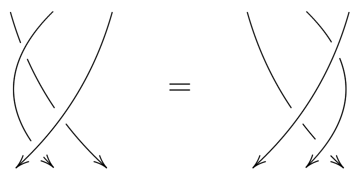
Contents

Introducción

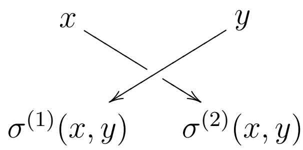
Una solución conjuntista de la ecuación de Yang-Baxter es un conjunto X provisto de una biyección $\sigma : X \times X \to X \times X$ que verifica

$$(\sigma \times \mathrm{Id})(\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id}) = (\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id})(\mathrm{Id} \times \sigma)$$

donde la igualdad es como biyección en $X \times X \times X$. Gráficamente, esta identidad se la suele visualizar como



donde, si $\sigma(x, y) = (\sigma^{(1)}(x, y), \sigma^{(2)}(x, y))$, en cada cruce se introducen etiquetas con elementos en X con la regla



Dada una solución conjuntista de la ecuación de Yang Baxter y un anillo conmutativo k, el k-módulo libre con base X: $V := \oplus_{x \in X} kx$ tiene una trenza

$$c : V \otimes V \to V \otimes V$$

definida por $c(x \otimes y) := z \otimes t$ si $\sigma(x, y) = (z, t)$, pues verifica

$$(c \otimes \mathrm{Id})(\mathrm{Id} \otimes c)(c \otimes \mathrm{Id}) = (\mathrm{Id} \otimes c)(c \otimes \mathrm{Id})(\mathrm{Id} \otimes c).$$

Estas soluciones juegan un rol central en el problema de clasificación de álgebras de Nichols, clasificación de álgebras de Hopf, y sus aplicaciones al estudio de categorías tensoriales, con aplicaciones en topología de dimensiones bajas como invariantes de nudos. Un caso particular muy estudiado es cuando la trenza es de la forma

$$\sigma(x, y) = (y, \sigma^{(2)}(x, y))$$

que denominaremos "solución tipo rack". Esta tesis contiene generalizaciones de construcciones del caso tipo rack al caso general, y construcciones nuevas en el caso general que tienen aplicaciones nuevas tanto en el caso general como el de tipo rack.

La teoría de (co)homología para soluciones conjuntistas de la ecuación de Yang-Baxter fue considerada simultáneamente por matemáticos del área de álgebras de Hopf (e.g. Andruskiewitsch-Graña para el caso rack) como por topólogos interesados en invariantes de nudos y superficies anudadas (Carter, Saito, Elhamdadi, Fenn, Rourke, Sanderson, Ceniceros, Green, Nelson, entre otros). Por un lado, si $c(x \otimes y) = z \otimes t$ es la trenza en V que proviene de linealizar la de X, y si $f : X \times X \to k^{\times}$ es una función a valores en las unidades del anillo k, la noción de 2-cociclo (en notación multiplicativa) puede verse como la condición sobre f para que la aplicación $c_f : V \otimes V \to V \otimes V$ dada por

$$c_f(x \otimes y) = f(x, y) z \otimes t$$

verifique la ecuación de trenzas.

Desde el punto de vista de invariantes de nudos, la condición de cociclo se obtiene pidiendo cierta invarianza ante movimientos de Reidemeister, de manera tal que, dado un 2-cociclo $f : X \times X \to A$, donde A es un grupo abeliano, el procedimiento conocido como "state-sum invariant" produce un invariante de nudos a valores en el álgebra de grupo $\mathbb{Z}[A]$.

Uno de los primeros objetos algebraicos asociados a una solución (X, σ) es el llamado grupo envolvente G_X, que es el grupo con generadores en X y con la relación

$$xy \sim zt \ \text{ si } \sigma(x, y) = (z, t)$$

y también podemos considerar la k-algebra de semigrupo como la k-algebra con generadores en X y las mismas relaciones. En el Capítulo 2, dado (X, σ) solución de la ecuación de Yang-Baxter, se define una biálgebra diferencial graduada (B, Δ, d) tal que al considerar cierta estructura de bimódulo sobre $A = k\langle X\rangle/\langle xy - zt : x, y \in X,\ (z, t) = \sigma(x, y)\rangle = k[M]$, se recuperan los complejos asociados a soluciones conjuntistas de la ecuación de Yang-Baxter definidos en [CES2] tomando respectivamente productos tensoriales sobre A u Hom_{A-A} (ver Teorema 32). La estructura algebraica descubierta en B (i.e. biálgebra diferencial graduada) permite demostrar de manera sencilla y puramente algebraica la existencia de un producto "cup" en cohomología (ver Proposición 42). Este hecho era conocido para el caso rack con demostración topológica (ver [C]), pero desconocido para el caso general.

Como segunda aplicación de la existencia de la bialgebra B se obtienen morfismos de comparación con la cohomología de Hochschild del álgebra del semigrupo envolvente de X (ver Teorema 59), y un estudio detallado de un posible morfismo de comparación muestra una relación nueva con el álgebra de Nichols asociada a $-\sigma$ (donde σ es una solución general de YBeq).

En el Capítulo 3, dado un biquandle (X, σ), definimos un 2-cociclo trenzado no conmutativo f que generaliza la noción análoga para quandles dada en [AG]. Dada una elección de biquandle X, un 2-cociclo trenzado no conmutativo f y un link orientado L, se definen pesos para cada cruce y un conjunto de clases de conjugacíon de ciertos productos de los antes mencionados pesos. Se muestra que estas clases de conjugación son invariantes por movimientos de Reidemeister. Es decir, se prueba que estas clases de conjugación definen un invariante de nudos/links. En una segunda instancia se define un grupo universal que gobierna todos los 2-cociclos para un biquandle X dado, es decir, un grupo $U_{nc}(X)$ junto con un 2-cociclo $\pi : X \times X \to U_{nc}(X)$ tal que si $f : X \times X \to G$ es un 2-cociclo no conmutativo trenzado a valores en un grupo G, entonces existe un único morfismo de grupos $\widetilde{f} : U_{nc}(X) \to G$ tal que $f = \widetilde{f}\pi$. Por ejemplo, si $U_{nc}(X)$ es el grupo trivial, entonces lo es todo 2-cociclo. Por otro lado, si $U_{nc}(X)$ es no trivial, este grupo universal contiene toda la información que algún grupo podría tener usando 2-cociclos.

Más adelante se define una versión reducida del grupo U_{nc}. Dada una aplicación $\gamma : X \to U_{nc}(X)$ se construye un cociente llamado $U_{nc}^{\gamma}(X)$, que en particular es un grupo, y existe un 2-cociclo $\pi_\gamma : X \times X \to U_{nc}^{\gamma}(X)$ con la siguiente propiedad: (Teorema 111): Si $f : X \times X \to G$ es un 2-cociclo, entonces existe un 2-cociclo cohomólogo (ver Definición 92) $f_\gamma : X \times X \to G$ y un morfismo de grupos $\widetilde{f_\gamma}$ tal que

$$f_\gamma = \widetilde{f}_\gamma \pi_\gamma.$$

Como el invariante definido no cambia bajo cociclos cohomólogos (Proposición 102), el invariante producido por f es el mismo que el proveniente de f_γ. Este paso al cociente permite mejorar y facilitar los cálculos y resultados en ejemplos concretos.

Sobre el final del capítulo se muestran ejemplos de cálculos concretos de invariantes de nudos y links encontrados con este procedimiento.

En el Capítulo 4 mostramos algunos de los programas utilizados en la búsqueda de ejemplos concretos.

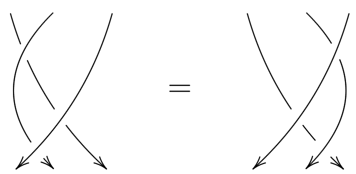
El primer capítulo de la tesis contiene las definiciones generales. Los aportes originales contenidos en esta tesis se encuentran en los Capítulos 2 y 3 que corresponden mayoritariamente a los trabajos *A differential bialgebra associated to a set theoretical solution of the Yang-Baxter equation* ArXiv:math/1508.07970, y *Link and knot invariants from non-abelian Yang-Baxter 2-cocycles*, ArXiv:math/1507.02232.

Introduction

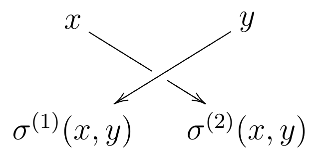
A set-theoretic solution of the Yang-Baxter equation is a set X provided with a bijection $\sigma : X \times X \to X \times X$ that verifies

$$(\sigma \times \mathrm{Id})(\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id}) = (\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id})(\mathrm{Id} \times \sigma)$$

for every $(x, y, z) \in X \times X \times X$. Graphically



where $\sigma(x, y) = (\sigma^{(1)}(x, y), \sigma^{(2)}(x, y))$, in each crossing labels are introduced with elements in X following the rule



For each set-theoretical solution of the Yang-Baxter equation and a commutative ring k, the free k-module with basis X: $V := \oplus_{x \in X} kx$ has a braiding

$$c : V \otimes V \to V \otimes V$$

defined by $c(x \otimes y) := z \otimes t$ si $\sigma(x, y) = (z, t)$, as verifies

$$(c \otimes \mathrm{Id})(\mathrm{Id} \otimes c)(c \otimes \mathrm{Id}) = (\mathrm{Id} \otimes c)(c \otimes \mathrm{Id})(\mathrm{Id} \otimes c)$$

This kind of solutions play a central role in classifying Nichols algebras, Hopf algebras, and their application to the study of tensor categories, low dimension topologies such as knot invariants. A (well known) particular case is when the braiding is like

$$\sigma(x, y) = (y, \sigma^{(2)}(x, y))$$

which are called "rack solutions". This thesis contains generalizations of constructions made for rack case and new constructions, with new applications for both, general case and rack case.

Cohomology theory for set theoretic solutions of the Yang-Baxter equation was considered simultaneously by mathematicians coming from Hopf algebra's area (e.g. Andruskiewitsch and Graña for rack case) and topologist interested in knot invariants or knoted surfaces (Carter, Saito, Elhamdadi, Fenn, Rourke, Sanderson, Ceniceros, Green, Nelson, among others). On one hand, if $c(x\otimes y)=z\otimes t$ is a braiding on V coming from linearizing the braid in X, and if $f:X\times X\to k^*$ where k^* is the set of units of a ring k, then the notion of 2-cocycle (multiplicative notation) could be written as a condition on f, such that $c_f:V\otimes V\to V\otimes V$ given by

$$c_f(x\otimes y)=f(x,y)z\otimes t$$

verifies braid equation.

From the knot invariant point of view, the cocycle condition is obtained by invariance under Reidemeister moves, so given a 2-cocycle $f:X\times X\to A$, where A is an abelian group, the "state-sum invariant" procedure gives a knot invariant in coefficients of the group algebra $\mathbb{Z}[A]$.

One of the most common objects associated to a solution (X,σ) is the enveloping group G_X, which is generated by X and relations

$$xy\sim zt \ \text{ if } \sigma(x,y)=(z,t).$$

It also can be considered the semigroup k-algebra as the k-algebra with generators in X and same relations. In Chapter 2, given (X,σ) a solution of Yang-Baxter equation, we define a differential graded bialgebra (B,Δ,d) such that, when considering certain bimodule structure on $A=k\langle X\rangle/\langle xy-zt : x,y\in X,\ (z,t)=\sigma(x,y)\rangle=k[M]$, the complexes associated to set theoretic solutions of the Yang-Baxter equation defined in [CES2] are recovered, taking tensor products on A, or Hom_{A-A} (see Theorem 32). The algebraic structure shown in B (i.e. differential graded bialgebra) allowed us to prove the existence of a "cup" product in cohomology, using purely algebraic methods (see Theorem 42). This fact was known for the rack case with topological proof ([C]), but unknown for the general case.

As a second application of the existence of the bialgebra B, comparison morphisms with Hochschild's cohomology of the enveloping semigroup of X, (see Theorem 59) are obtained. A detailed study of a possible comparison morphism shows a new relation with the Nichols algebra associated to $-\sigma$.

In Chapter 3, given (X,σ) we define a braided noncommutative 2-cocycle f that generalizes the one defined for quandles given in [AG]. Given a biquandle X, a braided noncommutative 2-cocycle f and an oriented link L, for each crossing we define weights and a set of conjugacy classes of certain products of the earlier mentioned weights. We show this conjugacy classes are invariant under Reidemeister moves, then define a link/knot invariant. Secondly we define a universal group, that governs all 2-cocycles for a given biquandle X, meaning, a group $U_{nc}(X)$ together with a 2-cocycle $\pi:X\times X\to U_{nc}(X)$ such that if $f:X\times X\to G$ is a braided noncommutative 2-cocycle to coefficients in G, then a unique group morphism exists $\widetilde{f}:U_{nc}(X)\to G$ such that $f=\widetilde{f}\pi$. For example, if $U_{nc}(X)$ is the trivial group, then 2-cocycle is also trivial. On the other hand, if $U_{nc}(X)$ is not trivial, this universal group contains all information any group could have using 2-cocycles.

Later we define a new and reduced version of this group, U_{nc}. Given an application $\gamma:X\to U_{nc}(X)$ we construct a quotient called $U^{\gamma}_{nc}(X)$, which is a group and a 2-cocycle

$\pi_\gamma : X \times X \to U_{nc}^{\gamma}(X)$ with the following property: (Theorem 111): If $f : X \times X \to G$ is a 2-cocycle, then there exists a cohomologous 2-cocycle (see Definition 92) $f_\gamma : X \times X \to G$ and a group morphism $\widetilde{f_\gamma}$ such that

$$f_\gamma = \widetilde{f_\gamma}\pi_\gamma.$$

As the defined invariant does not change under cohomologous cocycles (Proposition 102), the invariant produced by f is the same as the one coming from f_γ. This quotient improves calculations and lead to more concrete results.

Over the end of the chapter concrete examples of link/knot invariants are shown.

In Chapter 4 some of the programs used to find concrete examples are shown.

The first chapter of this thesis contains general definitions. The original results included in this thesis are included in chapters 2 and 3 that are based mostly on *A differential bialgebra associated to a set theoretical solution of the Yang-Baxter equation* ArXiv:math/1508.07970, and *Link and knot invariants from non-abelian Yang-Baxter 2-cocycles*, ArXiv:math/1507.02232.

Chapter 1

Preliminaries

Introducción al capítulo

En este capítulo introduciremos los objetos de estudio que utilizamos en esta tesis. Después de definir soluciones conjuntistas de la ecuación de Yang-Baxter y mostrar brevemente algunas familias de soluciones (ej: de tipo rack, soluciones de (bi)Alexander), presentamos las teorías de homología y cohomología de este tipo de estructuras. Estas teorías de (co)homología fueron principalmente desarrolladas por topólogos (con el objetivo de producir invariantes de nudos/links o superficies anudadas) y expertos en álgebras de Hopf (con el objetivo de producir álgebras Nichols). Terminaremos el capítulo describiendo un invariante llamado state-sum, asociado a un quandle Q y a un 2-cociclo (conmutativo) f. Ninguno de los contenidos de este capítulo es original y está incluido en la tesis a modo de contexto: en el Capítulo 2 probaremos resultados referentes a la cohomología usando un nuevo punto vista mientras que en el Capítulo 3 proponemos una definición de 2-cociclo no conmutativo para biquandles (generalizando la definición preexistente para quandles) y probamos que da lugar a un invariante de nudos/links.

Introduction to the chapter

In this chapter we introduce the objects of study of this thesis. After defining the set theoretical solutions of the Yang-Baxter equation and briefly showing some examples (e.g. of rack type, (bi)Alexander solutions), we present the homology and cohomology theories for this algebraic structure. These (co)homology theories were mainly developed by topologists (in order to produce invariants of links/knots or knoted surfaces) and Hopf-theorists (in order to produce Nichols algebras). We end this chapter by describing the so-called state-sum invariant associated to a quandle Q and a (commutative) 2-cocycle f. None of the contents of this chapter is new, we present them in order to put our results in context: in Chapter 2 we prove general results about cohomology using a new point of view, and in Chapter 3 we propose a definition of non-commutative 2-cocycles for biquandles (generalizing a previous definition for quandles) and prove that they provide knot/links invariants.

1.1 The Yang-Baxter equation

A set theoretical solution of the Yang-Baxter equation (YBeq) is a pair (X, σ) where $\sigma : X \times X \to X \times X$ is a bijection satisfying

$$(\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id})(\mathrm{Id} \times \sigma) = (\sigma \times \mathrm{Id})(\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id}) : X \times X \times X \to X \times X \times X$$

The equation above is called the Yang-Baxter or braid equation. If σ is a bilinear bijective map satisfying YBeq then it is called a braiding on V.

1.1.1 First Examples: Racks and quandles

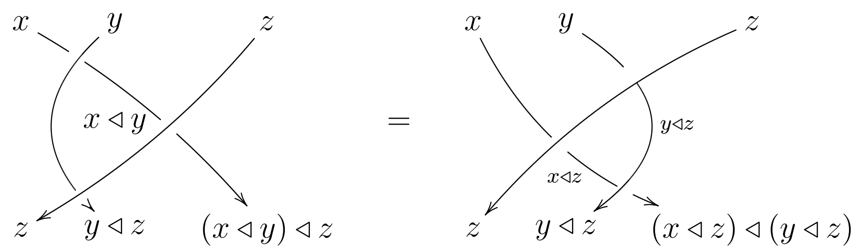
Let us consider a set X and a special type of solution of the form

$$\sigma(x, y) = (y, \sigma^{(2)}(x, y))$$

Since $\sigma^{(2)}(x, y)$ depends on x and y, one may see it as a binary operation, usually denoted by $\triangleleft : X \times X \to X$:

$$x \triangleleft y := \sigma^{(2)}(x, y)$$

If one writes the braid equation in this notation one gets:



This motivates the following definition:

Definition 1. A set X with a binary operation $\triangleleft : X \times X \to X$ is called a **rack** if

- $- \triangleleft x : X \to X$ is a bijection $\forall x \in X$ and
- $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ $\forall x, y, z \in X$.

$x \triangleleft y$ is usually denoted by x^y.
If X also verifies that $x \triangleleft x = x$ then X is called a *quandle*.

It is clear that $(X, \triangleleft)$ is a rack if and only if

$$\sigma_\triangleleft(x, y) := (y, x \triangleleft y)$$

is a set theoretical solution of the YBeq.
An important example of rack is G a group, with $x \triangleleft y = y^{-1}xy$, since

- $- \triangleleft y : X \to X$ is a bijection with inverse $(- \triangleleft^{-1} y)(x) = yxy^{-1}$ and
- $(x \triangleleft y) \triangleleft z = z^{-1}y^{-1}xyz$ and
  $(x \triangleleft z) \triangleleft (y \triangleleft z) = (z^{-1}xz) \triangleleft (z^{-1}yz) = (z^{-1}y^{-1}z)(z^{-1}xz)(z^{-1}yz)$

Generalizing for a given subset $X \subset G$, stable under conjugation, is also a rack with the operation given by conjugation.

Example 2. Let $D_n = \{0, 1, 2, \dots, n-1\}$ and define $i \triangleleft j := 2j - i \mod(n)$, then D_n is a quandle. It can be identified with the set of reflections of $\mathbb{D}_n$ (the dihedral group with $2n$ elements) via $j \leftrightarrow sr^j$. In Hopf literature it is called "dihedral rack".

Example 3. Let R be a ring, t a unit in R and M an R-module, then

$$x \triangleleft y := tx + (1-t)y$$

gives a quandle structure on M. This structure is known as "Alexander quandle" (see afine racks in [AG]).

Remark 4. Not every rack or quandle can be viewed as a subset of a group with conjugacy operation. The easiest way to see that is the following property of groups: if G is a group and $x, y \in G$, then x commutes with y if and only if y commutes with x. If a quandle Q could be embedded in a group G with conjugation as operation $\triangleleft$, then $x \triangleleft y = y$ would imply $y \triangleleft x = y$.

Consider, for example, $Q_3 := (\mathbb{Z}/3\mathbb{Z}, \triangleleft)$ with $\triangleleft$ defined as follows

$$x \triangleleft y = -x - xy^2.$$

One can check directly that Q_3 is a quandle, and it is clear that $0 \triangleleft y = 0$ for all y, that is, "y commutes with 0 for all y", but $y \triangleleft 0 = -y$, so for $y \in \{1, 2\}$ "y does not commute with 0".

1.1.2 Biquandles

In [KR], a generalization of quandles is proposed (we recall it with different notation), a solution of YBeq (X, σ) is called non-degenerated, or *birack* if in addition,

1. for any $x, z \in X$ there exists a unique y such that $\sigma^{(1)}(x, y) = z$, (if this is the case, $\sigma^{(1)}$ is called *left invertible*), and
2. for any $y, t \in X$ there exists a unique x such that $\sigma^{(2)}(x, y) = t$, (if this is the case, $\sigma^{(2)}$ is called *right invertible*).

Diagrammatically:

A birack is called *biquandle* if, given $x_0 \in X$, there exists a unique $y_0 \in X$ such that $\sigma(x_0, y_0) = (x_0, y_0)$. In other words, if there exists a bijective map $s : X \to X$ such that

$$\{(x, y) : \sigma(x, y) = (x, y)\} = \{(x, s(x)) : x \in X\}$$

Remark 5. Every quandle is a biquandle (with $\sigma(x, y) = (y, x \triangleleft y)$). Moreover, a rack type solution is a biquandle if and only if the rack is a quandle.

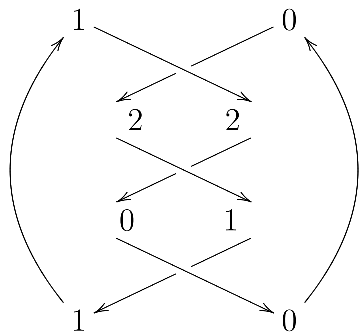
Remark 6. When painting (see Definition 8) a knot-link with a biquandle, knowing the colors of two semiarcs (excluding the diagonals) means knowing all colors of the crossing.

1.1.3 Some examples

1. Wada: given a group G,
$$\sigma(x,y) = (xy^{-1}x^{-1}, xy^2)$$
is a biquandle with $s(x) = x^{-1}$. In case G is abelian $\sigma(x,y) = (-y, x+2y)$.

2. Alexander biquandle: Let R be a ring, $s, t \in R$ two commuting units, and M an R-module, then
$$\sigma(x,y) = (s \cdot y, t \cdot x + (1-st) \cdot y), \quad (x,y) \in M \times M$$
is a biquandle, with function $s(x) = (s^{-1}) \cdot x$. In the particular case $s = -1$, $t = 1$ one gets the abelian Wada's solution. If $s = 1$ then one gets the solution induced by the Alexander rack.

Example 7. A possible coloring of the trefoil knot using Wada biquandle:



1.1.4 Examples of Small size

The cardinal of the rack-quandle-biquandle X is called size.

Here we transcribe Bartholomew's lists. Which are presented in a plain text format as two matrices one for the up action (U) and one for the down action (D). Each matrix is specified row by row. All of the lists are presented as biracks, that is with both U and D specified; for the quandles and racks, the D action is the identity. Writen with previous notation: $\sigma(x,y) = (D_{xy}, U_{yx})$ where $U_{xy} = x \triangleleft y$.

Biracks of size 2:

Number	*type*	*U*	*D*
1	(Trivial) Quandle	12 12	12 12
2	Rack (not quandle)	21 21	12 12
3	Biquandle (not quandle)	21 21	21 21

Biracks of size 3:

$Number$	$type$	U	D
1	(Trivial) Quandle	123 123 123	123 123 123
2	Quandle	132 123 123	123 123 123
3	Quandle	132 321 213	123 123 123
4	Rack (not quandle)	312 312 312	123 123 123
5	Rack (not quandle)	321 123 321	123 123 123
6	Rack (not quandle)	321 321 321	123 123 123
7	Biquandle (not quandle)	123 132 132	123 132 132
8	Biquandle (not quandle)	123 132 132	132 132 132
9	Biquandle (not quandle)	123 312 231	132 132 132
10	Biquandle (not quandle)	123 123 213	123 123 213
11	Biquandle (not quandle)	132 132 132	132 132 132
12	Biquandle (not quandle)	213 132 321	231 231 231
13	Biquandle (not quandle)	231 231 231	312 312 312
14	Birack (not biquandle, not rack)	123 132 132	132 123 123
15	Birack (not biquandle, not rack)	132 123 123	132 132 132
16	Birack (not biquandle, not rack)	231 231 231	231 231 231

(1.1)

1.2 Known Rack homologies and cohomologies

In [CJKLS] rack/quandle (co)homology is defined as follows:

Let $C_n^{\mathrm{R}}(X)$ be the free abelian group generated by n-tuples $(x_1, \dots, x_n)$ of elements of a quandle X. Define a homomorphism $\partial_n : C_n^{\mathrm{R}}(X) \to C_{n-1}^{\mathrm{R}}(X)$ by

$$\begin{aligned}
&\partial_n(x_1, x_2, \dots, x_n) \\
&\quad = \sum_{i=2}^{n} (-1)^i \left[(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n)\right. \\
&\quad \left. - (x_1 \triangleleft x_i, x_2 \triangleleft x_i, \dots, x_{i-1} \triangleleft x_i, x_{i+1}, \dots, x_n)\right]
\end{aligned} \tag{1.2}$$

for $n \geq 2$ and $\partial_n = 0$ for $n \leq 1$. Then $C_\bullet^{\mathrm{R}}(X) = \{C_n^{\mathrm{R}}(X), \partial_n\}$ is a chain complex.

Let $C_n^{\mathrm{D}}(X)$ be the subset of $C_n^{\mathrm{R}}(X)$ generated by n-tuples $(x_1, \dots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \dots, n-1\}$ if $n \geq 2$; otherwise let $C_n^{\mathrm{D}}(X) = 0$. If X is a quandle, then

$$\partial_n(C_n^{\mathrm{D}}(X)) \subset C_{n-1}^{\mathrm{D}}(X)$$

and $C_\bullet^{\mathrm{D}}(X) = \{C_n^{\mathrm{D}}(X), \partial_n\}$ is a sub-complex of $C_\bullet^{\mathrm{R}}(X)$. Put $C_n^{\mathrm{Q}}(X) = C_n^{\mathrm{R}}(X)/C_n^{\mathrm{D}}(X)$ and $C_\bullet^{\mathrm{Q}}(X) = \{C_n^{\mathrm{Q}}(X), \partial'_n\}$, where ∂'_n is the induced homomorphism. Henceforth, all boundary maps will be denoted by ∂_n.

For an abelian group G, define the chain and cochain complexes

$$C_\bullet^{\mathrm{W}}(X; G) = C_\bullet^{\mathrm{W}}(X) \otimes G, \qquad \partial = \partial \otimes \mathrm{id}; \tag{1.3}$$

$$C_{\mathrm{W}}^\bullet(X; G) = \mathrm{Hom}(C_\bullet^{\mathrm{W}}(X), G), \qquad \delta = \mathrm{Hom}(\partial, \mathrm{id}) \tag{1.4}$$

in the usual way, where W = D, R, Q.

The groups of cycles and boundaries are denoted respectively by

$$\ker(\partial) = Z_n^{\mathrm{W}}(X;G) \subset C_n^{\mathrm{W}}(X;G)$$

and

$$\mathrm{Im}(\partial) = B_n^{\mathrm{W}}(X;G) \subset C_n^{\mathrm{W}}(X;G)$$

while the cocycles and coboundaries are denoted respectively by

$$\ker(\delta) = Z^n_{\mathrm{W}}(X;G) \subset C^n_{\mathrm{W}}(X;G)$$

and

$$\mathrm{Im}(\partial) = B^n_{\mathrm{W}}(X;G) \subset C^n_{\mathrm{W}}(X;G).$$

In particular, a quandle 2-cocycle is an element $\phi \in Z^2_{\mathrm{Q}}(X;G)$, and the equalities

$$\begin{aligned} \phi(x,z) + \phi(x \lhd z, y \lhd z) &= \phi(x \lhd y, z) + \phi(x,y) && (1.5)\\ \text{and} \quad \phi(x,x) &= 0 && (1.6) \end{aligned}$$

are satisfied for all $x, y, z \in X$.

The n^{th} *quandle homology group* and the n^{th} *quandle cohomology group* [CJKLS] of a quandle X with coefficient group G are

$$\begin{aligned} H_n^{\mathrm{Q}}(X;G) &= H_n(C_*^{\mathrm{Q}}(X;G)) = Z_n^{\mathrm{Q}}(X;G)/B_n^{\mathrm{Q}}(X;G),\\ H^n_{\mathrm{Q}}(X;G) &= H^n(C^*_{\mathrm{Q}}(X;G)) = Z^n_{\mathrm{Q}}(X;G)/B^n_{\mathrm{Q}}(X;G). \end{aligned} \quad (1.7)$$

Definition 8. A *coloring* of an oriented classical knot diagram by a quandle X is a function $\mathcal{C} : R \to X$, where R is the set of arcs in the diagram, satisfying the condition depicted in next figure.

Note that locally the colors do not depend on the orientation of the under-arc. The quandle element $\mathcal{C}(r)$ assigned to an arc r by a coloring $\mathcal{C}$ is called a *color* of the arc. This definition of colorings on knot diagrams has been known, see [FR, F] for example. Henceforth, all the quandles that are used to color diagrams will be finite.

Remark 9. When painting Redemeister type III move (see **??**), the bottom arcs show the second quandle axiom (self-distributivity).

Definition 10. Let an oriented knot diagram, a quandle X, and a quandle 2-cocycle $\phi \in Z^2_{\mathrm{Q}}(X;A)$ be given. A *(Boltzmann) weight*, $B(\tau, \mathcal{C})$ (that depends on ϕ), at a crossing τ is defined as follows. Let $\mathcal{C}$ denote a coloring. Define

$$B(\tau, \mathcal{C}) = \phi(x,y)^{\epsilon(\tau)}$$

where x, y are the incoming arcs in case the crossing is positive or x, y are the outcoming arcs in case the crossing is negative. Definition 8, the leftmost is positive and the rightmost is negative. We will recall this notion in Definition 20.

The *partition function*, or a *state-sum*, is the expression

$$\sum_{\mathcal{C}} \prod_{\tau} B(\tau, \mathcal{C}).$$

The product is taken over all crossings of the given diagram, and the sum is taken over all possible colorings. The values of the partition function are taken to be in the group ring $\mathbb{Z}[A]$ where A is the coefficient group written multiplicatively. The partition function depends on the choice of 2-cocycle ϕ. This is proved [CJKLS] to be a knot invariant (see Definition 14), called the *(quandle) cocycle invariant*.

1.2.1 Twisted quandle homology

In [CES1] there is a generalization of quandle homology that considers coefficients in Alexander quandles. Here we recall such structure.

Let $\Lambda = \mathbb{Z}[T, T^{-1}]$, and let $C_n^{\mathrm{TR}}(X) = C_n^{\mathrm{TR}}(X; \Lambda)$ be the free module over Λ generated by n-tuples $(x_1, \ldots, x_n)$ of elements of a quandle X. Define a homomorphism $\partial = \partial_n^T : C_n^{\mathrm{TR}}(X) \to C_{n-1}^{\mathrm{TR}}(X)$ by

$$\begin{aligned} &\partial_n^T(x_1, x_2, \ldots, x_n) \\ &\quad = \sum_{i=1}^{n} (-1)^i \left[T(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \right. \\ &\quad \left. - (x_1 \triangleleft x_i, x_2 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, x_{i+1}, \ldots, x_n) \right] \end{aligned} \tag{1.8}$$

for $n \geq 2$ and $\partial_n^T = 0$ for $n \leq 1$. Note that the $i = 1$ terms contribute $(1-T)(x_2, \ldots, x_n)$. Then

$$C_*^{\mathrm{TR}}(X) = \{C_n^{\mathrm{TR}}(X), \partial_n^T\}$$

is a chain complex. For any Λ-module A, let

$$C_*^{\mathrm{TR}}(X; A) = \{C_n^{\mathrm{TR}}(X) \otimes_\Lambda A, \partial_n^T\}$$

be the induced chain complex, where the induced boundary operator is represented by the same notation. Let

$$C^n_{\mathrm{TR}}(X; A) = \mathrm{Hom}_\Lambda(C_n^{\mathrm{TR}}(X), A)$$

and define the coboundary operator

$$\delta = \delta^n_{\mathrm{TR}} : C^n_{\mathrm{TR}}(X; A) \to C^{n+1}_{\mathrm{TR}}(X; A)$$

by

$$(\delta f)(c) = (-1)^n f(\partial c)$$

for any $c \in C_n^{\mathrm{TR}}(X)$ and $f \in C^n_{\mathrm{TR}}(X; A)$. Then

$$C^*_{\mathrm{TR}}(X; A) = \{C^n_{\mathrm{TR}}(X; A), \delta^n_{\mathrm{TR}}\}$$

is a cochain complex. The n-th homology and cohomology groups of these complexes are called *twisted rack homology group* and *twisted rack cohomology group*, and are denoted by $H_n^{\mathrm{TR}}(X; A)$ and $H^n_{\mathrm{TR}}(X; A)$, respectively.

Let $C_n^{\rm TD}(X;A)$ be the subset of $C_n^{\rm TR}(X;A)$ generated by n-tuples $(x_1,\dots,x_n)$ with $x_i=x_{i+1}$ for some $i\in\{1,\dots,n-1\}$ if $n\geq 2$; otherwise let $C_n^{\rm TD}(X;A)=0$. If X is a quandle, then

$$\partial_n^T(C_n^{\rm TD}(X;A))\subset C_{n-1}^{\rm TD}(X;A)$$

and

$$C_*^{\rm TD}(X;A)=\{C_n^{\rm TD}(X;A),\partial_n^T\}$$

is a sub-complex of $C_*^{\rm TR}(X;A)$. Similar subcomplexes $C^*_{\rm TD}(X;A)=\{C^n_{\rm TD}(X;A),\delta^n_T\}$ are defined for cochain complexes.

The n-th homology and cohomology groups of these complexes are called *twisted degeneracy homology group* and *cohomology group*, and are denoted by $H_n^{\rm TD}(X;A)$ and $H^n_{\rm TD}(X;A)$, respectively.

Take

$$C_n^{\rm TQ}(X;A)=C_n^{\rm TR}(X;A)/C_n^{\rm TD}(X;A)$$

and

$$C_*^{\rm TQ}(X;A)=\{C_n^{\rm TQ}(X;A),\partial_n^T\},$$

where all the induced boundary operators are denoted by $\partial=\partial_n^T$. A cochain complex

$$C^*_{\rm TQ}(X;A)=\{C^n_{\rm TQ}(X;A),\delta^n_T\}$$

is similarly defined. Note again that all boundary and coboundary operators will be denoted by $\partial=\partial_n^T$ and $\delta=\delta^n_T$, respectively. The n-th homology and cohomology groups of these complexes are called *twisted homology group* and *cohomology group*, and are denoted by

$$H_n^{\rm TQ}(X;A)=H_n(C_*^{\rm TQ}(X;A)),\quad H^n_{\rm TQ}(X;A)=H^n(C^*_{\rm TQ}(X;A)). \tag{1.9}$$

The groups of (co)cycles and (co)boundaries are denoted using similar notations.

For $\rm W=D,R$, or $\rm Q$ (denoting the degenerate, rack or quandle case, respectively), the groups of twisted cycles and boundaries are denoted (resp.) by

$$\ker(\partial)=Z_n^{\rm TW}(X;A)\subset C_n^{\rm TW}(X;A)$$

and

$${\rm Im}(\partial)=B_n^{\rm TW}(X;A)\subset C_n^{\rm TW}(X;A).$$

The twisted cocycles and coboundaries are denoted respectively by

$$\ker(\delta)=Z^n_{\rm TW}(X;A)\subset C^n_{\rm TW}(X;A)$$

and

$${\rm Im}(\partial)=B^n_{\rm TW}(X;A)\subset C^n_{\rm TW}(X;A).$$

Thus the (co)homology groups are given as quotients:

$$\begin{aligned} H_n^{\rm TW}(X;A) &= Z_n^{\rm TW}(X;A)/B_n^{\rm TW}(X;A),\\ H^n_{\rm TW}(X;A) &= Z^n_{\rm TW}(X;A)/B^n_{\rm TW}(X;A). \end{aligned}$$

The following list provides some known facts ([C]):

- In [EG] a formula is proved for the rank of $H_n^R(X;\mathbf{Q})$ for X a finite rack. In particular for a connected quandle (a quandle with only one orbit) these dimensions are all one, as they are for the one point rack. This means that is important to study finite characteristic case.

- In [Mo] (for corrections see [Ma]) the third cohomology group is computed for Alexander quandles associated to a finite field k where T is multiplication by some $w \in k^*$.

- In [LN] it is proved that the torsion subgroup of $H_n^R(X)$ is annihilated by d^n for X a rack ($|X| = d$) with homogeneous orbits. For example: Alexander racks. In particular, all torsion in the homology of R_p (R_n is the set of reflections in the dihedral group of order $2n$) is p-primary.

- For $p = 3$ the torsion in the homology of R_p is of exponent p (see [NP]). It is true for general p (see [C]).

- There exists a monomorphism $h_a\colon H_n^Q(R_p) \to H_{n+2}^Q(R_p)$ for small n and p (see [NP]). The ranks of these groups form a delayed (shifted) Fibonacci sequence (see [C]).

1.2.2 Biquandle and quandle homology (with trivial coefficients)

Let (X, R) be a Yang-Baxter set (that is a set X and $R : X \times X \to X \times X$ a solution of YBeq). In [CES2] there are some computations of (low degree) boundary homomorphisms, we transcribe here those computations. $R_i(x, y) = \sigma^{(i)}(x, y)$ for $i \in \{1, 2\}$.

1. The boundary homomorphism in case ∂_2 is given by:

$$\partial_2(x, y) = (x) + (y) - (R_1(x, y)) - (R_2(x, y)).$$

2. For ∂_3:

$$\begin{aligned}
&\partial_3(x, y, z)\\
&\quad= \quad (x, y) + (R_2(x, y), z) + (R_1(x, y), R_1(R_2(x, y), z))\\
&\quad- \quad \{(y, z) + (x, R_1(y, z)) + (R_2(x, R_1(y, z)), R_2(y, z))\}
\end{aligned}$$

3. For ∂_4:

$$\begin{aligned}
&\partial_4(x_1, x_2, x_3, x_4)\\
&\quad= \quad (x_1, x_2, x_3) + (R_2(x_1, R_1(x_2, x_3)), R_2(x_2, x_3), x_4)\\
&\qquad\quad +(x_1, R_1(x_2, x_3), R_1(R_2(x_2, x_3), x_4)) + (x_2, x_3, x_4)\\
&\quad- \quad \{(R_1(x_1, x_2), R_1(R_2(x_1, x_2), x_3), R_1(R_2(R_2(x_1, x_2), x_3), x_4)) + (R_2(x_1, x_2), x_3, x_4)\\
&\qquad\quad +(x_1, x_2, R_1(x_3, x_4)) + (R_2(x_1, R_1(x_2, R_1(x_3, x_4))), R_2(x_2, R_1(x_3, x_4)), R_2(x_3, x_4))\}.
\end{aligned}$$

In Chapter 2, we give a general formula for boundary homomorphisms.

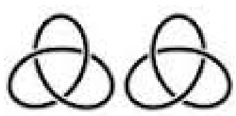
1.3 State-sum invariant: a brief survey of knots and links

In this chapter we introduce some general notions and definitions used in knot theory.

We begin with a colloquial definition. Take a piece of string, tie a knot in it and glue the ends together. A *knot* is just a knoted loop of string, except we think of the string as having no thickness, its cross-section being a single point.

We will not distinguish between the original closed knotted curve and all the deformations of that curve through space that do not allow the curve to pass through itself.

Definition 11. A *knot* (in $\mathbb{R}^3$) is a continuous and injective function $f : S^1 \to \mathbb{R}^3$ where $S^1 = \{x \in \mathbb{R}^2 / ||x|| = 1\}$. In the set of knots an equivalence relation is defined, two knots given by f and g are called *isotopic* if and only if exists a continuous function $H : S^1 \times [0,1] \to \mathbb{R}^3$ such that $H_t : z \mapsto H(z,t)$ is a knot for every $t \in [0,1]$, $H_0 = f$ and $H_1 = g$.

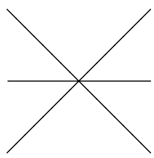


Trefoil knot and mirror image

A knot is called *tame* when is isotopic to another one given by a finite piecewise linear function. To avoid pathologies we will only consider tame knots.

An orientation in S^1 induces an orientation on the knot. In this work, a knot will always be an oriented knot.

Given a knot K, there are many pictures/projections (orthogonal projections to planes of $\mathbb{R}^2$) of K. A projection is called general if there are no triple intersections like the one in the following diagram:



A projection includes information of every crossing as in the following picture:

Given a projection of a knot-link, the isotopy class of the knot-link can be recovered.

In 1926, Reidemeister gave a combinatorial method to change the projection of a knot-link without changing the isotopy class. There are three (non oriented) Reidemeister moves, which are shown in next figure.

Theorem 12. *Two knots are equivalent if and only if given a (any) pair of projections, they are connected by a finite number of Reidemeister moves.*

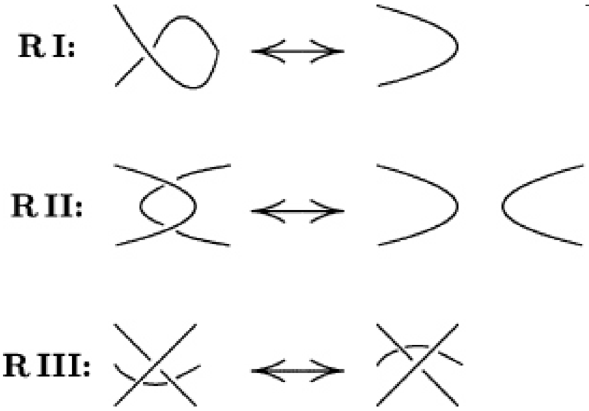


Figure 1.1: Reidemeister moves

Proof. See, for example, Theorem 1.14 [BZG]. $\square$

A Link is a set of knotted loops tied up together, formally:

Definition 13. A link is a set of functions $f_1, \cdots, f_n$ such that every f_i defines a knot and $\mathrm{Im} f_i \cap \mathrm{Im} f_j \neq$ for every $i \neq j$.

A knot will be considered a link of one component.

In the figure 1.3 many non isotopic knots and links are shown.

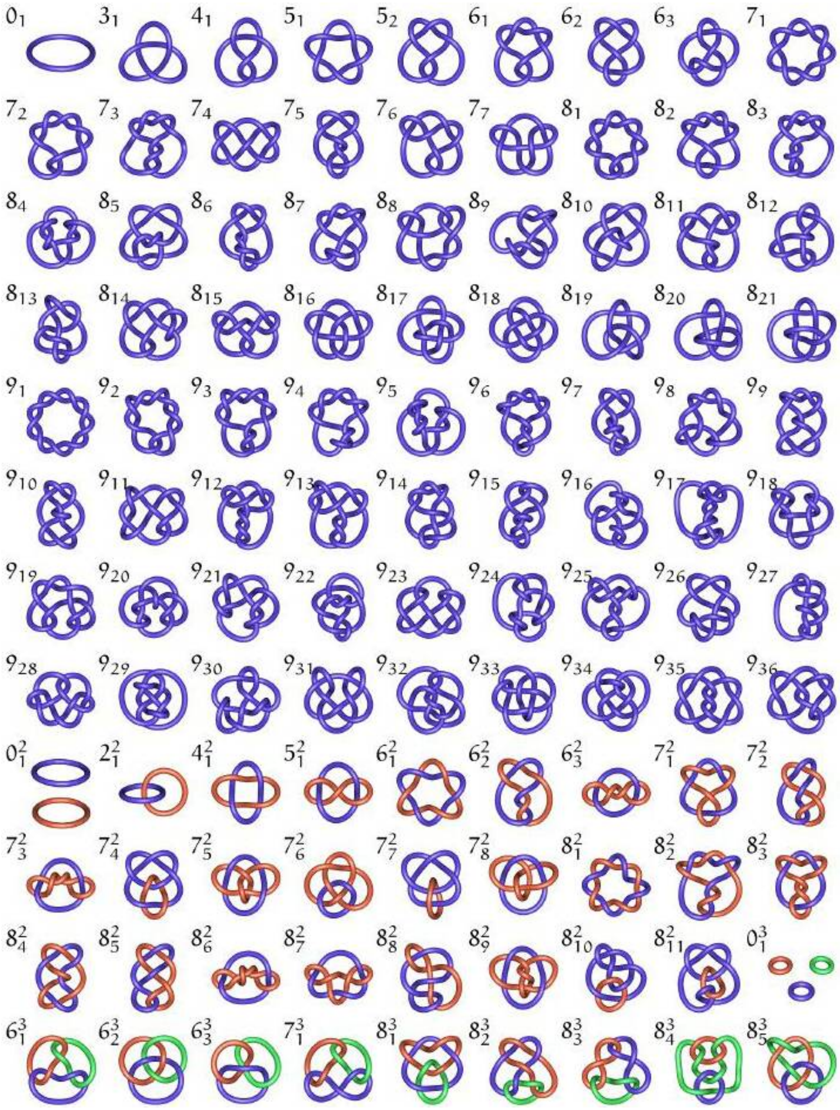


Figure1.3

One of the most important problems in knot theory is to distinguish knots. A knot invariant is a function that associates the same value to every knot of the same isotopy class.

Definition 14. A knot invariant is a *property* that does not change under ambient isotopy.

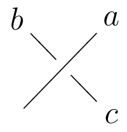
A very well known invariant is the knot group.

Definition 15. For a knot K, $\pi_1(\mathbb{R}^3 - K, *)$ ($\simeq \pi_1(S^3 - K, *)$), the fundamental group of the complement of the knot in $\mathbb{R}^3$ (or S^3) is called the group of the knot or knot group of K where $*$ denotes a base point. Since two groups with different base points are conjugated, $*$ is usually omited.

Another simple and well known way to distinguish links is based on colorings.

In 1956 R. Fox defined a generalization to the well known "three coloring". Here we recall the generalization.

A knot is said to be *p-colorable* for $2 \leq p$ a prime, if each arc in a given projection of a knot can be assigned with a number of $\mathbb{Z}_p = \{0, \ldots, p-1\}$ such that in every crossing



the equation

$$2a - b - c = 0$$

stands.

A classical result is the following:

Theorem 16. *The total amount of p-colorings is a knot invariant.*

Example 17. The trefoil knot (see 3_1 in 1.3) is 3-colorable (using more than one color). The unknot (see 0_1 in 1.3) is not colorable using more than one color. This fact tells both knots are different.

A generalization of colorings led to the definition of *quandles*. Quandles model Reidemeister's moves. This new structure allowed Joyce and Matveev [J, M] to define another knot invariant, the fundamental quandle, yielding a complete invariant of oriented classical knots.

Let K be a knot with n arcs $(a_1, \ldots, a_n)$ and m crossings. Label the arcs as in the next figure and consider, for each crossing τ, the relation

$$r_\tau : a_j \triangleleft a_i = a_k. \tag{1.10}$$

The **fundamental quandle** of the knot K is the following quandle:

$$Q(K) = \langle a_1, a_2, \ldots, a_n : r_1, \ldots, r_m \rangle,$$

where the relations $r_1, \ldots, r_m$ are given by the formula (1.10).

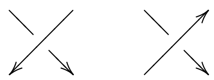
Theorem 18. *The fundamental quandle is a knot invariant.*

Proof. See [J, M]. □

Definition 19. Given two quandles Q_1, Q_2, a map $f : Q_1 \to Q_2$ is called a *quandle morphism* if $\forall x, y \in Q_1$ $f(x \triangleleft_{Q_1} y) = f(x) \triangleleft_{Q_2} f(y)$ is verified.

A coloring of a knot K by the quandle X is a quandle morphism $Q(K) \to X$. The total amount of such quandle morphisms is a knot invariant denoted $Col_X(K)$.

Definition 20. A sign is associated to every crossing in an oriented diagram of a knot according to the following:



positive negative

Given a link of two strands (conected components) L_1, L_2 there is an invariant based on the signs of the crossings which is called the *linking number*.

The linking number is defined as follows: calculate the sign for every crossing where both components are involved, the total number of positive crossings n_1 minus the total number of negative crossings n_2 is equal to twice the linking number. That is:

$$l(L_1, L_2) = \frac{n_1 - n_2}{2}$$

Definition 21. Given a quandle $(X, \triangleleft)$,

$$G_X = F_X / \langle xy = (x \triangleleft y),\ x, y \in X \rangle,$$

is the *enveloping group of the quandle* where F_X is the free group with the elements of X as a base. In 1982 Joyce showed that the enveloping group of the fundamental quandle satisfies $G_{Q(K)} \simeq \pi_1(K)\ \forall K$ (see 15).

In [CJKLS] invariants based on 2-cocycles are defined. We recall some constructions here (same as in formula 1.5 but with multiplicative notation).

Definition 22. [CJKLS] Let X be a quandle, A an abelian group. A function $f : X \times X \to A$ is called an *abelian 2-cocyle* if

- $f(x, y) f(x \triangleleft y, z) = f(x, z) f(x \triangleleft z, y \triangleleft z)$
- $f(x, x) = 1$

$\forall x, y, z \in X$.

This definition (if written in additive notation) corresponds to a 2-cocycle in 1.5.

Definition 23. [CJKLS] Let f and g be two 2-cocycles, f and g are called *cohomologous* (or equivalent) if there exists $\gamma \colon X \to A$ such that

$$f(x, y) = \gamma(x) g(x, y) \gamma(x \triangleleft y)^{-1}$$

for all $x, y \in X$.

In Chapter 3 we generalize these constructions to non abelian 2-cocycles.

Let K be a knot, X a quandle and A an abelian group. Given a 2-cocycle

$$f : X \times X \to A$$

the *partition function* $\Phi_{X,f}(K)$ is defined by

$$\Phi_{X,f}(K) = \sum_{\mathcal{C}} \prod_{\chi} \omega_f(\mathcal{C}, \chi), \tag{1.11}$$

where the product is taken over all crossings χ and the sum is taken over all colorings of K.

The formula (1.11) defines an element in $\mathbb{Z}[A]$.

Theorem 24. $\Phi_{X,f}$ *defines a knot invariant (called state sum).*

This notion of two cocycle was generalized for non commutative groups (see [AG] in the context of Hopf Algebras) and used in [CEGS] for constructing knot/link invariants. We generalize the non commutative version to general biquandles and produce knot/link invariants. Moreover, we produce a universal group with universal cocycle, and if one considers its abelianization, then one gets (usually nontrivial) commutative cocycles that can be used in the state sum procedure.

Chapter 2

A d.g. bialgebra associated to (X, σ)

Introducción al capítulo:

En este capítulo, para cada solución conjuntista de la ecuación de Yang Baxter (X, σ), definimos un álgebra diferencial graduada $B = B(X, \sigma)$, que contiene al álgebra de semigrupo $A = A_{\sigma(X)} = k\{X\}/\langle xy = zt : \sigma(x,y) = (z,t)\rangle$, tal que $k \otimes_A B \otimes_A k$ y $\mathrm{Hom}_{A-A}(B, k)$ son respectivamente las estructuras de homología y cohomología usuales asociados a una solución conjuntista de la ecuación Yang-Baxter. Probamos que esta estructura de álgebra diferencial graduada tiene una estructura natural de *biálgebra* diferencial graduada (Teorema 26). También, dependiendo de las propiedades de la solución (X, σ) (libre de cuadrados, de tipo quandle, biquandle, involutiva,...) esta biálgebra d.g. B tiene cocientes naturales que dan lugar a los subcomplejos standard al calcular cohomología de quandle, cohomología de biquandle, etc.

Como primera consecuencia de nuestra construcción, damos una prueba simple y puramente algebraica de la existencia de un producto cup en cohomología. Esto era conocido para cohomología de rack (ver [C]), la prueba está basada en métodos topológicos, y era desconocido para biquandles o soluciones generales de la ecuación de Yang-Baxter. Como segunda consecuencia, mostramos la existencia de un morfismo de comparación entre la (co)homología de Yang-Baxter y la (co)homología de Hochschild del álgebra de semigrupo A. Mirando cuidadosamente, este morfismo de comparación, probamos que se factoriza a través de un complejo de tipo $A \otimes \mathfrak{B} \otimes A$, donde $\mathfrak{B}$ es el álgebra de Nichols asociada a la solución $(X, -\sigma)$. Este resultado nos lleva a nuevas preguntas, por ejemplo cuando (X, σ) es involutiva (eso es cuando $\sigma^2 = \mathrm{Id}$) y la característica es cero mostramos que este complejo es acíclico (Proposición 51), nos preguntamos si esto es cierto en alguna otra característica, y no necesariamente en soluciones involutivas.

En característica cero, la cohomología de racks y quandles está completamente determinada por la cantidad de componentes conexas del rack (resp. quandle). Para soluciones arbitrarias de la ecuación de YB esto no es sabido. De cualquier manera, la mayoría de las técnicas para demostrar el resultado en el caso rack se generalizan, y hemos probado una cota superior para los números de Betti de la cohomología de YB.

Finalmente consideramos soluciones involutivas de la ecuación de YB y probamos que otro complejo natural asociado a este tipo de soluciones es isomorfo al complejo usual de soluciones de Yang-Baxter. En particular, la cota para los números de Betti de este complejo asociado a soluciones involutivas es una consecuencia del resultado general para cohomología de YB.

Introduction to the chapter:

In this chapter, for a set theoretical solution of the Yang-Baxter equation (X,σ), we define a d.g. algebra $B = B(X,\sigma)$, containing the semigroup algebra $A = k\{X\}/\langle xy = zt : \sigma(x,y) = (z,t)\rangle$, such that $k \otimes_A B \otimes_A k$ and $\mathrm{Hom}_{A-A}(B,k)$ are respectively the standard homology and cohomology complexes attached to general set theoretical solutions of the Yang-Baxter equation. We prove that this d.g. algebra has a natural structure of d.g. *bialgebra* (Theorem 26). Also, depending on properties of the solution (X,σ) (square free, quandle type, biquandle, involutive,...) this d.g. bialgebra B has natural (d.g. bialgebra) quotients, giving rise to the standard sub-complexes computing quandle cohomology (as sub-complex of rack homology), biquandle cohomology, etc.

As a first consequence of our construction, we give a very simple and purely algebraic proof of the existence of a cup product in cohomology. This was known for rack cohomology (see [C]), the proof was based on topological methods, but it was unknown for biquandles or general solutions of the Yang-Baxter equation. A second consequence is the existence of a comparison map between Yang-Baxter (co)homology and Hochschild (co)homology of the semigroup algebra A. Looking carefully this comparison map we prove that it factors through a complex of "size" $A \otimes \mathfrak{B} \otimes A$, where $\mathfrak{B}$ is the Nichols algebra associated to the solution $(X,-\sigma)$. This result leads to new questions, for instance when (X,σ) is involutive (that is $\sigma^2 = \mathrm{Id}$) and the characteristic is zero we show that this complex is acyclic (Proposition 51), we wander if this is true in any other characteristic, and for non necessarily involutive solutions.

Let $M = M_X$ be the monoid freely generated in X with relations

$$xy = zt$$

$\forall x,y,z,t$ such that

$$\sigma(x,y) = (z,t).$$

Denote G_X the group with the same generators and relations. For example, when $\sigma = \text{flip}$ then $M = \mathbb{N}_0^{(X)}$ and $G_X = \mathbb{Z}_0^{(X)}$. If $\sigma = \mathrm{Id}$ then M is the free (non abelian) monoid in X. If σ comes from a rack $(X,\triangleleft)$ then M is the monoid with relations $xy = y(x \triangleleft y)$ and G_X is the group with relations $x \triangleleft y = y^{-1}xy$.

In characteristic zero, rack and quandle cohomology is completely determined by the connected components of the rack (resp. quandle). For arbitrary solutions of the YBeq this is not known, however most of the techniques for the rack case can be generalized, and we were able to prove an upper bound for the Betti numbers of YB cohomology.

Finally we consider involutive solutions of the YBeq and we prove that another complex associated to these type of solutions is actually isomorphic to the standard one, that is, the one considered for the general case. In particular, the bound for the Betti numbers of the complex attached to involutive solutions is a consequence of the bound for the general YB cohomology.

2.1 A d.g. bialgebra associated to (X, σ)

Let k be a commutative ring with 1. Fix X a set, and $\sigma : X \times X \to X \times X$ a solution of the YBeq. Let $A_\sigma(X)$ (denoted simply by A if X and σ are understood) the quotient of the free k algebra on generators X modulo the ideal generated by elements of the form $xy - zt$ whenever $\sigma(x, y) = (z, t)$:

$$A = k\langle X\rangle/\langle xy - zt : x, y \in X,\ (z, t) = \sigma(x, y)\rangle = k[M]$$

It can be easily seen that A is a k-bialgebra declaring x to be grouplike for any $x \in X$, since A agrees with the semigroup-algebra on M (the monoid freely generated by X with relations $xy \sim zt$). If one considers G_X the group freely generated by X with relations $xy = zt$, then $k[G_X]$ is the (non commutative) localization of A, where one has inverted the elements of X. An example of an A-bimodule that will be used later, which is actually a $k[G_X]$-module, is k with the A-action determined on generators by

$$x\lambda y = \lambda,\ \forall x, y \in X$$

Note that is the trivial action given by the counit.

Definition 25. We define $\widehat{B}(X, \sigma)$ (also denoted by $\widehat{B}$) the algebra freely generated by three copies of X, denoted x, e_x and x', with relations as follows: whenever $\sigma(x, y) = (z, t)$ we have

- $xy' \sim z't$,
- $xe_y \sim e_z t$,
- $e_x y' \sim z' e_t$

and we define $B(X, \sigma)$ (also denoted by B) as the quotient of $\widehat{B}$ by the additional relations

- $xy \sim zt$,
- $x'y' \sim z't'$.

The main object of study of this chapter is the algebra B. Declaring

$$|x| = |x'| = 0,\quad |e_x| = 1,$$

the relations defining B are homogeneous, so B is a graded algebra.

Theorem 26. *The algebra B admits the structure of a differential graded bialgebra, with d the unique superderivation satisfying*

$$d(x) = d(x') = 0,\quad d(e_x) = x - x'$$

and comultiplication determined by

$$\Delta(x) = x \otimes x,\ \Delta(x') = x' \otimes x',\ \Delta(e_x) = x' \otimes e_x + e_x \otimes x,$$

the counit is given by

$$\varepsilon(x) = 1 = \varepsilon(x'),\ \varepsilon(e_x) = 0.$$

Remark 27. By differential graded bialgebra we mean that

- the algebra structure on $B \otimes B$ is the one of super vector spaces, that is
$$(b \otimes c)(b' \otimes c') = (-1)^{|b'||c|} bb' \otimes cc',$$
- the differential is both a derivation with respect to multiplication, and coderivation with respect to comultiplication.

Proof. In order to see that d is well-defined as super derivation, one must check that the relations are compatible with d. The first relations are easier since

$$d(xy - zt) = d(x)y + xd(y) - d(z)t - zd(t) = 0 + 0 - 0 - 0 = 0$$

and similar for the others (this implies that d is A-linear and A'-linear), for the others:

$$d(xe_y - e_z t) = xd(e_y) - d(e_z)t = x(y - y') - (z - z')t$$
$$= xy - zt - (xy' - z't) = 0$$
$$d(e_x y' - z'e_t) = (x - x')y' - z'(t - t') = xy' - z't - (x'y' - z't') = 0.$$

It is clear now that $d^2 = 0$ since d^2 vanishes on generators. In order to see that Δ is well-defined, we compute

$$\Delta(xe_y - e_z t) = (x \otimes x)(y' \otimes e_y + e_y \otimes y) - (z' \otimes e_z + e_z \otimes z)(t \otimes t)$$
$$= xy' \otimes xe_y + xe_y \otimes xy - z't \otimes e_z t - e_z t \otimes zt$$

and using the relations we get

$$= xy' \otimes xe_y + xe_y \otimes xy - xy' \otimes xe_y - xe_y \otimes xy = 0$$

similarly

$$\Delta(x'e_y - e_z t') = (x' \otimes x')(y' \otimes e_y + e_y \otimes y) - (z' \otimes e_z + e_z \otimes z)(t' \otimes t')$$
$$= x'y' \otimes x'e_y + x'e_y \otimes x'y - z't' \otimes e_z t' - e_z t' \otimes zt'$$
$$= x'y' \otimes x'e_y + x'e_y \otimes x'y - x'y' \otimes x'e_y - x'e_y \otimes x'y = 0.$$

This proves that B is a bialgebra, and d is (by construction) a derivation. Let us see that it is also a coderivation:

$$(d \otimes 1 + 1 \otimes d)(\Delta(x)) = (d \otimes 1 + 1 \otimes d)(x \otimes x) = 0 = \Delta(0) = \Delta(dx)$$

for x' is the same. For e_x:

$$(d \otimes 1 \pm 1 \otimes d)(\Delta(e_x)) = (d \otimes 1 \pm 1 \otimes d)(x' \otimes e_x + e_x \otimes x)$$
$$= dx' \otimes e_x + x' \otimes de_x + de_x \otimes x - e_x \otimes dx$$
$$= x' \otimes (x - x') + (x - x') \otimes x = x' \otimes x - x' \otimes x' + x \otimes x - x' \otimes x$$
$$= -x' \otimes x' + x \otimes x = \Delta(x - x') = \Delta(de_x)$$

$\square$

It is enough to check cosassociativity on generators because Δ is an algebra map and d is a derivation.

Notice that Δ is coassociative since x, x' are grouplike and e_x are skew-primitives.

Let us denote $\sigma(x,y) = (\sigma^{(1)}(x,y), \sigma^{(2)}(x,y))$ and $\sigma_i : X^l \to X^l$ where $\sigma_i := id_X^{i-1} * \sigma * \mathrm{Id}_X^{l-i-1}$

Remark 28. In particular

$$d(e_{x_1} \dots e_{x_n}) = \sum_{i=1}^{n} (-1)^{i+1} e_{x_1} \dots e_{x_{i-1}} d(e_{x_i}) e_{x_{i+1}} \dots e_{x_n}$$

$$= \sum_{i=1}^{n} (-1)^{i+1} e_{x_1} \dots e_{x_{i-1}} (x_i - x'_i) e_{x_{i+1}} \dots e_{x_n}$$

$$= \overbrace{\sum_{i=1}^{n} (-1)^{i+1} e_{x_1} \dots e_{x_{i-1}} x_i e_{x_{i+1}} \dots e_{x_n}}^{I}$$

$$- \overbrace{\sum_{i=1}^{n} (-1)^{i+1} e_{x_1} \dots e_{x_{i-1}} x'_i e_{x_{i+1}} \dots e_{x_n}}^{II}$$

Using the relations in B one has

$$I = \sum_{i=1}^{n} (-1)^{i+1} e_{x_1} \dots e_{x_{i-1}} e_{y^{(1)}_{i+1,i}} \dots e_{y^{(1)}_{n,i}} y^{(2)}_{n,i}$$

where

$$\begin{aligned}
y_{i+1,i} &= (\sigma^{(1)}(x_i, x_{i+1}), \sigma^{(2)}(x_i, x_{i+1})), \\
y_{i+2,i} &= (\sigma^{(1)}(y^{(2)}_{i+1,i}, x_{i+2}), \sigma^{(2)}(y^{(2)}_{i+1,i}, x_{i+2})), \\
y_{i+3,i} &= (\sigma^{(1)}(y^{(2)}_{i+2,i}, x_{i+3}), \sigma^{(2)}(y^{(2)}_{i+2,i}, x_{i+3})), \\
\vdots \quad & \quad \vdots \\
y_{n,i} &= (\sigma^{(1)}(y^{(2)}_{n-1,i}, x_n), \sigma^{(2)}(y^{(2)}_{n-1,i}, x_n))
\end{aligned}$$

and similarly

$$II = \sum_{i=1}^{n} (-1)^{i+1} (z^{(1)}_{1,i})' e_{z^{(2)}_{1,i}} \dots e_{z^{(2)}_{i-2,i}} e_{z^{(2)}_{i-1,i}} e_{x_i+1} \dots e_{x_n}$$

where

$$\begin{aligned}
z_{i-1,i} &= (\sigma^{(1)}(x_{i-1}, x_i), \sigma^{(2)}(x_{i-1}, x_i)), \\
z_{i-2,i} &= (\sigma^{(1)}(x_{i-2}, z^{(1)}_{i-1,i}), \sigma^{(2)}(x_{i-2}, z^{(1)}_{i-1,i})), \\
\vdots \quad & \quad \vdots \\
z_{1,i} &= (\sigma^{(1)}(x_1, z^{(1)}_{2,i}), \sigma^{(2)}(x_1, z^{(1)}_{2,i})).
\end{aligned}$$

$$\partial f(x_1, \dots, x_n) = f(d(e_{x_1} \dots e_{x_n})) =$$

$$\sum_{i=1}^{n} (-1)^{i+1} \left(f(x_1, \dots, x_{i-1}, y^{(1)}_{i+1,i}, \dots, y^{(1)}_{n,i}) y^{(2)}_{n,i} - (z^{(1)}_{1,i})' f(z^{(2)}_{1,i}, \dots, z^{(2)}_{i-1,i}, x_{i+1}, \dots, x_n) \right)$$

Example 29. In low degrees we have

- $d(e_x) = x - x'$
- $d(e_xe_y) = (e_zt - e_xy) - (x'e_y - z'e_t)$, where as usual $\sigma(x,y) = (z,t)$.
- $d(e_{x_1}e_{x_2}e_{x_3}) = A_I - A_{II}$

  where

  $A_I = e_{\sigma^{(1)}(x_1,x_2)}e_{\sigma^{(1)}(\sigma^{(2)}(x_1,x_2),x_3)}\sigma^{(2)}(\sigma^{(2)}(x_1,x_2),x_3) - e_{x_1}e_{\sigma^{(1)}(x_2,x_3)}\sigma^{(2)}(x_2,x_3) + e_{x_1}e_{x_2}x_3$

  $A_{II} = x_1'e_{x_2}e_{x_3} - \sigma^{(1)}(x_1,x_2)'e_{\sigma^{(2)}(x_1,x_2)}e_{x_3} + \sigma^{(1)}(x_1,\sigma^{(1)}(x_2,x_3))'e_{\sigma^{(2)}(x_1,\sigma^{(1)}(x_2,x_3))}e_{\sigma^{(2)}(x_2,x_3)}$

  In particular, if $f : B \to k$ is an $A' - A$ linear map, then

  $$f(d(e_{x_1}e_{x_2}e_{x_3})) =$$

  $$f(e_{\sigma^{(1)}(x_1,x_2)}e_{\sigma^{(1)}(\sigma^{(2)}(x_1,x_2),x_3)}) - f(e_{x_1}e_{\sigma^{(1)}(x_2,x_3)}) + f(e_{x_1}e_{x_2})$$
  $$-f(e_{x_2}e_{x_3}) + f(e_{\sigma^{(2)}(x_1,x_2)}e_{x_3}) - f(e_{\sigma^{(2)}(x_1,\sigma^{(1)}(x_2,x_3))}e_{\sigma^{(2)}(x_2,x_3)})$$

  Erasing the e's we notice the relationship with the cohomological complex given in [CES2], see Theorem 32 below.

If X is a rack and $\sigma(x,y) = (y, x \triangleleft y) = (x, x^y)$, then

- $d(e_x) = x - x'$
- $d(e_xe_y) = (e_yx^y - e_xy) - (x'e_y - y'e_{x^y})$
- $d(e_xe_ye_z) = e_xe_yz - e_xe_zy^z + e_ye_zx^{yz} - x'e_ye_z + y'e_{x^y}e_z - z'e_{x^z}e_{y^z}$.
- In general, expressions I and II are

  $$I = \sum_{i=1}^{n}(-1)^{i+1}e_{x_1}\dots e_{x_{i-1}}e_{x_{i+1}}\dots e_{x_n}x_i^{x_{i+1}\dots x_n}$$

  $$II = \sum_{i=1}^{n}(-1)^{i+1}x_i'e_{x_1^{x_i}}\dots e_{x_{i-1}^{x_i}}e_{x_{i+1}}\dots e_{x_n}$$

  then

  $$\partial f(x_1,\dots,x_n) = f(d(e_{x_1}\dots e_{x_n})) =$$

  $$\sum_{i=1}^{n}(-1)^{i+1}\left(f(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)x_i^{x_{i+1}\dots x_n} - x_i'f({x_1}^{x_i},\dots,x_{i-1}^{x_i},x_{i+1},\dots,x_n)\right)$$

  Let us consider $k \otimes_{k[M']} B \otimes_{k[M]} k$. Then d represents the canonical differential of rack homology and $\partial f(e_{x_1}\dots e_{x_n}) = f(d(e_{x_1}\dots e_{x_n}))$ gives the traditional rack cohomology structure.

In particular, taking trivial coefficients:

$$\partial f(x_1,\dots,x_n)=f(d(e_{x_1}\dots e_{x_n}))=$$

$$\sum_{i=1}^{n}(-1)^{i+1}\left(f(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)-f({x_1}^{x_i},\dots,x_{i-1}^{x_i},x_{i+1}\dots,x_n)\right)$$

Proposition 30. *If $\sigma : X\times X\to X\times X$ is a set theoretical solution of the YBeq then, as k-modules,*

$$\widehat{B}\cong TX'\otimes TE\otimes TX$$

and

$$B\cong A'\otimes TE\otimes A$$

where $A'=TX'/(x'y'=z't' : \sigma(x,y)=(z,t))$ and $A=TX/(xy=zt : \sigma(x,y)=(z,t))$.

Remark 31. The isomorphism is actually as braided (tensor product of) algebras. More easily, we will only use the fact that the second isomorphism is as left A'-module and right A-module.

As a corollary we get our first main theorem in this chapter. Proposition 30 will be proven after the proof of the following theorem.

Theorem 32. *Taking k the trivial A'-A-bimodule, the complexes associated to set theoretical Yang-Baxter solutions defined in [CES2] can be recovered as*

$$(C_\bullet(X,\sigma),\partial)\simeq(k\otimes_{A'}B_\bullet\otimes_A k,\partial=id_k\otimes_{A'}d\otimes_A id_k),$$

$$(C^\bullet(X,\sigma),\partial^*)\simeq(\mathrm{Hom}_{A'-A}(B,k),\partial^*=d^*).$$

Assuming Proposition (30) we show a proof for Theorem 32.

Proof. In this setting every expression in x,x',e_x, using the relations defining B, can be written as $x'_{i_1}\dots x'_{i_n}e_{x_1}\dots e_{x_k}x_{j_1}\dots x_{j_l}$, tensorizing leaves the expression

$$1\otimes e_{x_1}\dots e_{x_k}\otimes 1$$

This shows that $T=k\otimes_{k[M']}B\otimes_{k[M]}k\simeq T\{e_x\}_{x\in X}$, where $\simeq$ means isomorphism of k-modules. This also induces isomorphisms of complexes

$$(C_\bullet(X,\sigma),\partial)\simeq(k\otimes_{A'}B_\bullet\otimes_A k,\partial=id_k\otimes_{A'}d\otimes_A id_k),$$

$$(C^\bullet(X,\sigma),\partial^*)\simeq(\mathrm{Hom}_{A'-A}(B,k),d^*).$$

□

To obtain a proof for Proposition 30, some notions have to be defined:

Call $Y = \langle x, x', e_x \rangle_{x \in X}$ the free monoid in three copies of X with unit 1, $k\langle Y \rangle$ the k algebra associated to Y. Let us define $w_1(x, y') = xy'$, $w_2(x, y) = xe_y$ and $w_3(x, y') = e_x y'$. Let $S = \{r_1^{x,y'}, r_2^{x,y}, r_3^{x,y'}\}_{x \in X\ y' \in X'}$ be the reduction system defined as follows: $r_i^{-,-}$: $k\langle Y \rangle \to k\langle Y \rangle$ are k-module endomorphisms defined on monoids by the following rule: $r_1^{x,y'}$ fix all elements except monoids of the form $Aw_1(x, y')B = Axy'B$ where $r_1^{x,y'}()$

$r_1(xy') = z't$, $r_2(xe_y) = e_z t$ and $r_3(e_x y') = z'e_t$.

Note that S has more than 3 elements, each r_i is a family of reductions.

Definition 33. A reduction r_i *acts trivially* on an element a if w_i does not appear in a, ie: let V, W be words in Y, if Vw_iW appears with coefficient 0.

Following [B], $a \in k\langle Y \rangle$ is called *irreducible* if Vw_iW does not appear in any monomial of a for $i \in \{1, 2, 3\}$. Call $k_{irr}\langle Y \rangle$ the k submodule of irreducible elements of $k\langle Y \rangle$. A finite sequence of reductions is called *final* in a if $r_{i_n} \circ \cdots \circ r_{i_1}(a) \in k_{irr}(Y)$. An element $a \in k\langle Y \rangle$ is called *reduction-finite* if for every sequence of reductions r_{i_n} acts trivially on $r_{i_{n-1}} \circ \cdots \circ r_{i_1}(a)$ for sufficiently large n. If $a \in K(Y)$ is reduction-finite, then any maximal sequence of reductions, such that each r_{i_j} acts non trivially on $r_{i_{(j-1)}} \ldots r_{i_1}(a)$, will be finite, and hence a final sequence. It follows that the reduction-finite elements form a k-submodule of $k\langle Y \rangle$. An element $a \in k\langle Y \rangle$ is called *reduction-unique* if is reduction finite and its image under every finite sequence of reductions is the same. This common value will be denoted $r_s(a)$.

Definition 34. Given a monomial $a \in k\langle Y \rangle$ we define the disorder degree of a,

$$\mathrm{disdeg}(a) = \sum_{i=1}^{n_x} rp_i + \sum_{i=1}^{n_{x'}} lp_j$$

where rp_i is the position of the i-th letter "x" counting from right to left, lp_i is the position of the i-th letter "x'" counting from left to right, n_x and $n_{x'}$ are the number of letters x and x' in a.

If $a = \sum_{i=1}^{n} k_i a_i$ where a_i are monomials in letters of X, X', e_X and $k_i \in K - \{0\}$,

$$\mathrm{disdeg}(a) := \sum_{i=1}^{n} \mathrm{disdeg}(a_i)$$

Example 35.

- $\mathrm{disdeg}(x_1 e_{y_1} x_2 z_1' x_3 z_2') = (2 + 4 + 6) + (4 + 6) = 22$
- $\mathrm{disdeg}(x e_y z') = 3 + 3 = 6$ and $\mathrm{disdeg}(x' e_y z) = 1 + 1$
- $\mathrm{disdeg}(\prod_{i=1}^{n} x_i' \prod_{i=1}^{m} e_{y_i} \prod_{i=1}^{k} z_i) = \frac{n(n+1)}{2} + \frac{k(k+1)}{2}$

The reduction r_1 lowers disorder degree in two and reductions r_2 and r_3 lowers disorder degree in one.

Remark 36.

- $k_{irr}(Y) = \{\sum A'e_B C : A'$ word in X', e_B word in e_x, C word in $X\}$.
- $k_{irr} \simeq TX' \otimes TE \otimes TX$

Take for example $a = xe_yz'$, there are two possible sequences of final reductions: $r_3\circ r_1\circ r_2$ or $r_2\circ r_1\circ r_3$. The result will be $a = A'e_BC$ and $a = D'e_EF$ respectively, where

$A = \sigma^{(1)}\left(\sigma^{(1)}(x,y),\sigma^{(1)}(\sigma^{(2)}(x,y),z)\right)$,
$B = \sigma^{(2)}\left(\sigma^{(1)}(x,y),\sigma^{(1)}(\sigma^{(2)}(x,y),z)\right)$,
$C = \sigma^{(2)}\left(\sigma^{(2)}(x,y),z\right)$,
$D = \sigma^{(1)}\left(x,\sigma^{(1)}(y,z)\right)$,
$E = \sigma^{(1)}\left(\sigma^{(2)}(x,\sigma^{(1)}(y,z),\sigma^{(2)}(y,z))\right)$,
$F = \sigma^{(2)}\left(\sigma^{(2)}(x,\sigma^{(1)}(y,z),\sigma^{(2)}(y,z))\right)$.

We have $A = D$, $B = E$ and $C = F$ as σ is a solution of YBeq.
Then $r_3\circ r_1\circ r_2(xe_yz') = r_2\circ r_1\circ r_3(xe_yz')$.

A monomial a in $k\langle Y\rangle$ is said to have an *overlap ambiguity* if $a = ABCDE$ with $w_i = BC$ and $w_j = CD$. We shall say the overlap ambiguity is *resolvable* if there exist compositions of reductions, r, r' such that $r(Ar_i(BC)DE) = r'(ABr_j(CD)E)$. Notice that it is enough to take $r = r_s$ and $r' = r_s$.

Remark 37. In our case, there is only one type of overlap ambiguity and is the one we solved previously. That is because there is no rule with x' on the left nor rule with x on the right, so there will be no overlap ambiguity including the family r_1. There is only one type of ambiguity involving reductions r_2 and r_3.

Now we know that every element is reduction unique and finite.
r_s is a projector and $I = \langle xy' - z't, xe_y - e_zt, e_xy' - z'e_t\rangle$ is trivially included in the kernel.

Remark 38. Notice $\ker(r_s) = I$

An explanation of this fact would be: r_s is a projector, an element $a\in\ker(r_s)$ must be $a = b - r_s(b)$ where $b\in k\langle Y\rangle$. It is enough to prove it for monomials b.

- if $a = 0$ the result follows trivially.
- if not, then take a monomial b of a where at least one of the products xy', xe_y or e_xy' appear. Let us suppose b has a factor xy' (the rest of the cases are analogous).

  $b = Axy'B$ where A or B may be empty words. $r_1(b) = Ar_1(xy')B = Az'tB$. Now we can rewrite:

  $b - r_s(b) = \underbrace{Axy'B - Az'tB}_{\in I} + Az'tB - r_s(b)$. As r_1 lowers disdeg in two, we have $\mathrm{disdeg}(Az'tB - r_s(b)) < \mathrm{disdeg}(b - r_s(b))$ then in a finite number of steps we get $b = \sum_{k=1}^{N} i_k$ where $i_k\in I$. It follows that $b\in I$.

The following corollary ends the proof of Proposition 30.

Corollary 39. *The proyector r_s induces a k-linear isomorphism:*

$$k\langle Y\rangle/\langle xy' - z't, xe_y - e_zt, e_xy' - z'e_t\rangle \to TX'\otimes TE\otimes TX.$$

Returning to the bialgebra B, taking quotients and noticing that

$$\overline{x_1\dots x_n} = \overline{\prod \beta_m\circ\cdots\circ\beta_1(x_1,\dots,x_n)}$$

where

$$\beta_i = \sigma_{j_i}^{\pm 1}$$

analogously with $\overline{x'_1 \dots x'_n}$, follows the proof of Proposition 30.

This ends the proof of Theorem 32.

Example 40.

- If the coefficients are trivial, $f \in C^1(X,k)$ and we identify $C^1(X,k) = k^X$, then

$$(\partial f)(x,y) = f(d(e_x e_y)) = -f(x) - f(y) + f(z) + f(t)$$

where $\sigma(x,y) = (z,t)$ (If instead of considering $\mathrm{Hom}_{A'-A}$, we consider $\mathrm{Hom}_{A-A'}$ then

$$(\partial f)(x,y) = f(d(e_x e_y)) = f(x) + f(y) - f(z) - f(t)$$

but with $\sigma(z,t) = (x,y)$).

- Again with trivial coefficients, and $\Phi \in C^2(X,k) \cong k^{X^2}$, then

$$(\partial \Phi)(x,y,z) = \Phi(d(e_x e_y e_z)) = \Phi\left(\overbrace{x e_y e_z}^{I} - \overbrace{x' e_y e_z}^{II} - \overbrace{e_x y e_z}^{III} + \overbrace{e_x y' e_z}^{IV} + \overbrace{e_x e_y z}^{V} - \overbrace{e_x e_y z'}^{VI} \right)$$

If considering $\mathrm{Hom}_{A'-A}$ then, using the relations defining B, the terms I, III, IV and VI changes leaving

$$\partial\Phi(x,y,z) = \Phi(\sigma^{(1)}(x,y), \sigma^{(1)}(\sigma^{(2)}(x,y),z)) - \Phi(y,z) - \Phi(x, \sigma^{(1)}(y,z)) +$$
$$\Phi(\sigma^{(2)}(x,y),z) + \Phi(x,y) - \Phi(\sigma^{(2)}(x,\sigma^{(1)}(y,z)), \sigma^{(2)}(y,z))$$

- If M is a $k[T]$-module (notice that T need not be invertible as in [CES1]) then M can be viewed as an $A' - A$-bimodule via

$$x'm = m, \quad mx = Tm.$$

The actions are compatible with the relations defining B:

$$(mx)y = T^2 m \ , \quad (mz)t = T^2 m$$

and

$$x'(y'm) = m \ , \quad z'(t'm) = m$$

and using these coefficients we get twisted cohomology as in [CES1] but for general YB solutions. If one takes the special case of (X,σ) being a rack, namely $\sigma(x,y) = (y, x \triangleleft y)$, then the general formula gives

$$\partial f(x_1, \dots, x_n) = f(d(e_{x_1} \dots e_{x_n})) =$$
$$\sum_{i=1}^{n} (-1)^{i+1} \left(Tf(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) - f({x_1}^{x_i}, \dots, x_{i-1}^{x_i}, x_{i+1}, \dots, x_n) \right)$$

that agrees with the differential of the twisted cohomology defined in [CES1].

Next remark is well-known but we include it here because we will use it very often.

Remark 41. Let σ be a solution of YBeq. If $c(x \otimes y) = f(x,y)\sigma^{(1)}(x,y) \otimes \sigma^{(2)}(x,y)$, then c is a solution of YBeq if and only if f is a 2-cocycle.

$$c_1 \circ c_2 \circ c_1(x \otimes y \otimes z) =$$

$$= a\, \overbrace{\sigma^{(1)}\left(\sigma^{(1)}(x,y), \sigma^{(1)}(\sigma^{(2)}(x,y),z)\right) \otimes \sigma^{(2)}\left(\sigma^{(1)}(x,y), \sigma^{(1)}(\sigma^{(2)}(x,y),z\right) \otimes \sigma^{(2)}\left(\sigma^{(2)}(x,y),z\right)}^{I}$$

where

$$a = f(x,y) f\left(\sigma^{(2)}(x,y),z\right) f\left(\sigma^{(1)}(x,y), \sigma^{(1)}(\sigma^{(2)}(x,y),z)\right)$$

$$c_2 \circ c_1 \circ c_2(x \otimes y \otimes z) =$$

$$b\, \overbrace{\sigma^{(1)}(x, \sigma^{(1)}(y,z)) \otimes \sigma^{(1)}\left(\sigma^{(2)}(x, \sigma^{(1)}(y,z)), \sigma^{(2)}(y,z)\right) \otimes \sigma^{(2)}\left(\sigma^{(2)}(x, \sigma^{(1)}(y,z), \sigma^{(2)}(y,z))\right)}^{II}$$

where

$$b = f(y,z) f\left(x, \sigma^{(1)}(y,z)\right) f\left(\sigma^{(2)}(x, \sigma^{(1)}(y,z)), \sigma^{(2)}(y,z)\right)$$

Writing YBeq with this notation leaves:

$$\sigma \textit{ is a braiding} \Leftrightarrow I = II \tag{2.1}$$

Take f a two-cocycle, then

$$0 = \partial f(x,y,z) = f(d(e_x e_y e_z)) = f((x - x')e_y e_z - e_x(y - y')e_z + e_x e_y(z - z'))$$

is equivalent to the following equality

$$f(x e_y e_z) + f(e_x y' e_z) + f(e_x e_y z) = f(x' e_y e_z) + f(e_x y e_z) + f(e_x e_y z')$$

using the relations defining B we obtain

$$\begin{aligned} & f\left(e_{\sigma^{(1)}(x,y)} e_{\sigma^{(1)}(\sigma^{(2)}(x,y),z)} \sigma^{(2)}(\sigma^{(2)}(x,y)z)\right) + f\left(\sigma^{(1)}(x,y)' e_{\sigma^{(2)}(x,y)} e_z\right) + f\left(e_x e_y z\right) \\ & = f\left(x' e_y e_z\right) + f\left(e_x e_{\sigma^{(1)}(y,z)} \sigma^{(2)}(y,z)\right) + f\left(\sigma^{(1)}(x, \sigma^{(1)}(y,z))' e_{\sigma^{(2)}(x,\sigma^{(1)}(y,z))} e_{\sigma^{(2)}(y,z)}\right). \end{aligned}$$

If G is an abelian multiplicative group and $f : X \times X \to (G, \cdot\,)$ then last formula says

$$\begin{aligned} & f\left(e_{\sigma^{(1)}(x,y)} e_{\sigma^{(1)}(\sigma^{(2)}(x,y),z)} \sigma^{(2)}(\sigma^{(2)}(x,y)z)\right) f\left(\sigma^{(1)}(x,y)' e_{\sigma^{(2)}(x,y)} e_z\right) f\left(e_x e_y z\right) \\ & = f\left(x' e_y e_z\right) f\left(e_x e_{\sigma^{(1)}(y,z)} \sigma^{(2)}(y,z)\right) f\left(\sigma^{(1)}(x, \sigma^{(1)}(y,z))' e_{\sigma^{(2)}(x,\sigma^{(1)}(y,z))} e_{\sigma^{(2)}(y,z)}\right) \end{aligned}$$

which is exactly the condition $a = b$, when the action is trivial.

Notice that if the action is trivial, then the equation above simplifies giving

$$\begin{aligned} & f\left(e_{\sigma^{(1)}(x,y)} e_{\sigma^{(1)}(\sigma^{(2)}(x,y),z)}\right) f\left(e_{\sigma^{(2)}(x,y)} e_z\right) f\left(e_x e_y\right) \\ & = f\left(e_y e_z\right) f\left(e_x e_{\sigma^{(1)}(y,z)}\right) f\left(e_{\sigma^{(2)}(x,\sigma^{(1)}(y,z))} e_{\sigma^{(2)}(y,z)}\right) \end{aligned} \tag{2.2}$$

which is precisely the formula on [CES2] for Yang-Baxter 2-cocycles (with R_1 and R_2 instead of $\sigma^{(1)}$ and $\sigma^{(2)}$).

2.2 1st application: multiplicative structure on cohomology

Proposition 42. *The coproduct Δ induces an associative product in* $\mathrm{Hom}_{A'-A}(B,k)$ *(the graded Hom).*

Proof. It is clear that Δ induces an associative product on $\mathrm{Hom}_k(B,k)$ (the graded Hom), and $\mathrm{Hom}_{A'-A}(B,k)\subset \mathrm{Hom}_k(B,k)$ is a k-submodule. We will show that it is in fact a subalgebra.

Consider the $A'-A$ diagonal structure on $B\otimes B$ (i.e. $x_1'.(b_1\otimes b_2).x_2 = x_1'b_1x_2\otimes x_1'b_2x_2$) and denote $B\otimes^D B$ the k-module $B\otimes B$ considered as $A'-A$-bimodule in this diagonal way. We claim that $\Delta : B\to B\otimes^D B$ is a morphism of $A'-A$-modules:

$$\Delta(x_1'yx_2)=x_1'yx_2\otimes x_1'yx_2 = x_1'(y\otimes y)x_2$$

same with y', and with e_x:

$$\Delta(x_1'e_yx_2)=(x_1'\otimes x_1')(y'\otimes e_y+e_y\otimes y)(x_2\otimes x_2)=x_1'\Delta(e_y)x_2.$$

Dualizing Δ one gets:

$$\Delta^* : \mathrm{Hom}_{A'-A}(B\otimes^D B,k)\to \mathrm{Hom}_{A'-A}(B,k).$$

Consider the natural map

$$\iota : \mathrm{Hom}_k(B,k)\otimes \mathrm{Hom}_k(B,k)\to \mathrm{Hom}_k(B\otimes B,k)$$

$$\iota(f\otimes g)(b_1\otimes b_2)=f(b_1)g(b_2)$$

and denote $\iota|$ by

$$\iota| = \iota|_{\mathrm{Hom}_{A'-A}(B,k)\otimes \mathrm{Hom}_{A'-A}(B,k)}$$

Let us see that

$$Im(\iota|)\subset \mathrm{Hom}_{A'-A}(B\otimes B,k)\subset \mathrm{Hom}_k(B\otimes B,k)$$

If $f,g : B\to k$ are two $A'-A$-module morphisms (recall k has trivial actions, i.e. $x'\lambda=\lambda$ and $\lambda x = x$), then

$$\iota(f\otimes g)(x'(b_1\otimes b_2)) = f(x'b_1)g(x'b_2) = (x'f(b_1))(x'g(b_2))$$

$$= f(b_1)g(b_2) = x'\iota(f\otimes g)(b_1\otimes b_2)$$

$$\iota(f\otimes g)((b_1\otimes b_2)x) = f(b_1x)g(b_2x) = (f(b_1)x)(g(b_2)x)$$

$$= (f(b_1)g(b_2))x = \iota(f\otimes g)(b_1\otimes b_2)x.$$

So, it is possible to compose $\iota|$ and Δ^* $(\Delta\circ\iota)$, and obtain in this way an associative multiplication in $\mathrm{Hom}_{A'-A}(B,k)$. $\square$

Now we will describe several natural quotients of B, each of them gives rise to a subcomplex of the cohomological complex of X with trivial coefficients that are not only subcomplexes but also subalgebras; in particular they are associative algebras.

2.2.1 Square free case

A solution (X,σ) of YBeq satisfying $\sigma(x,x)=(x,x)\ \forall x\in X$ is called *square free.* For instance, if X is a rack, then this condition is equivalent to X being a quandle.

In the square free situation, we add the condition $e_xe_x\sim 0$. In a similar way as before, we have the following:

If (X,σ) is a square-free solution of the YBeq, let us denote sf the two sided ideal of B generated by $\{e_xe_x\}_{x\in X}$.

Proposition 43. *The ideal sf is differential Hopf. More precisely,*

$$d(e_xe_x)=0 \text{ and } \Delta(e_xe_x)=x'x'\otimes e_xe_x+e_xe_x\otimes xx.$$

Remark 44. Recall B is graded and we use the Kosul signed convention. This proposition is false if taking the not signed tensor product structure in $B\otimes B$.

In particular B/sf is a differential graded bialgebra. We may identify $\mathrm{Hom}_{A'-A}(B/sf,k)\subset \mathrm{Hom}_{A'-A}(B,k)$ as the elements f such that $f(\dots,x,x,\dots)=0$. If X is a quandle, these construction leads to the quandle-complex. We have that $\mathrm{Hom}_{A'-A}(B/sf,k)\subset \mathrm{Hom}_{A'-A}(B,k)$ is not only a subcomplex, but also a subalgebra.

2.2.2 Biquandles

If (X,σ) is a biquandle, for all $x\in X$ we add in B the relation $e_xe_{s(x)}\sim 0$. Let us denote bQ the two sided ideal of B generated by $\{e_xe_{s(x)}\}_{x\in X}$.

Proposition 45. *bQ is a differential Hopf ideal. More precisely, $d(e_xe_{s(x)})=0$ and $\Delta(e_xe_{s(x)})=x's(x)'\otimes e_xe_{s(x)}+e_xe_{s(x)}\otimes xs(x)$.*

In particular B/bQ is a differential graded bialgebra. We may identify

$$\mathrm{Hom}_{A'-A}(B/bQ,k)\cong\{f\in \mathrm{Hom}_{A'-A}(B,k) : f(\dots,x,s(x),\dots)=0\}\subset \mathrm{Hom}_{A'-A}(B,k)$$

In [CES2], the condition $f(\dots,x_0,s(x_0),\dots)=0$ is called the *type 1 condition.* A consequence of the above proposition is that $\mathrm{Hom}_{A'-A}(B/bQ,k)\subset \mathrm{Hom}_{A'-A}(B,k)$ is not only a subcomplex, but also a subalgebra. Before proving this proposition we will review some other similar constructions.

2.2.3 Identity case

The two cases above may be generalized in the following way:

Consider $S\subseteq X\times X$ a subset of elements verifying $\sigma(x,y)=(x,y)$ for all $(x,y)\in S$. Define idS to be the two sided ideal of B given by $idS=\langle e_xe_y/(x,y)\in S\rangle$.

Proposition 46. *idS is a differential Hopf ideal. More precisely, $d(e_xe_y)=0$ for all $(x,y)\in S$ and $\Delta(e_xe_y)=x'y'\otimes e_xe_y+e_xe_y\otimes xy$.*

In particular B/idS is a differential graded bialgebra.
If one identifies $\mathrm{Hom}_{A'-A}(B/idS,k)\subset \mathrm{Hom}_{A'-A}(B,k)$ as the elements f such that

$$f(\dots,x,y,\dots)=0\ \forall (x,y)\in S$$

we have that $\mathrm{Hom}_{A'-A}(B/idS,k)\subset \mathrm{Hom}_{A'-A}(B,k)$ is not only a subcomplex, but also a subalgebra.

2.2.4 Flip case

Consider the condition $e_xe_y+e_ye_x\sim 0$ for all pairs such that $\sigma(x,y)=(y,x)$. For such a pair (x,y) we have the equations $xy=yx$, $xy'=y'x$, $x'y'=y'x'$ and $xe_y=e_yx$. Note that there is no equation for e_xe_y. The two sided ideal $D=\langle e_xe_y+e_ye_x:\sigma(x,y)=(y,x)\rangle$ is a differential and Hopf ideal.

Notice the flip is involutive, the above ideal has the following generalization:

2.2.5 Involutive case

Assume $\sigma^2(x,y)=(x,y)$. This case is called *involutive* in [ESS]. Define $Invo$ the two sided ideal of B given by $Invo=\langle e_xe_y+e_ze_t:(x,y)\in X,\sigma(x,y)=(z,t)\rangle$.

Proposition 47. *$Invo$ is a differential Hopf ideal. More precisely, $d(e_xe_y+e_ze_t)=0$ for all $(x,y)\in X$ (with $(z,t)=\sigma(x,y)$) and if $\omega=e_xe_y+e_ze_t$ then $\Delta(\omega)=x'y'\otimes\omega+\omega\otimes xy$.*

In particular $B/Invo$ is a differential graded bialgebra. If one identifies

$$\mathrm{Hom}_{A'-A}(B/Invo,k)\subset\mathrm{Hom}_{A'-A}(B,k)$$

then

$$\mathrm{Hom}_{A'-A}(B/Invo,k)\subset\mathrm{Hom}_{A'-A}(B,k)$$

is not only a subcomplex, but a subalgebra.

The following conjecture is true in characteristic zero, and also true in any characteristic considering the flip.

Conjecture 48. *The bialgebra $B/Invo$ is acyclic in positive degrees.*

Example 49. If $\sigma=flip$ and $X=\{x_1,\dots,x_n\}$ then $A=k[x_1,\dots,x_n]=SV$, the symmetric algebra on $V=\oplus_{x\in X}kx$. In this case $(B/Invo,d)\cong(S(V)\otimes\Lambda V\otimes S(V),d)$ gives the Koszul resolution of $S(V)$ as $S(V)$-bimodule.

Example 50. If $\sigma=Id$, $X=\{x_1,\dots,x_n\}$ and $V=\oplus_{x\in X}kx$, then $A=TV$ the tensor algebra. If $\frac{1}{2}\in k$, then $(B/invo,d)\cong TV\otimes(k\oplus V)\otimes TV$ gives the Koszul resolution of TV as TV-bimodule. Notice that we don't really need $\frac{1}{2}\in k$, one could replace $invo=\langle e_xe_y+e_xe_y:(x,y)\in X\times X\rangle$ by $idXX=\langle e_xe_y:(x,y)\in X\times X\rangle$.

Proposition 51. *If $\mathbb{Q}\subseteq k$, then $B/Invo$ is acyclic in positive degrees.*

Proof. In $B/Invo$ it can be defined h as the unique (super)derivation such that:

$$h(e_x)=0;h(x)=e_x,h(x')=-e_x$$

Let us see that h is well-defined:

$$h(xy-zt)=e_xy+xe_y-e_zt-ze_t=0,$$

$$h(xy'-z't)=e_xy'-xe_y+e_zt-z'e_t=0,$$

$$h(x'y'-z't')=-e_xy'-x'e_y+e_zt'+z'e_t=0,$$

$$h(xe_y - e_z t) = e_x e_y + e_z e_t = 0.$$

In particular these equations show that h is not well-defined on B.

$$h(e_x y' - z' e_t) = e_x e_y + e_z e_t = 0,$$

$$h(zt' - x'y) = e_z t' - z e_t + e_x y - x' e_y = 0,$$
$$h(z e_t - e_x y) = e_z e_t + e_x e_y = 0,$$
$$h(e_z t' - x' e_y) = e_z e_t + e_x e_y = 0,$$
$$h(e_x e_y + e_z e_t) = 0.$$

Notice that $[h,d] = hd + dh$ is also a derivation. One easily computes

$$h(e_x) = 2e_x,\ h(x) = x - x',\ h(x') = x' - x$$

or equivalently

$$h(e_x) = 2e_x,\ h(x + x') = 0,\ h(x - x') = 2(x - x').$$

One can also easily see that $B/Invo$ is generated by $e_x, x_\pm$, where $x_\pm = x \pm x'$, and that their relations are homogeneous. We see that $hd + dh$ is nothing but the Euler derivation with respect to the grading defined by

$$\deg e_x = 2,\ \deg x_+ = 0,\ \deg x_- = 2.$$

We conclude automatically that the homology vanishes for positive degrees of the e_x's (and similarly for the x_-'s). □

Next, we generalize Propositions 43, 45, 46 and 47.

2.2.6 Braids of order N

Let $(x_0, y_0) \in X \times X$ such that $\sigma^N(x_0, y_0) = (x_0, y_0)$ for some $N \geq 1$. If $N = 1$ we have the "identity case" and all subcases, if $N = 2$ we have the "involutive case". Denote

$$(x_i, y_i) := \sigma^i(x_0, y_0)$$

$$1 \leq i \leq N - 1$$

Notice that the following relations hold in B:

- ⋆ $x_{N-1} y_{N-1} \sim x_0 y_0,\ x_{N-1} y'_{N-1} \sim x'_0 y_0,\ x'_{N-1} y'_{N-1} = x'_0 y'_0,$
- ⋆ $x_{N-1} e_{y_{N-1}} \sim e_{x_0} y_0,\ e_{x_{N-1}} y'_{N-1} \sim x'_0 e_{y_0},$

and for $1 \leq i \leq N - 1$:

- ⋆ $x_{i-1} y_{i-1} \sim x_i y_i,\ x_{i-1} y'_{i-1} \sim x'_i y_i,\ x'_{i-1} y'_{i-1} = x'_i y'_i,$
- ⋆ $x_{i-1} e_{y_{i-1}} \sim e_{x_i} y_i,\ e_{x_{i-1}} y'_{i-1} \sim x'_i e_{y_i}.$

Take $\omega = \sum_{i=0}^{N-1} e_{x_i} e_{y_i}$, then we claim that

$$d\omega = 0,$$

and

$$\Delta\omega = x_0 y_0 \otimes \omega + \omega \otimes x_0' y_0'.$$

For that, we compute

$$d(\omega) = \sum_{i=0}^{N-1} (x_i - x_i') e_{y_i} - e_{x_i}(y_i - y_i') =$$

$$\sum_{i=0}^{N-1} (x_i e_{y_i} - e_{x_i} y_i) - \sum_{i=0}^{N-1} (x_i' e_{y_i} - e_{x_i} y_i') = 0$$

For the comultiplication, we recall that

$$\Delta(ab) = \Delta(a)\Delta(b)$$

where the product on the right hand side is defined using the Koszul sign rule:

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 b_1 \otimes a_2 b_2.$$

So, in this case we have

$$\Delta(\omega) = \sum_{i=0}^{N-1} \Delta(e_{x_i} e_{y_i}) =$$

$$\sum_{i=0}^{N-1} (x_i' y_i' \otimes e_{x_i} e_{y_i} - x_i' e_{y_i} \otimes e_{x_i} y_i + e_{x_i} y_i' \otimes x_i e_{y_i} + e_{x_i} e_{y_i} \otimes x_i y_i)$$

the middle terms cancel telescopically, giving

$$= \sum_{i=0}^{N-1} (x_i' y_i' \otimes e_{x_i} e_{y_i} + e_{x_i} e_{y_i} \otimes x_i y_i)$$

and the relation $x_i y_i \sim x_{i+1} y_{i+1}$ gives

$$= x_0' y_0' \otimes (\sum_{i=0}^{N-1} e_{x_i} e_{y_i}) + (\sum_{i=0}^{n-1} e_{x_i} e_{y_i}) \otimes x_0 y_0$$

$$= x_0' y_0' \otimes \omega + \omega \otimes x_0 y_0.$$

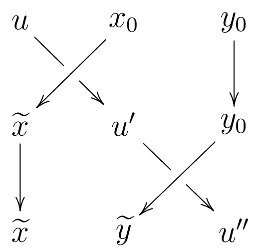
Then the two-sided ideal of B generated by ω is a Hopf ideal. If instead of a single ω we have several $\omega_1 \dots \omega_n$, we simply remark that the sum of differential Hopf ideals is also a differential Hopf ideal.

Remark 52. If X is finite then for every (x_0, y_0) there exists $N > 0$ such that $\sigma^N(x_0, y_0) = (x_0, y_0)$.

Remark 53. Let us suppose $(x_0, y_0) \in X \times X$ is such that $\sigma^N(x_0, y_0) = (x_0, y_0)$ and $u \in X$ an arbitrary element. Consider the element

$$(\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id})(u, x_0, y_0) = (\widetilde{x}_0, \widetilde{y}_0, u'')$$

graphically



then $\sigma^N(\widetilde{x}_0, \widetilde{y}_0) = (\widetilde{x}_0, \widetilde{y}_0)$. An explanation could be:

$$(\sigma^N \times id)(\widetilde{x}_0, \widetilde{y}_0, u'') = (\sigma^N \times id)(id \times \sigma)(\sigma \times id)(u, x_0, y_0) =$$

$$(\sigma^{N-1} \times id)(\sigma \times id)(id \times \sigma)(\sigma \times id)(u, x_0, y_0) =$$

using YBeq

$$(\sigma^{N-1} \times id)(id \times \sigma)(\sigma \times id)(id \times \sigma)(u, x_0, y_0) =$$

repeating the procedure $N - 1$ times leaves

$$(id \times \sigma)(\sigma \times id)(id \times \sigma^N)(u, x_0, y_0) = (id \times \sigma)(\sigma \times id)(u, x_0, y_0) = (\widetilde{x}_0, \widetilde{y}_0, u'')$$

2.3 2nd application: Comparison with Hochschild cohomology

B is a differential graded algebra, and on each degree n it is isomorphic to $A \otimes (TV)_n \otimes A$, where $V = \oplus_{x \in X} k e_x$. In particular B_n, is free as A^e-module. We have *for free* the existence of a comparison map

$$\begin{tikzcd}
\cdots \arrow[r] & B_n \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & B_2 \arrow[r, "d"] \arrow[d, equal] & B_1 \arrow[r, "d"] \arrow[d, equal] & B_0 \arrow[d, equal] & & \\
\cdots \arrow[r] & A'(TX)_n A \arrow[r] \arrow[d, equal, "\cong"] & \cdots \arrow[r] & \oplus_{x,y \in X} A' e_x e_y A \arrow[r, "d"] \arrow[d, equal, "\cong"] & \oplus_{x \in X} A' e_x A \arrow[r, "d"] \arrow[d, equal] & A'A \arrow[d, equal, "\cong"] & & \\
\cdots \arrow[r] & A \otimes V^{\otimes n} \otimes A \arrow[r] \arrow[d, "\widetilde{\mathrm{Id}}"] & \cdots \arrow[r] & A \otimes V^{\otimes 2} \otimes A \arrow[r, "d_2"] \arrow[d, "\widetilde{\mathrm{Id}}"] & A \otimes V \otimes A \arrow[r, "d_1"] \arrow[d, "\widetilde{\mathrm{Id}}"] & A \otimes A \arrow[r, "m"] \arrow[d, "\mathrm{Id}"] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
\cdots \arrow[r] & A \otimes A^{\otimes n} \otimes A \arrow[r] & \cdots \arrow[r] & A \otimes A^{\otimes 2} \otimes A \arrow[r, "b'"] & A \otimes A \otimes A \arrow[r, "b'"] & A \otimes A \arrow[r, "m"] & A \arrow[r] & 0
\end{tikzcd}$$

Corollary 54. *For all A-bimodule M, there exist natural maps*

$$\widetilde{\mathrm{Id}}_* : H_\bullet^{YB}(X, M) \to H_\bullet(A, M)$$

$$\widetilde{\mathrm{Id}}^* : H^\bullet(A, M) \to H^\bullet_{YB}(X, M)$$

that are the identity in degree zero and 1.

Moreover, one can choose an explicit map with extra properties. For that we recall some definitions: there is a set theoretical section to the canonical projection from the Braid group to the symmetric group

$$\begin{tikzcd} \mathbb{B}_n \arrow[r, two heads] & \mathbb{S}_n \arrow[l, dotted, bend right] \end{tikzcd}$$

$$T_s := \sigma_{i_1} \dots \sigma_{i_k} \longleftarrow\!\shortmid s = \tau_{i_1} \dots \tau_{i_k}$$

where

- $\tau \in S_n$ are transpositions of neighboring elements i and $i+1$, so-called simple transpositions,
- σ_i are the corresponding generators of $\mathbb{B}_n$,
- $\tau_{i_1} \dots \tau_{i_k}$ is one of the shortest words representing s.

This inclusion factorizes trough

$$\mathbb{S}_n \hookrightarrow \mathbb{B}_n^+ \hookrightarrow \mathbb{B}_n.$$

It is a set inclusion not preserving the monoid structure.

The following three definitions are well-known and will be used in our next theorem.

Definition 55. The permutation sets

$$\mathrm{Sh}_{p_1,\dots,p_k} := \left\{ s \in \mathbb{S}_{p_1+\dots+p_k} / s(1) < \dots < s(p_1), \cdots, s(p+1) < \dots < s(p+p_k) \right\},$$

where $p = p_1 + \dots + p_{k-1}$, are called *shuffle sets.*

Remark 56. It is well-known that a braiding σ gives an action of the positive braid monoid B_n^+ on $V^{\otimes n}$, i.e. a monoid morphism

$$\rho : B_n^+ \to End_{\mathbb{K}}(V^{\otimes n})$$

defined on generators σ_i of B_n^+ by

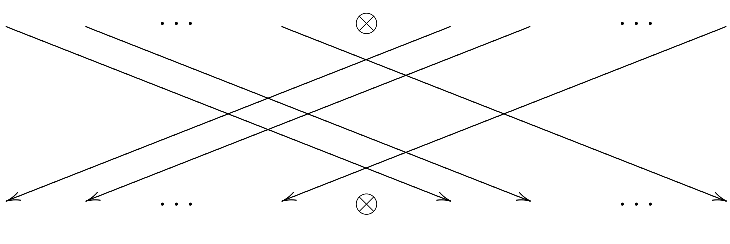
$$\sigma_i \mapsto \mathrm{Id}_V^{\otimes(i-1)} \otimes \sigma \otimes \mathrm{Id}_V^{\otimes(n-i+1)}.$$

Then there exist a natural extension of a braiding in V to a braiding in $T(V)$.

$$\sigma(v \otimes w) = (\sigma_k \dots \sigma_1) \circ \dots \circ (\sigma_{n+k-2} \dots \sigma_{n-1}) \circ (\sigma_{n+k-1} \dots \sigma_n)(vw) \in V^k \otimes V^n$$

for $v \in V^{\otimes n}$, $w \in V^k$ and vw being the concatenation.

Graphically



Definition 57. The quantum shuffle multiplication on the tensor space $T(V)$ of a braided vector space (V, σ) is the k-linear extension of the map

$$\sqcup\!\sqcup_\sigma = \sqcup\!\sqcup_\sigma^{p,q} : V^{\otimes p} \otimes V^{\otimes q} \to V^{\otimes(p+q)}$$

$$\overline{v} \otimes \overline{w} \mapsto \overline{v} \sqcup\!\sqcup_\sigma \overline{w} := \sum_{s \in Sh_{p,q}} T_s^\sigma(\overline{vw}).$$

Notation: T_s^σ stands for the lift $T_s \in \mathbb{B}_n^+$ acting on $V^{\otimes n}$ via the braiding σ. The algebra $Sh_\sigma(V) := (TV, \sqcup\!\sqcup_\sigma)$ is called the quantum shuffle algebra on (V, σ).

It is well-known that $\sqcup\!\sqcup_\sigma$ is an associative product on TV(see for example [Le] for details) that makes it a Hopf algebra with deconcatenation coproduct.

Definition 58. Let V be a braided vector space, then the quantum symmetrizer map $QS_\sigma : V^{\otimes n} \to V^{\otimes n}$ is defined by

$$QS_\sigma(v_1 \otimes \cdots \otimes v_n) = \sum_{\tau \in \mathbb{S}_n} T_\tau^\sigma(v_1 \otimes \cdots \otimes v_n)$$

where T_τ^σ is the lift $T_\tau^\sigma \in \mathbb{B}_n^+$ of τ, acting on $V^{\otimes n}$ via the braiding σ.

In terms of shuffle products the quantum symmetrizer can be computed as

$$\omega \sqcup\!\sqcup_\sigma \eta := \sum_{\tau \in \mathrm{Sh}_{p,q}} T_\tau^\sigma(\omega \otimes \eta)$$

The quantum symmetrizer map can also be defined as

$$QS_\sigma(v_1 \otimes \cdots \otimes v_n) = v_1 \sqcup\!\sqcup_\sigma \cdots \sqcup\!\sqcup_\sigma v_n$$

With this notation, our next result reads as follows:

Theorem 59. *The A'-A-linear quantum symmetrizer map*

$$\begin{array}{ccc} A'V^{\otimes n}A & \xrightarrow{\widetilde{\mathrm{Id}}} & A \otimes A^{\otimes n} \otimes A \\ & & \\ a_1' e_{x_1} \cdots e_{x_n} a_2 & \longmapsto & a_1 \otimes (x_1 \sqcup\!\sqcup_{-\sigma} \cdots \sqcup\!\sqcup_{-\sigma} x_n) \otimes a_2 \end{array}$$

is a chain map lifting the identity. Moreover, $\widetilde{\mathrm{Id}} : B \to (A \otimes TA \otimes A, b')$ is a differential graded algebra map, where in TA the product is $\sqcup\!\sqcup_{-\sigma}$, and in $A \otimes TA \otimes A$ the multiplicative structure is not the usual tensor product algebra, but the braided one. In particular, this map factors through $A \otimes \mathfrak{B} \otimes A$, where $\mathfrak{B}$ is the Nichols algebra associated to the braiding $\sigma'(x \otimes y) = -z \otimes t$, where $x, y \in X$ and $\sigma(x, y) = (z, t)$.

Remark 60. The Nichols algebra $\mathfrak{B}$ is the quotient of TV by the ideal generated by (skew)primitives that are not in V, so the result above explains the good behavior of the ideals *invo*, *idS*, or in general the ideal generated by elements of the form $\omega = \sum_{i=0}^{N-1} e_{x_i} e_{y_i}$ where $\sigma(x_i, y_i) = (x_{i+1}, y_{i+1})$ and $\sigma^N(x_0, y_0) = (x_0, y_0)$. It would be interesting to know the properties of $A \otimes \mathfrak{B} \otimes A$ as a differential object, since it appears to be a candidate of Koszul-type resolution for the semigroup algebra A (or similarly the group algebra $k[G_X]$).

The rest of the chapter is devoted to the proof of Theorem 59.

Lemma 61. *Let σ be a braid in the braided (sub)category that contains two associative algebras A and C, meaning there exist bijective functions*

$$\sigma_A : A \otimes A \to A \otimes A,\ \sigma_C : C \otimes C \to C \otimes C,\ \sigma_{C,A} : C \otimes A \to A \otimes C$$

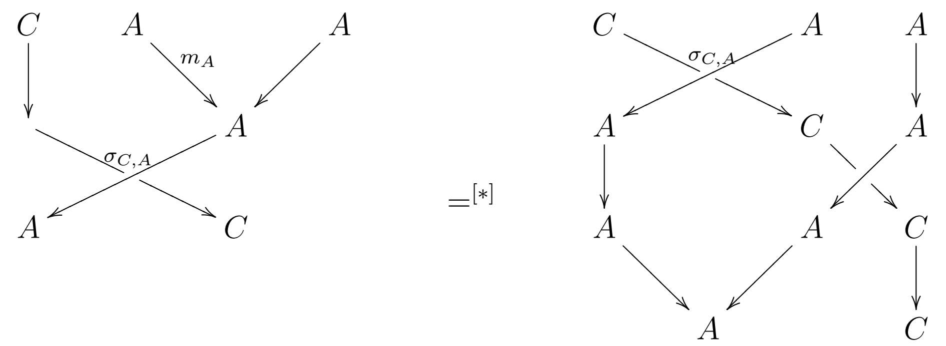
such that

$$\sigma_*(1, -) = (-, 1) \text{ and } \sigma_*(-, 1) = (1, -) \quad \text{for } * \in \{A, C; C, A\}$$
$$\sigma_{C,A} \circ (1 \otimes m_A) = (m_A \otimes 1)(1 \otimes \sigma_{C,A})(\sigma_{C,A} \otimes 1)$$

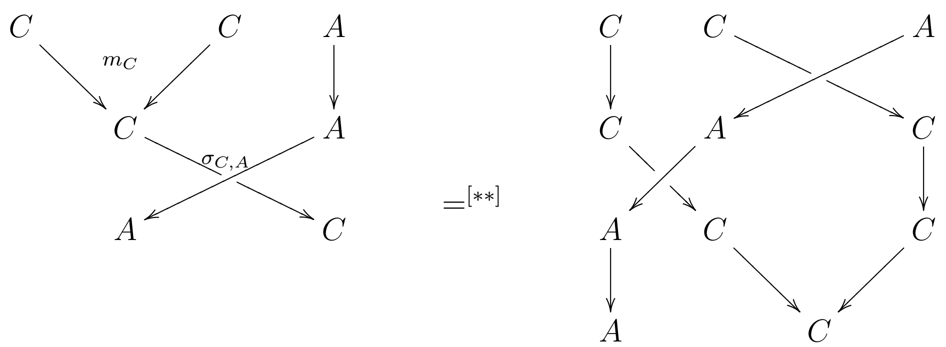
and

$$\sigma_{C,A} \circ (m_C \otimes 1) = (1 \otimes m_C)(\sigma_{C,A} \otimes 1)(1 \otimes \sigma_{C,A})$$

Diagrammatically



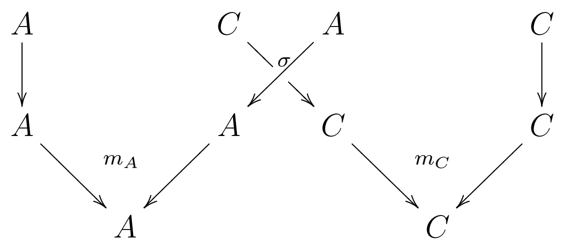
and



Assume that they satisfy the braid equation with any combination of σ_A, σ_C or $\sigma_{A,C}$. Then, $A \otimes_\sigma C = A \otimes C$ with product defined by

$$(m_A \otimes m_C) \circ (\mathrm{Id}_A \otimes \sigma_{C,A} \otimes \mathrm{Id}_C) \colon (A \otimes C) \otimes (A \otimes C) \to A \otimes C$$

is an associative algebra. In diagram:



Proof. Take $m \circ (1 \otimes m)((a_1 \otimes c_2) \otimes ((a_2 \otimes c_2) \otimes (a_3 \otimes c_3))$ use [$*$], associativity in A, associativity in C then [$**$] and the result follows. $\square$

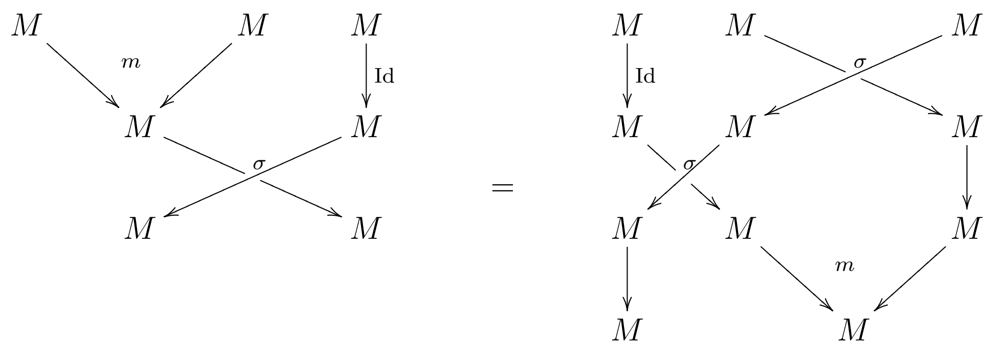
Lemma 62. *Let M be the monoid freely generated by X module the relation $xy = zt$ where $\sigma(x,y) = (z,t)$, then, $\sigma : X \times X \to X \times X$ naturally extends to a braiding in M and verifies both*

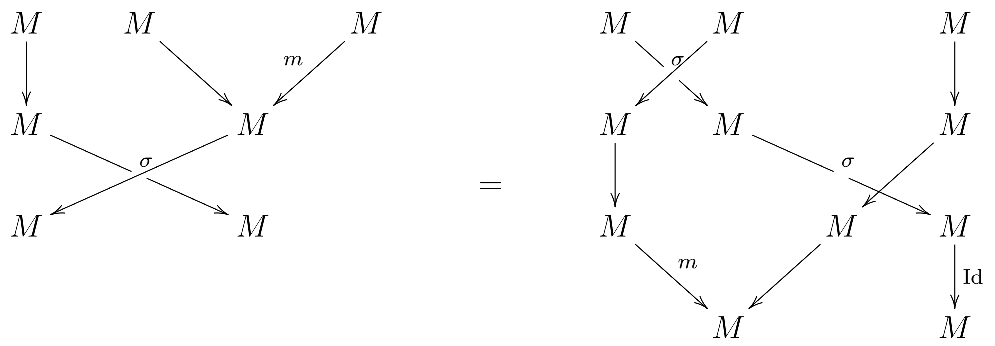
$$\sigma \circ (m \otimes \mathrm{Id}) = (\mathrm{Id} \otimes m) \circ (\sigma \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \sigma)$$

and

$$\sigma \circ (\mathrm{Id} \otimes m) = (m \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \sigma) \circ (\sigma \otimes \mathrm{Id}).$$

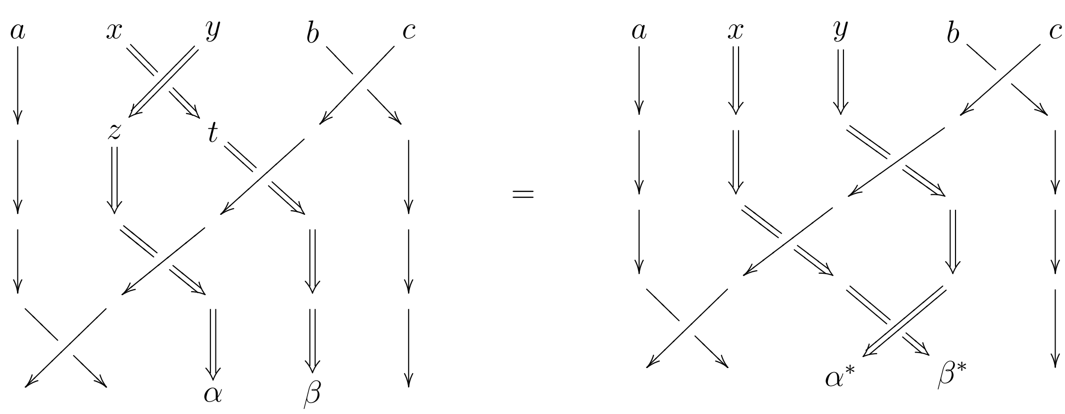
Graphically:





Proof. It is enough to prove that the extension mentioned before is well-defined in the quotient. Inductively, it will be enough to see that $\sigma(axyb, c) = \sigma(aztb, c)$ and $\sigma(c, axyb) = \sigma(c, aztb)$ where $\sigma(x,y) = (z,t)$, and this follows immediately from the braid equation:

A diagram for the first equation is the following:



As $\alpha\beta = \alpha^*\beta^*$ the result follows.

□

Lemma 63. *$m \circ \sigma = m$, diagrammatically:*

$$
\begin{tikzcd}
M \arrow[drr] & & M \arrow[dll] \\
M \arrow[dr] & {} \arrow[d, phantom, "m"] & M \arrow[dl] \\
& M &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
M \arrow[dr] & {} \arrow[d, phantom, "m"] & M \arrow[dl] \\
& M \arrow[d, "\mathrm{Id}"] & \\
& M &
\end{tikzcd}
$$

Proof. Using successively that $m \circ \sigma_i = m$, we have:

$$
\begin{aligned}
m \circ \sigma(x_1 \dots x_n, y_1 \dots y_k) &= m\left((\sigma_k \dots \sigma_1) \dots (\sigma_{n+k-1} \dots \sigma_n)_{(x_1 \dots x_n y_1 \dots y_k)}\right) \\
&= m\left((\sigma_{k-1} \dots \sigma_1) \dots (\sigma_{n+k-1} \dots \sigma_n)_{(x_1 \dots x_n y_1 \dots y_k)}\right) = \dots \\
&= m(x_1 \dots x_n, y_1 \dots y_k).
\end{aligned}
$$

□

Corollary 64. *If one considers $A = k[M]$, then the algebra A verifies all diagrams in previous lemmas.*

Lemma 65. *If $T = (TA, \sqcup\!\sqcup_\sigma)$ there are bijective functions*

$$\sigma_{T,A} := \sigma|_{T\otimes A} : T \otimes A \to A \otimes T$$

and

$$\sigma_{A,T} := \sigma|_{A\otimes T} : A \otimes T \to T \otimes A$$

that verifies the hypothesis of Lemma 61, and the same for $(TA, \sqcup\!\sqcup_{-\sigma})$.

Corollary 66. *$A \otimes (TA, \sqcup\!\sqcup_{-\sigma}) \otimes A$ is an algebra.*

Proof. Use Lemma 61 twice and the result follows. □

Corollary 67. *Taking $A = k[M]$, then the standard resolution of A as A-bimodule has a natural algebra structure defining the braided tensor product as follows:*

$$A \otimes TA \otimes A = A \otimes_\sigma (T^cA, \sqcup\!\sqcup_{-\sigma}) \otimes_\sigma A.$$

Recall the differential of the standard resolution is defined as $b' : A^{\otimes n+1} \to A^{\otimes n}$

$$b'(a_0 \otimes \dots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n$$

for all $n \geq 2$. If A is a commutative algebra then the Hochschild resolution is an algebra viewed as $\oplus_{n\geq 2} A^{\otimes n} = A \otimes TA \otimes A$, with right and left A-bilinear extension of the shuffle product on TA, and b' is a (super) derivation with respect to that product (see for instance Prop. 4.2.2 [L]). In the braided-commutative case we have the analogous result:

Lemma 68. *b' is a derivation with respect to the product mentioned in Corollary 67.*

Proof. Recall the commutative proof as in Prop. 4.2.2 [L]. Denote $*$ the product

$$(a_0 \otimes \ldots \otimes a_{p+1}) * (b_0 \otimes \ldots \otimes b_{q+1}) = a_0 b_0 \otimes ((a_1 \ldots \otimes a_p) \sqcup (b_1 \otimes \ldots \otimes b_q)) \otimes a_{p+1} b_{q+1}$$

Since $\oplus_{n \geq 2} A^{\otimes n} = A \otimes TA \otimes A$ is generated by $A \otimes A$ and $1 \otimes TA \otimes 1$, we check on generators. For $a \otimes b \in A \otimes A$, $b'(a \otimes b) = 0$, in particular, it satisfies Leibnitz rule for elements in $A \otimes A$. Also, b' is A-linear on the left, and right-linear on the right, so

$$b'\big((a_0 \otimes a_{n+1}) * (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)\big) = b'(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1})$$

$$= a_0 b'(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) a_{n+1} = (a_0 \otimes a_{n+1}) * b'(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$$

$$= 0 + (a_0 \otimes a_{n+1}) * b'(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$$

$$= b'(a_0 \otimes a_{n+1}) * (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) + (a_0 \otimes a_{n+1}) * b'(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1).$$

Now consider $(1 \otimes a_1 \otimes \ldots \otimes a_p \otimes 1) * (1 \otimes b_1 \otimes \ldots \otimes b_q \otimes 1)$, it is a sum of terms where two consecutive tensor terms can be of the form (a_i, a_{i+1}), or (b_j, b_{j+1}), or (a_i, b_j) or (b_j, a_i). When one computes b', multiplication of two consecutive tensor factors will give, respectively, terms of the form

$$\cdots \otimes a_i a_{i+1} \otimes \cdots \ , \ \cdots \otimes b_j b_{j+1} \otimes \cdots \ , \ \cdots \otimes a_i b_j \otimes \cdots \ , \cdots \otimes b_j a_i \otimes \cdots$$

The first type of terms will recover

$$b'((1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)) * (1 \otimes b_1 \otimes \cdots \otimes b_q \otimes 1)$$

and the second type of terms will recover

$$\pm (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) * b'((1 \otimes b_1 \otimes \cdots \otimes b_q \otimes 1)).$$

On the other hand, the difference between the third and forth type of terms is just a single trasposition so they have different signs, while $a_i b_j = b_j a_i$ because the algebra is commutative, if one take the *signed* shuffle then they cancel each other.

In the *braided* shuffle product, the summands are indexed by the same set of shuffles, so we have the same type of terms, that is, when computing b' of a (signed) shuffle product, one may do the product of two elements in coming form the first factor, two elements of the second factor. or a mixed term. For the mixed terms, they will have the form

$$\cdots \otimes A_i B_j \otimes \cdots, \text{ or } \cdots \otimes \sigma^{(1)}(A_i, B_j) \sigma^{(2)}(A_i, B_j) \otimes \cdots$$

As in the algebra A we have $A_i B_j = \sigma^{(1)}(A_i, B_j) \sigma^{(2)}(A_i, B_j)$ then this terms will cancel leaving only the terms corresponding to

$$b'(1 \otimes a_1 \otimes \cdots \otimes a_p \otimes 1) \sqcup_{-\sigma} (1 \otimes b_1 \otimes \cdots \otimes b_q \otimes)$$

and

$$\pm (1 \otimes a_1 \otimes \cdots \otimes a_p \otimes 1) \sqcup_{-\sigma} b'(1 \otimes b_1 \otimes \cdots \otimes b_q \otimes 1)$$

respectively.

□

Corollary 69. *There exist a comparison morphism $f:(B,d)\to(A\otimes TA\otimes A,b')$ which is a differential graded algebra morphism, $f(d)=b'(f)$, simply defining it on e_x $(x\in X)$ and verifying $f(x'-x)=b'(f(e_x))$.*

Proof. Define f on e_x, extend k-linearly to V, multiplicatively to TV, and A'-A linearly to $A'\otimes TV\otimes A=B$. In order to see that f commutes with the differential, by A'-A-linearity it suffices to check on TV, but since f is multiplicative on TV it is enough to check on V, and by k-linearity we check on basis, then we only need $f(de_x)=b'f(e_x)$. □

Corollary 70. *The function $f|_{TX}$ is the quantum symmetrizer map, and therefore* $\mathrm{Ker}(f)\cap TX\subset B$ *defines the Nichols ideal associated to $-\sigma$.*

Proof.

$$f(e_{x_1}\cdots e_{x_n})=f(e_{x_1})*\cdots*f(e_{x_n})=(1\otimes x_1\otimes 1)*\cdots*(1\otimes x_n\otimes 1)=1\otimes(x_1\sqcup\cdots\sqcup x_n)\otimes 1$$

□

The previous corollary explains why $\mathrm{Ker}(\mathrm{Id}-\sigma)\subset B_2$ gives a Hopf ideal and also ends the proof of Theorem 59.

Question 71. $Im(f)=A\otimes\mathfrak{B}\otimes A$ is a resolution of A as a A-bimodule? Namely, is $(A\otimes\mathfrak{B}\otimes A,d)$ acyclic?

This is the case for involutive solutions in characteristic zero, but also for $\sigma=$flip in any characteristic, and $\sigma=\mathrm{Id}$ (notice this Id-case gives the Koszul resolution for the tensor algebra). If the answer to that question is yes, and $\mathfrak{B}$ is finite dimensional then A have necessarily finite global dimension. Another interesting question is how to relate generators for the relations defining $\mathfrak{B}$ and cohomology classes for X.

2.4 Yang-Baxter cohomology in characteristic zero

2.4.1 Action of G_X in cohomology

Let k be the $A'-A$-bimodule, with trivial $A'-A$ actions, $C^n(X,k)\cong\mathrm{Hom}_{A'-A}(B_n,k)$ is also a G_X-G_X-bimodule with

$$(f\cdot y^{-1})(e_{\underline{x}})=f(e_{\underline{x}}y')$$

and

$$(x^{-1}\cdot f)(e_{\underline{x}})=f(xe_{\underline{x}})$$

where $e_{\underline{x}}=e_{x_1}\dots e_{x_{n-1}}$.

Notice none of the actions are of diagonal type.

Let us define

$$R_y(f)(e_{x_1}\dots e_{x_{n-1}})=(-1)^{n-1}f(e_{x_1}\dots e_{x_{n-1}}e_y)$$

and

$$L_x(f)(e_{x_1}\dots e_{x_{n-1}})=f(e_xe_{x_1}\dots e_{x_{n-1}})$$

for $f : B_n \to k$.

For $c : X^n \to k$ and $y \in X$, denote ${}_y c(x_1, \ldots, x_{n-1}) = c(y, x_1, \ldots, x_{n-1})$ (respectively $c_y(x_1, \ldots, x_{n-1}) = (-1)^{n-1} c(x_1, \ldots, x_{n-1}, y)$).

Then we have:

$$(R_y \partial + \partial R_y)(f)(e_{\underline{x}}) = R_y \partial f(e_{\underline{x}}) + \partial R_y f(e_{\underline{x}}) =$$

$$R_y(f)(d(e_{\underline{x}})) + \partial((-1)^{n-1} f(e_{\underline{x}} e_y)) = (-1)^{n-2} f(d(e_{\underline{x}}) e_y) + (-1)^{n-1} f(d(e_{\underline{x}} e_y)) =$$

$$(-1)^{n-2} f(d(e_{\underline{x}}) e_y - d(e_{\underline{x}} e_y)) = (-1)^{n-2} f\left(d(e_{\underline{x}}) e_y - (d(e_{\underline{x}}) e_y) + (-1)^{n-1} e_{\underline{x}} d(e_y)\right) =$$
$$f\left(e_{\underline{x}}(y - y')\right) = f(e_{\underline{x}} y) - f(e_{\underline{x}} y') = f(e_{\underline{x}}) - (f \cdot y^{-1})(e_{\underline{x}})$$

$$(R_y \partial + \partial R_y)(f)(e_{\underline{x}}) = f(e_{\underline{x}}) - (f \cdot y^{-1})(e_{\underline{x}}). \tag{2.3}$$

Analogously for L_x we have:

$$\left(L_x \partial + \partial L_x\right)(f)(e_{\underline{x}}) = (x^{-1} \cdot f)(e_{\underline{x}}) - f(e_{\underline{x}}).$$

This fact allow us to generalize Lemma 3.1, [EG], where the calculations where made for X a rack.

Lemma 72. *(1) The coboundary operator $\partial : C^n(X, k) \to C^{n+1}(X, k)$ is a map of $G_X - G_X$-bimodules. In particular, there are natural actions of G_X on the groups of cocycles $Z^n(X, k)$, coboundaries $B^n(X, k)$, and cohomology $H^n(X, k)$.*

(2) $H^n(X, k)$ is a trivial $G_X - G_X$-bimodule.

Proof. (1) Straightforward.

(2) From the above calculations we have

$$(R_y \partial + \partial R_y)(f)(e_{\underline{x}}) = -f(e_{\underline{x}}) + (f \cdot y^{-1})(e_{\underline{x}})$$

as $f \in Z^n(X, k)$ then $[f \cdot y^{-1}] = [f]$ in $H^n(X, k)$.

Analogously,

$$\left(L_x \partial + \partial L_x\right)(f)(e_{\underline{x}}) = (x^{-1} \cdot f)(e_{\underline{x}}) - f(e_{\underline{x}})$$

as $f \in Z^n(X, k)$ then $[x^{-1} \cdot f] = [f]$ in $H^n(X, k)$.

□

By Lemma 72 we consider the subcomplex

$$C^\bullet_{binv}(X, k) = {}^{G_X} C^\bullet(X, k)^{G_X}$$

and define the braided invariant cohomology

$$H^\bullet_{binv}(X, k) = H^\bullet(C^\bullet_{binv}(X, k)$$

In $\mathrm{char}(k) = 0$, the natural map

$$\xi : H^\bullet_{binv}(X, k) \to H^\bullet(X, k)$$

is a quasi isomorphism of complexes.

Remark 73. If $f \in Z^n_{binv}(X,k)$, by Lemma 72 part (2), we have $R_y(f)$and $L_x(f)$ are elements in $Z^{n-1}(X,k)$ $\forall x, y \in X$.

The following lemma will show that the "naive" product is well-defined in cohomology, if one multiply (bi)invariant cocycles.

$$C^a(X,k) \times C^b(X,k) \to C^{a+b}(X,k).$$

$$(f \otimes g)(x_1 \dots x_a x_{a+1} \dots x_{a+b}) = f(x_1 \dots x_a) g(x_{a+1} \dots x_{a+b})$$

This map will be denoted by $(f,g) \mapsto f \otimes g$.

Lemma 74. *Suppose that k is a trivial G_X-bimodule. Then for any $f \in C^i_{binv}(X,k)$, $g \in C^j_{binv}(X,k)$ one has*

$$d(f \otimes g) = df \otimes g + (-1)^i f \otimes dg.$$

Proof. Note that, as f is invariant, we have

$$f(e_{x_1} \dots e_{x_i} x_{i+1}) - f(e_{x_1} \dots e_{x_i} x'_{i+1}) = 0$$

analogously for g we have

$$g(e_{x_{i+1}} \dots e_{x_{i+j+1}} x_{i+j+1}) - g(e_{x_{i+1}} \dots e_{x_{i+j+1}} x'_{i+j+1}) = 0.$$

$$d(f \otimes g)(x_1, \dots, x_{i+j+1}) = (f \otimes g)\left(d(e_{x_1} \dots e_{x_{i+j+1}})\right) =$$

$$(f \otimes g)\left(\sum_{k=1}^{i+j+1} (-1)^{k+1} e_{x_1} \dots e_{x_{k-1}} (x_k - x'_k) e_{x_{k+1}} \dots e_{x_{i+j+1}}\right) =$$

$$(f \otimes g)\left(\sum_{k=0}^{i} (-1)^{k+1} a_k + b_{i+1} + \sum_{k=i+2}^{i+j+1} (-1)^{k+1} c_k\right)$$

where a_k, b_{i+1} and c_k are the corresponding normal forms of the elements

$$e_{x_1} \dots e_{x_{k-1}} (x_k - x'_k) e_{x_{k+1}} \dots e_{x_{i+j+1}}$$

As both, $f, g \in C_{binv}(X,k)$ we have that

$$f \otimes g(b_{i+1}) = 0$$

$$\sum_{k=0}^{i} (-1)^{k+1} (f \otimes g)(a_k) = (df \otimes g)(x_1 \dots x_{i+j+1})$$

and

$$\sum_{k=i+2}^{i+j+1} (-1)^{k+1} (f \otimes g)(c_k) = (-1)^i (f \otimes dg)(x_1 \dots x_{i+j+1}).$$

□

Last lemma shows that if $f \in Z^i_{binv}(X,k)$ and $g \in Z^j_{binv}(X,k)$ then $f \otimes g \in Z^{i+j}_{binv}(X,k)$. Furthermore, by the same lemma, the cohomology class of $f \otimes g$ depends only of the cohomology classes of f and g. Thus, we have a product

$$H^\bullet_{binv}(X,k) \times H^\bullet_{binv}(X,k) \to H^\bullet_{binv}(X,k).$$

We will call this product structure "naive product".

2.4.2 Cohomology of finite biracks

In this section we will assume that X is a finite birack. Let k be the trivial G_X-bimodule where $char(k) = 0$.

For each n, $C^n(X, k)$ is a left and right G_X-module, and this action factors through some finite group $G \subseteq S(X^n)$, hence, on each term of this complex we have a projector given by

$$P = \frac{1}{|G|} \sum_{g \in G} g$$

which projects to G_X-(bi)invariants. This projector commutes with the differential, so the complex $C^\bullet(X, k)$ is representable as a direct sum of complexes:

$$C^\bullet(X, k) = C^\bullet_{binv}(X, k) \oplus (1 - P)C^\bullet(X, k).$$

Let $\mathrm{Orb}(X) = G_X \backslash X / G_X$ be the set of G_X-(bi)orbits on X, and $m = |\operatorname{Orb}(X)|$. The main result in this section is

Theorem 75. *Under these conditions, identifying $H^\bullet \cong H^\bullet_{inv}$, and considering the naive product structure, there is an injective algebra morphism*

$$\rho : H^\bullet(X, k) \to \mathrm{Fun}(\mathrm{Orb}(X)^\bullet, k)$$

In particular $dim_k H^n(X, k) \leq m^n$

Remark 76. This result generalizes the result for racks obtained by [CJKS].

In [EG] rack case is analyzed, and give an isomorphism instead of an injection. We follow the arguments in [EG].

Proof. Since $k = H^0(X, k)$, we have an obvious multiplication mapping

$$\mu : T^\bullet(H^1(X, k)) \to H^\bullet(X, k)$$

which is compatible with the algebra and module structures. Thus, all we have to show is that μ is an isomorphism.

By induction in degree, let us show that μ is injective. As the base of induction is clear, suppose the statement is known in degrees $< n$.

Take $c \in \mathrm{Fun}(\mathrm{Orb}(X)^n, k)$ such that $\mu(c) = 0$ in H^n. This means that the pullback $f : X^n \to M$ of the function c is a coboundary: $f = \mu(c) = dg$ where $g \in C^{n-1}_{binv}$. Because f is (bi)invariant, and

$$C^\bullet = C^\bullet_{binv} \oplus (1 - P)C^\bullet$$

we can assume that g is (bi)invariant. This means that for any $x, y \in X$, we have

$$f_y = (dg)_y = -d(g_y)$$

(analogously ${}_xf = {}_x(dg) = -d({}_xg)$).

But f_y is a pullback of a function

$$c_y \in \mathrm{Fun}(\mathrm{Orb}(X)^{n-1}, k)$$

so by the inductive assumption $c_y = 0$ in H^n $\forall y$. Hence $c = 0$.

□

Conjecture 77. *ρ is an isomorphism.*

In rack case ρ is known to be an isomorphism.

2.5 Homology for cyclic sets and involutive solutions

Definition 78. [R] A *cycle set, or right-cyclic quasigroup, is a set* X *with a binary operation* $\cdot$ *satisfying*

$$(a \cdot b) \cdot (a \cdot c) = (b \cdot a) \cdot (b \cdot c) \tag{2.4}$$

and having all the left translations $a \mapsto b \cdot a$ *bijective, the inverse operation being denoted by* $a \mapsto b * a$*. As pointed out by Rump [R], these give rise to involutive braidings*

$$\sigma(a,b) = ((b * a) \cdot b, b * a)$$

and all right non-degenerate involutive braidings can be obtained this way.

In [LV] a complex associated to the structure mentioned above is considered. Here we recall definitions:

Definition 79. The *cycles/boundaries/homology groups of a cycle set* $(X, \cdot)$ with coefficients in an abelian group A are the cycles/ boundaries/ homology groups of the chain complex $C_n^{cs}((X,\cdot),A) = A[X^{\times n}]$, $n \geq 0$, with

$$\partial_n(x_1,\ldots,x_n) = \sum\nolimits_{i=1}^{n-1} (-1)^i ((x_1,\ldots,\widehat{x_i},\ldots,x_n) \\ - (x_i \cdot x_1, \ldots, x_i \cdot x_{i-1}, x_i \cdot x_{i+1}, \ldots, x_i \cdot x_n)),$$

where $(x_1,\ldots,\widehat{x_i},\ldots,x_n) = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$, and $\partial_1 = 0$. The *cocycle* is defined by $\partial(f) = f(\partial)$, as usual, on $C^\bullet_{cs}(X,A) = Hom_{\mathbb{Z}}(C^{cs}_\bullet(X,\mathbb{Z}),A) \simeq Fun(X^n,A)$.

One of the main results of this section is the following:

Theorem 80. *Consider* $(X,\cdot)$ *a cyclic set,* σ *its corresponding involutive solution of the YBeq, and* k *a commutative ring. There exists explicit isomorphisms of complexes*

$$C^{cs}_\bullet((X,\cdot),k) \cong C^{YB}_\bullet(X,k)$$

$$C_{cs}^\bullet((X,\cdot),k) \cong C_{YB}^\bullet(X,k)$$

In order to prove the theorem above we are going to define a modification of the previously defined bialgebra B.

Definition 81. Given a biquandle (X,σ), we may define $\overline{B}(X,\sigma)$ (also denoted by $\overline{B}$) the algebra freely generated by four copies of X, denoted x, x', $\overline{x'}$ (the inverse of x') and e_x, with relations as follows: whenever $\sigma(x,y) = (z,t)$ we have

- $x'\,\overline{x'} \sim 1$ and $\overline{x'}\,x' \sim 1$,
- $xy' \sim z't$,
- $t\,\overline{y'} \sim \overline{z'}\,x$,
- $xe_y \sim e_z t$,
- $e_x y' \sim z' e_t$,

- $e_t\overline{y'} \sim \overline{z'}e_x$,
- $xy \sim zt$,
- $x'y' \sim z't'$, $t'\overline{y'} = \overline{z'}x'$ and $\overline{y'}\ \overline{x'} \sim \overline{t'}\ \overline{z'}$.

As before, we can construct $\overline{B}$ in steps. First define

$$Y := \langle x, x', \overline{x'}, e_x \rangle_{x\in X}$$

the free monoid in X with unit 1, $k\langle Y\rangle$ the k-algebra associated to Y. Take the k-module

$$B_1 := k\langle Y\rangle / \langle x'\ \overline{x'} = 1 = \overline{x'}x' \rangle$$

We will show that $B_1/Y \simeq T\{(X^{\pm 1})'\} \otimes TE \otimes TX$.

Define $w_1 = xy'$, $w_2 = xe_y$, $w_3 = e_x y'$ (as before) and $w_4 = t\ \overline{y'}$, $w_5 = e_t\ \overline{y'}$. Let $S = \{r_1, r_2, r_3, r_4, r_5\}$ be the reduction system defined as follows: $r_i : B_1 \to B_1$ the families of k-module endomorphisms such that r_i fix all elements except w_i: $r_1(xy') = z't$, $r_2(xe_y) = e_z t$, $r_3(e_x y') = z'e_t$ (as before), $r_4(t\ \overline{y'}) = \overline{z'}\ x$ and $r_5(e_t\ \overline{y'}) = \overline{z'}\ x$ where $\sigma(x, y) = (z, t)$.

Following the construction given for B, a reduction r_i *acts trivially* on an element a if w_i does not appear in any monomial of a, ie: Aw_iB appears with coefficient 0. An element $a \in B_1$ is called *irreducible* if Aw_iB does not appear for $i \in \{1, 2, 3\}$.

The irreducible elements will be the ones such that every reduction acts trivially. Definitions such as "final reduction" (denoted by $r_s(a)$), "reduction finite", "reduction unique", etc, will be analogous.

A generalization for disdeg would be the following:

Definition 82. Given a monomial $a \in B_1$ we define the disorder degree of a,

$$\text{disdeg'}(a) = \sum_{i=1}^{n_x} rp_i + \sum_{i=1}^{n_{x'}} lp_j$$

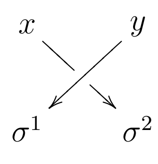
where rp_i is the position of the i-th letter "x" counting from right to left, and lp_i is the position of the i-th letter "x' or $\overline{x'}$" counting from left to right.

The extension of this definition for a finite sum of monomials is obvious.

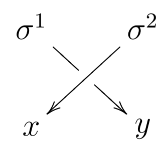
Now the reductions r_1, r_4 reduce disorder degree in two and reductions r_2, r_3 and r_5 reduce disorder degree in one. This implies that every element will be reduction finite but we have to check overlap ambiguities, there are only two of them and we have analyzed one already.

The "new" overlap ambiguity is $a = xe_y\ \overline{z'}$ and there are two possible sequences of final reductions: $r_2 \circ r_4 \circ r_5$ or $r_5 \circ r_4 \circ r_2$.

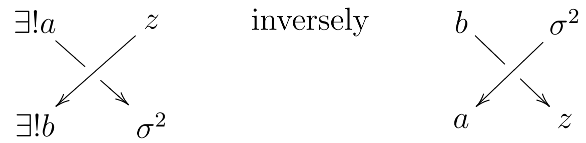
Let us start with $r_5 \circ r_4 \circ r_2(xe_y\ \overline{z'})$, calculate $r_2(xe_y\overline{z'})$:



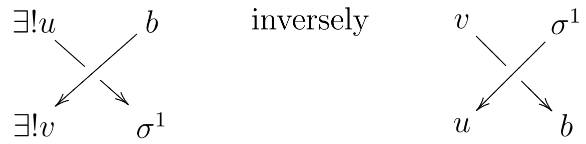
inversely



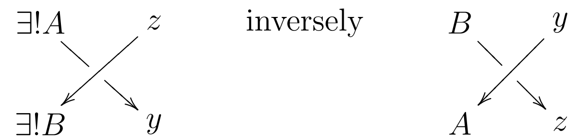
Now apply r_4:



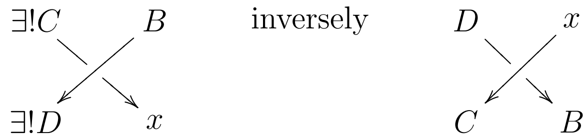
Finally r_5:



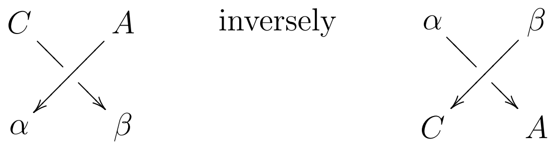
Take the first diagram on the left, and the following two from the right. Join them together in the only composable way, you get one of the diagrams of the third Reidemeister move. Now consider $r_2 \circ r_4 \circ r_5(xe_y\ \overline{z'})$, calculate $r_5(xe_y\ \overline{z'})$:



Apply r_4:

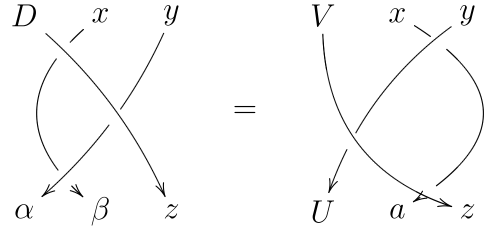


Finally r_2:



Take the fist two diagrams on the right, and the following from the left. Join them together in the only composable way, and get the remaining diagram of the third Reidemeister move.

Both diagrams are the same considering that X is a biquandle and 3 of the entries (x, y, z) are the same (this fact leads all the out coming arcs are the same). Graphically:



Remark 83. All overlap ambiguities are solvable.

Proof. There is no rule with $x', \overline{x'}$ on the left nor rule with x on the right, so there will be no overlap ambiguity including the family r_1, r_4. $\square$

As before, r_s is a proyector and the kernel is

$$I = \langle xy' - z't, xe_y - e_zt, e_xy' - z'e_t, t\ \overline{y'} - \overline{z'}\ x, e_t\ \overline{y'} - \overline{z'}\ e_x\rangle$$

.

Corollary 84. *The map r_s induces a k-linear isomorphism:*

$$B_1/I \to T\{X', \overline{X'}\} \otimes TE \otimes TX$$

Call $B_2 := B_1/I$, then

$$\overline{B} = B_2/\langle xy - zt, x'y' - z't', \overline{y'}\ \overline{x'} - \overline{t'}\ \overline{z'}, t'\ \overline{y'} - \overline{z'}\ x'\rangle$$

.

Remark 85. If we add $\overline{x} = x^{-1}$, say $\{x, x', e_x, \overline{x}, \overline{x'}\}_{x\in X}$ the set of generators of the differential graded bialgebra $\overline{B}$ defined similarly as B, then $\overline{\overline{B}}$ is a differential graded Hopf bialgebra

$$\overline{\overline{B}} = k[G'_X] \otimes TX \otimes k[G_X]$$

As an application of $\overline{B}$ we will prove that the complex considered by [LV] for (left invariant) involutive solutions is actually isomorphic to the YB-standard complex.

Same as before, it is possible to check that $\overline{B}$ is a differential graded bialgebra adding the following definitions: $d(\overline{x'}) = 0$ and $\Delta(\overline{x'}) = \overline{x'} \otimes \overline{x'}$

Definition 86. Call $J_x := \overline{x'}\ e_x$

Another set of generators of $\overline{B}$ can be obtained using $\{x, x', \overline{x'}, J_x\}_{x\in X}$.

Remark 87.
(1) The subalgebra generated by $\{x', \overline{y'}\}_{x,y\in X}$ is isomorphic to $k[G_X]$ (instead of $k[M_X]$ as before).
(2) A normal form can be defined (same as before), i.e. $\overline{B} \simeq k[G_{X'}] \otimes TX \otimes A$.
(3) k is a $k[G_{X'}] - k[M_X]$ trivial (bi)module.

In the *involutive case* we have the following simplification of the commuting relations:

Remark 88. Recall the notation $\sigma(x, y) = ((y\cdot x)*y, y\cdot x)$, we have the following relations:

$$yJ_x = J_{y\cdot a}y,$$

and

$$J_t\overline{y'} = \overline{y'}J_{y\cdot t}.$$

Proof. □

Notice that if $J_x := \overline{x'}e_x$ and $d(J_x) = \overline{x'}x - 1$ then

$$d(J_{x_1}\cdots J_{x_n}) = \sum_{i=1}^{n}(-1)^{i+1}J_{x_1}\cdots J_{x_{i-1}}(\overline{x'}x-1)J_{x_{i+1}}\cdots J_{x_n}$$

$$= \sum_{i=1}^{n}(-1)^{i+1}\left(\overline{x'}_i J_{x_i\cdot x_1}\cdots J_{x_i\cdot x_{i-1}}J_{x_i\cdot x_{i+1}}\cdots J_{x_i\cdot x_n}x_i - J_{x_1}\cdots J_{x_{i-1}}J_{x_{i+1}}\cdots J_{x_n}\right)$$

Which is, after tensoring with trivial coefficients, the definition of homology groups given in [LV], and Theorem 80 is proved.

As a corollary of our last theorem we give a (upper) bound for Betti numbers, [LV] conjecture the opposite bound, which is clearly implied by Conjecture 77.

Chapter 3

An oriented knot/link invariant based on Yang-Baxter solutions

Introducción al capítulo:

La primera parte de este capítulo consiste en una generalización a biquandles y de la noción de 2-cociclo no conmutativo dada en [AG] para quandles. Es también una generalización al caso no conmutativo de parte del trabajo sobre cociclos conmutativos en [CEGS]. En este sentido, obtuvimos en principio nuevos invariantes basados en biquandles que no provienen de quandles, admitiendo 2-cociclos no conmutativos, eso es, cuyo grupo universal (ver sección 2 o 3) es no abeliano.

En la segunda sección, definimos un grupo universal que gobierna todos los 2-cociclos para un biquandle X. Eso es, un grupo $U_{nc}(X)$ junto con un 2-cociclo $\pi : X \times X \to U_{nc}(X)$ tal que si $f : X \times X \to G$ es 2-cociclo no conmutativo a valores en un grupo G, entonces existe un único morfismo de grupos $\widetilde{f} : U_{nc}(X) \to G$ tal que $f = \widetilde{f}\pi$. Por ejemplo, si $U_{nc}(X)$ es el grupo trivial, entonces todo 2-cociclo lo es. Por otro lado, si $U_{nc}(X)$ es no trivial, esta propiedad universal dice que contiene toda la información que cualquier grupo podría dar usando 2-cociclos.

En la tercer sección, damos una versión reducida del grupo U_{nc} que depende de una aplicación $\gamma : X \to U_{nc}(X)$. El conjunto construido se nota $U_{nc}^{\gamma}(X)$, en particular es un grupo y existe un 2-cociclo $\pi_\gamma : X \times X \to U_{nc}^{\gamma}(X)$ con la siguiente propiedad (Teorema 111): si $f : X \times X \to G$ es un 2-cociclo, entonces existe un 2-cociclo cohomólogo (ver definición 92) $f_\gamma : X \times X \to G$ y un morfismo de grupos $\widetilde{f_\gamma}$ tal que $f_\gamma = \widetilde{f_\gamma}\pi_\gamma$. Como el invariante definido en la primer sección no cambia por cociclos cohomólogos (Proposición 102), el invariante producido por f es el mismo que el proveniente de f_γ, de este modo todos los invariantes son gobernados por el grupo U_{nc}^{γ} que, en general, es más chico que U_{nc}.

En la última sección mostramos algunos ejemplos obtenidos con GAP. Una observación interesante es que, si (X, σ) es una solución de la ecuación de Yang-Baxter, entonces también lo es $\overline{\sigma} := \sigma^{-1}$, y si σ otorga estructura de biquandle a X, también $\overline{\sigma}$ dará estructura de biquandle. Uno podría sospechar que $\overline{\sigma}$ es, en algún sentido, equivalente a σ y que probablemente no dé nueva información, pero este no es el caso. En la sección 4 hay un ejemplo (con X de cardinal 3) tal que σ siempre da invariante trivial, mientras que $\overline{\sigma}$ es no trivial.

Introduction to the chapter:

The first part of this chapter consists of a generalization to biquandles and of the notion of non-commutative 2-cocycle given in [AG] for quandles. It is also a generalization to the non-commutative case of part of the work in [CEGS] for commutative cocycles. In this way, we obtain in principle new invariants based on biquandles that do not come from quandles, admiting non-commutative 2-cocycles, that is, whose universal group (see section 2 or 3) is non abelian.

In the second section we define a universal group governing all 2-cocycles for a given biquandle X, that is, a group $U_{nc}(X)$ together with a 2-cocycle $\pi : X \times X \to U_{nc}(X)$ such that if $f : X \times X \to G$ is a noncommutative 2-cocycles with values in a group G, then there is a unique group homomorphism $\widetilde{f} : U_{nc}(X) \to G$ such that $f = \widetilde{f}\pi$. For instance, if $U_{nc}(X)$ is the trivial group, then every 2-cocycle is trivial. On the opposite, if $U_{nc}(X)$ is non-trivial, this universal property says that it carries all information that any group could give using 2-cocycles.

In the third section, we give a reduced version of U_{nc}, it depends on a map $\gamma : X \to U_{nc}(X)$. The constructed set is called $U^{\gamma}_{nc}(X)$, in particular it is a group and there is given a 2-cocycle $\pi_\gamma : X \times X \to U^{\gamma}_{nc}(X)$ with the following property (Theorem 111): if $f : X \times X \to G$ is a 2-cocycle, then there exists a cohomologous (see definition 92) 2-cocycle $f_\gamma : X \times X \to G$ and a group homomorphism $\widetilde{f_\gamma}$ such that $f_\gamma = \widetilde{f_\gamma}\pi_\gamma$. Since the invariant defined in section 1 is unchanged for cohomologous cocycles (Proposition 102), the invariant produced with f is the same as the one coming from f_γ, so we see that all invariants are governed by the group U^{γ}_{nc}, which is, in general, much smaller than U_{nc}.

In section 4 we exhibit some examples of computations. An interesting observation is that, if (X, σ) is a solution of the Yang-Baxter equation, then also is $\overline{\sigma} := \sigma^{-1}$, and if σ makes X into a biquandle (see definition below), then also $\overline{\sigma}$ gives a biquandle structure. One may suspect that $\overline{\sigma}$ is, in a sense, equivalent to σ and probably gives no new information, but this is not the case: the first examples of section 4 are made of a solution (with X of cardinality 3) such that σ always give trivial invariant, but $\overline{\sigma}$ give non-trivial.

3.1 Non-abelian 2-cocycles

In the following definition we generalize the notion of non commutative 2-cocycles given in [CEGS] from quandle case, to biquandle case.

Let (X, σ) be a biquandle and H a (not necessarily abelian) group.

Definition 89. A function $f : X \times X \to H$ is a *braid non-commutative 2-cocycle* if

- $f\big(x_1, x_2\big) f\big(\sigma^{(2)}(x_1, x_2), x_3\big) = f\big(x_1, \sigma^{(1)}(x_2, x_3)\big) f\big(\sigma^{(2)}(x_1, \sigma^{(1)}(x_2, x_3)), \sigma^{(2)}(x_2, x_3)\big)$, and
- $f\big(\sigma^{(1)}(x_1, x_2), \sigma^{(1)}(\sigma^{(2)}(x_1, x_2), x_3)\big) = f\big(x_2, x_3\big)$

are satisfied for any $x_1, x_2, x_3 \in X$.

Note that (in abelian case) if multiply the right-hand sides of both equations and similarly left-hand sides then we obtain Definition 2.2.

Definition 90. If f further satisfies $f(x, s(x)) = 1$ for all $x \in X$ then it will be called of *type I.*

Remark 91. If f is a braided noncommutative 2-cocycle and $\lambda : X \to H$ is an arbitrary function such that $\lambda(y) = \lambda(\sigma^{(1)}(x, y))$, then

$$f'(x, y) = \lambda(x)f(x, y)\lambda^{-1}(\sigma^{(2)}(x, y))$$

is also a braided noncommutative 2-cocycle. If moreover f is of type I, and $\lambda(x) = \lambda(s(x))$ for all $x \in X$, then f' is also of type I.

Definition 92. Two cocycles f, f' are *cohomologous* ($f \sim f'$) if there is a function $\gamma : X \to H$ such that $\gamma(x) = \gamma(s(x))$, $\gamma(y) = \gamma(\sigma^{(1)}(x, y))$ and

$$f'(x, y) = \gamma(x)f(x, y)[\gamma(\sigma^{(2)}(x, y))]^{-1},\ \forall x, y \in X.$$

This notion of cohomology does not come necessarily from a chain complex, for example when H is noncommutative the product of cocycles does not necessarily give a cocycle as a result.

Remark 93. It is easy to see that $\sim$ is an equivalence relation.

An equivalence class is called a cohomology class. The *set* of cohomology classes is denoted by

$$H^2_{NC}(X, H)$$

This definitions, in case $(X, \triangleleft)$ is a quandle and considering $\sigma(x, y) = (y, x \triangleleft y)$, agree with the ones in [CEGS], since in this case the second condition of Definition 89 is trivial. As in the rack/quandle case, if H is not commutative, $H^2_{NC}(X, H)$ need not to be a group, it is just a set.

Remark 94. If H happens to be commutative and $f : X \times X \to H$ is a 2-cocycle in the non commutative sense, then f is necessarily a special (invariant under the action of G_X) type of 2-cocycle with trivial coefficients in the sense of [CES2], but our definition is more restrictive, because we ask for a set of equations of the form $ab = a'b'$ and $c = c'$ (plus being type I), while in the usual abelian 2-cocycles the equation is of the form $abc = a'b'c'$ (plus being type I).

Remark 95. The first condition of Definition 89 is invariant under inverting σ, namely, f satisfies it for σ if and only if f does it for σ^{-1}. On the other hand, the second condition is not invariant under inverting σ. For example, if $(X, \triangleleft)$ is a rack and $\sigma(x, y) = (y, x \triangleleft y)$, then the second condition is trivially satisfied for any function f (and hence, this definition is equivalent, in this setting, to the one given in [CEGS]), while $\overline{\sigma}(x, y) = (y \triangleleft^{-1} x, x)$ means that f must be invariant under the action of the Inner group associated to the rack X.

3.1.1 Weights

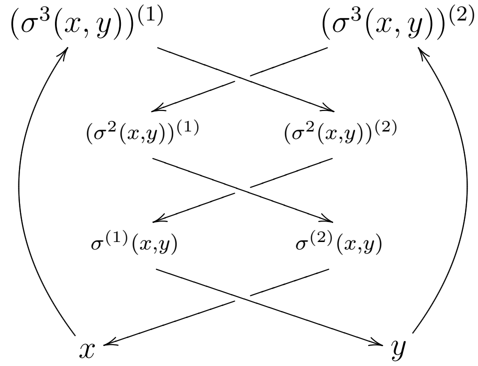
Let X be a biquandle, H a group, $f : X \times X \to H$ a braided non-abelian 2-cocycle. Let $L = K_1 \cup \cdots \cup K_r$ be a classical oriented link diagram on the plane, where $K_1, \ldots, K_r$ are connected components, for some positive integer r.

Definition 96. A *coloring* of L by X is a rule that assigns an element of X to each semi-arc of L, in such a way that for every crossing

we have $(z, t) = \sigma(x, y)$ if the crossing is positive, and $(c, d) = \sigma^{-1}(a, b)$ if the crossing is negative.

$Col_X(L)$ will denote the set of all possible colorings of L by X.

Example 97. The following diagram represents a projection of the trefoil knot, colored by a biquandle (X, σ)



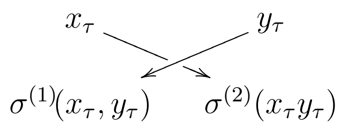
Then, the trefoil is colorable by (X, σ) if and only if there exists $(x, y) \in X \times X$ such that $\sigma^3(x, y) = (x, y)$.

Let $\mathcal{C} \in Col_X(L)$ be a coloring of L by X and $(b_1, \ldots, b_r)$ a set of base points on the components $(K_1, \ldots, K_r)$. Let $\tau^{(i)}$, for $i = 1, \ldots, r$ the set of crossings such that the under-arc is from the component i. Let $(\tau_1^{(i)}, \ldots, \tau_{k_{(i)}}^{(i)})$ be the crossings in $\tau^{(i)}$, $i = 1, \ldots, r$ such that appear in this order when one travels K_j in the given orientation.

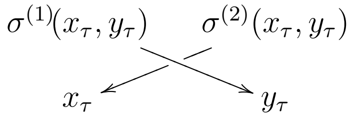
Definition 98. At a positive crossing τ, let x_τ, y_τ be the color on the incoming arcs. The *Boltzmann weight* at τ is $B_f(\tau, \mathcal{C}) = f(x_\tau, y_\tau)$. At a negative crossing τ, denote $\sigma(x_\tau, y_\tau)$ the colors on the incoming arcs. The *Boltzmann weight* at τ is

$$B_f(\tau, \mathcal{C}) = [f(x_\tau, y_\tau)]^{-1}$$

$B_{f,\tau} = f(x_\tau, y_\tau)$:



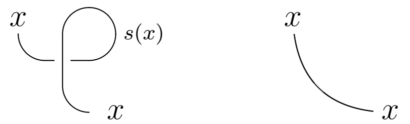
$B_{f,\tau} = [f(x_\tau, y_\tau)]^{-1}$:



We will show that a convenient product of these weights is invariant under Reidemeister moves.

3.1.2 Reidemeister type I moves

First notice that $\sigma(x, s(x)) = (x, s(x))$ implies $\sigma^{-1}(x, s(x)) = (x, s(x))$, so, adding any orientation to the diagram



the condition $[f(x, s(x))]^{\pm 1} = 1$ assures that the factor due to this crossing does not count.

3.1.3 Reidemeister type II moves

We consider several cases:

Case 1:

Case 2:

Case 3: In this case and the following, start naming the top arcs of the diagrams on the left, the rest of the arcs are known as X is a biquandle.

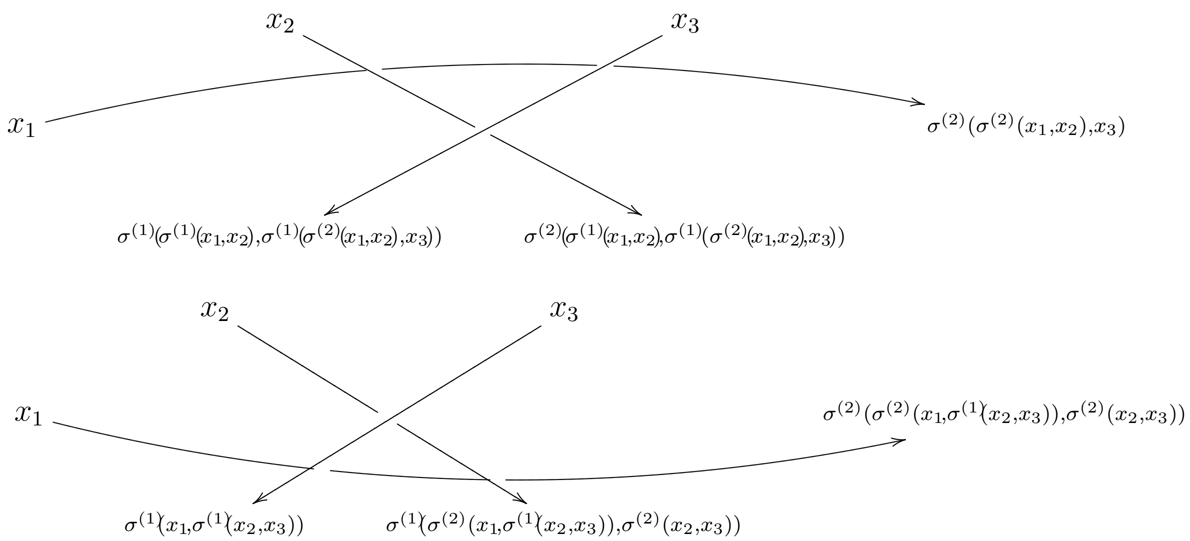
Case 4:

The product of weights (following the orientation of the underarc) corresponding to the diagrams on the left in cases 1 and 3 is $[f(x,y)]^{-1}f(x,y) = 1$, in cases 2 and 4 is $f(x,y)[f(x,y)]^{-1} = 1$.

3.1.4 Reidemeister type III moves

While there are eight oriented Reidemeister type III moves, only four of them are different.

Case 1: Start by naming the incoming-arcs x_1, x_2, x_3. In case 1, as well as in the rest of the cases, once chosen three arcs in both diagrams the remaining arcs are respectively equal as σ is a solution of YBeq.



The product of the weights following the horizontal under-arc, in the first diagram, is:

$$I = f(x_1,x_2)f(\sigma^{(2)}(x_1,x_2),x_3)$$

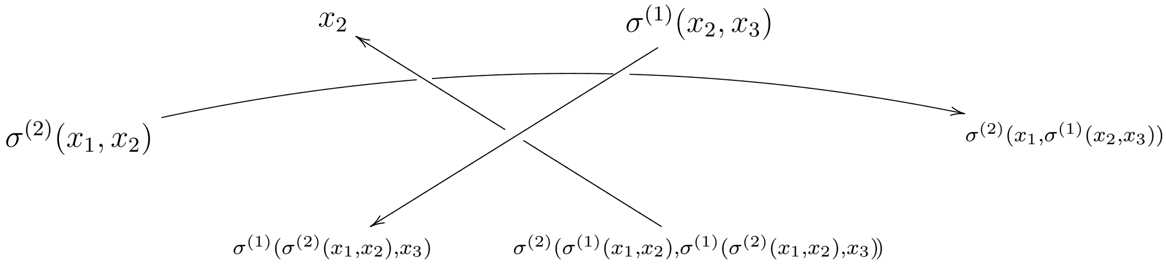
and in the second, is:

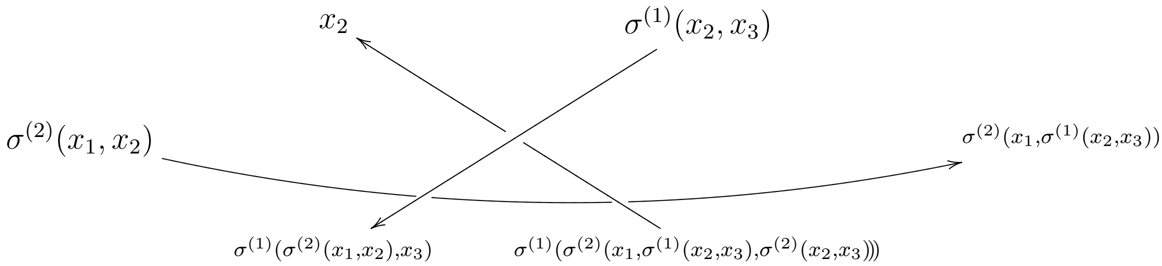
$$II = f(x_1,\sigma^{(1)}(x_2,x_3))f(\sigma^{(2)}(x_1,\sigma^{(1)}(x_2,x_3)),\sigma^{(2)}(x_2,x_3))$$

$I = II$ is one of the equalities defining 2-cocycles, the other equation defining 2-cocycles affirms that the weights given to the remaining crossings are the same. Notice that in the quandle coloring this condition is trivial, but in the biquandle coloring it is not.

Case 2: Start by naming the arcs $\sigma^{(2)}(x_1, x_2)$, x_2 and $\sigma^{(1)}(x_2, x_3)$ in both diagrams. The remaining arcs are known using the fact that X is a biquandle and (due to the braid equation):

$$\sigma^{(1)}\left(\sigma^{(1)}(x_1, x_2), \sigma^{(1)}(\sigma^{(2)}(x_1, x_2), x_3)\right) = \sigma^{(1)}(x_1, \sigma^{(1)}(x_2, x_3))$$





The product (always multiplying to the right) of weights for the horizontal line (which is the underarc in both crossings) in the first diagram is

$$I = [f(x_1, x_2)]^{-1} f(x_1, \sigma^{(1)}(x_2, x_3))$$

and for the second diagram is

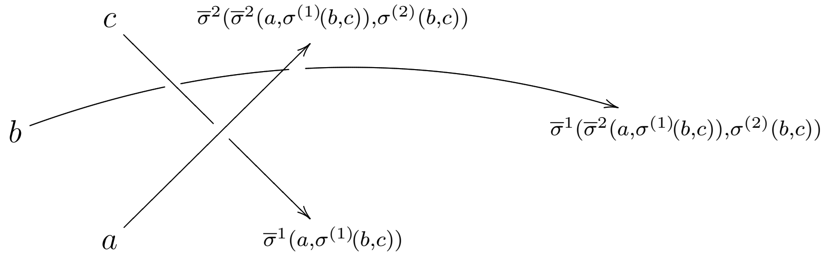
$$II = f(\sigma^{(2)}(x_1, x_2), x_3)[f(\sigma^{(2)}(x_1, \sigma^{(1)}(x_2, x_3)), \sigma^{(2)}(x_2, x_3))]^{-1}.$$

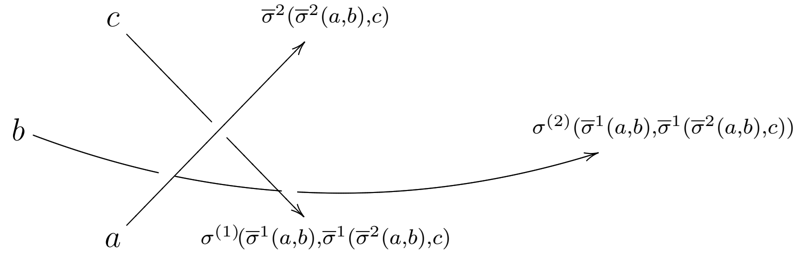
The remaining weights in both diagrams are $a = [f(\sigma^{(1)}(x_1, x_2), \sigma^{(1)}(\sigma^{(2)}(x_1, x_2), x_3))]^{-1}$ and $b = [f(x_2, x_3)]^{-1}$. As f is a 2-cocycle, $a = b$.

Case 3: Name the incoming arcs by a, b and c.

Remark 99. YBeq is equivalent to the following equation, which explains the equality of the out-coming arcs in both diagrams.

$$(\sigma \times 1)(1 \times \overline{\sigma})(\overline{\sigma} \times 1) = (1 \times \overline{\sigma})(\overline{\sigma} \times 1)(1 \times \sigma) \tag{3.1}$$





The product of weights for the horizontal line in the first diagram is

$$I = f(b,c)[f\left(\overline{\sigma}(\overline{\sigma}^2(a,\sigma^{(1)}(b,c)),\sigma^{(2)}(b,c))\right)]^{-1}$$

and for the second diagram is

$$II = [f(\overline{\sigma}(a,b))]^{-1} f\left(\overline{\sigma}^1(a,b),\overline{\sigma}^1(\overline{\sigma}^2(a,b),c)\right).$$

Using (3.1) in I:

$$I = f(b,c) f\left((\sigma^{(2)}(\overline{\sigma}^1(a,b),\overline{\sigma}^1(\overline{\sigma}^2(a,b),c)),\overline{\sigma}^2(\overline{\sigma}^2(a,b),c))\right)$$

Take the changes of variables $(x_1,d) = \overline{\sigma}(a,b)$ and $(x_2,x_3) = \overline{\sigma}(d,c)$. Then

$$I = f(\sigma^{(2)}(x_1,\sigma^{(1)}(x_2,x_3)),\sigma^{(2)}(x_2,x_3))[f(\sigma^{(2)}(x_1,x_2),x_3)]^{-1}$$

$$II = [f(x_1,\sigma^{(1)}(x_2,x_3))]^{-1} f(x_1,x_2).$$

We see that if f is a non-commutative 2 cocycle then $I = II$.

The weights that correspond to the other crossings are:

$$III = [f(\overline{\sigma}(a,\sigma^{(1)}(b,c)))]^{-1},\ IV = [f(\overline{\sigma}(\overline{\sigma}^2(a,b),c))]^{-1}$$

changing variables and composing (3.1) with $1 \times \delta$:

$$III = [f(\sigma^{(1)}(x_1,x_2),\sigma^{(1)}(\sigma^{(2)}(x_1,x_2),x_3))]^{-1},\ IV = [f(x_2,x_3)]^{-1}$$

$III = IV$ is verified as f is a 2-cocycle.

Case 4: We only exhibit the diagram corresponding to this case, the computations are similar to the previous case.

This shows, not only, that the product of the weights does not change under Reidemeister moves but the remaining weights stay the same.

For a group element $h \in H$, denote $[h]$ the conjugacy class to which h belongs.

Definition 100. The set of conjugacy classes

$$\overrightarrow{\Psi}(L,f) = \overrightarrow{\Psi}_{(X,f)}(L) = \{[\Psi_i(L,\mathcal{C},f)]\}_{\substack{1\leq i\leq r \\ \mathcal{C}\in Col_X(L)}}$$

where $\Psi_i(L,\mathcal{C},f) = \prod_{j=1}^{k(i)} B_f(\tau_j^{(i)},\mathcal{C})$ (the order in this product is following the orientation of the component) is called the *conjugacy biquandle cocycle invariant* of the link.

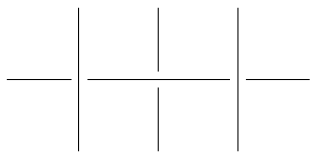
Theorem 101. *The conjugacy biquandle cocycle invariant* Ψ *is well defined.*

Proof. The fact that Ψ does not change under Reidemeister moves for fixed base points was proven earlier. A change of base points causes cyclic permutations of Boltzmann weights, and hence the invariant is defined up to conjugacy. □

Proposition 102. *If* f,g *are two cohomologous non-commutative 2-cocycle functions then* $[\Psi_i(L,\mathcal{C},f)] = [\Psi_i(L,\mathcal{C},g)]$.

Proof. Take an oriented link L. Take the i^{th} component of the link, start at a base point and continue traveling the component using the orientation of the component. Every crossing where the underarc belongs to the i^{th} component will contribute a factor to Ψ_i.

Let us suppose $f(x_1,x_2) = \gamma(x_1)g(x_1,x_2)[\gamma(\sigma^{(2)}(x_1,x_2))]^{-1}$. Take a string like the horizontal line depicted in the following figure, if reach an over crossing like the one in the middle:

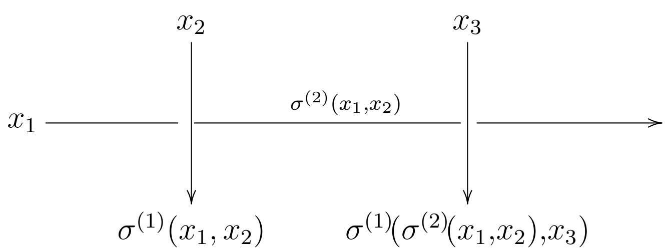


Notice that $\gamma(x_2) = \gamma(\sigma^{(1)}(x_1,x_2))$ implies $\gamma(x_2) = \gamma(\overline{\sigma}^{(1)}(x_1,x_2))$, then every over crossing will change the label on the semiarc but γ will remain the same.

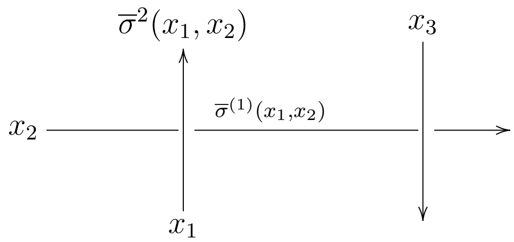
Will be enough to consider a concatenation of under crossings.

There are four cases to analyze:

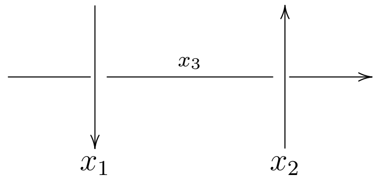
(1)



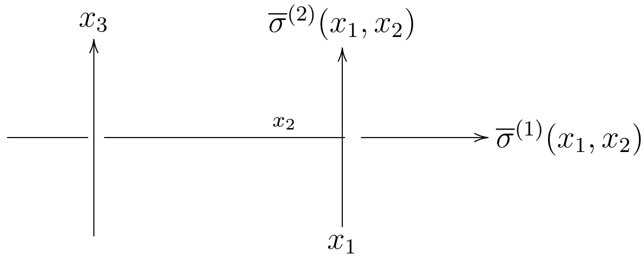
(2)



(3)



(4)



In case 1), the product of weights for the horizontal line is: $f(x_1, x_2)f(\sigma^{(2)}(x_1, x_2), x_3) =$
$= \gamma(x_1)g(x_1, x_2)\gamma(\sigma^{(2)}(x_1, x_2))^{-1}\gamma(\sigma^{(2)}(x_1, x_2))g(\sigma^{(2)}(x_1, x_2), x_3)\gamma(\sigma^{(2)}(\sigma^{(2)}(x_1, x_2), x_3))^{-1}$

In case 2):

$$[f(\overline{\sigma}(x_1, x_2))]^{-1} f(\overline{\sigma}^{(1)}(x_1, x_2), x_3) = ab$$

where

$$a = \gamma(\sigma^{(2)}(\overline{\sigma}(x_1, x_2)))[g(\overline{\sigma}(x_1, x_2))]^{-1}\gamma(\overline{\sigma}^{(1)}(x_1, x_2))^{-1}$$
$$b = \gamma(\overline{\sigma}^{(1)}(x_1, x_2))g(\overline{\sigma}^{(1)}(x_1, x_2), x_3)\gamma(\sigma^{(2)}(\overline{\sigma}^{(1)}(x_1, x_2), x_3))^{-1}$$

Note that: $\sigma^{(2)}(\overline{\sigma}(x_1, x_2)) = x_2$.

In case 3):

$$f(\overline{\sigma}(x_1, x_3))[f(\overline{\sigma}(x_2, x_3))]^{-1} =$$
$$\gamma(\overline{\sigma}^{(1)}(x_1, x_2))g(\overline{\sigma}(x_1, x_3))\gamma(\sigma^{(2)}(\overline{\sigma}(x_1, x_3)))^{-1}\gamma(\sigma^{(2)}(\overline{\sigma}(x_2, x_3)))g(\overline{\sigma}(x_2, x_3))[\gamma(\overline{\sigma}(x_2, x_3))]^{-1}$$

Note that: $\sigma^{(2)}(\overline{\sigma}(x_1, x_3))) = \sigma^{(2)}(\overline{\sigma}(x_2, x_3)) = x_3$

And finally, in case 4): $[f(x_2, x_3)]^{-1}[f(\overline{\sigma}(x_1, x_2))]^{-1} =$

$$\gamma(\sigma^{(2)}(x_2, x_3))[g(x_2, x_3)]^{-1}\gamma(x_2)^{-1}\gamma(\sigma^{(2)}(\overline{\sigma}(x_1, x_2)))[g(\overline{\sigma}(x_1, x_2))]^{-1}\gamma(\overline{\sigma}^1(x_1, x_2))^{-1}$$

Note that: $x_2 = \sigma^{(2)}(\overline{\sigma}(x_1, x_2))$.

□

3.2 Universal noncommutative 2-cocycle

Given a biquandle (X, σ) and a group H, recall a noncommutative 2-cocycle is a function $f : X \times X \to H$ satisfying

$$f(x, y)f(\sigma^{(2)}(x, y), z) = f(x, \sigma^{(1)}(y, z))f(\sigma^{(2)}(x, \sigma^{(1)}(y, z)), \sigma^{(2)}(y, z))$$

and

$$f(\sigma^{(1)}(x, y), \sigma^{(1)}(\sigma^{(2)}(x, y), z)) = f(y, z)$$

for any $x, y, z \in X$, and is called type I if in addition $f(x, s(x)) = 1$.

Definition 103. We define $U_{nc} = U_{nc}(X, \sigma)$, the Universal biquandle 2-cocycle group, as the group freely generated by symbols $(x, y) \in X \times X$ with relations

(Unc1) $(x, y)(\sigma^{(2)}(x, y), z) = (x, \sigma^{(1)}(y, z))(\sigma^{(2)}(x, \sigma^{(1)}(y, z)), \sigma^{(2)}(y, z))$

(Unc2) $(\sigma^{(1)}(x, y), \sigma^{(1)}(\sigma^{(2)}(x, y), z)) = (y, z)$

(Unc3) $(x, s(x)) = 1$

The following is immediate from the definitions:

Proposition 104. *Let (X, σ) be a biquandle:*

- *Denote $[x, y]$ the class of (x, y) in U_{nc}. The map*

$$\begin{array}{rcl} \pi\colon X \times X & \to & U_{nc} \\ (x, y) & \mapsto & [x, y] \end{array}$$

  is a type I non commutative 2-cocycle.

- *Let H be a group and $f : X \times X \to H$ a type I non commutative 2-cocycle, then there exists a unique group homomorphism $\overline{f} : U_{nc} \to H$ such that $f = \overline{f}\pi$.*

$$\begin{tikzcd} X \times X \arrow[r, "f"] \arrow[d, "\pi"'] & H \\ U_{nc} \arrow[ur, dashed, "\exists!\ \overline{f}"'] & \end{tikzcd}$$

In particular, given (X, σ), there exists non trivial 2-cocycles if and only if U_{nc} is a non trivial group.

Proposition 105. *U_{nc} is functorial. That is, if $\phi : (X, \sigma) \to (Y, \tau)$ is a morphism of set theoretical solutions of the YBeq, namely ϕ satisfy*

$$(\phi \times \phi)\sigma(x, x') = \tau(\phi x, \phi x')$$

then, ϕ induces a (unique) group homomorphism $U_{nc}(X) \to U_{nc}(Y)$ satisfying

$$[x, x'] \mapsto [\phi x, \phi x']$$

Proof. We need to prove that the assignment $(x, x') \mapsto (\phi x, \phi x')$ is compatible with the relations defining $U_{nc}(X)$ and $U_{nc}(Y)$ respectively, and that is clear since $(\phi \times \phi) \circ \sigma = \tau \circ (\phi \times \phi)$. □

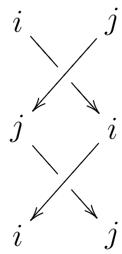
Remark 106. In order to produce an invariant of knots or links, given a solution (X, σ), we need to produce a coloring of the knot-link by X, and then find a non commutative 2-cocycle, but since U_{nc} is functorial, given X we always have the universal 2-cocycle $X \times X \to U_{nc}$, and hence, we only need to consider all different colorings.

Also, if $\phi : X \to X$ is a bijection commuting with σ, then, given a coloring and its invariant calculated with the universal cocycle, we may apply ϕ to each color and get another coloring, and this will produce the same invariant pushed by ϕ in U_{nc}.

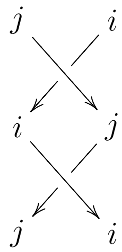
Proposition 107. *Given a link L of two strands colored using (both) colors $\{1,2\}$, the invariant obtained is $\Psi_i(L, \mathcal{C}, f) = [i,j]^{ln(i,j)}$ $i,j \in \{1,2\}$ and $i \neq j$, where $ln(i,j)$ is the linking number between the two strands.*

Proof. First notice that every component must be colored by a single color.

To every underarc of the component i with itself will correspond a $[i,i] = 1$ as weight. Then these crossings will not change the product. Then one can think that each component is unknoted with itself. It is well known that any two closed curves in space, if allowed to pass through themselves but not each other, can be moved into a concatenation of the following standard positions:



This diagram will contribute a factor $[i,j]^{\mathbf{1}}$ to Ψ_i ((j,i) to Ψ_j) and if trying to calculate the linking number will add **1** for each pair of crossings. Analogously in the next diagram:



so, the invariant Ψ_i will be $[i,j]^{\frac{a-b}{2}} = [i,j]^{ln(i,j)}$ where a, b are the total amount of positive and negative crossings $\square$

Example 108. The Whitehead (see 5_1^2 in 1.3) link has linking number zero, the same happens taking the link consisting of two unknots. If you paint these links using $Wada(\mathbb{Z}_3)$ (see example below), Whitehead has only 3 possibilities, while there are 9 ways to paint the pair of unknots.

3.2.1 Some examples of biquandles of small cardinality

We first list some well-known general constructions generating biquandle solutions:

1. If $(X, \triangleleft)$ is a rack, one may consider two different solutions of the YBeq:

$$\sigma(x,y) = (y, x \triangleleft y), \quad \text{and} \quad \overline{\sigma}(x,y) := \sigma^{-1}(x,y) = (y \triangleleft^{-1} x, x)$$

these solutions are biquandles if and only if $(X, \triangleleft)$ is a quandle, namely $x \triangleleft x = x$ for all $x \in X$, in this case the function s is the identity: $s(x) = x$.

When considering n.c. 2-cocycles, condition Unc2 is not preserved (in general) if one changes σ with σ^{-1}, so it is relevant to see σ and σ^{-1} as different biquandles.

2. Let $\tau : X \times X \to X \times X$ denote the flip, namely $\tau(x,y) = (y,x)$. Let $\mu, \nu : X \to X$ be two bijections of X. then

$$(\mu \times \nu)\tau(x,y) = (\mu(y), \nu(x))$$

satisfies YBeq if and only if $\mu\nu = \nu\mu$, and this solution is a biquandle if and only if $\nu = \mu^{-1}$, in this case, the function $s : X \to X$ is equal to μ^{-1}. In this way, the set of bijections of X maps injectively into the set of biquandle structures on X, each conjugacy class of a given bijection maps into an isomorphism class of biquandle structures. Notice that every biquandle structure obtained in this way is involutive, namely $\sigma = (\mu \times \mu^{-1}) \circ \tau$ verifies $\sigma^2 = \mathrm{Id}$.

3. Wada: if G is a group, then the formula $\sigma(x,y) = (xy^{-1}x^{-1}, xy^2)$ is a biquandle, with $s(x) = x^{-1}$. As a particular case, if G is abelian and with additive notation we have $\sigma(x,y) = (-y, x+2y)$.

4. Alexander biquandle or Alexander switch:

   Let R be a ring, $s,t \in R$ two commuting units, and M an R-module, then

$$\sigma(x,y) = (s \cdot y, t \cdot x + (1-st) \cdot y), \quad (x,y) \in M \times M$$

   is a biquandle, with function $s(x) = (s^{-1}) \cdot x$. In the particular case $s = -1$, $t = 1$ one gets the abelian Wada's solution. If $s = 1$ then one gets the solution induced by the Alexander rack.

These general constructions are enough when considering solutions of small cardinality. For instance, if $|X| = 2$, call $X = \{0,1\}$, one have the flip, satisfying $s(1) = 1$ and $s(0) = 0$, and this condition fully characterize this solution. If $s(0) \neq 0$ then $s(0) = 1$ and necessarily $s(1) = 0$; this forces $\sigma(0,0) = (1,1)$ and $\sigma(1,1) = (0,0)$. This is actually a biquandle coming from the bijection construction

$$\sigma(x,y) = (y+1, x-1) : x,y \in \mathbb{Z}/2\mathbb{Z}.$$

We will call this solution "the antiflip".

If $|X| = 3$, we call the elements $X = \{0,1,2\}$ and identify $X = \mathbb{Z}/3\mathbb{Z}$. The above constructions give the following list:

1. There are three isomorphism classes of quandles of 3 elements:

   (a) the trivial quandle ($x \triangleleft y = x$ for all x,y), this gives the flip solution (number 1 in 1.1).

   (b) D_3: $x \triangleleft y = 2y - x$, for $x,y \in \mathbb{Z}/3\mathbb{Z}$ (number 3 in 1.1), which gives two solutions

$$\sigma(x,y) = (y, x \triangleleft y) = (y, 2y-x)$$

   and its inverse

$$\overline{\sigma}(x,y) = (x \triangleleft^{-1} y, x) = (2x-y, x)$$

(c) another quandle which we call Q_3 (number 2 in 1.1), with operation given by $-\triangleleft 0 = (12)$ (the permutation $1 \leftrightarrow 2$), and $-\triangleleft 1 = -\triangleleft 2 = \mathrm{Id}$. The solution

$$\sigma(x,y) = (y, x \triangleleft y)$$

behaves like the flip for $\{x,y\} = \{1,2\}$, but

$$\sigma(0,1) = (1,0),\ \sigma(1,0) = (0,2)$$

$$\sigma(0,2) = (2,0),\ \sigma(2,0) = (0,1).$$

One can check that this equalities can be achieved with the formula

$$\sigma(x,y) = (y, -x - xy^2) = (y, -x(1+y^2)) : x, y \in \mathbb{Z}/3\mathbb{Z}$$

We also have the inverse solution.

2. If $X = \{0\} \coprod \{1,2\}$, with $\sigma(0,i) = (i,0)$ and $\sigma(i,0) = (0,i)$, then the flip on $\{1,2\}$ produces again the flip on three elements, but the other solution produce a new solution (number 7 in 1.1) of the YBeq:

$$\sigma(1,2) = (1,2),\ \sigma(2,1) = (2,1),$$

$$\sigma(1,1) = (2,2),\ \sigma(2,2) = (1,1),$$

$$\sigma(0,i) = (i,0),\ \sigma(i,0) = (0,i).$$

One may check that this equalities are given by $\sigma(x,y) = (y + x^2y, x + y^2x)$.

3. Wada's construction for $\mathbb{Z}_3$ gives the example $\sigma(x,y) = (-y, x-y)$ (number 9 in 1.1) and its inverse: $\overline{\sigma}(x,y) = (y-x, -x)$.

4. Bijection biquandles:

   (a) Using the bijection $\mu(x) = -x$ (number 11 in 1.1) we have the solution $\sigma(x,y) = (-y,-x)$,

   (b) if $\mu(x) = x+1$ then we have the solution $\sigma(x,y) = (y+1, x-1)$ (the inverse solution is number 13 in 1.1). One can check that all bijections $\neq \mathrm{Id}$ are conjugated to one of these.

For $M = R = \mathbb{Z}_3$, the units of R are ± 1: the Alexander biquandle gives Wada's for $s = t = -1$, the Dihedral quandle solution for $s = 1$ and $t = -1$, the flip for $s = t = 1$, and the bijection solution $\sigma(x,y) = (-y,-x)$ when $s = t = -1$, so we have no new solutions in this small cardinality considering the Alexander biquandle.

In this way, we obtain 10 solutions of the YBeq that are biquandles, if we don't count σ in case we already have σ^{-1} (for instance, if each quandle counts by one and not by two), we obtain $7 = 3$ quandles $+ 4$ biquandles that are not quandles, in agreement with A. Bartholomew and R. Finn's classification list (see [BF]). One may check that these are non-isomorphic to each other, so we conclude the list is exhaustive.

3.2.2 Computations of U_{NC}

We will explicitly describe the group U_{nc} for some examples, mainly in $\mathbb{Z}_3$:

From now on, abusing notation, elements in U_{NC} will be denoted by (x,y) instead of $[x,y]$.

The flip

Take $X = \mathbb{Z}/m\mathbb{Z} = \{0,1,2,\dots,n-1\}$ and $\sigma(x,y) = (y,x)$. In this case, conditions (Unc1-3) of Definition 103 mean

$$\begin{cases} (x,y)(x,z) &= (x,z)(x,y) \\ (y,z) &= (y,z) \\ (y,y) &= 1 \end{cases}$$

The second condition is trivial, while the first means that the subgroup $A_x \subset U_{nc}$ defined by $A_x = \langle (x,y) : y \in X \rangle$ is free abelian group with only one relation $(x,x) = 1$. We conclude

$$U_{nc} = *_{x\in X}\mathbb{Z}^{X\setminus\{x\}},$$

the free product of $|X|$ copies of the free abelian with $|X|$-1 generators. In particular, if $X = \{0,1\}$ then U_{nc} is the free group on 2 generators $a = (0,1)$ and $b = (1,0)$.

Bijection biquandle

Let $\mu : X \to X$ be a bijection, and $\sigma(x,y) = (\mu(y), \mu^{-1}(x))$. Now conditions (Unc1-3) mean

$$\begin{cases} (x,y)(\mu^{-1}(x),z) &= (x,\mu z)(\mu^{-1}(x),\mu^{-1}(y)) \\ (\mu(y),\mu(z)) &= (y,z) \\ (\mu(y),y) &= 1 \end{cases}$$

or, equivalently (call $z' = \mu(z)$): $\begin{cases} (x,y)(x,z') &= (x,z')(x,y) \\ (\mu(y),\mu(z)) &= (y,z) \\ (\mu(y),y) &= 1 \end{cases}$

So, for each $x \in X$ we consider as before $A_x = \langle (x,y) : y \in X \rangle \cong \mathbb{Z}^{X\setminus\{\mu^{-1}(x)\}}$ it is free abelian on $X \setminus \{\mu^{-1}x\}$, so U_{nc} is a quotient of the free product over X:

$$U_{nc} = (*_{x\in X}A_x)/((x,y) \sim (\mu x, \mu y))$$

As a particular case, one can fully characterize the following example:

Involutive $\mathbb{Z}_m$

Let $X = \mathbb{Z}_m$ and $\sigma(x,y) = (y+1, x-1)$, then U_{nc} is the free abelian on $m-1$ generators.

Proof. The bijection being $\mu(x) = x+1$, so $(x,y) \sim (x+1,y+1)$ implies that the inclusion map $A_0 \to U_{nc}$ is onto, hence $U_{nc} = A_0 \cong \mathbb{Z}^{m-1}$ □

For example, if $m = 2$ we have the "antiflip" solution, $U_{nc} \cong \mathbb{Z}$ in this case, if $m = 3$ then $U_{nc} = \langle (0,0),(0,1) \rangle \cong \mathbb{Z}^2$.

The example $\sigma(x,y)=(-y,-x)$

This is also a bijection biquandle with $\mu(x)=-x$, so the group is

$$U_{nc}=A_0*A_1*A_2/(x,y)\sim(-x,-y)$$

where $A_x=\langle (x,y) : y\in\mathbb{Z}/3\mathbb{Z}, y\neq -x\rangle$. The list of generators of U_{nc} is $a=(0,1)=(0,2)$, $b=(1,0)=(2,0)$, $c=(1,1)=(2,2)$. The relation

$$(x,y)(x,z)=(x,z)(x,y)$$

gives $cb=bc$, so $U_{nc}=Free(a,b,c)/(bc\sim cb)$.

Quandle solutions

If $(X,\triangleleft)$ is a quandle and $\sigma(x,y)=(y,x\triangleleft y)$, then the cocycle condition (Unc2) is trivial, the group relations are

$$\begin{cases} (x,y)(x\triangleleft y,z) &= (x,z)(x\triangleleft z,y\triangleleft z)\\ (x,x) &= 1\end{cases}$$

This group is the same as the subgroup of units of the algebra $\Omega(X)$ generated by $\eta_{x,y}$, $x,y\in X$, defined by Andruskiewitsch-Graña in [AG]. Notice that these equations imply (setting $x=z$):

$$(x,y)(x\triangleleft y,x)=(x,y\triangleleft x)$$

The situations $x=y$ or $y=z$ give trivial equations.

On the other hand, if we consider the *inverse* solution: $\overline{\sigma}(x,y)=(y\triangleleft^{-1}x,x)$ then condition (Unc2) is not trivial, we have the relations

$$(x,y)(x,z)=(x,z\triangleleft^{-1}y)(x,y)$$
$$(y\triangleleft^{-1}x,z\triangleleft^{-1}x)=(y,z)$$
$$(x,x)=1$$

We see that, in presence of the second identity, the first one can be modified into

$$(x,y)(x,z)=(x\triangleleft y,z)(x,y) \qquad (\mathrm{Unc}\overline{\mathrm{Q}})$$

or also

$$(x\triangleleft z,y\triangleleft z)(x,z)=(x\triangleleft y,z)(x,y)$$

so, $U_{nc}(\overline{\sigma})$ have the same generators as $U_{nc}(\sigma)$ but with "opposite relations", together with the *additional* relation $(x,y)=(x\triangleleft z,y\triangleleft z)$. Notice that (Unc$\overline{Q}$), with $x=z$, says

$$(x,y)(x,x)=(x\triangleleft y,x)(x,y)\Rightarrow(x,y)=(x\triangleleft y,x)(x,y)\Rightarrow 1=(x\triangleleft y,x).$$

This equation, for $x=y$, gives $1=(x\triangleleft x,x)=(x,x)$, that is, $1=(x\triangleleft y,x)$ implies the type I condition. So, we may list a set of relations for $U_{nc}(\overline{\sigma})$ in the following way

$$\begin{cases} (x,y)(x,z) &= (x\triangleleft y,z)(x,y) & (Unc\overline{Q}1)\\ (x\triangleleft y,x) &= 1 & (Unc\overline{Q}2)\\ (x\triangleleft z,y\triangleleft z) &= (x,y) & (Unc\overline{Q}3)\end{cases}$$

Corollary 109. *Let Q be a quandle and consider the biquandle solution*

$$\overline{\sigma}(a,b) = (b \triangleleft^{-1} a, a).$$

If Q is such that for every $z \in Q$ there exists y with $z = x \triangleleft y$, then $U_{nc}(\overline{\sigma}) = 1$.

Proof. Given (z,x), let y be such that $z = x \triangleleft y$, then $(z,x) = (x \triangleleft y, x) = 1$. $\square$

Example 110. If $(X, \triangleleft) = D_n = (\mathbb{Z}/n\mathbb{Z}, x \triangleleft y = 2y - x)$ with n is odd then $U_{nc}(\overline{\sigma}) = 1$.
When z is even take x an even element, when z is odd take x any odd element in X.

The quandle D_3 ($X = \mathbb{Z}_3$)

For $x, y \in \{0,1,2\}$ consider $\sigma(x,y) = (y, 2y - x)$. We have the relations

$$\begin{cases} (x,y)(2y-x,z) &= (x,z)(2z-x,2z-y) \\ (x,x) &= 1 \end{cases}$$

If $x = z$ we get $(x,y)(2y - x, x) = (x, 2x - y)$. When x, y, z are all different, then $2y - x = z$, $2z - x = y$ and $2z - y = x$ so we get

$$(x,y)(2y-x,z) = (x,z)(2z-x,2z-y) \iff (x,y)(z,z) = (x,z)(y,x)$$

$$\iff (x,y) = (x, 2y-x)(y,x)$$

which is equivalent to the previous case. For instance, $x = 0$, $y = 1$, $z = 2$ gives $(0,1) = (0,2)(1,0)$, and because every permutation of the set X is an automorphism of the quandle D_3, we conclude that the full list of relations is the following:

$$(0,1) = (0,2)(1,0); \quad (1,2) = (1,0)(2,1)$$

$$(2,0) = (2,1)(0,2); \quad (0,2) = (0,1)(2,0)$$

$$(2,1) = (2,0)(1,2); \quad (1,0) = (1,2)(0,1)$$

Let us call $a := (0,2)$, $b := (1,0)$ and $c := (2,1)$, with this notation we have

$$(0,1) = ab, \ (1,2) = bc, \ (2,0) = ca$$

$$a = abca, \ c = cabc, \ b = bcab$$

so $abc = 1$, we get $c = (ab)^{-1}$. It follows that U_{nc} is free on generators a, b.

For the inverse solution $\overline{\sigma}(x,y) = (2x - y, x)$, Corollary 109 says $U_{nc}(\overline{\sigma}) = 1$.

The quandle Q_3 ($X = \mathbb{Z}_3$)

Consider the solution induced by the quandle operation

$$x \triangleleft y = -x - xy^2$$

We have $\sigma(x,y) = (y, -x - xy^2) = (y, -x(1+y^2))$, so the relations of U_{nc} are

$$\begin{cases} (x,y)(-x-xy^2, z) & = & (x,z)(-x-xz^2, -y-yz^2) \\ (x,x) & = & 1 \end{cases}$$

If $x = z$ then $(x,y)(-x-xy^2, x) = (x, -y-yx^2)$. This equality, for $x = 0$ gives

$$(0,y)(0,0) = (0,-y) \;\Rightarrow\; a := (0,1) = (0,2)$$

while for $y = 0$ gives

$$(x,0)(-x,x) = (x,0) \;\Rightarrow 1 = (-x,x) = (1,-1) = (-1,1)$$

Define $b := (1,0)$ and $c := (-1,0)$. It is long but trivial to check there are no extra relations, so $U_{nc} = Free(a,b,c)$.

If we consider the solution $\overline{\sigma} = \sigma^{-1}$ then equations (Unc$\overline{Q}$1-3) are

$$\begin{array}{lrcl} (Unc\overline{Q}1) & (x,y)(x,z) & = & (-x-xy^2, z)(x,y) \\ (Unc\overline{Q}2) & (-x-xy^2, x) & = & 1 \\ (Unc\overline{Q}3) & (-x-xz^2, -y-yz^2) & = & (x,y) \end{array}$$

(Unc$\overline{Q}$2) says $1 = (x,x) = (-x,x)$, so $1 = (0,0) = (1,1) = (2,2) = (1,-1) = (-1,1)$. Equation (Unc$\overline{Q}$3) says $(-x,-y) = (x,y)$, so we only have 2 generators: $a := (1,0) = (-1,0)$ and $b := (0,1) = (0,-1)$. Equation (Unc$\overline{Q}$1) is trivial if $x = 0$ or $y = 0$ or $z = 0$, so we conclude $U_{nc}(\overline{\sigma}) = Free(a,b)$.

(anti-flip)$\coprod\{0\}$: $\sigma(x,y) = (y + x^2y, x + xy^2)$

The fixed points are $(0,0)$, $(2,1)$ and $(1,2)$, so (Unc3) says $1 = (0,0) = (2,1) = (1,2)$. (Unc2) gives

$$(y + x^2y, z + (x+xy^2)^2 z) = (y,z)$$

Notice that in $\mathbb{Z}_3$ we have $(1+y^2)^2 = 1 - y^2 + y^4 = 1$, so $z + (x+xy^2)^2 z = z + x^2 z$; hence, this relation says $(y(1+x^2), z(1+x^2)) = (y,z)$, which is equivalent to

$$(-y,-z) = (y,z)$$

Finally (Unc1) gives $(x,y)(x(1+y^2), z) = (x, z(1+y^2))(x(1+z^2), y(1+z^2))$, or equivalently

$$(x,y)(x,(1+y^2)z) = (x, z(1+y^2))(x,y)$$

So, the list of relations is

$$\begin{cases} (x,y)(x,z') & = & (x,z')(x,y) \\ (x,y) & = & (-x,-y) \\ 1 & = & (0,0) = (1,2) = (2,1) \end{cases}$$

We conclude that the list of generators of U_{nc} is $a = (0,1) = (0,2)$, $b = (1,0) = (2,0)$ and $c = (1,1) = (2,2)$, and the relation is $bc = cb$: $U_{nc} = Free(a,b,c)/(bc = cb)$.

Wada: $\sigma(x,y)=(-y,x-y)$ $(X=\mathbb{Z}_3)$

The fixed points are $(x,-x)$, that is $(0,0),(1,2),(2,1)$, so $(0,0)=(1,2)=(2,1)=1$. Conditions (Unc1-2) are:

$$(x,y)(x-y,z)=(x,-z)(x+z,y-z)$$

$$(-y,-z)=(y,z)$$

so, for $x=z=0$ we have

$$(0,y)(-y,0)=(0,0)(0,y)\ \Rightarrow(-y,0)=1\ \Rightarrow(1,0)=(-1,0)=1$$

It remains to consider $a:=(0,1)=(0,-1)$ and $b:=(1,1)=(-1,-1)$. If $z=-x$ then

$$(x,y)(x-y,-x)=(x,x)(0,y+x)$$

so, for $y=0$ we get $(x,0)=(x,x)(0,x)$, and setting $x=1$ we have $1=a.b$. We conclude $U_{nc}=\langle a\rangle\cong\mathbb{Z}$.

For the inverse solution: $\overline{\sigma}(x,y)=(y-x,-x)$, the fixed points are the same, so $(0,0)=(1,2)=(2,1)=1$. Conditions Unc1-2 are

$$\begin{cases}(x,y)(-x,z) & = & (x,z-y)(-x,-y)\\ (y-x,z+x) & = & (y,z)\end{cases}$$

Unc2 says that $(x,y)\sim(0,y+x)$, so we have two generators: $a:=(1,0)=(0,1)=(2,2)$ and $b:=(1,1)=(0,2)=(2,0)$, and condition Unc1 is equivalent to

$$(0,y+x)(0,z-x)=(0,z-y+x)(0,-y-x)$$

If $x=z$ then $(0,y+x)(0,0)=(0,-y-x)(0,-y-x)$, so $a=b^2$ and $b=a^2$, hence

$$a^4=a\ \Rightarrow a^3=1$$

One can check that there are no extra relations, so $U_{nc}=\langle a:a^3=1\rangle$.

3.2.3 Tables

We collect the information in the following tables:

X	$\sigma(x,y)$	$order$	$\#\Delta\cap(X\times X)^\sigma$	U_{nc}
$\mathbb{Z}_n$	(y,x)	2	n	$*_n(\mathbb{Z}^{n-1})$
$\mathbb{Z}_n$	$(y+1,x-1)$	2	0	$\mathbb{Z}^{n-1}$

In particular, for cardinal 2, $U_{nc}(\text{Flip})\cong\mathbb{Z}*\mathbb{Z}$, $U_{nc}(\text{a-Flip})\cong\mathbb{Z}$. For cardinal 3, $X=\mathbb{Z}/3\mathbb{Z}=\{0,1,2\}=\{0,1,-1\}$, some examples are:

name		*formula* $\sigma(x,y) =$	*order of* σ	U_{nc}	$\#\Delta \cap (X\times X)^\sigma$
flip	BQ^3_1	(y,x)	2	$\mathbb{Z}^2 * \mathbb{Z}^2 * \mathbb{Z}^2$	3
a-flip $\cup \{0\}$	BQ^3_2	$(y+x^2y, x+xy^2)$	2	$\mathbb{Z} * \mathbb{Z}^2$	1
$\mu(x) = -x$	BQ^3_7	$(-y,-x)$	2	$\mathbb{Z} * \mathbb{Z}^2$	1
involutive $\mathbb{Z}_3$	BQ^3_{10}	$(y+1, x-1)$	2	$\mathbb{Z}^2$	0
σ_{D_3}	BQ^3_8	$(y, x\triangleleft y)$ $(y,-y-x)$	3	$\mathbb{Z}*\mathbb{Z}$	3
$\overline{\sigma}_{D_3}$	BQ^{3*}_8	$(y\triangleleft^{-1} x, x)$ $(-y-x,x)$	3	1	3
σ_{Wada}	BQ^3_4	$(-y, x-y)$	3	$\mathbb{Z}$	1
$\overline{\sigma}_{Wada}$	BQ^{3*}_4	$(y-x,-x)$	3	$\langle a : a^3 = 1\rangle$	1
σ_{Q_3}	BQ^3_6	$(y,-x-xy^2)$	4	$\mathbb{Z}*\mathbb{Z}*\mathbb{Z}$	3
$\overline{\sigma}_{Q_3}$	BQ^{3*}_6	$(-y-x^2y, x)$	4	$\mathbb{Z}*\mathbb{Z}$	3

Notice that in this table, U_{nc} distinguish σ from $\overline{\sigma}$ (as isomorphism class of solution of the YBeq) in all cases where $\sigma^{(2)} \neq \mathrm{Id}$.

Also we have described a procedure that can be implemented in a computer program:

1. Add to the set $X\times X$ a new element "1" and begin to define an equivalence relation $(x, s(x)) \sim 1$.

2. from the second condition, add $(y,z) \sim (\sigma^{(1)}(x,y), \sigma^{(1)}(\sigma^{(2)}(x,y),z))$ to the equivalence relation.

   More precisely, given a list of subsets of $(X\times X)\coprod\{1\}$ whose union is $(X\times X)\coprod\{1\}$ (if this is not the case we add the sets $\{(x,y)\}$ to the list) one can easily give an algorithm producing the partition of $(X\times X)\cup\{1\}$ corresponding to the equivalence relation generated by the list of subsets: for each pair of subsets of the list, with non-trivial intersection, we replace these two subsets by their union, run over all different pairs, and iterate until saturate. We call *classes* this list of subsets.

3. From the data *classes*, choose representatives (if the list of subsets is ordered and their members are ordered, just pick the first member for each element of the list). Write down *all* cocycle equations, in terms of these representatives.

4. Eliminate the trivial equations, and

   - for any cocycle equation where 1 appears, in case one found $a.1$, replace it by $1.a$, so we do not count twice the same equation.
   - For any cocycle equation of the form $ac = bc$ or $ca = cb$, add $a \sim b$ and recalculate the equivalence relation that it generates.

   With the new data *classes* go to step 3, and iterate the process until it stabilizes.

The set of *classes* containing 1 is called S, this is a list of trivial elements in U_{nc}. A set of representatives of the other *classes* of elements gives a set of generators of U_{nc}. The remaining non-trivial 2-cocycle equations, written in terms of these representatives, give a set of relations. This algorithm produces a relatively small set of generators, and all the relations between them. We have implemented this algorithm in G.A.P.

Taking the list of biquandles of cardinality 3 from Bartholomew and Fenn's list, adding the inverse solutions (when they are not isomorphic), we obtain the table below.

We remark that the procedure gives not only the number of generators, but the full equivalence class, we omit the full data in the table just for space considerations. We also add to the table the order of σ, and the number of fixed points on the diagonal $\Delta := \{(x,x) : x \in X\}$, for instance, $\Delta^\sigma = \Delta$ if X is a quandle.

name	σ	*generators of* U_{nc}	*equations*	*order of* σ	$\#\Delta^\sigma$
flip	BQ^3_1	6	$f_2f_1 = f_1f_2, f_4f_3 = f_3f_4,$ $f_6f_5 = f_5f_6,$	2	3
a-*flip* $\cup \{1\}$	BQ^3_2	3	$f_3f_2 = f_2f_3,$	2	1
	BQ^3_3	3	–	4	1
	BQ^{3*}_3	3	–	4	1
$Wada(\mathbb{Z}_3)$	BQ^3_4	2	$f_2f_1 = 1,$	3	1
inv. $Wada(\mathbb{Z}_3)$	BQ^{3*}_4	2	$f_1f_1 = f_2,\ f_2f_2 = f_1$	3	1
	BQ^3_5	3	$f_2f_1 = f_1f_2,$	2	3
Q_3	BQ^3_6	3	–	4	3
inverse Q_3	BQ^{3*}_6	3	–	4	3
$(x,y) \mapsto (\text{-}y,\text{-}x)$	BQ^3_7	3	$f_3f_2 = f_2f_3,$	2	1
D_3	BQ^3_8	6	$f_1f_5 = f_2, f_2f_3 = f_1, f_3f_6 = f_4,$ $f_4f_1 = f_3, f_5f_4 = f_6, f_6f_2 = f_5,$	3 3	3
inverse D_3	BQ^{3*}_8	0	–	3	3
	BQ^3_9	2	$f_2f_2 = f_1, f_1f_1 = f_2,$	3	0
	BQ^{3*}_9	0	–	3	0
involutive$(\mathbb{Z}_3)$	BQ^3_{10}	2	$f_1f_2 = f_2f_1$	2	0

We remark that for some cases (e.g.: $BQ_{4,8,9}$) the invariant U_{nc} distinguishes between σ and $\overline{\sigma}$. For BQ^3_3, the generators are the same in the strong sense that the equivalent classes of generators (as equivalent classes in $X \times X$) are the same, the relations are also the same (no relation at all), so they will give the same knot/link invariant, even though σ and $\overline{\sigma}$ are non isomorphic biquandle solutions.

The computer program gives the set of generators and relations in a reasonable human time for biquandles of cardinality 12 or less. As a matter of numerical experiment, the groups associated to the inverse solution of bialexander solution on $\mathbb{Z}_m$, for $s = -1$, and $t = 1$, are cyclic of order m (in a non trivial way) if $m = 3$, 5, 7, 11, 13 (and much more complicated groups for $m = 4, 6, 8, 9, 10, 12$). We don't know if this is a general fact for all primes p.

Inverse quandle solutions

If $(X, \triangleleft)$ is a quandle then $\sigma(x,y) = (y, x \triangleleft y)$ is a biquandle, and condition Unc2 is trivial. But if we consider the *inverse* solution: $\overline{\sigma}(x,y) = (y \triangleleft^{-1} x, x)$ then condition (Unc2) is not trivial, we have the relations

$$(x,y)(x,z) = (x, z \triangleleft^{-1} y)(x,y)$$

$$(y \triangleleft^{-1} x, z \triangleleft^{-1} x) = (y,z)$$

$$(x,x) = 1$$

We see that, in presence of the second identity, the first one can be modified into

$$(x,y)(x,z) = (x \triangleleft y, z)(x,y) \qquad (\mathrm{Unc}\overline{\mathrm{Q}})$$

or also $(x \triangleleft z, y \triangleleft z)(x,z) = (x \triangleleft y, z)(x,y)$. Notice that $(\mathrm{Unc}\overline{Q})$, with $x = z$, says

$$(x,y)(x,x) = (x \triangleleft y, x)(x,y) \Rightarrow (x,y) = (x \triangleleft y, x)(x,y) \Rightarrow 1 = (x \triangleleft y, x)$$

This equation, for $x = y$, gives $1 = (x \triangleleft x, x) = (x,x)$. That is, $1 = (x \triangleleft y, x)$ implies the type I condition. So, we may list a set of relations for $U_{nc}(\overline{\sigma})$ in the following way

$$\begin{cases} (x,y)(x,z) &= (x \triangleleft y, z)(x,y) & (Unc\overline{Q}1) \\ (x \triangleleft y, x) &= 1 & (Unc\overline{Q}2) \\ (x \triangleleft z, y \triangleleft z) &= (x,y) & (Unc\overline{Q}3) \end{cases}$$

3.3 The reduced U_{NC}

We begin with the trefoil example: first, if one wants to color the trefoil, one needs a solution (X, σ) having elements (x_0, y_0) such that $\sigma^3(x_0,y_0) = (x_0,y_0)$. One may use the solution given by D_3, but if we color using this quandle the induced invariant in $U_{nc}(D_3) = (1,2)(2,3)(3,1) = 1 \in U_{nc}$. If one changes the colors (always in D_3) one always get the trivial invariant in U_{nc} and that is because $\pi : D_3 \times D_3 \to U_{nc}$ is a coboundary. This well-know fact can be seen as a general result that is helpful for simplifying the computation of U_{nc}, replacing it by another (in general much smaller) group, with a similar universal property.

We recall that if $f : X \times X \to G$ is a (type I) cocycle, $\gamma : X \to G$ is a function satisfying $\gamma(x) = \gamma(sx)$ and $\gamma(y) = \gamma(\sigma^1(x,y))$, then $f_\gamma(x,y) := \gamma(x) f(x,y) \gamma(\sigma^2(x,y))^{-1}$ is also a (type I) 2-cocycle, and the knot/link invariant produce by f is the same as the one produced by f. In particular, one can consider the universal 2-cocycle $\pi : X \times X \to U_{nc}$ and try to see if there is a cohomologous one, simpler that π. This procedure leads to a construction that we call *reduced* universal group:

Theorem 111. *Let $\gamma : X \to U_{nc}$ be a (set theoretical) map such that $\gamma(x) = \gamma(s(x))$, $\gamma(y) = \gamma(\sigma^1(x,y))$ and $\pi_\gamma : X \times X \to U_{nc}$ given by*

$$\pi_\gamma(x,y) = \gamma(x)\pi(x,y)\gamma(\sigma^{(2)}(x,y))^{-1}.$$

Define $S = \{(x,y) \in X \times X : \pi_\gamma(x,y) = 1 \in U_{nc}\} \subseteq X \times X$ and consider the group U^γ_{nc} defined by

$$U^\gamma_{nc} := U_{nc}/ < \pi(x,y)/(x,y) \in S >$$

Denote $\overline{[x,y]} \in U^\gamma_{nc}$ the class of (x,y) and $p : X \times X \to U^\gamma_{nc}$ the map $p(x,y) = \overline{[x,y]}$. The map p has the following universal property:

- *p is a 2-cocycle.*
- *for any group G and 2-cocycle $f : X \times X \to G$, there exists a cohomologous map f^Γ and a group homomorphism $f^\gamma : U^\gamma_{nc} \to G$ such that f^Γ factorizes through p, that is $f^\Gamma = f^\gamma \circ p$.*

Proof. The fact that p is a 2-cocycle is immediate. By Proposition (104) we obtain the existence of the unique group morphism $\overline{f}$ such that

$$\begin{tikzcd}
X \times X \arrow[r, "f"] \arrow[d, "\pi"'] & G \\
U_{nc} \arrow[ur, dashed, "\exists!\ \overline{f}"'] &
\end{tikzcd}$$

commutes. Define $f^\Gamma := \overline{f} \circ \pi_\gamma$; in diagram:

$$\begin{tikzcd}
X \times X \arrow[r, "\pi_\gamma"] \arrow[dr, "f^\Gamma"'] & U_{nc} \arrow[d, "\overline{f}"'] \\
& G
\end{tikzcd}$$

We have

$$f^\Gamma(x,y) = \overline{f} \circ \pi_\gamma(x,y) = \overline{f} \circ \gamma(x) \overline{f} \circ \pi(x,y) \left(\overline{f} \circ \gamma(\sigma^2(x,y))\right)^{-1}$$

so f^Γ and $\overline{f} \circ \pi = f$ are cohomologous. Using again the universal property of U_{nc}, f^Γ factorizes through U_{nc}, hence there exists a group homomorphism $\overline{f^\Gamma} : U_{nc} \to G$ such that $f^\Gamma = \overline{f^\Gamma} \circ \pi$.

On the other hand, since $\pi_\gamma(S) = 1$ we have $f^\Gamma(S) = \overline{f}(\pi_\gamma(S)) = \overline{f}(1) = 1$, but also $f^\Gamma(S) = \overline{f^\Gamma}(\pi(S))$, so the group homomorphism $\overline{f^\Gamma} : U_{nc} \to G$ induces a map

$$f^\gamma : U^\gamma_{nc} = U_{nc}/\pi(S) \to G$$

such that, if

$$p' : U_{nc} \to U_{nc}/\pi(S)$$

is the canonical group projection to the quotient ($p = p' \circ \pi$), then $\overline{f^\Gamma} = f^\gamma \circ p'$. In diagram:

$$\begin{tikzcd}
X \times X \arrow[r, "\pi"] \arrow[rr, bend left, "p"] \arrow[dr, "f^\Gamma"'] & U_{nc} \arrow[r, "p'"] \arrow[d, "\exists!\overline{f^\Gamma}"'] & U_{nc}/\pi(S) \arrow[r, equal] \arrow[dl, dashed, "\exists f^\gamma"] & U^\gamma_{nc} \\
& G & &
\end{tikzcd}$$

Clearly $f^\Gamma = f^\gamma \circ p$.

□

For a given γ, the associated U_{nc}^{γ} is called the *reduced* universal group.

Corollary 112. *Given a biquandle X, if there exists $\gamma : X \to U_{nc}$ such that $U_{nc}^{\gamma} = 1$ then every 2-cocycle in X is trivial.*

An example of the above situation is given by some Alexander biquandles.

3.3.1 The Alexander biquandle

Let $A = \mathbb{Z}[s, t, s^{-1}, t^{-1}]$, X an A-module and $\sigma : X \times X \to X \times X$ given by the matrix

$$\begin{pmatrix} 0 & t \\ s & (1 - st) \end{pmatrix}$$

equivalently $\sigma(x, y) = (sy, tx + (1 - st)y)$. The condition of being a fixed point is $x = sy$:

$$\sigma(sy, y) = ((sy), t(sy) + (1 - st)y) = (sy, y)$$

Cocycle conditions are

$$\begin{cases} (x, y)(tx + (1 - st)y, z) & = & (x, sz)(tx + (1 - st)sz, ty + (1 - st)z) \\ (sy, sz) & = & (y, z) \\ (sy, y) & = & 1 \end{cases}$$

Following M. Graña, we can adapt to the biquandle situation the proof for the quandle case (see Lemma 6.1 of [G]). Consider $\gamma : X \to U_{nc}$ given by $\gamma(x) = (0, cx)$, where $c = (1 + st)^{-1}$. Notice that c is an endomorphism commuting with s, and $(sy, sz) = (y, z) \in U_{nc}$, so

$$\gamma(x) = (0, cx) = (s0, scx) = (0, csx) = \gamma(sx)$$

hence, we can use γ in order to get another 2-cocycle, cohomologous to π.

$$\pi_{\gamma}(x, y) := \gamma(x)(x, y)\gamma(\sigma^{(2)}(x, y))^{-1}$$

where $\sigma(x, y) = (sy, tx + (1 - st)y)$, so

$$\pi_{\gamma}(x, y) = (0, cx)(x, y)\big(0, c(tx + (1 - st)y)\big)^{-1}$$

in particular

$$\pi_{\gamma}(0, y) = (0, 0)(0, y)(0, c(1 - st)y)^{-1} = (0, y)(0, y)^{-1} = 1$$

so, the class of $(0, y) = 1$ in U_{nc}^{γ}.

Lemma 113. *Let X be an Alexander birack such that $(1 - st)$ is invertible in* End(X). *If we define γ as above, then, the following identities hold in U_{nc}^{γ}:*

1. $(x, 0) = 1$ *for all x, and*

2. $(a, b) = (a, b + a)$ *for all $a, b \in X$.*

Proof. 1. From the cocycle condition,

$$(x,y)(tx+(1-st)y,z)=(x,sz)(tx+(1-st)sz,ty+(1-st)z)$$

taking $x=0=z$ we get

$$(0,y)((1-st)y,0)=(0,0)(0,ty)$$

but we know that $(0,*)=1$ in U^{γ}_{nc}, so $((1-st)y,0)=1$, and because $(1-st)$ is a unity we conclude $(x,0)=1$ for all x.

2. Using the cocycle condition

$$(x,y)(tx+(1-st)y,z)=(x,sz)(tx+(1-st)sz,ty+(1-st)z)$$

and clear z from $tx+(1-st)sz=0$, that is, set $z=\frac{-t}{(1-st)s}x$, then

$$(x,y)\Big(tx+(1-st)y,\frac{-t}{(1-st)s}x\Big)=\Big(x,s\frac{-t}{(1-st)s}x\Big)(0,ty+(1-st)z)$$

or

$$(x,y)\Big(tx+(1-st)y,\frac{-t}{(1-st)s}x\Big)=\Big(x,\frac{-t}{(1-st)}x\Big)\qquad(*)$$

clearing $y=\frac{-t}{1-st}x$, get

$$\Big(x,\frac{-t}{1-st}x\Big)(0,z)=(x,sz)\Big(tx+(1-st)sz,t\frac{-t}{1-st}x+(1-st)z\Big)$$

or

$$\Big(x,\frac{-t}{1-st}x\Big)=(x,sz)\Big(tx+(1-st)sz,\frac{-t^2}{1-st}x+(1-st)z\Big)\qquad(\dagger)$$

in particular, using RHS of (*) = LHS of (†) with $y=sz$ we get

$$(x,sz)\Big(tx+(1-st)sz,\frac{-t}{(1-st)s}x\Big)=(x,sz)\Big(tx+(1-st)sz,\frac{-t^2}{1-st}x+(1-st)z\Big)$$

so

$$\Big(tx+(1-st)sz,\frac{-t}{(1-st)s}x\Big)=\Big(tx+(1-st)sz,\frac{-t^2}{1-st}x+(1-st)z\Big)$$

Now we simply change variables. Call $a=tx+(1-st)sz$, then $(1-st)z=a-\frac{t}{s}x$, replacing

$$\Big(a,\frac{-t}{(1-st)s}x\Big)=\Big(a,\frac{-t^2}{1-st}x+a-\frac{t}{s}x\Big)$$

or

$$\Big(a,\frac{-t}{(1-st)s}x\Big)=\Big(a,\frac{-t}{(1-st)s}x+a\Big)$$

Call $b=\frac{-t}{(1-st)s}x$ (notice that $(x,y)\mapsto(a,b)$ is bijective) and get $(a,b)=(a,b+a)$. □

Inductively $(a,b)=(a,b+na)\ \forall n\in\mathbb{N}$; if a generates X additively then

$$(a,b)=(a,0)=1\ \forall b\in X.$$

Corollary 114. *If p is an odd prime, $X = \mathbb{F}_p$, and $s^{-1} \neq t \in \mathbb{F}_p \setminus \{0\}$, then every cocycle in the Alexander's birack in X is cohomologous to the trivial one. In other words, the reduced Universal group U_{nc}^{γ} is trivial. In particular every 2-cocycle in D_3 is trivial.*

Remark 115. This generalizes the result of Graña in [G] where he proves the quandle case, that is, the case $s = 1$.

A biquandle example that is not a quandle is the following:

Corollary 116. *Let $X = \mathbb{Z}_3$, then Wada's biquandle agree with bialexander biquandle ($s = -1$, $t = 1$, $1 - st = -1 \in \mathbb{Z}_3$), so $U_{nc}^{\gamma} = 1$ and every noncommutative 2-cocycle is trivial. In particular, for any coloring with this biquandle, the corresponding element in U_{nc} is trivial.*

Remark 117. For the inverse solution $\overline{\sigma}$ of of Wada's biquandle (with $G = \mathbb{Z}_3$), the group $U_{nc}(\overline{\sigma}) = \langle a : a^3 = 1 \rangle$ is not trivial.

Remark 118. In the previous corollary, the hypothesis $|X|$ being prime was essential, the smallest case where it fails is $X = \mathbb{Z}_4$, as an example of computation we calculate the reduced universal group for $X = \mathbb{Z}_4$, $s = -1$ and $t = 1$.

3.3.2 Reduced U_{nc} in computer

It is clear that the procedure that computes U_{nc} in the computer can be trivially adapted for the reduced version, just adding as input a given set $S_0 \subset X \times X$, and begin with $S = S_0 \cup \{(x, sx) : x \in S\}$, instead of simply $S = \{(x, sx) : x \in X\}$. The procedure will actually compute a list of generators and relations of the quotient group $U_{nc}/(S_0)$, so it could be also used to produce other quotients, not only U_{nc}^{γ}. The advantage of U_{nc}^{γ} is that it gives the same knot/link invariant as U_{nc}, so in order to find suitable S_0's one can do the following:

- In order to produce functions $\gamma : X \to U_{nc}$ with $\gamma(x) = \gamma(sx)$, consider the equivalence relation on X induced by s, that is the equivalence relation generated by $x \sim s(x)$. Denote $\overline{x}$ the class of x modulo s.

- for all pairs $(x, y) \in X \times X$, consider the coboundary relation

$$f_\gamma(x, y) = \gamma(x) f(x, y) \gamma(\sigma^{(2)}(x, y))^{-1}$$

  if $\overline{x} = \overline{\sigma^{(2)}(x, y)}$ then $f(x, y)$ is conjugated to $f_\gamma(x, y)$, so

$$f(x, y) = 1 \iff f_\gamma(x, y) = 1,$$

  so it is clear that (x, y) can not be included in S because of γ.

- if $\overline{x} \neq \overline{\sigma^{(2)}(x, y)}$ then we can choose $\gamma : X \to U_{nc}$ such that $\gamma(z) = \gamma(sz)$ for all z and $\gamma(x) f(x, y) = \gamma(\sigma^{(2)}(x, y))$.

By the above considerations, it is useful to list all tuples

$$(\overline{x}, (x,y), \overline{\sigma^{(2)}(x,y)})$$

with $\overline{x} \neq \overline{\sigma^{(2)}(x,y)}$. For any of these elements, add the pair $(\overline{x}, \overline{\sigma^{(2)}(x,y)})$ to a set f "used" elements, so we continue with the others with $(\overline{x}, (x,y), \overline{\sigma^{(2)}(x,y)})$ with $\overline{x} \neq \overline{\sigma^{(2)}(x,y)}$ but $(\overline{x}, \overline{\sigma^{(2)}(x,y)})$ not "used". This procedure is easily implemented in G.A.P. For example, for the Dihedral quandle gives

$$[0,[0,1],2], \ \ [0,[0,2],1], \ \ [1,[1,0],2]]$$

so we can choose $\gamma(0) = 1$, $\gamma(2) = [0,1]$, $\gamma(1) = [0,2]$ and hence define $S_0 := \{[0,1],[0,2]\}$. With this entry, the procedure computing U^{γ}_{nc} gives $S = X \times X$, that is $U^{\gamma}_{nc}(D_3)$ is trivial. We give the list of generators and relations of U^{γ}_{nc} for biquandles of cardinality 3, with the corresponding S_0.

name	σ	*generators of* U^{γ}_{nc}	*equations*	S_0
flip	BQ^3_1	6	$f_2f_1 = f_1f_2, f_4f_3 = f_3f_4,$ $f_6f_5 = f_5f_6,$	–
a-flip$\{2,3\} \cup \{1\}$	BQ^3_2	3	$f_3f_2 = f_2f_3,$	–
	BQ^3_3	3	–	–
	BQ^{3*}_3	3	–	–
$Wada(\mathbb{Z}_3)$	BQ^3_4	0	–	$\{[1,2]\}$
inv. $Wada(\mathbb{Z}_3)$	BQ^{3*}_4	1	$f_1^3 = 1$	–
	BQ^3_5	2	–	$\{[1,3]\}$
Q_3	BQ^3_6	2	–	$\{[2,1]\}$
inverse Q_3	BQ^{3*}_6	3	–	–
$(x,y) \mapsto (\text{-}y, \text{-}x)$	BQ^3_7	3	$f_3f_2 = f_2f_3,$	–
D_3	BQ^3_8	0	–	$\{[1,2],[1,3]\}$
inverse D_3	BQ^{3*}_8	0	–	–
	BQ^3_9	1	$f_1^3 = 1,$	–
	BQ^{3*}_9	0	–	–
involutive$(\mathbb{Z}_3)$	BQ^3_{10}	2	$f_1f_2 = f_2f_1$	–

3.3.3 Reduced group for Alexander biquandle in $\mathbb{Z}_4$

Take $s = -1$ and $t = 1$. We will compute the reduce universal group for $X = (\mathbb{Z}_4, \sigma)$ where $\sigma(x,y) = (-y, x+2y)$. Cocycle conditions are

$$\begin{cases} (x,y)(x+2y,z) &= (x,-z)(x-2z,y+2z) \\ (-y,-z) &= (y,z) \\ (-x,x) &= 1 \end{cases}$$

The fixed points are $\{(0,0),(3,1),(2,2),(1,3)\}$. Considering

$$\pi_\gamma(x,y) = \gamma(x)(x,y)\gamma(x+2y)^{-1}$$

we have, in particular

$$\pi_\gamma(0,y) = \gamma(0)(0,y)\gamma(2y)^{-1}$$

Define

$$\gamma(0) := 1,\ \gamma(2) := (0,1)$$

and $\gamma(1) = \gamma(3)$ (so that $\gamma(x) = \gamma(sx)$). We get $\pi_\gamma(0,1) = 1$, hence $(0,1) = 1$ in U^γ_{nc}, and consequently $(s0, s1) = (0,3) = 1$. The values of (x,y) in U^γ_{nc}, considering also the equality $(x,y) = (sx, sy)$ are of the form

(x,y)	0	1	2	3
0	1	1	f	1
1	c	b	d	1
2	e	a	1	a
3	c	1	d	b

The cocycle condition for $x = z = 0$ gives $(0,y)(2y,0) = (0,y)$, so $(2,0) = 1$.
The cocycle condition for $x = 2$ gives

$$(2,y)(2+2y,z) = (2,-z)(2-2z, y+2z).$$

If also $y = 0$ y $z = 1$

$$(2,0)(2,1) = (2,-1)(0,2)$$

hence $(2,1) = (2,-1)(0,2)$. But $(u,v) = (-u,-v)$, then $(2,1) = (2,-1)$ and so $(0,2) = 1$.

The cocycle condition is trivial for $y = -x$ and for $z = -y$. If we omit this cases, considering the equalities $(0,1) = (0,3) = (2,0) = (0,2) = 1$ and replacing $(0,3)$ by $(0,1)$, $(2,3)$ by $(2,1)$ and $(3,3)$ by $(1,1)$, the complete list of equations is

$$(1,0) = (1,1)(1,2),$$

$$(1,0)(1,1) = (1,2),$$

$$(1,2) = (1,1)(1,0),$$

$$(1,2)(1,0) = (1,0)(1,2),$$

$$(1,2)(1,1) = (1,0),$$

$$(1,1)(1,0) = (1,0)(1,1),$$

$$(1,1)(1,2) = (1,2)(1,1).$$

Calling $a := (2,1)$, $b := (1,1)$ and $d := (1,2)$ we get

$$(1,0) = bd,$$

$$(1,0)b = d,$$

$$d = b(1,0),$$

$$d(1,0) = (1,0)d,$$

$$db = (1,0),$$

$$b(1,0) = (1,0)b,$$

$$bd = db.$$

Clearly the equations $(1,0) = bd$ and $bd = db$, but also, plugging $(1,0) = bd$ into the second (or third) equation, we get $b^2 d = d$, so $b^2 = 1$. We conclude $U^{\gamma}_{nc} = free(a,b,d)/(bd = db, b^2)$, and the table with the values of the cocycle is

(x,y)	0	1	2	3
0	1	1	1	1
1	bd	b	d	1
2	1	a	1	a
3	bd	1	d	b

3.3.4 Reduced Universal group of 4-cycles in S_4

Another example of application of U^{γ}_{nc} is the following: consider the quandle

$$Q = \{(1,2,3,4),(1,2,4,3),(1,3,4,2),(1,3,2,4),(1,4,3,2),(1,4,2,3)\}$$

that is, 4-cycles in S_4, with quandle operation $x \triangleleft y = y^{-1}xy$. Recall that $f : Q \times Q \to U_{nc}(Q)$ is cohomologous to f_γ if there exists a function $\gamma : Q \to U_{nc}$ such that

$$f_\gamma(x,y) = \gamma(x) f(x,y) \gamma(x \triangleleft y)^{-1}$$

If we list $(x,(x,y),\sigma^2(x,y))$ without repeating "used" pairs $(x, x \triangleleft y)$, we get

$$[1,[1,2],4],[1,[1,3],6],[1,[1,4],3],[1,[1,6],2],[2,[2,1],6],[2,[2,5],4],$$

$$[2,[2,6],5],[3,[3,1],4],[3,[3,4],5],[3,[3,5],6],[4,[4,2],5],[5,[5,2],6]$$

If we define $\gamma(1) = 1$, $\gamma(4) = [1,2]$, $\gamma(6) = [1,3]$, $\gamma(3) = [1,4]$, $\gamma(2) = [1,6]$, $\gamma(5) = \gamma(2)[2,6]$ then $S_0 = \{[1,2],[1,3],[1,4],[1,6],[2,6]\}$. If we compute U_{nc} using our algorithm, it gives 30 generators with with 108 equations, while U^{γ}_{nc} has only 5 generators with 20 equations

$$1 = f_1 f_3, f_2 f_4 = 1, f_3 f_1 = 1, f_5 f_1 = 1, f_1 f_5 = 1,$$

$$f_1 f_1 = f_2, f_1 f_1 = f_4, f_1 f_1 = f_3 f_5, f_5 = f_1 f_4, f_1 = f_2 f_5,$$

$$f_1 = f_3 f_4, f_2 f_1 = f_3, f_4 f_3 = f_1, f_4 f_1 = f_5, f_5 f_2 = f_1,$$

$$f_1 f_2 = f_3, f_1 f_2 = f_2 f_1, f_1 f_3 = f_3 f_1, f_1 f_4 = f_4 f_1, f_1 f_5 = f_5 f_1,$$

Call $a := f_1$, then $f_3 = f_5 = a^{-1}$, $f_2 = f_4 = a^2$, and replacing these values into the 20 equations, the only remaining condition is $a^4 = 1$, we conclude $U^{\gamma}_{nc} = \langle a : a^4 = 1 \rangle$

This quandle is interesting because it distinguish (using U^{γ}_{nc} and its canonical cocycle) the trefoil from its mirror image: there are 30 colorings, 6 of them give trivial invariant both for the trefoil and its mirror (these are the 6 constant colorings), but the other 24 colorings gives a^{-1} for the trefoil and a for its mirror, and clearly $a \neq a^{-1}$ in $\langle a : a^4 = 1 \rangle$.

3.4 Some knots/links and their n.c. invariants

There are 3 quandles of size 3, *none of them* give nontrivial invariant for knots up to 11 crossings. On the other hand, using the biquandle BQ_2^3=aflip$\coprod\{1\}$, from the list of 84 knots with less or equal to 10 crossings, all of them have exactly 3 different colorings, but there are 44 with nontrivial invariant. For instance, figure eight has nontrivial invariant for tree biquandles of size 3: BQ_2^3=aflip$\coprod\{1\}$, BQ_7^3: $\sigma(x,y)$=$(-y,-x)$, and BQ_9^3.

3.4.1 Alexander biquandle on $\mathbb{Z}_4, \mathbb{Z}_8$

The Borromean link has trivial linking number, but has only 3 colorings using D_3, so we distinguish from three separated unknots. The Unc invariant are trivial for all biquandles of size 3.

On the other hand, for the biAlexander biquandle on $\mathbb{Z}_4$ with $s=-1$ and $t=1$, even though there are 64 colorings, they give non trivial invariants:

Recall Unc=$Free(a,b,f_4)/(b^2=1, ab=ba)$, the invariant for the Borromean link is trivial in 40 colorings, but gives twice $(\alpha,\alpha,1)$, $(\alpha,1,\alpha)$, $(1,\alpha,\alpha)$, $(1,\alpha,\alpha^{-1})$, $(\alpha,1,\alpha^{-1})$, $(\alpha,\alpha^{-1},1)$ on the others, with $\alpha=a$ and $\alpha=a^{-1}$.,

In a similar way, Whitehead's link has trivial linking number, give trivial invariant for all biquandles of size 3 (even though non-trivial number of colorings), with bialexander on $\mathbb{Z}_4$ also give trivial invariant, but with with biAlexander on $\mathbb{Z}_8$ one has non trivial invariants. First we compute U_{nc}^{γ} for $\mathbb{Z}_8$, $t=1$, $s=-1$, with subset $S_0=\{[1,2],[1,3],[2,2]\}$ (it may be seen that this is a subset corresponding to a convenient γ). The algorithm gives as answer that U_{nc}^{γ} has 4 generators:

$$f_1=(2,1)=(2,7)=(4,1)=(4,7)=(6,1)=(6,3)=(8,1)=(8,3),$$

$$f_2=(2,3)=(2,5)=(4,3)=(4,5)=(6,5)=(6,7)=(8,5)=(8,7),$$

$$f_3=(2,4)=(2,6)=(4,2)=(4,4)=(6,6)=(6,8)=(8,4)=(8,6),$$

$$f_4=(3,2)=(3,4)=(3,6)=(3,8)=(7,2)=(7,4)=(7,6)=(7,8),$$

Trivial elements are

$$1=[1,1]=[1,2]=[1,3]=[1,4]=[1,5]=[1,6]=[1,7]=[1,8]=[2,2]=[2,8]=[3,1]$$

$$=[3,3]=[3,5]=[3,7]=[4,6]=[4,8]=[5,1]=[5,2]=[5,3]=[5,4]=[5,5]=[5,6]$$

$$=[5,7]=[5,8]=[6,2]=[6,4]=[7,1]=[7,3]=[7,5]=[7,7]=[8,2]=[8,8]$$

with relations

$$f_1f_3 = f_2,\ f_3f_1 = f_2,\ f_3f_2 = f_1,\ f_3f_3 = 1,$$

$$f_2f_1 = f_1f_2,\ f_2f_2 = f_1f_1,\ f_2f_3 = f_1,\ f_3f_1 = f_1f_3,\ f_3f_2 = f_2f_3$$

Calling $a := f_1$, $b := f_3$, we get

$$ab = f_2, f_2a = af_2, f_2f_2 = aa, f_2b = a, ba = f_2, ba = ab, bf_2 = a, bf_2 = f_2b, bb = 1$$

so $b^2 = 1$, and $f_2 = ab = ba$. we conclude $U^\gamma_{nc} = Free(a, b, f_4)/(b^2 = 1, ab = ba)$.

If we use this biquandle with Whitehead's link we get 64 colorings, 32 of them give trivial invariant, 16 colorings give $(b, 1)$ and 16 colorings give as invariant $(1, b)$.

3.5 Final comments

In the examples we saw, very often the group U^γ_{nc} is non commutative, but we haven't found a knot/link with genuine non commutative invariant, that is, for example a commutator of two non commuting elements of U^γ_{nc}. Also, sometimes U^γ_{nc} have pairs of commuting elements and other non commuting, for instance,

$$U^\gamma_{nc}(biAlex(Z_8)) = Free(a, b, f_4)/(b^2 = 1, ab = ba),$$

but using this biquandle, computing the invariants for knots and links with less than 11 crossings, the elements a and b do not "mix" with f_4. We don't know if this is a general fact or not, that is, if the invariant obtained is the same if we use the abelianization of U^γ_{nc}.

If U^γ_{nc} happens to be abelian, then the information we get with the non commutative invariant is essentially the state-sum invariant for the canonical cocycle $\pi_\gamma : X \times X \to U^\gamma_{nc}$. If this is the case (or if one considers the abelianization of U^γ_{nc}), then our construction can be seen as a natural and nontrivial way to produce interesting 2-cocycles, so that satesum invariant becomes a procedure with input only a biquandle, and not a biquandle *plus* a 2-cocycle, because a natural 2-cocycle is always present when one gives a biquandle.

Another natural question about state-sum invariant for biquandles is how to generalize it for 2-cocycles with values in nontrivial coefficients, which is known for quandles, but unknown for biquandles. In order to answer this question, it should be convenient to have an action of some group (to be defined) into the abelian group of coefficients where the 2-cocycle takes values, and if one imitates the quandle case, one should define, for each crossing, an exponent (in this group) that twist the value of the cocycle at that crossing. If the exponent is well-defined, that is, for instance it remains unchanged under Reidemeister moves of other crossings, then essentially it must be a non commutative 2-cocycle. The group U_{nc} was the candidate, and in fact this was origin of the present work. In the quandle case there is a natural map $U_{nc}(X) \to G_X$, where G_X is the group generated by X with relations $xy = zt$ if $\sigma(x, y) = (z, t)$; the map $U_{nc}(X) \to G_X$ is simply determined by $(x_1, x_2) \mapsto x_2$. So, for quandles, G_X-modules are natural candidates for coefficients (see [CEGS]), or also $U_{nc}(X)$-modules, or quandle-modules as considered in [AG]. We hope 2-cocycles with values in U_{nc}-modules will allow to define more general state-sum invariants, but at the moment we don't know how, we end remarking that for biquandles, there is no general well-defined map $U_{nc}(X) \to G_X$, and $U_{nc}(X)$ sometimes is the trivial group.

Chapter 4

Gap

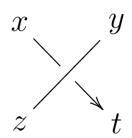
En este capítulo mostramos algunos programas/funciones escritos el lenguaje GAP que utilizamos a la hora de buscar ejemplos.

In this chapter we show some of the programs we wrote in Gap.

In [FG] one can found the GAP programs computing colorings, U_{nc}^{γ}, and invariants for knots and links given as planar diagrams.

In Gap, comments are preceded by # .

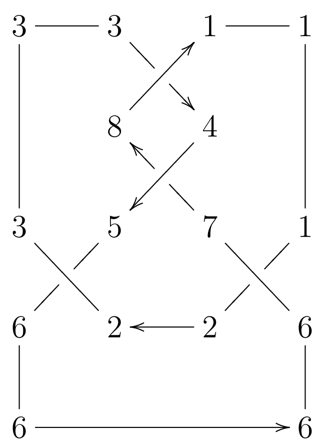
Given an oriented projection of a knot, a planar diagram (pd) can be constructed as follows: start labeling a (any) semiarc with number one and continue labeling (2,3, and so on til return to 1) in the orientation of the knot. For every crossing c (start looking al the incoming underarc and continue reading the crossing counterclockwise)



write $[x, z, t, y]$ to get a not signed pd, or $[\text{sign(c)}, [x, z, t, y]]$. To get a pd, make a list with all the crossings.

Note that the orientation of the over arc does not change the (not signed) planar notation of the crossing.

Some examples of knot pd's:



$$N4_1 := [[1, [1, 7, 2, 6]], [1, [5, 3, 6, 2]], [-1, [3, 8, 4, 1]], [-1, [7, 4, 8, 5]]];$$

Another example:

$$N5_1 := [[-1, [1, 6, 2, 7]], [-1, [7, 2, 8, 3]], [-1, [3, 8, 4, 9]], [-1, [5, 10, 6, 1]],$$

$$[-1, [9, 4, 10, 5]]];$$

A biquandle is a function $r : X \times X \to X \times X$ that in particular (due to non degeneracy condition) can be given as a list of permutations. If

$$r(x, y) = (\sigma_x(y), \tau_y(x))$$

then $\{\sigma_x\}_{x\in X}$ is a list of permutations, called lperms, and similarly rperms for τ_y. For example, if $r(x, y) = (y, x)$ (the flip) then $\mathrm{rperm}_x = \mathrm{lperm}_x = id$ for all x. Then the data of this solution, for $X = \{1, 2, 3\}$ is encoded by

```
flip:=rec(lperms=[[1,2,3],[1,2,3],[1,2,3]],rperms=[[1,2,3],[1,2,3],
[1,2,3]])
```

Given a group G, then $r(x, y) = (xy^{-1}x^{-1}, xy^2)$ is a biquandle solution, due to Wada. The function "wada" computes this biquandle. For example

```
gap> wada(CyclicGroup(3));
rec(lperms:=[[1,3,2],[1,3,2],[1,3,2]],
rperms:=[[1,2,3],[3,1,2],[2,3,1]],size:=3,
labels:=[<identity>of...,f1,f1^2],s:=[1,3,2])
```

The Wada function is given by:

```
#Wadda's solution
wada := function(group)
  local x, y, e, s, lperms, rperms;
  e := Elements(group);
  s := [];
  lperms := NullMat(Size(group), Size(group));
  rperms := NullMat(Size(group), Size(group));
  for x in group do
    for y in group do
      lperms[Position(e,x)][Position(e,y)] :=Position(e,x*Inverse(y)*
      Inverse(x));
      rperms[Position(e,y)][Position(e,x)] :=Position(e,x*y^2);
    od;
  od;
  for x in e do
    Add(s, Position(e, Inverse(x)));
  od;
return rec(lperms:=lperms,rperms:=rperms,size:=Order(group),labels:=e,
 s:=s);
end;
```

If R is a ring, s, t two commuting units, and M is an R-module then M is a biquandle via $r(x, y) = (sy, tx + (1 - st)y)$, it is called the Alexander biquandle. Function "bialexander" computes this biquandle for $M = R = \mathbb{Z}/m\mathbb{Z}$. The command is

```
bialexander(m,s,t)
```

For example:

```
gap>bialexander(4,-1,1);
rec(lperms:=[[1,4,3,2],[1,4,3,2],[1,4,3,2],[1,4,3,2]],
rperms:=[[1,2,3,4],[3,4,1,2],[1,2,3,4],[3,4,1,2]],
size:=4,labels:=<enumeratorof(Integersmod4)>,
s:=[1,4,3,2])
```

This function is given by

```
#Bialexander biquandle
bialexander := function(n, s, t)
  local e, lperms, rperms, x, y, ss;
  s := s*One(ZmodnZ(n));
  t := t*One(ZmodnZ(n));
  e := Enumerator(ZmodnZ(n));
  lperms := NullMat(Size(e), Size(e));
  rperms := NullMat(Size(e), Size(e));
  for x in e do
    for y in e do
      lperms[Position(e,x)][Position(e,y)]:=Position(e,s*y);
      rperms[Position(e,y)][Position(e,x)]:=Position(e,t*x+(1-s*t)*y);
    od;
  od;
  ss := [ ];
  for x in e do
    Add(ss, Position(e, Inverse(s)*x));
  od;
return rec(lperms:=lperms,rperms:=rperms,size:=n,labels:=e,s:=ss);
end;
```

If C is a subset of a group G, stable under conjugation, then $r(x, y) = (y, y^{-}1xy)$ is a (quandle) solution, we call it "conj".

As an example:

```
gap> conj(ConjugacyClass(AlternatingGroup(4),(1,2,3)));
rec(lperms:=[[1,2,3,4],[1,2,3,4],[1,2,3,4],[1,2,3,4]],
rperms:=[[1,4,2,3],[3,2,4,1],[4,1,3,2],[2,3,1,4]],
size:=4,labels:=[(2,4,3),(1,2,3),(1,3,4),(1,4,2)],
s:=[1,2,3,4])
```

Function "conj" is given by:

```
 conj := function(conjclass)
  local x, y, e, s, lperms, rperms;
  e := Elements(conjclass);
  s := [];
  lperms := NullMat(Size(conjclass), Size(conjclass));
  rperms := NullMat(Size(conjclass), Size(conjclass));
  for x in conjclass do
    for y in conjclass do
      lperms[Position(e, x)][Position(e, y)] := Position(e, y);
      rperms[Position(e, y)][Position(e, x)] := Position(e,
   Inverse(y)*x*y);
    od;
  od;
  for x in e do
    Add(s, Position(e, x));
  od;

  return rec(lperms:=lperms, rperms:=rperms, size := Size(conjclass),
  labels := e, s :=  s);
end;
```

If (X, r) is a biquandle, then the inverse function r^{-1} is also a biquandle. The function "inversebiquandle" computes r^{-1} as biquandle: An example:

```
gap> inversebiquandle(oficialbiquandle(listbiquandles5[47]));
rec(lperms:=[[1,2,4,3,5],[1,2,4,3,5],[1,5,3,4,2],[1,5,3,4,2],
[1,2,4,3,5]],
rperms:=[[1,5,3,4,2],[5,2,3,4,1],[1,5,3,4,2],[1,5,3,4,2],
[2,1,3,4,5]],
size:=5,labels:=[1,2,3,4,5],s:=[1,2,3,4,5])
```

First define a function that will be used for "inversebiquandle":

```
r:=function (biq,x,y)
return [biq.lperms [x][y], biq.rperms [y][x]];
end;
```

Function "inversebiquandle" is defined as:

```
inversebiquandle:=function(biquandle)
local x,y,u,v,n,L,R;
n:=biquandle.size;
L := NullMat(n,n);
R := NullMat(n,n);
for x in [1..n] do
  for y in [1..n] do
    for u in [1..n] do
      for v in [1..n] do
```

```
        if r(biquandle, x,y)=[u,v] then
         L[u][v] := x;
         R[v][u] := y;
        fi;
      od;
    od;
  od;
od;
return rec(lperms:=L, rperms := R, size := n, labels := biquandle.labels,
s := biquandle.s);
end;
```

Function “colorings” considers all possible colorings of the semiarcs for a link/knot over a given biquandle and eliminates the non compatible ones.

```
 colorings := function(pd, biquandle)
  local e, i, crossing, p, candidato, max, coloreos;
  e := [1..biquandle.size];
  i := 0;
  max := Maximum(Flat(pd));
  coloreos := [];
  crossing := true;

  for candidato in Iterator(Tuples(e, max)) do
    if ForAny(pd, x->check_equation(x, candidato, biquandle)=false) then
      continue;
    else
      Add(coloreos, candidato);
    fi;
  od;
  return coloreos;
end;
```

Function “colormejor” does the same thing but coloring one crossing at a time and adding one by one the rest of crossings, eliminating non compatible ones on the way. Gives the same answer but is (a lot) faster. In both cases, the knot/link must be given by a signed planar diagram.

Example:

```
gap>trefoil:=[[-1,[1,4,2,5]],[-1,[3,6,4,1]],[-1,[5,2,6,3]]];
[[-1,[1,4,2,5]],[-1,[3,6,4,1]],[-1,[5,2,6,3]]]
gap>colormejor(trefoil,bialexander(3,1,-1));
[[1,1,1,1,1,1],[3,1,1,2,2,3],[2,1,1,3,3,2],
[3,2,2,1,1,3],[2,2,2,2,2,2],[1,2,2,3,3,1],
[2,3,3,1,1,2],[1,3,3,2,2,1],[3,3,3,3,3,3]]
```

Function “colormejor” is given by:

```
colormejor := function(pd, biquandle)
  local  lista, n, c, colorprevio, colorviejo, candidatos, precandidatos,
  candidatoacumulado2,e, colnuevo, solocolor, ppd, coloreos,
  semiarcospasados, x, y, i, j, cn, candi_acum;
  semiarcospasados:=[];
  colorprevio:=[];
  colnuevo:=[];
  colorviejo:=[];
  candi_acum:=[];
  candidatoacumulado2:=[];
  e := [1..biquandle.size];
  #pait first crossing
  candidatos:=[];
  precandidatos:= [];
  i:=0;
  for x in e do
    for y in e do
      if pd[1][1]=1 then
        precandidatos:=[
[pd[1][2][1],x],
[pd[1][2][2],r(biquandle, x,y)[1] ],
[pd[1][2][3],r(biquandle, x,y)[2] ],
[pd[1][2][4],y]
];
      else
        precandidatos:=[
[pd[1][2][1],r(biquandle, x,y)[2] ],
[pd[1][2][2],r(biquandle, x,y)[1] ],
[pd[1][2][3],x],
[pd[1][2][4],y]
];
      fi;
      i:=i+1;
      candidatos[i]:=precandidatos;
      precandidatos:=[];
    od;
  od;
  semiarcospasados:=pd[1][2];
  lista:=candidatos;
  ppd:=Difference(pd,[pd[1]]);
  for c in ppd do;
  #esto pinta localmente cada cruce
  j:=0;
  candidatos:=[];
  precandidatos:= [];
    for x in e do
```

```
      for y in e do
#this colors every crossing satisfying r
#without taking care if the semiarc appears in more than one crossing
        if c[1]=1 then
        precandidatos:=[
[c[2][1],x],
[c[2][2],r(biquandle, x,y)[1] ],
[c[2][3],r(biquandle, x,y)[2] ],
[c[2][4],y]
];
      else
        precandidatos:=[
[c[2][1],r(biquandle, x,y)[2] ],
[c[2][2],r(biquandle, x,y)[1] ],
[c[2][3],x],
[c[2][4],y]
];
        fi;
      j:=j+1;
      colnuevo[j]:=precandidatos;
      precandidatos:=[];
      od;
    od;
  #finished painting with "colnuevo"
    n:=Size(IntersectionSet(semiarcospasados,c[2]));
    i:=0;
    candidatoacumulado2:=[];
    for colorviejo in lista do
      for cn in colnuevo do
        if Size(IntersectionSet(cn,colorviejo))=n then
          i:=i+1;
          candidatoacumulado2[i]:=UnionSet(cn,colorviejo);
        fi;
      od;
    od;
    lista:=candidatoacumulado2;
    candidatoacumulado2:=[];
    semiarcospasados:=UnionSet(semiarcospasados,c[2]);
  od;
#transforms [[semicarco,color]..] in [[color],..]
  solocolor:=[];
  coloreos:=[];
for x in lista do
  solocolor:=[];
  for j in [1..Size(semiarcospasados)] do
    for i in [1..Size(semiarcospasados)] do
```

```
      if x[i][1]=j then
        solocolor[j]:=x[i][2];
       fi;
    od;
  od;
  Add(coloreos,solocolor);
od;
return(coloreos);
end;
```

The function "componentesconexas" gives the number of connected component of a link up to four connected components.

Function $n1$ gives the first semiarc of the 2^{nd} connected component, and function "ns" gives the pair [first semiarc of the second component, first semiarc of the third component]. The entry is always a planar diagram with sign.

Given a list of sets, "relationgenerated", returns the relation generated (as a partition) on the union of these sets. As example:

```
gap>relationgenerated([[1],[2],[3,4],[5,6],[6,2],[7,1,4]]);

[[1,3,4,7],[2,5,6]]
```

Where the function is the following:

```
paste:=function(clases)
local i,j,c,cl,n;
cl:=clases;
for i in [1..Size(cl)] do
  cl[i]:=UnionSet(cl[i],[]);
 od;
cl:=UnionSet(cl,[]);
n:=Size(cl);
for i in [1..n] do
  for j in [i+1..n] do
   if j>Size(cl) then continue;
   else
    c:=IntersectionSet(cl[i],cl[j]);
    if Size(c)>0 then
     cl[i]:=UnionSet(cl[i],cl[j]);
     if cl[i]=cl[j] then continue;
     else cl:=Difference(cl, [cl[j]] );
     fi;
    fi;
   fi;
  od;
 od;
 return cl;
```

```
 end;

relationgenerated:=function(clase)
local i, cl;
 cl:=clase;
 for i in [1..Size(clase)] do
  cl:=paste(cl);
 od;
 return cl;
 end;
```

Function “unc” computes generators and relations of a given biquandle, together with a prescribed set of trivial elements in $X \times X$. Gives as answer the set of trivial elements S, the equivalence classes (where any cocycle takes the same value) and the equations (in terms of representatives of the classes) from the 2-cocycle condition.

First some auxiliary functions:

```
 #sistem of representatives in a biquandle module
repclase:=function(biquandle)
local x, reps, igualdades;
igualdades:=[];

for x in [1..biquandle.size] do
 Add( igualdades,[x,biquandle.s[x]] );
od;
reps:=relationgenerated(igualdades);
return reps;
end;


# this function has as input some equalites, some set of trivial
# elements S, and a set of cocycle equations. The function
# evaluates the elements of S in the cocycle equaion, identify elements
# that are equal, and get -if possible- new trivial elements
# (comming from equalities or from cocycle condition), and get
# new equalities (from cocycle condition of type ab=ac, ba=ca, or sa=bs
# with s in S, etc) returns an enlarged set S, a new set of equalities,
# and shorter list of cocycle conditions
New:=function(lista,clases)
local i, l, c, cl, lista2,lista3;
 cl:=clases;
 #find equalities cocycle eq.
 lista2:=[];
 for l in lista do #try to deduce new equalities
   if l[1]=l[3] then
     if l[2]=l[4] then continue;
```

```
     else
      cl:=relationgenerated(    UnionSet(cl,[[l[2],l[4]]]) );
      #use the new equalitye to generate the new classes
     fi;
   elif l[2]=l[4] then
      cl:=relationgenerated(UnionSet(cl,[[l[1],l[3]]]) );
   else
     lista2:=UnionSet(lista2,[l]);
   fi;
 od;
```

Returning to Unc, as input has a biquandle and a set S (so it can compute the reduced Unc) the procedure is to iterate the function above.

```
unc:=function(biquandle,S)
local c, clases, i, j, x, y, z, A, d1 , d2, lista, n, lis, cob, cl, S2,
gam1,gam2,usados, News;

lista:=[];
A:=[];
#generate clases using condition 2
clases:=[];
n:=biquandle.size;
for x in [1..n] do
  for y in [1..n] do
    for z in [1..n] do
clases:=
UnionSet(
clases,
 [[
 [y,z],[r(biquandle,x,y)[1],r(biquandle,r(biquandle,x,y)[2],z)[1] ]
]]);
    od;
  od;
od;
#Print(clases);
clases:=relationgenerated(clases);

# Add [] and S = { (x,s(x))} to the set of classes
S2:=UnionSet(S,[[]]);
n:=biquandle.size;
for x in [1..n] do
S2:=
UnionSet(S2, [[x,biquandle.s[x] ]]);
od;
clases:=UnionSet(clases,[S2]);
clases:=relationgenerated(clases);
```

```
#generates the list of equations of condition 1 (cocycle condition)
n:=biquandle.size;
for x in [1..n] do
 for y in [1..n] do
  for z in [1..n] do
   A[1]:=[x,y];
   A[2]:=[r(biquandle,x,y)[2],z];
   A[3]:=[x,r(biquandle,y,z)[1]];
   d1:=r(biquandle,x,r(biquandle,y,z)[1])[2];
   d2:=r(biquandle,y,z)[2];
   A[4]:=[d1,d2];
   for i in [1..4] do
    for c in clases do
     if A[i] in c then A[i]:=c[1]; fi; #choose representatives
    od;
   od;
   lista:=UnionSet(lista,[A]);
   A:=[];
  od;
 od;
od;


c:=Size(clases)+1;
lis:=Size(lista)-1;
for i in [1..60] do #repeat the procedure many times
 if c=Size(clases) and lis=Size(lista) then continue;
 else
  c:=Size(clases);
  lis:=Size(lista);

  News:=New(lista,clases);
  lista:=News.list;
  clases:=News.clases;
 fi;
od;


#just in case...
  News:=New(lista,clases);
  lista:=News.list;
  clases:=News.clases;
cob:=[];
usados:=[];
```

```
for c in clases do
if [] in c then S2:=c; fi;
od;
cl:=Difference(clases,[S2]);
n:=Size(repclase(biquandle));
for i in [1..Size(cl)] do
x:=cl[i][1][1];
y:=cl[i][1][2];
z:=r(biquandle,x,y)[2]; #sigma2(x,y)
for j in [1..n] do
 if x in repclase(biquandle)[j]  then
  gam1:= repclase(biquandle)[j][1];
 fi;
 if z in repclase(biquandle)[j]  then
  gam2:= repclase(biquandle)[j][1];
 fi;
od;
if gam1=gam2 then
  continue;
 elif [gam1,gam2] in usados then continue;
 else
  usados:=UnionSet(usados,[[gam1,gam2]]);
  Add(cob,
  [
  #clases[i][1],
  #=
  gam1,cl[i][1],gam2#^-1"
  ]);
 fi;
od;
return rec( S:=S2, clases := cl, equations := lista, cocycle := cob);
end;
```

Example:

```
gap>unc(bialexander(3,1,-1),[]);
rec(S:=[[],[1,1],[2,2],[3,3]],
clases:=[[[1,2]],[[1,3]],[[2,1]],[[2,3]],
[[3,1]],[[3,2]]],
equations:=[[[],[1,3],[1,2],[3,1]],
[[],[2,3],[2,1],[3,2]],[[],[3,2],[3,1],[2,3]]
,[[1,3],[2,1],[],[1,2]],[[2,3],
[1,2],[],[2,1]],[[3,2],[1,3],[],[3,1]]],
cocycle:=[[1,[1,2],3],[1,[1,3],2],[2,[2,1],3],
[2,[2,3],1],[3,[3,1],2],[3,[3,2],1]])
```

The equations should be read in the following way: every element $[a, b, c, d]$ means $ab = cd$, and $[\,] = 1$, so for example $[[\,], [1, 3], [1, 2], [3, 1]]$ means $[1, 3] = [1, 2][3, 1]$.

The cocycle part give a list of possible elements to consider cohomologically trivial. The notation is, given $\gamma : X \to G$, the values of a given 2-cocycle changes under gamma via $f_\gamma = \gamma(1)f(1,2)\gamma(3)^{-1}$, etc. So in the example above (Dihedral quandle) one can see that it is possible to trivialize [1, 2] and [1, 3] simultaneously. If we do so, we get

```
gap>unc(bialexander(3,1,-1),[[1,2],[1,3]]);
rec(S:=[[],[1,1],[1,2],[1,3],[2,1],[2,2],
[2,3],[3,1],[3,2],[3,3]],
clases:=[],equations:=[],cocycle:=[])
```

That is, no generators at all, the group is trivial, so every 2-cocycle is a coboundary.

The function print_equations is similar to unc but human friendly:

```
print_equations:=function(biquandle,S)
local i, j, clases, Unc, n, co,lista,l, trivial;

Unc:=unc(biquandle,S);

trivial:=Unc.S;
clases:=Unc.clases;
lista:=Unc.equations;

n:=Size(clases);
if n=1 then
 Print("Unc has ");Print(n);Print(" generator:");
elif n=0 then
 Print("Unc ={1}");
else
 Print("Unc have ");Print(n);Print(" generators:");
fi;
Print("\n");
for i in [1..Size(clases)] do
 Print("f_{"); Print(i);Print("}");
 for j in clases[i] do
  Print("=");
Print(j);
 od;
 Print(", ");
od;
Print("\n");
Print("trivial elements:");
Print("\n");
#Print("1");
for i in trivial do
  Print("=");
#Print("(");Print(i[1]);Print(",");Print(i[2]);Print(")");
Print(i);
```

```
od;
Print("\n");
Print("equations: ");
Print("\n");
n:=Size(clases);
for l in lista do
  for i in [1..4] do;
  if i=3 then Print("="); fi;
  if l[i] in trivial then continue;
    else
      for j in [1..n] do
        if l[i] in clases[j] then
          Print("f_{"); Print(j);Print("}");
        fi;
      od;
    fi;
  od;
    Print(", ");
od;
Print("\n");
Print("\n");
Print("coboundary conditions to be consider to eventually add
elements to S: ");
Print("\n");
for co in Unc.cocycle do
Print("\\gamma_");
Print(co[1]);
Print(co[2]);
Print("\\gamma_");
Print(co[3]);
Print("^{-1}, ");
od;
Print("\n");
end;
```

In the above examples:

```
gap>print_equations(bialexander(3,1,-1),[]);
Unc have 6 generators:
f_{1}=(1,2),f_{2}=(1,3),f_{3}=(2,1),f_{4}=(2,3),f_{5}=(3,1),
f_{6}=(3,2),
trivial elements:
=[]=[1,1]=[2,2]=[3,3]
equations:
f_{2}=f_{1}f_{5},f_{4}=f_{3}f_{6},f_{6}=f_{5}f_{4},f_{2}f_{3}=f_{1},
f_{4}f_{1}=f_{3},f_{6}f_{2}=f_{5},
```

Coboundary conditions to be considered to eventually add elements to S:
$\gamma_1[1,2]\gamma_3^{-1}, \gamma_1[1,3]\gamma_2^{-1}, \gamma_2[2,1]\gamma_3^{-1}, \gamma_2[2,3]\gamma_1^{-1}, \gamma_3[3,1]\gamma_2^{-1}, \gamma_3[3,2]\gamma_1^{-1},$

```
gap> print_equations(bialexander(3,1,-1),[[1,2],[1,3]]);
Unc ={1}
trivial elements:
=[]=[1,1]=[1,2]=[1,3]=[2,1]=[2,2]=[2,3]=[3,1]
=[3,2]=[3,3]
```

The following function computes the Boltzmann weight of a crossing, for a given coloring, a set of equivalent classes of pairs (where the cocycle takes the same values) and a group (Fnc) is given by generators.

```
BWgen := function(cruce, coloreo, clases,Fnc)
local bwg,i;
bwg:=One(Fnc);
  if cruce[1]=1 then
    for i in [1..Size(clases)] do
      if [coloreo[cruce[2][1]],coloreo[cruce[2][4]]] in clases[i] then
        bwg := GeneratorsOfGroup(Fnc)[i];
      fi;
    od;
  else
    for i in [1..Size(clases)] do
      if [coloreo[cruce[2][3]],coloreo[cruce[2][4]]] in clases[i] then
        bwg := Inverse(GeneratorsOfGroup(Fnc)[i]);
      fi;
    od;
  fi;
return bwg;
end;
```

"invariant.g" computes the universal nc-invariant for each coloring of a given biquandle, together with a subset $S \subset X\times$ that we declare that our cocycle is trivial. The sintaxis is

```
gap> invariantgen(trefoil,bialexander(3,1,-1),[]);
[[[1,1,1,1,1,1],],[[3,1,1,2,2,3],f1^-1*f4^-1*f5^-1],
[[2,1,1,3,3,2],f2^-1*f6^-1*f3^-1],[[3,2,2,1,1,3],
f3^-1*f2^-1*f6^-1],[[2,2,2,2,2,2],],
[[1,2,2,3,3,1],f4^-1*f5^-1*f1^-1],[[2,3,3,1,1,2],
f5^-1*f1^-1*f4^-1],
[[1,3,3,2,2,1],f6^-1*f3^-1*f2^-1],[[3,3,3,3,3,3],]]
```

Over all colorings, the following function computes the invariant (of each connected component) up to 3 connected components. The group is the one computed by "unc", that have an S as input, so it can (nearly) compute the reduced Unc, or also a quotient of Unc.

```
invariantgen:=function(pd,biquandle,S)
  local generators,
inv,max, W, W1,W2, W3, BWg, coloreo,   cruce, Fnc,
n, rk, N1, N2,i, j,k,l;
max := Maximum(Flat(pd));
inv:=[];
#first calculates equations of Unc and it's generators
unc(biquandle,S);
generators:=unc(biquandle,S).clases;
rk:=Size(generators);
Fnc := FreeGroup(rk);
# and calculate invariant for every coloring
n:=componentesconexas(pd);
if n=1 then
  for coloreo in colormejor(pd, biquandle) do
    W := One(Fnc);
    for i in [1..max] do
      for cruce in pd do
        if cruce[2][1]=i then
          W:=W*BWgen(cruce, coloreo, generators,Fnc);
        fi;
      od;
    od;
    Add(inv,[coloreo,W]);
  od;
  return(inv);
elif n=2 then
  N1:=n1(pd);
  for coloreo in colormejor(pd, biquandle) do
    W1 := One(Fnc);
    W2 := One(Fnc);
    for i in [1..N1-1] do
      for cruce in pd do
        if cruce[2][1]=i then
          W1:=W1*BWgen(cruce, coloreo, generators, Fnc);
        fi;
      od;
    od;
    for j in [N1..max] do
      for cruce in pd do
        if cruce[2][1]=j then
          W2:=W2*BWgen(cruce, coloreo, generators, Fnc);
        fi;
      od;
    od;
    Add(inv,[coloreo,[W1,W2]]);
```

```
  od;
return(inv);
elif n=3 then
  N1:=ns(pd)[1];
  N2:=ns(pd)[2];
  for coloreo in colormejor(pd, biquandle) do
    W1 := One(Fnc);
    W2 := One(Fnc);
    W3 := One(Fnc);
    for i in [1..N1-1] do
      for cruce in pd do
        if cruce[2][1]=i then
          W1:=W1*BWgen(cruce, coloreo, generators, Fnc);
        fi;
      od;
    od;
    for j in [N1..N2-1] do
      for cruce in pd do
        if cruce[2][1]=j then
          W2:=W2*BWgen(cruce, coloreo, generators, Fnc);
        fi;
      od;
    od;
    for k in [N2..max] do
      for cruce in pd do
        if cruce[2][1]=k then
          W3:=W3*BWgen(cruce, coloreo, generators, Fnc);
        fi;
      od;
    od;
    Add(inv,[coloreo,[W1,W2,W3]]);
  od;
return inv;
fi;
end;
```

Notice that the answer is given in the FREE group on same number of generators as unc, but no relations are included. So one has to look at "unc(biquandle,[S])" in order to see if these invariants are trivial or not. For instance, they are all trivial in this case. But with the biquandle conj(ConjugacyClass(SymmetricGroup(4),(1,2,3,4))) we get nontrivial invariant.

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