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Sistemas de traslaciones: generadores de marcos y aproximación

Tesis presentada para optar al título de Doctora de la Universidad de Buenos Aires en el área Ciencias Matemáticas

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Sistemas de traslaciones: generadores de marcos y aproximación

(Resumen)

Los espacios invariantes por traslaciones son espacios cerrados de $L^2(\mathbb{R}^d)$ que son invariantes por traslaciones enteras. Estos espacios juegan un rol muy importante en teoría de aproximación, análisis armónico, teoría de wavelets, muestreo y procesamiento de imágenes.

Un conjunto de funciones Φ es un conjunto de generadores de un espacio invariante por traslaciones V si la clausura en $L^2(\mathbb{R}^d)$ del espacio generado por las traslaciones enteras de las funciones de Φ coincide con V. Es importante saber cuándo es posible obtener un conjunto minimal de generadores de un conjunto de generadores dado en Φ. Con referencia a esta pregunta, Bownik y Kaiblinger en 2006, mostraron que puede obtenerse de Φ un conjunto minimal de generadores de V a través de combinaciones lineales de sus elementos. Como tomar combinaciones lineales de un número finito de funciones preserva propiedades tales como la suavidad, el decaimiento, tener soporte compacto, ser de banda limitada, etc., una consecuencia interesante del resultado de Bownik y Kaiblinger es que si los generadores de V tienen una propiedad adicional, existe un conjunto minimal de generadores que hereda esa propiedad.

En muchos problemas que involucran espacios invariantes por traslaciones, es importante que el sistema de traslaciones posea una estructura analítica particular como ser una base ortonormal, una base de Riesz o un marco. Luego, resulta interesante saber cuándo un conjunto minimal de generadores obtenido a través de combinaciones lineales de uno dado preserva la misma estructura que el sistema de generadores original. En esta tesis daremos respuesta a esta pregunta para el caso de marcos, bases de Riesz y bases ortonormales.

Los espacios invariantes por traslaciones enteras pueden tener además extra invariancia, es decir, pueden ser invariantes bajo otras traslaciones además de las enteras. Estos espacios con extra invariancia son importantes en aplicaciones especialmente en aquellas donde el error que se produce en ciertas aproximaciones es relevante.

En esta tesis, dado un conjunto de datos $\mathcal{F} = \{f_1, \ldots, f_m\} \subseteq L^2(\mathbb{R}^d)$, probamos la existencia y mostramos cómo construir un “espacio invariante por traslaciones enteras pequeño” V más cercano a $\mathcal{F}$ sobre cierta clase de subespacios cerrados de $L^2(\mathbb{R}^d)$. El espacio aproximante debe tener cierta extra-invariancia, es decir, debe ser invariante bajo traslaciones sobre un subgrupo aditivo de $\mathbb{R}^d$ que contenga estrictamente a $\mathbb{Z}^d$, ya fijado. Aquí con pequeño, nos referimos a que el subespacio solución esté generado por las traslaciones enteras de un pequeño número de generadores. Además, damos una expresión del error en función de los datos y construimos un marco de Parseval para el espacio óptimo.

Por otro lado, consideramos el problema de aproximar $\mathcal{F}$ por espacios de Paley-Wiener generalizados de $\mathbb{R}^d$, que están generados por las traslaciones enteras de un número finito de funciones. Estos espacios pueden verse como espacios invariantes por traslaciones enteras finitamente generados que son $\mathbb{R}^d$ invariantes.

Además mostramos la relación entre estos espacios y los conjuntos “multi-tile” de $\mathbb{R}^d$ y la conexión con resultados recientes acerca de bases de Riesz de exponenciales.

Palabras claves: espacios invariantes por traslaciones enteras; marcos; bases de Riesz; bases ortonormales; muestreo; extra invariancia; espacios de Paley-Wiener.

Systems of translates: frame generators and approximation

(Abstract)

Shift invariant spaces are closed subspaces of $L^2(\mathbb{R}^d)$ that are invariant under integer translations. They play an important role in approximation theory, harmonic analysis, wavelet theory, sampling and signal processing.

A set of functions Φ is a set of generators for a shift invariant space V if the closure of the space spanned by all integer translations of the functions in Φ agrees with V. It is interesting to know whether it is possible to obtain a minimal set of generators from the given generators in Φ. Concerning this question, Bownik and Kaiblinger in 2006, showed that a minimal set of generators for V can be obtained from Φ by linear combinations of its elements. Since linear combinations of a finite number of functions preserve properties such as smoothness, compact support, bandlimitedness, decay, etc, an interesting consequence of Bownik and Kaiblinger's result is that if the generators for V have some additional property, there exists a minimal set of generators that inherits this property.

In many problems involving shift invariant spaces, it is important that the system of translates bears a particular functional analytic structure such as being an orthonormal basis, a Riesz basis or a frame. Therefore, it is interesting to know when a minimal set of generators obtained by taking linear combinations of the original one has the same structure. In this thesis we answer this question completely, in the case of frames, Riesz bases and orthonormal bases.

Shift invariant spaces can also have extra invariance, that is they could be invariant under translates other than integers. Such spaces are important in applications specially in those where the error in approximations is an issue.

In this thesis, given an arbitrary finite data $\mathcal{F} = \{f_1, \ldots, f_m\} \subseteq L^2(\mathbb{R}^d)$, we prove the existence and show how to construct a "small shift invariant space" V that is the "closest" to the data $\mathcal{F}$ over certain class of closed subspaces of $L^2(\mathbb{R}^d)$. The approximating subspace is required to have extra-invariance properties, that is to be invariant under translations by a prefixed additive subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$. Here small means that our solution

subspace should be generated by the integer translates of small number of generators. We give an expression for the error in terms of the data and construct a Parseval frame for the optimal space.

We also consider the problem of approximating $\mathcal{F}$ from generalized Paley-Wiener spaces of $\mathbb{R}^d$, that are generated by the integer translates of finite number of functions. These spaces can be seen as finitely generated shift invariant spaces that are $\mathbb{R}^d$ invariant.

We show the relations between these spaces and multi-tile sets of $\mathbb{R}^d$, and the connections with recent results on Riesz basis of exponentials.

Key words: shift-invariant spaces; frames; Riesz bases; orthonormal bases; sampling; extra invariance; Paley-Wiener spaces.

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Introducción

(Resumen)

Los *espacios invariantes por traslaciones enteras* (EITE) son subespacios cerrados de $L^2(\mathbb{R}^d)$ que resultan invariantes bajo traslaciones por números enteros. Estos espacios juegan un rol importante en la teoría de aproximación, análisis armónico, teoría de wavelets, muestreo y procesamiento de imágenes [AG01, Gro01, HW96, Mal89]. Además, estos espacios sirven como modelos en varias aplicaciones en el procesamiento de señales e imágenes. Debido a su importancia tanto en la teoría como en las aplicaciones, la estructura de dichos espacios fue analizada en profundidad durante los últimos veinticinco años (ver por ejemplo [dBDVR94, dBDR94, Bow00, Hel64, RS95] y las referencias que en ellos se encuentran).

Cualquier espacio invariante por traslaciones enteras puede generarse por un conjunto de funciones Φ en $L^2(\mathbb{R}^d)$, en el sentido que el espacio coincide con la clausura del espacio generado por las traslaciones enteras de las funciones de Φ. En este caso, se dice que Φ es un conjunto de generadores de V. Cuando existe un conjunto finito Φ de generadores de V, decimos que V está finitamente generado. En ese caso, existe un número entero positivo ℓ, llamado la longitud de V, que se define como el mínimo número de funciones que generan V. Si Φ es un conjunto de generadores de V con ℓ elementos, decimos que Φ es un conjunto minimal de generadores.

Un problema interesante en el contexto de los espacios invariantes por traslaciones enteras finitamente generados es el siguiente: Supongamos que $\Phi = \{\phi_1, \cdots, \phi_m\}$ es un conjunto de generadores del espacio invariante por traslaciones V, ¿cuándo es posible obtener un conjunto minimal de generadores del conjunto de generadores dado Φ? Existen varios ejemplos para los cuales ningún subconjunto de Φ es un conjunto minimal de generadores. Luego, suprimir elementos de un conjunto de generadores dado Φ no parece ser un procedimiento exitoso.

Para dar solución a este problema, Bownik y Kaiblinger en [BK06], mostraron que puede obtenerse un conjunto de generadores minimal de V usando combinaciones lineales de los elementos del conjunto de generadores dado Φ. Más aún, los autores probaron que casi todo conjunto de ℓ funciones que son combinaciones lineales de $\{\phi_1, \cdots, \phi_m\}$ es un conjunto de generadores minimal de V (ver [BK06, Theorem 1]). Es importante recalcar aquí que por combinaciones lineales se entienden solo las que involucran a las funciones $\{\phi_1, \cdots, \phi_m\}$ y no a sus traslaciones.

Como las combinaciones lineales de un número finito de funciones preservan propiedades tales como la suavidad, el decaimiento, tener soporte compacto, ser de banda limitada, etc., una consecuencia interesante del resultado de Bownik y Kaiblinger es que si un conjunto de generadores de V tiene alguna propiedad adicional, existe un conjunto minimal de generadores del espacio que hereda dicha propiedad.

En muchos problemas que involucran a los espacios invariantes por traslaciones enteras, es importante que el sistema de traslaciones $\{T_k\phi_j : k \in \mathbb{Z}^d,\ j = 1, \cdots, m\}$ tenga una determinada estructura funcional como ser una base de Riesz, una base ortonormal o un marco. Luego, resulta interesante saber cuándo puede obtenerse un conjunto minimal de generadores por combinaciones lineales de uno dado que preserve la misma estructura. Más precisamente, supongamos que $\Phi = \{\phi_1, \cdots, \phi_m\}$ genera el espacio invariante por traslaciones enteras V de longitud ℓ, y asumamos además que el nuevo conjunto de generadores $\Psi = \{\psi_1, \cdots, \psi_\ell\}$ de V es producido tomando combinaciones lineales de las funciones de Φ. Es decir, supongamos que $\psi_i = \sum_{j=1}^m a_{ij}\phi_j$ para $i = 1, \cdots, \ell$ y para algunos escalares complejos a_{ij}. Sea $A = \{a_{ij}\}_{i,j} \in \mathbb{C}^{\ell\times m}$ la matriz formada por esos escalares, luego en notación matricial, podemos escribir $\Psi = A\Phi$. Queremos saber qué matrices A preservan la estructura de Φ en Ψ. Más precisamente, estudiamos el siguiente problema: Supongamos que $\{T_k\phi_i : k \in \mathbb{Z}^d,\ i = 1, \cdots, m\}$ es un marco de V. ¿ Cuándo será $\{T_k\psi_i : k \in \mathbb{Z}^d,\ i = 1, \cdots, \ell\}$ también un marco de V?

En Chapter 2, damos respuesta a esta pregunta. Como mencionamos anteriormente, la propiedad de ser un “conjunto de generadores” de un EITE V generalmente se preserva por la acción de una matriz A ([BK06]). Esto no sigue siendo válido para el caso de marcos.

Construímos un ejemplo de un espacio invariante por traslaciones enteras V con un conjunto de generadores Φ tal que sus traslaciones enteras $\{T_k\phi_i : k \in \mathbb{Z}^d,\ i = 1, \cdots, m\}$ forman un marco de V y tal que ninguna matriz A, de tamaño $\ell\times m$ con $\ell < m$, transforma a Φ en un nuevo conjunto de generadores que forme un marco de V.

En esta tesis, damos condiciones sobre la matriz A para que preserve la propiedad de marco. Estas condiciones las establecemos usando la noción de ángulo de Friedrich entre subespacios en espacios de Hilbert. Hay una relación geométrica particular que debe cumplirse entre el núcleo de A y el espacio columna de $G_\Phi(\omega)$, para casi todo ω. Para la demostración de estos resultados usamos otros acerca de los valores singulares de la composición de operadores. Además, damos una condición analítica entre la matriz A y $G_\Phi(\omega)$, equivalente a la anterior. A pesar de que estamos interesados en el caso $\ell = \ell(V)$ (la longitud del EITE V en estudio), la mayoría de nuestros resultados siguen valiendo aún cuando $\ell(V) \leq \ell \leq m$.

Por otro lado, incluimos los casos particulares de bases de Riesz y bases ortonormales, que son ya conocidos.

Luego, estudiamos un problema de aproximación usando espacios invariantes por traslaciones enteras finitamente generados.

Una suposición clásica en la teoría de muestreo es que las señales a ser muestreadas pertenezcan a cierto espacio invariante por traslaciones enteras. Por ejemplo, si el espacio está generado por $\phi(x) = \operatorname{sinc}(x)$, el espacio es el espacio de Paley-Wiener PW de funciones de banda limitada definido de la siguiente manera

$$PW = \left\{ f \in L^2(\mathbb{R}) \colon \operatorname{sop}(\widehat{f}) \subseteq [-1/2, 1/2] \right\}. \tag{0.1}$$

La razón para considerar dicho espacio se debe a un resultado clásico de Whittaker [Whit64] y Shannon [Sha49] que establece que una función $f \in PW$ puede reconstruirse exactamente a partir de sus muestras $\{f(k) \colon k \in \mathbb{Z}\}$ mediante la siguiente fórmula

$$f(k) = \sum_{k \in \mathbb{Z}} f(k) \operatorname{sinc}(x - k). \tag{0.2}$$

Como la función sinc tiene soporte infinito y decaimiento lento, el espacio PW de funciones de banda limitada no es muy aplicable en implementaciones numéricas. Luego, otros espacios son considerados para modelar señales. Por ejemplo, espacios invariantes por traslaciones enteras generados por funciones con soporte compacto o con buena localización en tiempo y frecuencia. Uno de los objetivos en el problema de muestreo en este contexto es buscar condiciones sobre los generadores de un espacio invariante por traslaciones enteras con el objetivo de que cualquier función del espacio pueda reconstruirse a partir de sus valores en una sucesión de puntos, como en el caso de las funciones de banda limitada. Varios autores [AG01, Sun05, Wal92, ZS99] han estudiado este problema en el contexto de los espacios invariantes por traslaciones enteras.

Nosotros consideramos el problema general de muestreo de señales que pertenecen a un conjunto finito de funciones $\mathcal{F} = \{f_1, \dots, f_m\} \subseteq L^2(\mathbb{R}^d)$. Ciertas suposiciones acerca de los espacios invariantes por traslaciones enteras son útiles en las aplicaciones, sin embargo, la elección de un espacio invariante por traslaciones enteras finitamente generado particular no se deduce tipicamente del conjunto de señales en $\mathcal{F}$. Por ejemplo, las hipótesis a priori pueden ser correctas pero el conjunto de datos puede verse perturbado por ruido.

Con el objetivo de modelar conjuntos de señales por espacios invariantes por traslaciones enteras, que puedan ser maleables computacionalmente, es natural buscar espacios invariantes por traslaciones enteras V de longitud ℓ (donde tipicamente ℓ es chico comparado con m) más cercanos a un conjunto de datos dado, en el sentido que V minimiza

$$\sum_{j=1}^{m} \|f_j - P_W f_j\|^2, \tag{0.3}$$

sobre todos los espacios invariantes por traslaciones enteras W que pueden generarse por a lo sumo ℓ generadores. Aquí denotamos con P_W a la proyección ortogonal sobre W.

En [ACHM07], los autores garantizan la existencia de dichos espacios invariantes por traslaciones enteras minimizantes. Dichos problemas de cuadrados mínimos pueden resolverse usando la descomposición en valores singulares (SVD). A pesar de que los espacios invariantes por traslaciones enteras son infinito dimensionales, y que la SVD no puede aplicarse en forma directa, los autores reducen dicho problema de optimización a un conjunto no contable de problemas de cuadrados mínimos finito dimensionales. Para dar solución a estos problemas, los autores consideran una adaptación del teorema de Eckart-Young [EY36, Sch07]. Luego, construyen los generadores para el espacio óptimo usando las soluciones de los problemas reducidos. Además dan una fórmula para el error en dicha aproximación, usando una matriz especial.

Por otro lado, en [ACM08], los autores consideran el caso de espacios múltiples en el caso finito dimensional, encontrando una unión de subespacios de dimensión pequeña que mejor aproximan a un conjunto de datos dado en $\mathbb{R}^d$. Además, encuentran un algoritmo para encontrar dicha unión. Este algoritmo es mejorado en [AACM11], usando técnicas de reducción dimensional.

En [AT11], los autores trabajan en el contexto general de espacios de Hilbert, encontrando condiciones necesarias y suficientes para la existencia de subespacios óptimos pero sin mostrar una manera de construir dichos espacios. En Chapter 4 repasamos estos resultados previos junto con otros, en forma más detallada.

El espacio de Paley-Wiener definido en (0.1) es un ejemplo de espacio invariante por traslaciones enteras en $\mathbb{R}$. El mismo tiene la propiedad de ser no sólo invariante por traslaciones enteras, si no de serlo además bajo cualquier traslación real. Un teorema de Wiener (ver [Hel64]) caracteriza completamente a estos espacios invariantes por traslaciones enteras en $L^2(\mathbb{R}^d)$ que son invariantes bajo cualquier traslación real, como aquellos de la forma

$$\{f \in L^2(\mathbb{R}^d) \colon \operatorname{sop} \widehat{f} \subseteq \Omega\},$$

donde $\Omega \subseteq \mathbb{R}^d$ es medible.

En muchas aplicaciones se busca que los espacios invariantes por traslaciones enteras tengan extra invariancia [CS03, Web00], es decir que los mismos sean además invariantes por traslaciones sobre un particular conjunto $M \subseteq \mathbb{R}^d$. En [ACHKM10] los autores caracterizan aquellos espacios invariantes por traslaciones enteras con extra invariancia en $L^2(\mathbb{R})$. y en [ACP11] los autores generalizan estos resultados, dando una caracterización de estos espacios en $L^2(\mathbb{R}^d)$. En Chapter 3 repasamos dichas caracterizaciones.

A pesar de que los autores en [ACHM07] muestran la existencia de un espacio invariante por traslaciones enteras finitamente generado V más cercano al conjunto de datos $\mathcal{F} \subseteq L^2(\mathbb{R}^d)$, en el sentido de que V minimiza (0.3), los autores no dan respuesta si se le pide a los espacios invariantes por traslaciones enteras que tengan algún tipo de extra invariancia. Luego, en esta tesis consideraremos este problema en un contexto más general, es decir, espacios invariantes por traslaciones enteras con extra invariancia M, donde M es un subgrupo cerrado de $\mathbb{R}^d$ que contiene exactamente a $\mathbb{Z}^d$.

En Chapter 4, primero consideramos el caso de M siendo un subgrupo propio de $\mathbb{R}^d$ conteniendo a $\mathbb{Z}^d$. En este caso probamos que para cualquier conjunto de datos finito $\mathcal{F} \subseteq L^2(\mathbb{R}^d)$, para cualquier subgrupo propio de $\mathbb{R}^d$ que contenga a $\mathbb{Z}^d$ y para cualquier $\ell \in \mathbb{N}$, existe un EITE V de longitud a lo sumo ℓ con extra invariancia M y tal que minimiza (0.3) sobre todos los espacios invariantes por traslaciones enteras de longitud menor o igual que ℓ que son M extra-invariantes. Más aún, construímos un conjunto de generadores para el espacio óptimo y damos una expresión para el error cometido en dicha aproximación.

Luego consideramos el caso $M = \mathbb{R}^d$, es decir, el caso de espacios invariantes por traslaciones enteras con extra invariancia total. Estos son espacios generados por las traslaciones enteras de un número finito de funciones que resultan $\mathbb{R}^d$−invariantes. En este caso, como mencionamos antes, usando un teorema de Wiener [Hel64] sabemos que el espacio es isométrico a $L^2(\Omega)$ para algún conjunto medible $\Omega \subseteq \mathbb{R}^d$. Estos son espacios de Paley-Wiener generalizados. Con el fin de caracterizar la clase de subespacios donde trabajaremos, probamos que, para los espacios de Paley-Wiener generalizados, si las traslaciones enteras de los generadores forman una base de Riesz, entonces el conjunto Ω asociado es un conjunto ℓ “multi-tile” de $\mathbb{R}^d$, es decir

$$\sum_{k\in\mathbb{Z}^d} \chi_\Omega(\omega - k) = \ell, \qquad \text{para casi todo } \omega \in \mathbb{R}^d.$$

Estudiamos el problema de aproximación para estos espacios de Paley-Wiener generalizados. Mostramos cómo construir un conjunto de generadores para el espacio óptimo que además da una base de Riesz de traslaciones en $\mathbb{Z}^d$. Además mostramos una conexión con resultados recientes acerca de la construcción de bases de exponenciales de $L^2(\Omega)$ cuando $\Omega \subseteq \mathbb{R}^d$ es un multi-tile (ver [GL14, Ko15]).

Finalmente consideramos un problema similar pero en el espacio de Hilbert $\ell^2(\mathbb{Z}^d)$. La clase de espacios aproximantes en este caso es elegido convenientemente. Probamos la existencia de un espacio óptimo y mostramos cómo construir un conjunto de generadores para dicho espacio.

El caso discreto resulta interesante en si mismo y es además relevante dada su conexión con el caso continuo en $L^2(\mathbb{R}^d)$, como veremos en el final de Chapter 4.

Esquema de la tesis

En Chapter 1 incluimos algunas definiciones básicas y resultados acerca de marcos y bases de Riesz en espacios de Hilbert que usaremos a lo largo de la tesis. Damos además ciertas propiedades y caracterizaciones de los espacios invariantes por traslaciones enteras. Definimos también el operador Gramiano, la función rango y la noción de fibras en espacios invariantes por traslaciones enteras.

En Chapter 2 estudiamos condiciones para que un conjunto de generadores minimal de un espacio invariante por traslaciones enteras finitamente generado obtenido por combinaciones lineales de uno dado preserve la misma estructura que el original.

Como mencionamos antes, Bownik y Kaiblinger en [BK06] mostraron que, dado un conjunto de generadores Φ de un EITE V, puede obtenerse un conjunto minimal de generadores para V. Además, los autores probaron que casi todo conjunto de ℓ funciones que son combinaciones lineales de los elementos de Φ resulta un conjunto minimal de generadores de V.

En varios problemas que involucran espacios invariantes por traslaciones enteras es importante que el sistema de traslaciones enteras de los elementos del conjunto de generadores Φ tenga una determinada estructura funcional (base de Riesz, base ortonormal, marco, etc). Luego, es interesante saber cuáles de estas propiedades se preservan al tomar combinaciones lineales. Es decir, supongamos que Φ es un conjunto de generadores de V y que el nuevo conjunto de generadores Ψ es obtenido tomando combinaciones lineales de las funciones de Φ, es decir, $\Psi = A\Phi$ en notación matricial, para una matriz A adecuada. En este capítulo damos condiciones para que la propiedad de marco sea preservada por una matriz A. Además tratamos el mismo problema para el caso de bases de Riesz y bases ortonormales.

En Chapter 3 repasamos algunas definiciones acerca de los espacios invariantes por traslaciones enteras con extra invariancia. Incluimos además resultados sobre la caracterización de dichos espacios en $L^2(\mathbb{R}^d)$.

Finalmente, en Chapter 4, dado un conjunto arbitrario de datos $\mathcal{F} = \{f_1, \ldots, f_m\} \subseteq L^2(\mathbb{R}^d)$ probamos la existencia y cómo construir un espacio de dimensión pequeña más cercano a $\mathcal{F}$. El espacio a construir debe ser invariante por traslaciones bajo un subgrupo aditivo fijado de $\mathbb{R}^d$ que contenga a $\mathbb{Z}^d$. Con pequeña nos referimos a que nuestro espacio solución debe estar generado por las traslaciones enteras de un pequeño número de generadores. Además, damos una expresión del error en términos del conjunto de datos dado.

Por otro lado consideramos el problema de aproximar $\mathcal{F}$ por espacios generalizados de Paley-Wiener en $\mathbb{R}^d$. En este caso, la solución resulta invariante bajo cualquier traslación. Mostramos la relación entre estos espacios y los conjuntos "multi-tile" de $\mathbb{R}^d$, y las conexiones con resultados recientes acerca de la construcción de bases de Riesz de exponenciales.

Además consideramos un problema similar pero en el caso discreto de $\ell^2(\mathbb{Z}^d)$ y sobre una cierta clase de subespacios apropiada.

Nuestros resultados pueden formularse en el contexto de grupos localmente compactos y abelianos. Este trabajo está en proceso y no es parte de esta tesis.

Contribuciones originales y publicaciones de la tesis

Las contribuciones originales de la tesis se encuentran en Chapter 2 y en Chapter 4. En Chapter 2 presentamos los resultados acerca de la preservación de la propiedad de marco por combinaciones line ales, ver Section 2.5. En Chapter 4, mostramos los resultados acerca de las aproximaciones con espacios invariantes por traslaciones enteros con extra invariancia, Section 4.3. Estos nuevos resultados han originado las siguientes publicaciones:

- C. Cabrelli, C. Mosquera and V. Paternostro, *Linear combinations of frame generators in systems of translates*, J. Math. Anal. Appl., **413**(2), 2014, 776–788.
- C. Cabrelli and C. Mosquera, *Subspaces with extra invariance nearest to observed data*, Preprint, 2015, `http://arxiv.org/abs/1501.03187`

Introduction

Shift invariant spaces (SISs) are closed subspaces of $L^2(\mathbb{R}^d)$ that are invariant under integer translations. They play an important role in approximation theory, harmonic analysis, numerical analysis, wavelet theory [AG01, Gro01, HW96, Mal89]. They also serve as models in many applications in signal and image processing.

Due to their importance in theory and applications, their structure has been deeply analyzed during the last twenty five years (see for example [dBDVR94, dBDR94, Bow00, Hel64, RS95] and the references therein).

Every shift invariant space V can be generated by a set Φ of functions in $L^2(\mathbb{R}^d)$ in the sense that it is the closure of the space spanned by all integer translations of the functions in Φ. In this case, the set Φ is called a set of generators for V. When there exists a finite set of generators Φ for V, we say that V is finitely generated. In this case, there exists a positive integer ℓ, called the length of V, that is defined as the minimal number of functions that generate V. If the length of V is ℓ, any set of generators with ℓ elements will be called a minimal set of generators.

Concerning finitely generated shift invariant spaces, a particular problem of interest is the following: suppose $\Phi = \{\phi_1, \ldots, \phi_m\}$ is a set of generators for a SIS V. When it is possible to obtain a minimal set of generators from the given generators in Φ? There are many examples with the property that no subset of Φ is a minimal set of generators. So, deleting elements from Φ may not be a successfull procedure.

To give solution to the above problem, Bownik and Kaiblinger in [BK06], showed that a minimal set of generators for V can be obtained from Φ by linear combinations of its elements. Moreover, they proved that almost every set of ℓ functions that are linear combinations of $\{\phi_1, \ldots, \phi_m\}$ is a minimal set of generators for V (see [BK06, Theorem 1]). We want to remark here that the linear combinations of the generators only involve the functions $\{\phi_1, \ldots, \phi_m\}$ and not their translations.

As a consequence of the above Bownik and Kaiblinger's result we have the following interesting observation. Suppose that a shift invariant space V has as a set of generators $\{\phi_1, \ldots, \phi_m\}$, which have some additional special properties, such as smoothness, compact support, bandlimitedness, decay, or membership in certain function spaces. Assume that we have that V can be generated by a fewer generators, that is, the length of V is $\ell < m$. Then a priori such ℓ new generators may not have the special properties of the original generators. Now by this result of Bownik and Kaiblinger we know that the answer is always affirmative. Moreover, we have that if the generators for V have some additional property then there exists a minimal set of generators that inherits this property.

In many problems involving shift invariant spaces, it is important that the system of translates $\{T_k\phi_j : k \in \mathbb{Z}^d,\ j = 1, \ldots, m\}$ bears a particular functional analytic structure such as being an orthonormal basis, a Riesz basis or a frame. Therefore, it is interesting to know when a minimal set of generators obtained by taking linear combinations of the original one has the same structure. More precisely, suppose that $\Phi = \{\phi_1, \ldots, \phi_m\}$ generates a shift invariant space V of length ℓ, and assume that a new set of generators $\Psi = \{\psi_1, \ldots, \psi_\ell\}$ for V is produced by taking linear combinations of the functions in Φ. That is, assume that $\psi_i = \sum_{j=1}^{m} a_{ij}\phi_j$ for $i = 1, \ldots, \ell$ for some complex scalars a_{ij}. If we collect the coefficients in a matrix $A = \{a_{ij}\}_{i,j} \in \mathbb{C}^{\ell\times m}$, then we can write in matrix notation $\Psi = A\Phi$. We would like to know, which matrices A transfer the structure of Φ over Ψ. Precisely, in this thesis we study the following question: If we know that $\{T_k\phi_i : k \in \mathbb{Z}^d,\ i = 1, \ldots, m\}$ is a frame for V, when will $\{T_k\psi_i : k \in \mathbb{Z}^d,\ i = 1, \ldots, \ell\}$ also be a frame for V?

In Chapter 2, we answer this question completely. As we mentioned before, the property of being a "set of generators" for a SIS V is generically preserved by the action of a matrix A ([BK06]). This is not anymore valid for the case of frames. More than that, we were able to construct a surprising example of a shift invariant space V of length ℓ with a set of generators Φ such that their integer translates $\{T_k\phi_i : k \in \mathbb{Z}^d,\ i = 1, \ldots, m\}$ form a frame for V and with the property that no matrix A, of size $\ell\times m$ with $\ell < m$, transform Φ into a new set of generators that form a frame for V.

We give exact conditions in order that the frame property is preserved by a matrix A. These conditions are established using the notion of Friedrichs angle between subspaces of a Hilbert space. The angle between the nullspace of A and the column space of the special matrix $G_\Phi(\omega)$ (the Gramian matrix associated to Φ) has to satisfy a particular geometrical relation for almost all ω. The proof of these conditions use recent results about singular values of the composition of operators. Also, we provide an equivalent analytic condition between A and $G_\Phi(\omega)$ in order that this same result holds.

For completeness we also include the particular case of Riesz bases and orthonormal bases that are known.

Then we studied an approximation problem using finitely generated shift invariant spaces. In many signal and image processing applications, images and signals are assumed to belong to some shift invariant space. For example, if the space is generated by $\phi(x) = \mathrm{sinc}(x)$, the space is the Paley-Wiener space PW of band-limited functions defined as follows

$$PW = \left\{ f \in L^2(\mathbb{R}) : \mathrm{supp}(\widehat{f}) \subseteq [-1/2, 1/2] \right\}. \tag{0.4}$$

The reason for consider this space is a classical result of Whittaker [Whit64] and Shannon [Sha49] which states that a function $f \in PW$ can be recovered exactly from its samples

$\{f(k) \colon k \in \mathbb{Z}\}$ by the following interpolation formula

$$f(k) = \sum_{k \in \mathbb{Z}} f(k) \operatorname{sinc}(x - k). \tag{0.5}$$

This result is fundamental in engineering and digital signal processing because it gives a framework for converting analog signals into sequences of numbers. These sequences can then be processed digitally and converted back to analog signals via (0.5).

Since the sinc function has infinite support and slow decay, the space PW of band-limited functions is often unsuitable for numerical implementations. Then, other spaces are considered as signal models. For example, shift invariant spaces generated by functions with compact support or with good time-frequency localization. One of the goals in the sampling problem in this context is finding conditions on the generators of a shift invariant space in order that every function of this space can be reconstructed from its values in a discrete sequence of samples as in the band-limited case. Several authors [AG01, Sun05, Wal92, ZS99] have studied the sampling problem in the context of shift invariant spaces.

We consider the general problem of sampling signals that belong to a finite set $\mathcal{F} = \{f_1, \ldots, f_m\} \subseteq L^2(\mathbb{R}^d)$. Although certain assumptions over shift invariant spaces are useful in applications, the choice of the particular finitely generated shift invariant space typically is not deduced from the set of signals in $\mathcal{F}$. For example, the a priori hypothesis can be correct but the data can be corrupted by noise. Also, for computational considerations, a shift invariant space of length ℓ could be modelled by a shift invariant space with length n much smaller than ℓ. For example, in Learning Theory, the problem of reducing the number of generators for a subspace of a reproducing kernel Hilbert space is also important for improving the efficiency and sparsity of learning algorithms (see [SZ04]).

In order to model sets of signals by finitely generated SIS in realistic cases, or to model these sets by a computationally tractable SIS, it is natural to search for a finitely generated shift invariant space V with length at most ℓ (where typically ℓ is chosen to be small compared to m) that is nearest to a set of some observed data, in the sense that V minimizes

$$\sum_{j=1}^{m} \|f_j - P_W f_j\|^2, \tag{0.6}$$

over all shift invariant spaces W that can be generated by ℓ or less generators. Here P_W denotes the orthogonal projection over W.

In [ACHM07], the authors guarantee the existence of such shift invariant space minimizer. Least square problems of the form above in finite dimensional spaces can be solved using the singular value decomposition (SVD). Although shift invariant spaces are infinite dimensional and SVD cannot be applied directly, the authors reduce the optimization problem into an uncountable set of finite dimensional least square problems, using the special structure of shift invariant spaces. To give solution for these problems, the authors consider an adaptation of the Eckart-Young's theorem [EY36, Sch07]. Then they construct the generators for an optimal shift invariant space for the original problem patching together the solutions of the reduced problems. Also, the authors give a formula for the error in this approximation using the eigenvalues of a special matrix that involves the functions in the set of data $\mathcal{F}$.

On the other hand, in [ACM08] the authors consider the case of multiple subspaces in the finitely dimensional case, finding a union of low dimensional subspaces that best fits a given set of data in $\mathbb{R}^d$. Besides they provide an algorithm to find it. This algorithm is improved in [AACM11], using dimensional reduction techniques. Further, in [AT11], the authors work in the general context of Hilbert spaces finding necessary and sufficient conditions for the existence of optimal subspaces but they do not provide a way to construct them. We review these result joint with another related works in a more detailed way in Chapter 4.

The Paley-Wiener space defined in (0.4) is an example of a shift invariant space in $\mathbb{R}$. It has the property that is not only invariant under integer translations but it is invariant under every real translation, that is, is a translation invariant SIS. A theorem of Wiener (see [Hel64]) completely characterizes the closed translation invariant subspaces of $L^2(\mathbb{R}^d)$ as being of the form

$$\{f \in L^2(\mathbb{R}^d) \colon \operatorname{supp} \widehat{f} \subseteq \Omega\},$$

where $\Omega \subseteq \mathbb{R}^d$ is measurable.

In several applications it is desirable to have shift invariant spaces that possess extra-invariance [CS03, Web00], that is SIS that are not only invariant under integer translates but are also invariant under some particular set of translations $M \subseteq \mathbb{R}^d$. In [ACHKM10] the authors characterize those invariant spaces with extra-invariance in $L^2(\mathbb{R})$ and in [ACP11] the authors generalizes these results giving a characterization of the extra-invariance for shift invariant spaces in $L^2(\mathbb{R}^d)$. Further, in [ACP10] the authors give a similar characterization for the context of shift invariant spaces on locally compact abelian (LCA) groups.

Although the authors in [ACHM07] show the existence of a finitely generated shift invariant space V nearest to a set of data $\mathcal{F} \subseteq L^2(\mathbb{R}^d)$, in the sense that V minimizes the expression (0.6), it does not provide an answer if we require the shift invariant space to have some type of extra-invariance. For this reason we consider this problem in this more general context, that is, SIS with M extra-invariance, where M is a closed subgroup of $\mathbb{R}^d$ that strictly contains $\mathbb{Z}^d$.

In Chapter 4, we first consider the case when M is a proper subgroup of $\mathbb{R}^d$ that contains $\mathbb{Z}^d$. In this case we prove that for any finite set of data $\mathcal{F} \subseteq L^2(\mathbb{R}^d)$, for any proper subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$ and for any $\ell \in \mathbb{N}$, there exists a shift invariant space V of length at most ℓ with extra-invariance M that minimizes (0.6) over all the SIS of length smaller or equal than ℓ that are M extra-invariant. Moreover, we construct a set of generators for the SIS that produces the solution and give an expression for the exact value of the error between the data $\mathcal{F}$ and the optimal subspace, using the eigenvalues of a special matrix.

It is worth to mention that a previous result in this direction appears in [AKTW12] where the authors consider the problem for principal shift invariant spaces in one variable and assuming that the space has an orthonormal generator (that is, a generator with orthonormal integer translates). These are keys for their proofs. So the techniques of this particular case do not apply to our general case.

Next we consider the case of $M = \mathbb{R}^d$, that is, the case of shift invariant spaces with total extra-invariance. They are spaces generated by integer translates of a finite number of functions that result $\mathbb{R}^d$−invariant. In this case, as we mentioned before, using a theorem of Wiener [Hel64] we know that the space is isometric to $L^2(\Omega)$ for some measurable subset $\Omega \subseteq \mathbb{R}^d$. These are generalized Paley-Wiener spaces. In order to characterize the class where we will consider the approximation problem, we prove for the generalized Paley-Wiener spaces that, if the integer translates of the generators form a Riesz basis then the associated set Ω is an ℓ multi-tile of $\mathbb{R}^d$, that is,

$$\sum_{k\in\mathbb{Z}^d} \chi_\Omega(\omega - k) = \ell, \qquad \text{for a.e. } \omega \in \mathbb{R}^d.$$

We study the approximation problem for these generalized Paley-Wiener spaces. We show how to construct a set of generators for the optimal SIS that also gives a Riesz basis of translates in $\mathbb{Z}^d$. Also, we show a connection with recent results concerning the construction of bases of exponentials for $L^2(\Omega)$ when $\Omega \subseteq \mathbb{R}^d$ is a multi-tile set (see [GL14, Ko15]).

Finally, we consider a similar problem but in the Hilbert space $\ell^2(\mathbb{Z}^d)$. The class of approximating subspaces will be chosen conveniently. We prove the existence of an optimal subspace and show how to construct a set of generators for this space, similarly to the results obtained for $L^2(\mathbb{R}^d)$.

Although this discrete case is interesting by itself, it is also relevant due to its connection with the continuous case in $L^2(\mathbb{R}^d)$, as we will show at the end of Chapter 4.

Thesis outline

In Chapter 1 we include basic definitions and results regarding frames and Riesz bases in Hilbert spaces that we will use throughout this thesis. We give some characterizations and properties of shift invariant spaces. We also define the Gramian operator, the range function and the notion of fibers for shift invariant spaces.

In Chapter 2 we study conditions in order that a minimal set of generators of a finitely generated shift invariant space obtained by taking linear combinations of the original one has the same structure.

As we mentioned above, Bownik and Kaiblinger in [BK06] showed that a minimal set of generators for a shift invariant space V can be obtained from the given set of generators Φ. Furthermore, they proved that almost every set of ℓ functions that are linear combinations of the elements of Φ is a minimal set of generator for V.

In several problems involving shift invariant spaces it is important that the system of integer translates of the elements of a generator set Φ bears a particular structure such as being an orthonormal basis, a Riesz basis or a frame. Then it is interesting to analize when these functional analytic structures are preserved by taking linear combinations of the given set of generators. More precisely, assume that Φ is a set of generators for V and suppose that a new set of generators Ψ is obtained by taking linear combinations of the functions in Φ, that is, in matrix notation, $\Psi = A\Phi$ for an appropriate matrix A. Then we give exact conditions in order that the frame property is preserved by a matrix A. Surprisingly the results are very different to the recently studied case when the property to be a frame is not required. Also, we include the case of Riesz bases and orthonormal bases that are known.

In Chapter 3 we give definitions and properties concerning shift invariant spaces with extra invariance. We review results about the characterization of shift invariant spaces that are not only invariant under integer translations, but are also invariant under a given closed subgroup of $\mathbb{R}^d$.

In Chapter 4, given an arbitrary finite set of data $\mathcal{F} = \{f_1, \dots, f_m\} \subseteq L^2(\mathbb{R}^d)$ we prove the existence and show how to construct a "small shift invariant space" that is the "closest" to the data $\mathcal{F}$ over certain class of closed subspaces of $L^2(\mathbb{R}^d)$. The approximating subspace is required to have extra-invariance properties, that is to be invariant under translations by a prefixed additive subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$. This is important for example in situations where we need to deal with jitter error of the data. Here small means that our solution subspace should be generated by the integer translates of small number of generators. We give an expression for the error in terms of the data and construct a Parseval frame for the optimal space.

We also consider the problem of approximating $\mathcal{F}$ from generalized Paley-Wiener spaces of $\mathbb{R}^d$, that are generated by the integer translates of a finite number of functions. These spaces can be seen as finitely generated shift invariant spaces that are $\mathbb{R}^d$ invariant.

We show the relations between theses spaces and multi-tile sets of $\mathbb{R}^d$, and the connections with recent results on Riesz basis of exponentials.

Further we consider the same approximation problem in a discrete case, that is, in the Hilbert space $\ell^2(\mathbb{Z}^d)$ and over a certain appropriated class of subspaces.

Original contributions and publications from this thesis

The original contributions of this thesis are in Chapter 2 and Chapter 4. In Chapter 2 we present the results about the preservation of the frame property by linear combinations, see Section 2.5. In Chapter 4, we have the results about approximations by shift invariant spaces with extra invariance, Section 4.3. These new results have originated the following publications:

- C. Cabrelli, C. Mosquera and V. Paternostro, *Linear combinations of frame generators in systems of translates*, J. Math. Anal. Appl., **413**(2), 2014, 776–788.

- C. Cabrelli and C. Mosquera, *Subspaces with extra invariance nearest to observed data*, Preprint, 2015, http://arxiv.org/abs/1501.03187

Capítulo 1: Preliminares

(Resumen)

En este capítulo presentamos las notaciones y definiciones que usaremos a lo largo de esta tesis. Algunos de los resultados que presentaremos son ya conocidos.

En la Sección 1.1, comenzamos dando definiciones y algunas propiedades importantes acerca de bases ortonormales, bases de Riesz y marcos en espacios de Hilbert.

En la Sección 1.2, presentamos la noción de operador Gramiano.

En la Sección 1.3 revisamos definiciones y algunas propiedades acerca de los espacios invariantes por traslaciones enteras.

En la Sección 1.4 presentamos las nociones de función rango, fibras y espacio fibrado en el contexto de espacios invariantes por traslaciones enteras.

En la Sección 1.5 caracterizamos cuándo un sistema de traslaciones enteras de un conjunto $\Phi \subseteq L^2(\mathbb{R}^d)$ es una familia de Bessel, un marco o una base de Riesz, en término de las fibras y del operador Gramiano.

1

Preliminaries

In this chapter we introduce the notation and some definitions used throughout this thesis. Some of the results presented below are well known, but we include them here for the sake of completeness.

In Section 1.1, we begin by giving definitions and some relevant properties about orthonormal bases, Riesz bases and frames in Hilbert spaces. In Section 1.2 we introduce the notion of Gramian operator. In Section 1.3 we present the definition and some properties of shift invariant spaces. In Section 1.4 we give the definitions of range function, fibers and fiber spaces in the context of shift invariant spaces. In Section 1.5 we characterize the system of integer translates of a set $\Phi \subseteq L^2(\mathbb{R}^d)$ as being a Bessel family, a frame, or a Riesz basis in terms of fibers and in terms of its Gramian operator.

1.1 Bases and frames in Hilbert spaces

In this section we collect some definitions about bases and frames in Hilbert spaces. For more details we refer the reader to [Chr03, Hei11] and the references therein.

Definition 1.1.1. Let H be a separable Hilbert space. A sequence of vectors $\{f_j\}_{j\in J}$ belonging to H is a (Schauder) basis for H if, for each $f \in H$, there exist unique scalar coefficients $\{c_j(f)\}_{j\in J}$ such that

$$f = \sum_{j\in J} c_j(f) f_j. \tag{1.1}$$

Sometimes we refer to (1.1) as the expansion of f in the basis $\{f_j\}_{j\in J}$ and this equation merely means that the series $f = \sum_{j\in J} c_j(f) f_j$ converges with respect to the chosen order of the elements.

Besides the existence of an expansion of each $f \in H$, the above definition asks for uniqueness. For example, if the basis $\{f_j\}_{j\in J}$ is orthonormal (that is, $\langle f_i, f_j\rangle = \delta_{i,j}$ for all $i, j \in J$), there is a unique representation of each $f \in H$ in the basis, being $c_j(f) = \langle f, f_j\rangle$ for all $j \in J$.

Theorem 1.1.2. *Let H be a separable Hilbert space and $\{f_j\}_{j\in J}$ be an orthonormal system in H. The following are equivalent:*

(i) $\{f_j\}_{j\in J}$ *is an orthonormal basis.*

(ii) $f = \sum_{j\in J}\langle f, f_j\rangle f_j$, *for all* $f \in H$.

(iii) $\sum_{j\in J} |\langle f, f_j\rangle|^2 = \|f\|^2$, *for all* $f \in H$ *(Parseval's identity).*

(iv) $\overline{span}\{f_j\}_{j\in J} = H$.

(v) If $\langle f, f_j\rangle = 0$, *for all* $j \in J$, *then* $f = 0$.

Theorem 1.1.3. *Every separable Hilbert space H has an orthonormal basis.*

The simplest case is the following.

Example 1.1.4. Let f_j be the sequence in $\ell^2(J)$ whose $j-$th entry is 1, and all other entries are zero. Then $\{f_j\}_{j\in J}$ is an orthonormal basis for $\ell^2(J)$ and it is called the canonical orthonormal basis.

Now we will introduce the definition of Riesz bases.

Definition 1.1.5. Let H be a separable Hilbert space and $\{f_j\}_{j\in J}$ be a sequence in H. The sequence $\{f_j\}_{j\in J}$ is said to be a Riesz basis for H if it is complete in H and if there exist $0 < \alpha \leq \beta$ such that

$$\alpha \sum_{j\in J} |c_j|^2 \leq \|\sum_{j\in J} c_j f_j\|^2 \leq \beta \sum_{j\in J} |c_j|^2, \quad \forall \{c_j\}_{j\in J} \in \ell^2(J). \tag{1.2}$$

The constants α and β are called Riesz bounds.

A sequence $\{f_j\}_{j\in J}$ satisfying (1.2) for all sequences $\{c_j\}_{j\in J}$ is called a Riesz sequence. Then, a Riesz sequence $\{f_j\}_{j\in J}$ is a Riesz basis for $\overline{\text{span}}\{f_j\}_{j\in J}$, which might just be a subspace of H.

The following theorem establishes a relationship between Riesz bases, bases and orthonormal bases. As a consequence of this result we can obtain that all orthonormal bases are Riesz bases.

Theorem 1.1.6. *Let $\{f_j\}_{j\in J}$ be a sequence in H. The following statements are equivalent.*

(i) $\{f_j\}_{j\in J}$ *is a Riesz basis for* H.

(ii) $\{f_j\}_{j\in J}$ *is a basis for* H, *and*

$$\sum_{j\in J} c_j f_j \text{ converges if and only if } \sum_{j\in J} |c_j|^2 < \infty.$$

(iii) There exist a bounded linear operator $T\colon H \to H$ and an orthonormal basis $\{x_j\}_{j\in J}$ of H such that $T(x_j) = f_j$ for all $j \in J$.

The following proposition gives a condition for a Riesz basis to be an orthonormal basis.

Proposition 1.1.7. *Assume that $\overline{span}\{f_j\}_{j\in J} = H$ and that*

$$\|\sum_{j\in J} c_j f_j\|^2 = \sum_{j\in J} |c_j|^2$$

for all finite scalar sequences $\{c_j\}_{j\in J}$. Then $\{f_j\}_{j\in J}$ is an orthonormal basis for H.

As a generalization of the basis concept, we will give the definition of frame.

The main feature of a basis $\{f_j\}_{j\in J}$ in a Hilbert space H is that every $f \in H$ can be represented as an (infinite) linear combination of the elements f_j in the basis. The coefficients $c_j(f)$ in (1.1) are unique. We now introduce the concept of frames. A frame is also a sequence of elements $\{f_j\}_{j\in J}$ in H, which allows every $f \in H$ to be written as in (1.1). However, the corresponding coefficients are not necessarily unique. Although the lack of uniqueness, it is usually enough to know the existence of some usable coefficients, together with a recipe for finding them. For example, in 1986, Daubechies, Grossmann and Meyer [DGM86] observed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$ which are very similar to the expansions using orthonormal bases.

Definition 1.1.8. The sequence $\{f_j\}_{j\in J}$ is said to be a frame for H if there exist $0 < \alpha \le \beta$ such that

$$\alpha\|f\|^2 \le \sum_{j\in J} |\langle f, f_j\rangle|^2 \le \beta\|f\|^2, \quad \forall f \in H. \tag{1.3}$$

The constants α and β are called frame bounds. They are not unique. The optimal upper frame bound is the infimum over all upper frame bounds, and the optimal lower frame bound is the supremum over all lower frame bounds.

When only the right hand side inequality in (1.3) is satisfied, we say that $\{f_j\}_{j\in J}$ is a Bessel sequence with Bessel bound β.

We say that a frame is tight if we can choose $\alpha = \beta$ as frame bounds, and it is a Parseval frame if $\alpha = \beta = 1$.

If a frame ceases to be a frame when an arbitrary element is removed, it is called an exact frame.

We say that $\{f_j\}_{j\in J}$ is a frame sequence if it is a frame for $\overline{\text{span}}\{f_j\}_{j\in J}$.

Example 1.1.9. Let $\{e_j\}_{j\in J}$ be an orthonormal basis for a separable Hilbert space H.

(a) By Plancherel's identity, $\{e_j\}_{j\in J}$ is a tight frame with $\alpha = \beta = 1$. Moreover, $\{e_j\}_{j\in J}$ is an exact frame since if we delete any element $e_{j'}$, then $\sum_{j\neq j'} |\langle e_j, e_{j'}\rangle|^2 = 0$, and therefore $\{e_j\}_{j\neq j'}$ cannot be a frame.

(b) $\{e_1, e_1, e_2, e_2, \dots\}$ is a tight inexact frame with bounds $\alpha = \beta = 2$, but it is not a basis, although it does contain an orthonormal basis. Similarly, if $\{f_j\}_{j\in J}$ is another orthonormal basis for H then $\{e_j\}_{j\in J} \cup \{f_j\}_{j\in J}$ is a tight inexact frame.

(c) $\{e_1, e_2/2, e_3/3, \dots\}$ is a complete orthogonal sequence and it is a basis for H, but it does not possess a lower frame bound and hence is not a frame.

Usually, in this thesis we will work with the above definitions in the following context. We will assume that H is a closed subspace of $L^2(\mathbb{R}^d)$ and the sequence $\{f_k\}_{k\in\mathbb{Z}} \subseteq H$ consists of integer translates of a fixed finite set of functions $\Phi \subseteq L^2(\mathbb{R}^d)$.

As we mention above, frames can be viewed as a generalization of Riesz bases.

Theorem 1.1.10. *A Riesz basis $\{f_j\}_{j\in J}$ for H is a frame for H, and the Riesz basis bounds coincide with the frames bounds.*

A frame $\{f_j\}_{j\in J}$ which is not a basis is said to be overcomplete, that is, there exist coefficients $\{c_j\}_{j\in J} \in \ell^2(J) \setminus \{0\}$ for which $\sum_{j\in J} c_j f_j = 0$. The next theorem gives necessary and sufficient conditions in order for a frame to be a Riesz basis.

Theorem 1.1.11. *Let $\{f_j\}_{j\in J}$ be a sequence in H. Then ,$\{f_j\}_{j\in J}$ is a Riesz basis if and only if it is an exact frame for H.*

Now we define some operators that we will use later.

Definition 1.1.12. Let $\{f_j\}_{j\in J}$ be a Bessel sequence in a Hilbert space H.

(a) The analysis operator for $\{f_j\}_{j\in J}$ is the continuous mapping $U\colon H \to \ell^2(J)$ defined by
$$Uf = (\langle f, f_j\rangle)_{j\in J} \text{ for } f \in H.$$

(b) The synthesis operator or the pre-frame operator of $\{f_j\}_{j\in J}$ is the continuous mapping $U^*\colon \ell^2(J) \to H$ defined by
$$U^*(\{c_j\}_{j\in J}) = \sum_{j\in J} c_j f_j \text{ for } \{c_j\}_{j\in J} \in \ell^2(J).$$

(c) The frame operator for $\{f_j\}_{j\in J}$ is the continuous mapping $S : H \to H$ defined by
$$Sf = U^*Uf = \sum_{j\in J} \langle f, f_j\rangle f_j, \quad f \in H.$$

Since $\{f_j\}_{j\in J}$ is a Bessel sequence, the synthesis operator is bounded from $\ell^2(J)$ to H.

Using the synthesis operator we can characterize frames as the following theorem shows.

Theorem 1.1.13. *A Bessel sequence $\{f_j\}_{j\in J}$ in H is a frame for H if and only if its synthesis operator $U^*\colon \ell^2(J) \to H$ is a well-defined surjective operator from $\ell^2(J)$ into H.*

For an arbitrary sequence $\{f_j\}_{j\in J}$ in a Hilbert space, we observe that $\overline{\text{span}}\{f_j\}_{j\in J}$ is itself a Hilbert space, where span $\{f_j\}_{j\in J}$ denotes the vector space consisting of all finite linear combinations of vectors f_j. Therefore, the above theorem leads to a statement about frame sequences.

Corollary 1.1.14. *A sequence $\{f_j\}_{j\in J}$ in H is a frame sequence if and only if its synthesis operator $U^*\colon \ell^2(J) \to H$ is a well-defined operator on $\ell^2(J)$ and has closed range.*

Remark 1.1.15. As a consequence of the above proposition we have that if $\{f_j\}_{j\in J}$ is a frame for $S = \overline{\text{span}}\{f_j\}_{j\in J}$, then

$$S = \left\{ \sum_{j\in J} c_j f_j \colon \{c_j\}_{j\in J} \in \ell^2(J) \right\}.$$

In the next theorem we will present properties about the frame operator that allow us characterize a frame.

Theorem 1.1.16. *Given a sequence $\{f_j\}_{j\in J}$ in a Hilbert space H, the following statements are equivalent.*

(a) $\{f_j\}_{j\in J}$ *is a frame with bounds α and β.*

(b) The frame operator S is bounded, invertible, self-adjoint, positive, and satisfies

$$\alpha I \leq S \leq \beta I.$$

(c) $\{S^{-1} f_j\}_{j\in J}$ *is a frame for H, with frame bounds $0 < \beta^{-1} < \alpha^{-1}$ and frame operator S^{-1}.*

(d) The following series converges unconditionally for each $f \in H$:

$$f = \sum_{j\in J} \langle f, S^{-1} f_j \rangle f_j = \sum_{j\in J} \langle f, f_j \rangle S^{-1} f_j. \tag{1.4}$$

(e) If the frame is tight, then $S = \alpha I$, $S^{-1} = \alpha^{-1} I$ and

$$f = \alpha^{-1} \sum_{j\in J} \langle f, f_j \rangle f_j \quad \forall f \in H.$$

Given $\{f_j\}_{j\in J}$ be a frame in H, a Bessel sequence $\{g_j\}_{j\in J}$ is said to be a dual frame of $\{f_j\}_{j\in J}$ if

$$f = \sum_{j\in J} \langle f, g_j \rangle f_j, \quad \forall f \in H.$$

The frame $\{S^{-1} f_j\}_{j\in J}$ is a dual frame of $\{f_j\}_{j\in J}$, called the canonical dual frame of $\{f_j\}_{j\in J}$. If $\{f_j\}_{j\in J}$ is a Riesz basis, the unique dual is $\{S^{-1} f_j\}_{j\in J}$. When the frame is overcomplete, that is, is not a Riesz basis, there exist dual frames which are different from the canonical dual.

The frame decomposition is the most important frame result. That is a natural reason to view a frame as a some kind of generalized basis. As (1.4) shows, if $\{f_j\}_{j\in J}$ is a frame for H, then every element in H has a representation as an infinite linear combination of the frame elements. Then, all the information about a given $f \in H$ is contained in the sequence $\{\langle f, S^{-1}f_j\rangle\}_{j\in J}$. The numbers $\langle f, S^{-1}f_j\rangle$ are called frame coefficients.

1.2 Gramian operator

In this section we introduce the notion of Gramian associated with a Besel sequence in a Hilbert space H.

Definition 1.2.1. Let $\{f_j\}_{j\in J}$ be a Bessel sequence in a Hilbert space H and let $U\colon H \to \ell^2(J)$ be the analysis operator of $\{f_j\}_{j\in J}$. The Gramian of $\{f_j\}_{j\in J}$ is defined by

$$G\colon \ell^2(J) \to \ell^2(J), \quad G := UU^*. \tag{1.5}$$

We can identify the Gramian with its matrix representation

$$(G)_{i,j} = \langle f_i, f_j\rangle, \quad \forall i, j \in J.$$

Also, we can define the dual Gramian of $\{f_j\}_{j\in J}$ by

$$\widetilde{G}\colon H \to H, \quad \widetilde{G} := U^*U.$$

We observe that one can consider the Gramian associated to any sequence $\{f_j\}_{j\in J}$ in H, but if we want to define a bounded operator on $\ell^2(J)$ we can not avoid the Bessel condition as the following lemma shows.

Lemma 1.2.2. *For a sequence $\{f_j\}_{j\in J}$ in H the following are equivalent:*

(a) $\{f_j\}_{j\in J}$ is a Bessel sequence with bound β.

(b) The Gramian associated to $\{f_j\}_{j\in J}$ defines a bounded operator on $\ell^2(J)$, with norm at most β.

Given a Hilbert space H and a bounded linear operator $T\colon H \to H$, we will denote by $\Sigma(T)$ the spectrum of T, that is,

$$\Sigma(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \text{ is not invertible}\},$$

where I denotes the identity operator of H.

We can relate the notion of frame sequence with the spectrum of its Gramian as follows.

Theorem 1.2.3. *Let $\{f_j\}_{j\in J} \subseteq H$ be a Bessel sequence and G be its Gramian operator. Then $\{f_j\}_{j\in J}$ is a frame sequence with constants α and β if and only if*

$$\Sigma(G) \subseteq \{0\} \cup [\alpha, \beta].$$

1.3 Shift invariant spaces

In this section, we will give some definitions and properties of shift invariant spaces. For more details we refer to [Bow00, dBDR94, dBDVR94, Hel64, RS95] and the references therein.

Definition 1.3.1. A closed subspace $V \subseteq L^2(\mathbb{R}^d)$ is shift invariant if $f \in V$ implies that $T_k f \in V$ for any $k \in \mathbb{Z}^d$. Here $T_y f(x) = f(x - y)$ is the translation by the vector $y \in \mathbb{R}^d$.

For any subset $\Phi \subseteq L^2(\mathbb{R}^d)$ we define

$$S(\Phi) = \overline{\text{span}}\{T_k\phi \colon \phi \in \Phi, k \in \mathbb{Z}^d\} \quad \text{and} \quad E(\Phi) = \{T_k\phi \colon \phi \in \Phi, k \in \mathbb{Z}^d\}.$$

We call $S(\Phi)$ the shift invariant space (SIS) generated by Φ. We call Φ a set of generators for $V(\Phi)$. If $\Phi = \{\phi\}$ we will also write $S(\phi) = S(\{\phi\})$. If $V = S(\Phi)$ for some finite set Φ we say that V is a finitely generated SIS, and a principal SIS if V can be generated by the integer translates of a single function.

For a finitely generated shift invariant space $V \subseteq L^2(\mathbb{R}^d)$ we define the length of V as

$$\ell(V) = \min\{n \in \mathbb{N} \colon \exists\ \phi_1, \dots, \phi_n \in V \text{ with } V = S(\phi_1, \dots, \phi_n)\}.$$

We say that Φ is a minimal set of generators for V if $V = S(\Phi)$ and Φ has exactly $\ell(V)$ elements.

A shift invariant space always has a Parseval frame of translates as the following theorem shows.

Theorem 1.3.2. *Given $V \subseteq L^2(\mathbb{R}^d)$ a SIS, there exists a subset $\Phi = \{\phi_j\}_{j\in J} \subseteq V$ such that $E(\Phi)$ is a Parseval frame for V. If V is finitely generated, the cardinal of J can be chosen to be the length of V.*

We should mention here that there are shift invariant spaces which have no Riesz bases of translates. That is one of the reasons for which it is important to consider frames instead of Riesz bases for studying the structure of shift invariant spaces.

1.4 Fiber space for shift invariant spaces

Helson in [Hel64] introduced range functions and used this notion to completely characterize shift invariant spaces. Later on, several authors have used this framework to describe and characterize frames and bases of these spaces. See for example [dBDVR94, dBDVR94, RS95, Bow00, CP10]. We are not going to review the complete theory of Helson. We will only mention the required definitions and properties that we need in this thesis. We refer to [Bow00] for a clear and complete description.

Let $\mathcal{U} \subseteq \mathbb{R}^d$ be a measurable set of representatives of the quotient $\mathbb{R}^d/\mathbb{Z}^d$ (we usually will take $\mathcal{U} = [-1/2, 1/2]^d$ or $\mathcal{U} = [0, 1]^d$). The Hilbert space of square integrable vector functions $L^2(\mathcal{U}, \ell^2(\mathbb{Z}^d))$ consists of all vectors valued measurable functions $F\colon \mathcal{U} \to \ell^2(\mathbb{Z}^d)$ with the norm

$$||F|| = \left(\int_{\mathcal{U}} ||F(x)||^2_{\ell^2(\mathbb{Z}^d)}\, dx\right)^{1/2} < \infty.$$

Proposition 1.4.1. *The function $\tau\colon L^2(\mathbb{R}^d) \to L^2(\mathcal{U}, \ell^2(\mathbb{Z}^d))$ defined for $f \in L^2(\mathbb{R}^d)$ by*

$$\tau f\colon \mathcal{U} \to \ell^2(\mathbb{Z}^d), \quad \tau f(\omega) \equiv \{\widehat{f}(\omega + k)\}_{k\in\mathbb{Z}^d},$$

is an isometric isomorphism between $L^2(\mathbb{R}^d)$ and $L^2(\mathcal{U}, \ell^2(\mathbb{Z}^d))$.

The sequence $\tau f(\omega) \equiv \{\widehat{f}(\omega + k)\}_{k\in\mathbb{Z}^d}$ is called the fiber of f at ω.

Definition 1.4.2. A range function is a mapping

$$J\colon \mathcal{U} \to \{\text{closed subspaces of } \ell^2(\mathbb{Z}^d)\}.$$

J is measurable if the associated orthogonal projections $P(\omega)\colon \ell^2(\mathbb{Z}^d) \to J(\omega)$ are weakly operators measurable. Using that in a Hilbert space measurability is equivalent to weak measurability, the measurability of J es equivalent to $\omega \mapsto P(\omega)a$ being vector measurable for each $a \in \ell^2(\mathbb{Z}^d)$.

The next proposition gives a characterization of shift invariant spaces in terms of the range function.

Proposition 1.4.3. *A closed subspace $V \subseteq L^2(\mathbb{R}^d)$ is shift invariant if and only if*

$$V = \{f \in L^2(\mathbb{R}^d)\colon \tau f(\omega) \in J_V(\omega) \text{ for a.e. } \omega \in \mathcal{U}\},$$

where J_V is a measurable range function. The correspondence between V and J_V is one-to-one under the convention that the range functions are identified if they are equal a.e. Furthermore, if $V = S(\Phi)$ for some countable $\Phi \subseteq L^2(\mathbb{R}^d)$, then

$$J_V(\omega) = \overline{span}\{\tau\varphi(\omega)\colon \varphi \in \Phi\}, \text{ for a.e. } \omega \in \mathcal{U}.$$

The subspace $J_V(\omega)$ is called the fiber space of V at ω.

We observe that if $V = S(\varphi_1, \ldots, \varphi_m) \subseteq L^2(\mathbb{R}^d)$ then

$$J_V(\omega) = \text{span}\,\{\tau\varphi_1(\omega), \ldots, \tau\varphi_m(\omega)\}.$$

Then the fiber spaces $J_V(\omega)$ are all finite dimensional subspaces of $\ell^2(\mathbb{Z}^d)$.

We have the following useful properties.

Lemma 1.4.4. *If $f \in L^2(\mathbb{R}^d)$, then*

(i) *the sequence* $\{\tau f(\omega)\}_k = \{\widehat{f}(\omega+k)\}_k$ *is a well-defined sequence in* $\ell^2(\mathbb{Z}^d)$ *a.e.* $\omega \in \mathbb{R}^d$.

(ii) $\|\tau f(\omega)\|_{\ell^2}$ *is a measurable function of* ω *and*

$$\|f\|^2 = \|\widehat{f}\|^2 = \int_{\mathcal{U}} \|\tau f(\omega)\|^2_{\ell^2}\, d\omega.$$

Lemma 1.4.5. *Let V be a finitely generated SIS in* $L^2(\mathbb{R}^d)$*. Then we have*

(i) $J_V(\omega)$ *is a closed subspace of* $\ell^2(\mathbb{Z}^d)$ *for a.e.* $\omega \in \mathcal{U}$.

(ii) $V = \{f \in L^2(\mathbb{R}^d) \colon \tau f(\omega) \in J_V(\omega)$ *for a.e.* $\omega \in \mathcal{U}\}$.

(iii) *For each* $f \in L^2(\mathbb{R}^d)$ *we have that* $\|\tau(P_V f)(\omega)\|_{\ell^2}$ *is a measurable function of the variable* ω *and*

$$\tau(P_V f)(\omega) = P_{J_V(\omega)}(\tau f(\omega)).$$

(iv) *Let* $\varphi_1, \ldots, \varphi_m \in L^2(\mathbb{R}^d)$*. We have that* $\{\varphi_1, \ldots, \varphi_m\}$ *is a set of generators of V, if and only if* $\{\tau\varphi_1(\omega), \ldots, \tau\varphi_m(\omega)\}$ *span* $J_V(\omega)$ *for a.e.* $\omega \in \mathcal{U}$.

Now we introduce the concept of dimension function of a shift invariant space.

Definition 1.4.6. The dimension function of V is a mapping given by

$$\dim_V \colon \mathcal{U} \to \mathbb{N} \cup \{0, \infty\}, \quad \dim_V(\omega) = \dim J_V(\omega),$$

where J_V is the range function associated with V.

We have the following property concerning the length of a finitely generated SIS.

Proposition 1.4.7. *Let* $V \subseteq L^2(\mathbb{R}^d)$ *be a finitely generated SIS. Then*

$$\textit{length}\ (V) = \textit{ess sup}\{\dim_V(\omega) \colon \omega \in \mathcal{U}\}.$$

1.5 Riesz bases and frames for shift invariant spaces

Next results give a characterization of system of translates $E(\Phi)$ as being a Bessel sequence, a frame, or a Riesz basis using the notion of fibers.

Theorem 1.5.1. *Suppose* $\Phi \subseteq L^2(\mathbb{R}^d)$ *is countable.*

(a) $E(\Phi)$ *is a Bessel sequence in* $L^2(\mathbb{R}^d)$ *with bound* β *if and only if* $\{\tau\varphi(\omega) \colon \varphi \in \Phi\}$ *is a Bessel sequence in* $\ell^2(\mathbb{Z}^d)$ *with bound* β *for a.e.* $\omega \in \mathcal{U}$.

(b) $E(\Phi)$ *is a frame for* $V = S(\Phi)$ *with bounds* α *and* β *if and only if* $\{\tau\varphi(\omega) \colon \varphi \in \Phi\}$ *is a frame for* $J_V(\omega)$ *with bounds* α *and* β *for a.e.* $\omega \in \mathcal{U}$.

(c) *$E(\Phi)$ is a Riesz basis for $V = S(\Phi)$ with bounds α and β if and only if $\{\tau\varphi(\omega) : \varphi \in \Phi\}$ is a Riesz basis for $J_V(\omega)$ with bounds α and β for a.e. $\omega \in \mathcal{U}$.*

We observe that the above theorem allows us to reduce the problem of checking whether $E(\Phi)$ is a frame or a Riesz basis in a big subspace of $L^2(\mathbb{R}^d)$ to analyzing the fibers in smaller subspaces of $\ell^2(\mathbb{Z}^d)$. Then, although the finitely generated SISs are infinite dimensional subspaces, some important properties can be translated into properties on the fibers of the spanning sets, working with finite dimensional subspaces of $\ell^2(\mathbb{Z}^d)$.

Now we define de Gramian operator for shift invariant spaces.

Definition 1.5.2. Let $\Phi = \{\varphi_j\}_{j\in J}$ be a countable collection of functions in $L^2(\mathbb{R}^d)$ such that $E(\Phi)$ is a Bessel sequence. The Gramian G_Φ of Φ at $\omega \in \mathcal{U}$ is $G_\Phi(\omega) : \ell^2(J) \to \ell^2(J)$,

$$[G_\Phi(\omega)]_{ij} = \langle \tau\varphi_j(\omega), \tau\varphi_i(\omega)\rangle_{\ell^2(\mathbb{Z}^d)} = \sum_{k\in\mathbb{Z}^d} \widehat{\varphi}_i(\omega+k)\overline{\widehat{\varphi}_j(\omega+k)}, \quad \forall i, j \in J. \tag{1.6}$$

The dual Gramian $\widetilde{G}_\Phi$ of Φ at $\omega \in \mathcal{U}$ is $\widetilde{G}_\Phi(\omega) : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$,

$$[\widetilde{G}_\Phi(\omega)]_{kl} = \sum_{j\in J} \widehat{\varphi}_j(\omega+k)\overline{\widehat{\varphi}_j(\omega+l)}, \quad \forall k, l \in \mathbb{Z}^d. \tag{1.7}$$

We observe that the above definition agrees with the Gramian operator associated to the Bessel sequence $\tau\Phi(\omega) = \{\tau\varphi_j(\omega)\}_{j\in J}$ in $\ell^2(\mathbb{Z}^d)$, with the notation of Definition 1.5.

The Gramian of Φ is determined a.e. by its values at $\omega \in \mathcal{U}$ and satisfies $G_\Phi(\omega)^* = G_\Phi(\omega)$ for a.e. $\omega \in \mathcal{U}$.

Now, using the notion of Gramian operator, we obtain the following characterization of frames and Riesz bases. This result can be proved using Theorem 1.5.1. For more details see [Bow00].

Theorem 1.5.3. *Let $\Phi = \{\varphi_j\}_{j\in J} \subseteq L^2(\mathbb{R}^d)$. Then:*

(1) *The following statements are equivalent:*

  (a) *$E(\Phi)$ is a Bessel sequence with bound β.*

  (b) *$\operatorname{ess\,sup}_{\omega\in\mathcal{U}} \|G_\Phi(\omega)\|_{op} \leq \beta$.*

(2) *The following statements are equivalent:*

  (a) *$E(\Phi)$ is a frame for $S(\Phi)$ with bounds α and β.*

  (b) *For almost all $\omega \in \mathcal{U}$,*

$$\alpha\langle G_\Phi(\omega)a, a\rangle \leq \langle G_\Phi^2(\omega)a, a\rangle \leq \beta\langle G_\Phi(\omega)a, a\rangle, \quad \forall a \in \ell^2(J).$$

  (c) *For almost all $\omega \in \mathcal{U}$, $\Sigma(G_\Phi(\omega)) \subseteq [\alpha, \beta] \cup \{0\}$.*

(3) The following statements are equivalent:

(a) $E(\Phi)$ *is a Riesz basis for* $S(\Phi)$ *with bounds* α *and* β*.*

(b) For almost all $\omega \in \mathcal{U}$*,*

$$\alpha \|a\|^2 \leq \langle G_\Phi(\omega)a, a\rangle \leq \beta\|a\|^2, \quad \forall a \in \ell^2(J).$$

(c) For almost all $\omega \in \mathcal{U}$*,* $\Sigma(G_\Phi(\omega)) \subseteq [\alpha, \beta]$*.*

(d) For almost all $\omega \in \mathcal{U}$*,* $\|G_\Phi(\omega)\|_{op} \leq \beta$ *and* $\|(G_\Phi(\omega))^{-1}\|_{op} \leq \frac{1}{\alpha}$*.*

For a finitely generated SIS V, the length of V can be expressed in terms of the Gramian as follows (see [Bow00, dBDVR94, TW14])

$$\ell(V) = \underset{\omega \in \mathcal{U}}{ess\,sup}\ \mathrm{rk}(G_\Phi(\omega)) \tag{1.8}$$

where $\mathrm{rk}(B)$ denotes the rank of a matrix B and Φ is a generator set for V.

Capítulo 2:
Combinaciones lineales de generadores en sistemas de traslaciones

(Resumen)

Sea V un espacio invariante por traslaciones enteras generado por las traslaciones enteras de $\Phi = \{\phi_1, \ldots, \phi_m\}$. Es decir, V es la clausura del conjunto de todas las funciones φ de la forma

$$\varphi(x) = \sum_{j=1}^{m} \sum_{k \in \mathbb{Z}^d} c_{j,k}\, \phi_j(x-k), \qquad x \in \mathbb{R}^d, \tag{1.9}$$

donde solo un número finito de los coeficientes $c_{j,k} \in \mathbb{C}$ son no nulos.

Resulta interesante saber cuándo es posible obtener un conjunto minimal de generadores del conjunto de generadores dado Φ. Es decir, cúando existen $\{\psi_1, \ldots, \psi_\ell\} \subseteq L^2(\mathbb{R}^d)$ con $\ell = \operatorname{long}(V) \leq m$ tales que

$$S(\phi_1, \ldots, \phi_m) = S(\psi_1, \ldots, \psi_\ell).$$

Hay varios ejemplos en los cuales ningún subconjunto de Φ resulta un conjunto de generadores minimal. Luego, suprimir elementos de Φ no parece ser un procedimiento exitoso.

En busca de dar solución a esta pregunta, Bownik y Kaiblinger en [BK06], mostraron que puede obtenerse de Φ un conjunto minimal de generadores para V usando *combinaciones lineales* de sus elementos. Más aún, los autores prueban que casi todo conjunto de ℓ funciones que son combinaciones lineales de $\{\phi_1, \ldots, \phi_m\}$ resulta un conjunto minimal de generadores de V (Ver Section 2.3). Es importante recalcar aquí que las combinaciones lineales sólo involucran a las funciones $\{\phi_1, \ldots, \phi_m\}$ y no a sus traslaciones.

En muchos problemas que involucran a los espacios invariantes por traslaciones enteras es importante que el sistema de traslaciones $\{T_k\phi_j : k \in \mathbb{Z}^d, \ j = 1, \ldots, m\}$ tenga una determinada estructura analítica como ser una base ortonormal, una base de Riesz o un marco. Luego, resulta interesante saber cuándo un conjunto minimal de generadores obtenido por combinaciones lineales de los elementos de uno dado preserva la misma estructura. Más precisamente, supongamos que $\Phi = \{\phi_1, \ldots, \phi_m\}$ genera un espacio invariante por traslaciones enteras V de longitud ℓ, y supongamos que el nuevo conjunto de generadores $\Psi = \{\psi_1, \ldots, \psi_\ell\}$ para V se forma tomando combinaciones lineales de las funciones de Φ. Es decir, supongamos que $\psi_i = \sum_{j=1}^{m} a_{ij}\phi_j$ para $i = 1, \ldots, \ell$ y para ciertos escalares complejos a_{ij}. Si armamos con los coeficientes la siguiente matriz $A = \{a_{ij}\}_{i,j} \in \mathbb{C}^{\ell\times m}$, podemos escribir matricialmente $\Psi = A\Phi$. Luego, queremos saber qué matrices A preservan la estructura que tenía Φ en Ψ. Concretamente, estudiamos la siguiente pregunta: Supongamos que $\{T_k\phi_i \colon k \in \mathbb{Z}^d,\ i = 1, \ldots, m\}$ es un marco de V. ¿Cuándo será $\{T_k\psi_i \colon k \in \mathbb{Z}^d,\ i = 1, \ldots, \ell\}$ también un marco de V?

En la Section 2.5 damos respuesta a la anterior pregunta.

Como mencionamos antes, la propiedad de ser un "conjunto de generadores" de un espacio invariante por traslaciones enteras V se preserva generalmente bajo la acción de una matriz A ([BK06]). Esto no sigue sucediendo para el caso de marcos, lo cual es bastante inesperado. Más aún, en este capítulo construímos un sorprendente ejemplo de un espacio invariante por traslaciones enteras V con un conjunto de generadores Φ tal que el sistema formado por sus traslaciones enteras $\{T_k\phi_i \colon k \in \mathbb{Z}^d,\ i = 1, \ldots, m\}$ forma un marco de V pero tal que ninguna matriz A, de tamaño $\ell \times m$ con $\ell < m$, transforma a Φ en un nuevo conjunto de generadores que forme un marco de V.

Nuestros principales resultados dan condiciones exactas sobre la matriz A para que dicha matriz preserve la propiedad de ser un marco del espacio. Estas condiciones están expresadas en función de una relación geométrica particular que debe cumplirse entre el núcleo de A y el espacio columna de $G_\Phi(\omega)$ para casi todo ω. En la demostración de nuestros resultados usamos otros recientes acerca de los valores singulares de la composición de operadores y también usamos la noción de ángulo de Friedrich entre subespacios de espacios de Hilbert. Además, establecemos una condición analítica entre A y $G_\Phi(\omega)$, que resulta equivalente a la anterior. A pesar de que nosotros estamos interesados en el caso $\ell = \ell(V)$ (la longitud del espacio invariante por traslaciones enteras V en estudio), la mayoría de nuestros resultados siguen valiendo cuando $\ell(V) \le \ell \le m$.

Por otro lado, incluimos los casos de bases de Riesz y bases ortonormales, que ya son conocidos.

Recientemente Paternostro [Pat14] obtuvo resultados similares a los probados por Bownik y Kaiblinger en [BK06] y a los obtenidos mencionados arriba de [CMP14], pero para espacios invariantes por traslaciones en un contexto más general que el de $L^2(\mathbb{R}^d)$.

Este capítulo está organizado de la siguiente manera. En la Section 2.2 incluimos las definiciones y resultados que necesitaremos para probar nuestros resultados. En la Section 2.3 presentamos los resultados de Bownik y Kaiblinger, mencionados antes, acerca de conjuntos minimales de generadores de espacios invariantes por traslaciones enteras en $L^2(\mathbb{R}^d)$. Incluimos algunos resultados acerca de los autovalores de matrices conjugadas en la Section 2.4. Estos resultados serán necesarios para las pruebas de nuestros resultados. En la Section 2.5 establecemos nuestros principales resultados con sus pruebas.

2

Linear combinations of generators in systems of translates

2.1 Introduction

Let V be a shift invariant space generated by the integer translates of $\Phi = \{\phi_1, \dots, \phi_m\}$. That is, V is the closure of the set of all functions φ of the form

$$\varphi(x) = \sum_{j=1}^{m} \sum_{k\in\mathbb{Z}^d} c_{j,k}\, \phi_j(x-k), \qquad x \in \mathbb{R}^d, \tag{2.1}$$

where finitely many $c_{j,k} \in \mathbb{C}$ are nonzero.

It is interesting to know whether it is possible to obtain a minimal set of generators from the given generators in Φ. That is whether there exists $\{\psi_1, \dots, \psi_\ell\} \subseteq L^2(\mathbb{R}^d)$ with $\ell =$ length$(V) \leq m$ such that

$$S(\phi_1, \dots, \phi_m) = S(\psi_1, \dots, \psi_\ell).$$

There are many examples with the property that no subset of Φ is a minimal set of generators. So, deleting elements from Φ may not be a successfull procedure.

Concerning this question, Bownik and Kaiblinger in [BK06], showed that a minimal set of generators for V can be obtained from Φ by linear combinations of its elements. Moreover, they proved that almost every set of ℓ functions that are linear combinations of $\{\phi_1, \dots, \phi_m\}$ is a minimal set of generators for V (see Section 2.3). We emphasize that the linear combinations only involve the functions $\{\phi_1, \dots, \phi_m\}$ and not their translations.

In many problems involving shift invariant spaces, it is important that the system of translates $\{T_k\phi_j \colon k \in \mathbb{Z}^d,\ j = 1, \dots, m\}$ bears a particular functional analytic structure such as being an orthonormal basis, a Riesz basis or a frame. Therefore, it is interesting to know when a minimal set of generators obtained by taking linear combinations of the original one has the same structure. More precisely, suppose that $\Phi = \{\phi_1, \dots, \phi_m\}$ generates a shift invariant space V of length ℓ, and assume that a new set of generators

$\Psi = \{\psi_1, \ldots, \psi_\ell\}$ for V is produced by taking linear combinations of the functions in Φ. That is, assume that $\psi_i = \sum_{j=1}^m a_{ij}\phi_j$ for $i = 1, \ldots, \ell$ for some complex scalars a_{ij}. If we collect the coefficients in a matrix $A = \{a_{ij}\}_{i,j} \in \mathbb{C}^{\ell\times m}$, then we can write in matrix notation $\Psi = A\Phi$. We would like to know, which matrices A transfer the structure of Φ over Ψ. Precisely, we study the following question: If we know that $\{T_k\phi_i : k \in \mathbb{Z}^d,\ i = 1, \ldots, m\}$ is a frame for V, when will $\{T_k\psi_i : k \in \mathbb{Z}^d,\ i = 1, \ldots, \ell\}$ also be a frame for V?

In Section 2.5 we answer this question completely. As we mentioned before, the property of being a "set of generators" for a SIS V is generically preserved by the action of a matrix A ([BK06]). This is not anymore valid for the case of frames. This is an unexpected result. More than that, we were able to construct a surprising example of a shift invariant space V with a set of generators Φ such that their integer translates $\{T_k\phi_i : k \in \mathbb{Z}^d,\ i = 1, \ldots, m\}$ form a frame for V and with the property that no matrix A, of size $\ell \times m$ with $\ell < m$, transform Φ into a new set of generators that form a frame for V.

Our main result gives exact conditions in order that the frame property is preserved by a matrix A. These conditions are in terms of a particular geometrical relation that has to be satisfied between the nullspace of A and the column space of $G_\Phi(\omega)$ for almost all ω. The proof uses recent results about singular values of composition of operators and involves the Friedrichs angle between subspaces in Hilbert spaces. We also provide an equivalent analytic condition between A and $G_\Phi(\omega)$ in order that this same result holds. Although we are interested in the case $\ell = \ell(V)$ (the length of the SIS V under study), most of our results are still valid when $\ell(V) \le \ell \le m$.

For completeness we include the particular case of Riesz bases and orthonormal bases that are known.

Recently Paternostro [Pat14] obtain similar results to [BK06] and [CMP14] but for shift invariant spaces considered in more general contexts than $L^2(\mathbb{R}^d)$.

This rest of this chapter is organized as follows. In Section 2.2 we set the definitions and results that we need for the case of $L^2(\mathbb{R}^d)$. In Section 2.3 we present the results of Bownik and Kaiblinger about minimal generator sets of finitely generated shift invariant spaces in $L^2(\mathbb{R}^d)$. We include some results about the eigenvalues of conjugated matrices in Section 2.4. In Section 2.5 we state and prove our main results.

2.2 Preliminaries

In this section we only include the notations, definitions and properties that we need for Section 2.3 and Section 2.5, in the context of $L^2(\mathbb{R}^d)$.

As we mentioned in the introduction, we are interested in when a set of linear combinations of the generators for a finitely generated SIS inherits some particular structure from the original generators. In order to make clear our exposition we use the following notation.

Let $\Phi = \{\phi_1, \ldots, \phi_m\}$ be a set of functions in $L^2(\mathbb{R}^d)$. By taking linear combinations of the elements of Φ, we construct a new set $\Psi = \{\psi_1, \ldots, \psi_\ell\}$ where $\psi_i = \sum_{j=1}^m a_{ij}\phi_j$ and $A = \{a_{ij}\}_{i,j} \in \mathbb{C}^{\ell\times m}$ is a matrix. Using a matrix notation we set $\Psi = A\Phi$. Then, we consider the following questions:

Let $V = S(\Phi)$ where $\Phi = \{\phi_1, \ldots, \phi_m\}$.

- If $E(\Phi)$ is a frame for V and $\ell(V) \leq \ell \leq m$, for which matrices $A \in \mathbb{C}^{\ell\times m}$, is $E(A\Phi)$ a frame for V?
- If $E(\Phi)$ is an orthonormal basis (Riesz basis) for V, for which square matrices A, is $E(A\Phi)$ an orthonormal basis (Riesz basis) for V?

Sometimes in the chapter by convenient abuse of notation, we will say that a set $\Phi = \{\phi_1, \ldots, \phi_m\}$ is a frame for a SIS V to indicate that actually $E(\Phi)$ forms a frame for V.

In order to study when $E(A\Phi)$ is a frame or a Riesz basis for V we will use Theorem 1.5.3. So, we need to know the Gramian associated to $A\Phi$.

Proposition 2.2.1. *Let $V = S(\Phi)$ be a SIS where $\Phi = \{\phi_1, \ldots, \phi_m\}$ and let $A = \{a_{ij}\}_{i,j} \in \mathbb{C}^{\ell\times m}$ be a matrix. Consider the set $\Psi = \{\psi_1, \ldots, \psi_\ell\}$ where $\Psi = A\Phi$. Then, the Gramian G_Ψ is a conjugation of G_Φ by A, i.e.*

$$G_\Psi(\omega) = AG_\Phi(\omega)A^*$$

for a.e. $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d$.

Proof. Let $i, j \in \{1, \cdots, \ell\}$ be fixed. Then,

$$\begin{aligned}
[G_\Psi(\omega)]_{ij} &= \Big\langle (\widetilde{\psi_i})_\omega, (\widetilde{\psi_j})_\omega \Big\rangle \\
&= \Big\langle \sum_{k=1}^m a_{ik}(\widetilde{\phi_k})_\omega, \sum_{r=1}^m a_{jr}(\widetilde{\phi_r})_\omega \Big\rangle \\
&= \sum_{k=1}^m \sum_{r=1}^m a_{ik}\overline{a_{jr}} \Big\langle (\widetilde{\phi_k})_\omega, (\widetilde{\phi_r})_\omega \Big\rangle \\
&= \sum_{k=1}^m \sum_{r=1}^m a_{ik}\overline{a_{jr}} [G_\Phi(\omega)]_{kr} \\
&= [AG_\Psi(\omega)A^*]_{ij}.
\end{aligned}$$

□

For a matrix $B \in \mathbb{C}^{\ell\times m}$ we denote by $\sigma(B)$ the smallest non-zero singular value of B. By $Ker(B)$ and $Im(B)$ we denote the nullspace and column space of B respectively, as an operator acting by right multiplication, i.e., matrix-vector multiplication.

For a squared positive-semidefinite matrix B such that $B = B^*$, the eigenvalues and its singular values agree. In particular, $\sigma(B) = \lambda_-(B)$ where $\lambda_-(B)$ denotes the smallest non-zero eigenvalue of B.

We need the notion of Friedrichs angle between subspaces that we will use in the following sections for stating our results. The Friedrichs angle can be defined for subspaces of a general Hilbert space (see [Deu95, HJ95, Ka84]). However, we will define it for subspaces of $\mathbb{C}^n$, since this is the context on which we will use it.

Definition 2.2.2. Let $N, M \neq \{0\}$ be subspaces of $\mathbb{C}^n$. The Friedrichs angle between M and N is the angle in $[0, \frac{\pi}{2}]$ whose cosine is defined by

$$\mathcal{G}[M,N] = \sup\{|\langle x, y\rangle| : x \in M \cap (M \cap N)^{\perp}, \|x\| = 1, y \in N \cap (M \cap N)^{\perp}, \|y\| = 1\}.$$

We define $\mathcal{G}[M,N] = 0$ if $M = \{0\}$, $N = \{0\}$, $M \subseteq N$ or $N \subseteq M$.

As usual, the sine of the Friedrichs angle is defined as $\mathcal{F}[M,N] = \sqrt{1 - \mathcal{G}[M,N]^2}$.

We have the following proposition. See [ACRS05] for more details.

Proposition 2.2.3. *Let $N, M \neq \{0\}$ be closed subspaces of $\mathbb{C}^n$. Then*

(i) $\mathcal{G}[M,N] = \mathcal{G}[N,M] = \mathcal{G}[M \cap (M \cap N)^{\perp}, N] = \mathcal{G}[M, N \cap (M \cap N)^{\perp}]$.

(ii) $\mathcal{G}[M,N] < 1$ *if and only if* $M + N$ *is closed.*

(iii) $\mathcal{G}[M,N] = \mathcal{G}[M^{\perp}, N^{\perp}]$.

(iv) $\mathcal{G}[M,N] = \|P_M P_{N\cap(M\cap N)^{\perp}]}\| = \|P_{M\cap(M\cap N)^{\perp}} P_N\| = \|P_M P_N P_{(M\cap N)^{\perp}}\| = \|P_M P_N - P_{M\cap N}\|$.

2.3 Minimal generator sets for finitely generated SIS in $L^2(\mathbb{R}^d)$

As we mentioned in the introduction, there are examples of sets of generators that do not contain any minimal subset of generators. For instance, consider the shift invariant space $V = S(\phi)$ where ϕ is such that $\widehat{\phi} = \chi_{[0,1]}$. (Here we denote by χ_M the characteristic function of a set M). It can be seen that $V = S(\phi_1, \phi_2)$ with ϕ_1, ϕ_2 such that $\widehat{\phi_1} = \chi_{[0,\frac{1}{2}]}$ and $\widehat{\phi_2} = \chi_{[\frac{1}{2},1]}$. However, neither ϕ_1 nor ϕ_2 generates V by itself.

An alternative to overcome this problem will be to try to obtain a smaller set of generators, for example a minimal one, considering instead linear combinations of the original generators. Bownik and Kaiblinger in [BK06] gave a solution to this question in this direction. Since the new generators $\psi_1, \ldots, \psi_\ell$ belong to V, they can be approximated in the L^2−norm by functions of the form (2.1), i.e., by finite sums of shifts of the original generators. However, they proved that at least one reduced set of generators can be obtained from a linear combination of the original generators without translations. In particular, no limit or infinite summation is required. Moreover, they proved that almost every set of ℓ functions that are linear combinations of $\{\phi_1, \ldots, \phi_m\}$ is a minimal set of generators for V, as the following theorem shows.

Theorem 2.3.1. *[BK06, Theorem 1] Let V be a finitely generated SIS with length $\ell(V)$, and $\Phi = \{\phi_1, \dots, \phi_m\} \subseteq L^2(\mathbb{R}^d)$ where $\ell(V) \leq m$ and such that $V = S(\Phi)$. For every $\ell(V) \leq \ell \leq m$, consider the set of matrices $\mathcal{R} = \{A \in \mathbb{C}^{\ell\times m} : V = S(\Psi), \Psi = A\Phi\}$. Then,*

(i) $\mathbb{C}^{\ell\times m} \setminus \mathcal{R}$ has zero Lebesgue measure.

(ii) $\mathcal{N} = \mathbb{C}^{\ell\times m} \setminus \mathcal{R}$ can be dense in $\mathbb{C}^{\ell\times m}$.

This result briefly says that given any set of generators of a finitely generated SIS, almost every matrix (of the right size) transforms it in a new set of generators. In particular, a *minimal* set of generators can be obtained with this procedure.

As an immediate consequence of Theorem 2.3.1 in case of principal shift invariant spaces we have the following corollary.

Corollary 2.3.2. *Let V be a principal SIS, and $\Phi = \{\phi_1, \dots, \phi_m\} \subseteq L^2(\mathbb{R}^d)$ such that $V = S(\Phi)$. Consider the set of vectors $\mathcal{R} = \{v \in \mathbb{C}^m : V = S(\Psi), \Psi = v\Phi\}$. Then,*

(i) $\mathbb{C}^m \setminus \mathcal{R}$ has zero Lebesgue measure.

(ii) $\mathcal{N} = \mathbb{C}^m \setminus \mathcal{R}$ can be dense in $\mathbb{C}^m$.

The following examples illustrate Corollary 2.3.2 and show that the set $\mathcal{N}$ can be a singleton (Example 2.3.3) or indeed be dense in $\mathbb{C}^m$ (Example 2.3.4). For a detailed proofs of the following examples we refer the reader to [BK06].

Example 2.3.3. The function sinc $\in L^2(\mathbb{R})$ is defined by

$$\operatorname{sinc}(t) = \frac{\sin \pi t}{\pi t}, \quad t \in \mathbb{R}.$$

Given $m \geq 1$, let $\phi_1, \dots, \phi_m \in L^2(\mathbb{R})$ be a collection of distinct translations of the sinc function. That is,

$$\phi_j(t) = \operatorname{sinc}(t - t_j), \quad t \in \mathbb{R}, j = 1, \dots, m,$$

where $t_1, \dots, t_m \in \mathbb{R}$ satisfy $t_j \neq t_k$, for $j \neq k$. Let $V = S(\phi_1, \dots, \phi_m)$.

Then V is the Paley-Wiener space of functions in $L^2(\mathbb{R})$ which are band-limited to $[-1/2, 1/2]$, so V is principal. For example, the sinc function itself or any of its shifts individually generate V. Indeed any linear combination of the original generators yields a single generator for V as well, unless all coefficients are zero. Hence, in this example the set $\mathcal{N}$ of Theorem 2.3.1 is

$$\mathcal{N} = \{0\}, \quad 0 = (0, \dots, 0) \in \mathbb{C}^m,$$

consisting of the zero vector only.

Example 2.3.4. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the largest integer less or equal x. We define a discretized version of the Archimedean spiral by $\gamma\colon [0,1) \to \mathbb{Z}^2$,

$$\gamma(x) = (\lfloor u\cos 2\pi u\rfloor, \lfloor u\sin 2\pi u\rfloor), \quad u = \tan\frac{\pi}{2}x, \quad x \in [0,1).$$

Next, let

$$\gamma^0(x) = \begin{cases} \frac{\gamma(x)}{|\gamma(x)|} & \text{if } \gamma(x) \neq 0, \\ 0 & \text{otherwise} \end{cases}, \quad x \in [0,1).$$

Now define $\phi_1, \phi_2 \in L^2(\mathbb{R})$ by their Fourier transforms, obtained from $\gamma^0 = (\gamma_1^0, \gamma_2^0)$ by

$$\widehat{\phi_j}(x) = \begin{cases} \gamma_j^0(x), & x \in [0,1), \\ 0, & x \in \mathbb{R} \setminus [0,1) \end{cases}, \quad j = 1,2.$$

Let $V = S(\phi_1, \phi_2)$.

Then V is principal. In fact, the function $\psi = \lambda_1\phi_1 + \lambda_2\phi_2$ is a single generator, $V = S(\psi)$, if and only if λ_1 and λ_2 are rationally linearly independent. So here the set $\mathcal{N}$ of Theorem 2.3.1 is

$$\mathcal{N} = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 \colon \lambda_1 \text{ and } \lambda_2 \text{ are rationally linear dependent}\}.$$

In particular, any rational linear combination of ϕ_1, ϕ_2 fails to generate V. This example illustrates Corollary 2.3.2 for the case of real coefficients. Namely, $\mathcal{N} \cap \mathbb{R}^2$ is a null set in $\mathbb{R}^2$ yet it contains $\mathbb{Q}^2$, so it is dense in $\mathbb{R}^2$. One can extend this example for the case of complex coefficients.

Since linear combinations of a finite number of functions preserve properties such as smoothness, compact support, bandlimitedness, decay, etc., an interesting consequence of Bownik and Kaiblinger's result is that if the generators for V have some additional property, there exists a minimal set of generators that inherits this property.

2.4 Eigenvalues of a conjugated matrix

According to Theorem 1.5.3, we will need to study the eigenvalues of the Gramian $G_\Psi(\omega)$ which is, as Proposition 2.2.1 shows, a conjugation of $G_\Phi(\omega)$ by A. The behaviour of the eigenvalues of the conjugation of a given matrix is in general not very well established. In our case, we will find uniform bounds for the eigenvalues of $G_\Psi(\omega)$.

We now state a recent result that we need for the next section. It is a particular case of the result stated in Remark 2.10 in [ACRS05] for square matrices.

Theorem 2.4.1. *Let A, B be non zero square matrices in $\mathbb{C}^{m\times m}$. Then,*

$$\sigma(A)\sigma(B)\mathcal{F}[Ker(A), Im(B)] \leq \sigma(AB) \leq \|A\|\|B\|\mathcal{F}[Ker(A), Im(B)].$$

In order to adapt Theorem 2.4.1 to our setting we need the following lemma.

Lemma 2.4.2. *Let $A \in \mathbb{C}^{\ell \times m}$ and $G \in \mathbb{C}^{m \times m}$ with $\ell \leq m$. If $rk(AGA^*) = rk(G)$ then $Ker(AG) = Ker(G)$.*

Proof. Clearly $Ker(G) \subseteq Ker(AG)$. Suppose that $\dim Ker(G) < \dim Ker(AG)$. Then $\mathrm{rk}(AG) < \mathrm{rk}(G)$. Now, $\mathrm{rk}(AGA^*) \leq \min\{\mathrm{rk}(AG), \mathrm{rk}(A)\} \leq \mathrm{rk}(AG) < \mathrm{rk}(G)$ which is a contradiction. Thus, we must have $\dim Ker(G) = \dim Ker(AG)$. □

Now, from Theorem 2.4.1 and Lemma 2.4.2 we obtain:

Proposition 2.4.3. *Let G be a positive-semidefinite matrix in $\mathbb{C}^{m \times m}$ such that $G = G^*$ and $A = \{a_{ij}\}_{i,j} \in \mathbb{C}^{\ell \times m}$ with $\ell \leq m$ such that $rk(G) = rk(AGA^*)$. Then,*

$$\sigma(A)^2 \lambda_-(G) \mathcal{F}[Ker(A), Im(G)]^2 \leq \lambda_-(AGA^*) \leq \|A\|^2 \|G\| \mathcal{F}[Ker(A), Im(G)].$$

Proof. Let $\widetilde{A} \in \mathbb{C}^{m \times m}$ be the matrix defined by $\widetilde{a}_{ij} = \begin{cases} a_{ij} & \text{if } 1 \leq i \leq \ell \\ 0 & \text{if } \ell < i \leq m \end{cases}$. Then,

$$\widetilde{A}G(\widetilde{A})^* = \left(\begin{array}{c|c} AGA^* & 0 \\ \hline 0 & 0 \end{array} \right),$$

and therefore $\lambda_-(AGA^*) = \lambda_-(\widetilde{A}G\widetilde{A}^*)$.

Now, using Theorem 2.4.1 and Lemma 2.4.2, we have

$$\begin{aligned} \lambda_-(\widetilde{A}G\widetilde{A}^*) &\geq \sigma(\widetilde{A}G)\sigma(\widetilde{A}^*)\mathcal{F}[Ker(\widetilde{A}G), Im(\widetilde{A}^*)] \\ &= \sigma(\widetilde{A}G)\sigma(\widetilde{A}^*)\mathcal{F}[Ker(G), Im(\widetilde{A}^*)] \\ &\geq \sigma(\widetilde{A})\lambda_-(G)\mathcal{F}[Ker(\widetilde{A}), Im(G)]\sigma(\widetilde{A}^*)\mathcal{F}[Ker(G), Im(\widetilde{A}^*)], \end{aligned}$$

and

$$\lambda_-(\widetilde{A}G\widetilde{A}^*) \leq \|\widetilde{A}G\|\|\widetilde{A}^*\|\mathcal{F}[Ker(\widetilde{A}G), Im(\widetilde{A}^*)] \leq \|\widetilde{A}\|^2\|G\|\mathcal{F}[Ker(G), Im(\widetilde{A}^*)].$$

By the properties of the sine of the Friedrichs angle it can be seen that $\mathcal{F}[Ker(G), Im(\widetilde{A}^*)] = \mathcal{F}[Ker(\widetilde{A}), Im(G)]$. Using that $\sigma(A) = \sigma(\widetilde{A}) = \sigma(\widetilde{A}^*)$ and $\|\widetilde{A}\| = \|A\|$, we finally obtain

$$\sigma(A)^2 \lambda_-(G) \mathcal{F}[Ker(\widetilde{A}), Im(G)]^2 \leq \lambda_-(\widetilde{A}G\widetilde{A}^*) \leq \|A\|^2 \|G\| \mathcal{F}[Ker(\widetilde{A}), Im(G)].$$

We finish the proof by observing that $Ker(A) = Ker(\widetilde{A})$. □

The last result of this section gives an equivalent condition for $\mathrm{rk}(AGA^*) = \mathrm{rk}(G)$ to hold, and we will use it in the next section.

Lemma 2.4.4. *Let $A \in \mathbb{C}^{\ell \times m}$ with $\ell \leq m$ and $G \in \mathbb{C}^{m \times m}$ such that G is positive-semidefinite and $G = G^*$. Then, $rk(AGA^*) = rk(G)$ if and only if $Ker(A) \cap Im(G) = \{0\}$.*

Proof. Note that, since G is positive semidefinite, and $G = G^*$, it is always true that $\text{rk}(AGA^*) = \text{rk}(AG^{1/2}G^{1/2}A^*) = \text{rk}(AG^{1/2})$. Then, $\text{rk}(AGA^*) = \text{rk}(G)$ if and only if $\dim(Ker(AG^{1/2})) = \dim(Ker(G))$. Thus, since $Im(G) = Im(G^{1/2})$ and then $Ker(G) = Ker(G^{1/2})$, we want to prove that $\dim(Ker(AG^{1/2})) = \dim(Ker(G^{1/2})$ if and only if $Ker(A) \cap Im(G^{1/2}) = \{0\}$. Now, using that $Ker(G^{1/2}) \subseteq Ker(AG^{1/2})$, we have $\dim(Ker(AG^{1/2})) = \dim(Ker(G^{1/2})$ if and only if $Ker(AG^{1/2}) = Ker(G^{1/2})$.

Finally, it is easy to check that the condition $Ker(AG^{1/2}) = Ker(G^{1/2})$ is equivalent to $Ker(A) \cap Im(G) = \{0\}$. □

2.5 Linear combinations of frame generators in systems of translates in $L^2(\mathbb{R}^d)$

As we mentioned in Section 2.3, Bownik and Kaiblinger proved that given any set of generators of a finitely generated SIS, almost every matrix (of a right size) transforms it in a minimal set of generators.

An interesting question arises here. When is the set of generators obtained, a set of *frame* generators? (i.e. the integer translations form a frame of the SIS?) More precisely we want to obtain new sets of generators of the form $\Psi = A\Phi$ with the additional property of being a frame.

First, we show that in order to $E(\Psi)$ to be a frame, $E(\Phi)$ needs to be a frame. More precisely we prove:

Proposition 2.5.1. *Let $\Phi = \{\phi_1, \dots, \phi_m\} \subseteq L^2(\mathbb{R}^d)$ be a generator set for a SIS V of length $\ell(V) \leq m$ and suppose that $E(\Phi)$ is a Bessel sequence but not a frame for V. Then, for each matrix $A \in \mathbb{C}^{\ell \times m}$, with $\ell(V) \leq \ell \leq m$, $E(\Psi)$ is not a frame for V where $\Psi = A\Phi$.*

After this result, the right question will be which matrices (of the right size) map frame generator sets into *new* frame generator sets? The answer of this question is not as direct as is the case of a plain set of generators and it will take us the rest of the section. Let us first start with the proof of Proposition 2.5.1

Proof of Proposition 2.5.1. Let $A \in \mathbb{C}^{\ell \times m}$. If $\Psi = A\Phi$ is not a generator set for V, then Ψ it is not a frame for V. Thus, suppose that Ψ is a set of generators for V.

Since $E(\Phi)$ is a Bessel sequence but not a frame for V, the lower frame inequality in (1.3) is not satisfied. Therefore, there exists $\{f_n\}_{n\in\mathbb{N}} \subseteq V$ such that

$$t_n(\Phi) := \sum_{j=1}^{m} \sum_{k\in\mathbb{Z}^d} |\langle f_n, T_k\phi_j\rangle|^2 \to 0, \text{ when } n \to +\infty.$$

Now,

$$\begin{aligned} t_n(\Psi) &= \sum_{i=1}^{\ell}\sum_{k\in\mathbb{Z}^d} |\langle f_n, T_k\psi_i\rangle|^2 \\ &= \sum_{i=1}^{\ell}\sum_{k\in\mathbb{Z}^d} |\sum_{j=1}^{m} \overline{a_{ij}}\langle f_n, T_k\phi_j\rangle|^2 \\ &\leq \sum_{i=1}^{\ell}\sum_{k\in\mathbb{Z}^d} \left(\sum_{j=1}^{m}|a_{ij}|^2\right)\left(\sum_{j=1}^{m}|\langle f_n, T_k\phi_j\rangle|^2\right) \\ &= \left(\sum_{i=1}^{\ell}\sum_{j=1}^{m}|a_{ij}|^2\right) t_n(\Phi). \end{aligned}$$

Then, $t_n(\Psi) \to 0$, when $n \to +\infty$ and thus, $E(\Psi)$ does not satisfy the lower frame inequality. □

When $E(\Phi)$ is not a Bessel sequence it can happen that for certain matrix A, $E(A\Phi)$ is a Bessel sequence. To construct an easy example take $\phi \in L^2(\mathbb{R})$ such that $E(\phi)$ is a frame for $S(\phi)$. Now, choose a second generator $\widetilde{\phi} \in S(\phi)$ such that $E(\widetilde{\phi})$ is not a Bessel sequence. Thus, if $\Phi = \{\phi, \widetilde{\phi}\}$, $S(\Phi) = S(\phi)$ and $E(\{\phi, \widetilde{\phi}\})$ is not a Bessel sequence. However, taking $A = (1, 0)$, we get that $E(A\Phi)$ is a Bessel sequence. The hypothesis in the above proposition of $E(\Phi)$ being a Bessel sequence it is not very restrictive and greatly simplifies the treatment.

We now point out a property about the result of Bownik and Kaiblinger that will be important in what follows.

From the proof of Theorem 2.3.1, it follows that the set $\mathcal{R}$ can be described in terms of the Gramian as

$$\mathcal{R} = \{A \in \mathbb{C}^{\ell\times m} \;:\; \mathrm{rk}(G_\Phi(\omega)) = \mathrm{rk}(AG_\Phi(\omega)A^*) \text{ for a.e. } \omega \in [-1/2, 1/2]^d\}, \tag{2.2}$$

where ℓ is a number between length of $S(\Phi)$ and m. Now, using Lemma 2.4.4

$$\mathcal{R} = \{A \in \mathbb{C}^{\ell\times m} \;:\; Ker(A) \cap Im(G_\Phi(\omega)) = \{0\} \text{ for a.e. } \omega \in [-1/2, 1/2]^d\}.$$

Remark 2.5.2. When ℓ is exactly the length of $S(\Phi)$, note that if $A \in \mathcal{R}$, by (1.8), $\ell = \mathrm{rk}(A)$ and then A is full rank.

As we have already discussed, a key point in our problem is the behavior of the eigenvalues of conjugated matrices. Here one wants to get a smaller set of generators from a given large set of generators. In terms of matrices, this translates in conjugating the Gramian by rectangular matrices. The behavior of the eigenvalues in this case is not very well understood. However we are able to exactly determine those matrices that yield frames, in terms of the Friedrichs angle, and using recent results by Antezana et al [ACRS05] on singular values of composition of operators.

The main result of this section is the following theorem:

Theorem 2.5.3. *Let $\Phi = \{\phi_1, \dots, \phi_m\} \subseteq L^2(\mathbb{R}^d)$ such that $E(\Phi)$ is a frame for $V = S(\Phi)$ and suppose that $\ell(V) \leq \ell \leq m$. Let $A \in \mathbb{C}^{\ell\times m}$ be a matrix and consider $\Psi = \{\psi_1, \dots, \psi_\ell\}$ where $\Psi = A\Phi$. Then, $E(\Psi)$ is a frame for V if and only if A satisfies the following two conditions*

1. *$A \in \mathcal{R}$ where $\mathcal{R}$ is as in (2.2).*
2. *There exists $\delta > 0$ such that $\mathcal{F}\,[Ker(A), Im(G_\Phi(\omega))] \geq \delta$ for a.e. $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d$.*

Proof of Theorem 2.5.3. Let $0 < \alpha \leq \beta$ be the frame bounds for $E(\Phi)$.

First suppose that $E(\Psi)$ is a frame for V and let $\beta' \geq \alpha' > 0$ be its frame bounds. Since, in particular, Ψ is a generator set for V, A belongs to $\mathcal{R}$.

By Proposition 2.4.3, we have that

$$\lambda_-(G_\Psi(\omega)) = \lambda_-(AG_\Phi(\omega)A^*) \leq \|A\|^2\|G_\Phi(\omega)\|\,\mathcal{F}[Ker(A), Im(G_\Phi(\omega))],$$

for a.e. $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d$.

Using Theorem 1.5.3, $\alpha' \leq \lambda_-(G_\Psi(\omega))$ and $\|G_\Phi(\omega)\| \leq \beta$. Thus,

$$\alpha' \leq \|A\|^2\beta\,\mathcal{F}[Ker(A), Im(G_\Phi(\omega))].$$

Then, item (2) is satisfied taking $\delta = \frac{\alpha'}{\|A\|^2\beta}$.

Conversely. Since $A \in \mathcal{R}$, $V = S(\Psi)$ and $\mathrm{rk}(AG_\Phi(\omega)A^*) = \mathrm{rk}(G_\Phi(\omega))$ for a.e. $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d$.

Now, for a.e. $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d$, we apply the lower inequality that Proposition 2.4.3 gives to $G_\Phi(\omega)$ and A. Then,

$$\lambda_-(AG_\Phi(\omega)A^*) \geq \sigma(A)^2\lambda_-(G_\Phi(\omega))\,\mathcal{F}[Ker(A), Im(G_\Phi(\omega))]^2 \geq \sigma(A)^2\alpha\delta^2.$$

On the other hand, by Theorem 1.5.3,

$$\|G_\Psi(\omega)\| = \|AG_\Phi(\omega)A^*\| \leq \|A\|^2\|G_\Phi(\omega)\| \leq \|A\|^2\beta$$

and from this it follows that the eigenvalues of $G_\Psi(\omega)$ are bounded above by $\|A\|^2\beta$.

Therefore, $\Sigma(G_\Psi(\omega)) \subseteq [\,\sigma(A)^2\alpha\delta^2, \|A\|^2\beta\,] \cup \{0\}$ for a.e. $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d$ and the result follows from Theorem 1.5.3. □

As a consequence of the above theorem, we have the following result. We impose more restrictive condition on $G_\Phi(\omega)$ than in Theorem 2.5.3. However the new hypothesis is easy to check and avoid the calculation of the sine of the Friedrichs angle.

Corollary 2.5.4. *Let $\Phi = \{\phi_1, \dots, \phi_m\} \subseteq L^2(\mathbb{R}^d)$ such that $E(\Phi)$ is a frame for $V = S(\Phi)$ and suppose that $\ell(V) \leq \ell \leq m$. Consider $A \in \mathbb{C}^{\ell\times m}$ such that $Ker(A) = Ker(G_\Phi(\omega))$ for a.e. $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d$, and $dim(Ker(A)) = m - \ell$. If $A \in \mathcal{R}$ where $\mathcal{R}$ is as in (2.2), and $\Psi = \{\psi_1, \dots, \psi_\ell\}$ where $\Psi = A\Phi$, then $E(\Psi)$ is a frame for V.*

Proof. Since $Ker(A) \subseteq Ker(G_\Phi(\omega))$ and $Ker(G_\Phi(\omega)) = Im(G_\Phi(\omega))^\perp$, it follows that $\mathcal{F}[Ker(A), Im(G_\Phi(\omega))] = 1$. □

Example 2.5.5. Let $V = S(\Phi)$ be the shift invariant space generated by $\Phi = \{\phi_1, \phi_2, \phi_3\} \subseteq L^2(\mathbb{R})$, where ϕ_1, ϕ_2, ϕ_3 are defined by

$$\begin{aligned}
\widehat{\phi_1}(\omega) &= -8\chi_{[-\frac{1}{2},\frac{1}{2}]}(\omega) + 4\chi_{[\frac{1}{2},\frac{3}{2}]}(\omega),\\
\widehat{\phi_2}(\omega) &= \chi_{[-\frac{1}{2},\frac{1}{2}]}(\omega) + 4\chi_{[\frac{3}{2},\frac{5}{2}]}(\omega),\\
\widehat{\phi_3}(\omega) &= \chi_{[\frac{1}{2},\frac{3}{2}]}(\omega) + 8\chi_{[\frac{3}{2},\frac{5}{2}]}(\omega).
\end{aligned}$$

Then, the associated Gramian is

$$G_\Phi(\omega) = \begin{pmatrix} 80 & -8 & 4 \\ -8 & 17 & 32 \\ 4 & 32 & 65 \end{pmatrix}.$$

It can be seen that $G^2_\Phi(\omega) = 81 G_\Phi(\omega)$ and $\mathrm{rk}(G_\Phi(w)) = 2$ for all $\omega \in [-\frac{1}{2}, \frac{1}{2}]$. Thus, by Theorem 1.5.3, $E(\Phi)$ is a frame for V and using that

$$\ell(V) = \operatorname*{esssup}_{\omega\in[-\frac{1}{2},\frac{1}{2}]^d} \mathrm{rk}(G_\Phi(\omega)),$$

we obtain $\ell(V) = 2$.

On the other side, $Ker(G_\Phi(\omega)) = \mathrm{span}\{(1, 8, -4)\}$. Then, V satisfy the hypothesis of Corollary 2.5.4. In what follows, we construct all possible matrices $A \in \mathbb{C}^{2\times 3}$ such that $Ker(A) = \mathrm{span}\{(1, 8, -4)\}$ and $A \in \mathcal{R}$. These matrices give frames $E(\Psi)$ with $\Psi = A\Phi$.

Since $Ker(A) = \mathrm{span}\{(1, 8, -4)\}$, A has the form

$$A = \begin{pmatrix} (4b - 8a) & a & b \\ (4d - 8c) & c & d \end{pmatrix},$$

with $a, b, c, d \in \mathbb{C}$.

Now, $A \in \mathcal{R}$ if and only if $\mathrm{rk}(AG_\Phi(\omega)A^*) = 2$. Now $\mathrm{rk}(AG_\Phi(\omega)A^*) = 2$ if and only if $\det(AG_\Phi(\omega)A^*) \neq 0$. Since $\det(AG_\Phi(\omega)A^*) = (81)^3(ad - bc)^2$, we conclude that $A \in \mathcal{R}$ if and only if $ad - bc \neq 0$. ■

Condition (2) in Theorem 2.5.3 is a geometric property that A and G_Φ need to satisfy in order to A preserves the frame property of $E(\Phi)$ over $E(\Psi)$. We now state a result on which we give an analytic way to express conditions (1) and (2) of Theorem 2.5.3. In this case, ℓ will be exactly the length of the SIS.

Theorem 2.5.6. *Let $\Phi = \{\phi_1, \ldots, \phi_m\} \subseteq L^2(\mathbb{R}^d)$ such that $E(\Phi)$ is a frame for $V = S(\Phi)$ and suppose that $\ell(V) = \ell \leq m$. Let $A \in \mathbb{C}^{\ell\times m}$ be a matrix and consider $\Psi = \{\psi_1, \ldots, \psi_\ell\}$ where $\Psi = A\Phi$. Then, $E(\Psi)$ is a frame for V if and only if the following condition between A and G_Φ is satisfied: AA^* is invertible and*

$$\operatorname*{esssup}_{\omega\in[-\frac{1}{2},\frac{1}{2}]^d} \|(I_m - A^*(AA^*)^{-1}A)G_\Phi(\omega)G^\dagger_\Phi(\omega)\| < 1. \tag{2.3}$$

Here, I_m is the identity in $\mathbb{C}^{m\times m}$ and $G^\dagger_\Phi(\omega)$ is the Moore-Penrose pseudoinverse of $G_\Phi(\omega)$.

Proof. Let us first prove that conditions (1) and (2) of Theorem 2.5.3 imply condition (2.3).

Note that condition (2) is equivalent to $\mathcal{G}[Ker(A), Im(G_\Phi(\omega))] \leq \sqrt{1-\delta^2}$ for a.e. $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d$. Now, since $Ker(A) \cap Im(G_\Phi(\omega)) = \{0\}$ for a.e $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d$, it follows from Proposition 2.2 in [ACRS05] that

$$\mathcal{G}[Ker(A), Im(G_\Phi(\omega))] = \|P_{Ker(A)} P_{Im(G_\Phi(\omega))}\| \quad \text{for a.e. } \omega \in [-\frac{1}{2}, \frac{1}{2}]^d,$$

where $P_{Ker(A)}$ and $P_{Im(G_\Phi(\omega))}$ denote the orthogonal projection onto $Ker(A)$ and $Im(G_\Phi(\omega)$ respectively. Using the Moore-Penrose pseudoinverse we can write

$$\|P_{Ker(A)} P_{Im(G_\Phi(\omega))}\| = \|(I_m - A^\dagger A) G_\Phi(\omega) G_\Phi^\dagger(\omega)\|.$$

Finally, since A is full rank, we replace $A^\dagger = A^*(AA^*)^{-1}$ and then the assertion follows.

Conversely. Suppose that (2.3) holds. Then, we have that

$$\|P_{Ker(A)} P_{Im(G_\Phi(\omega))}\| \leq \gamma < 1 \quad \text{for all } \omega \in [-\frac{1}{2}, \frac{1}{2}]^d \setminus Z,$$

where Z is a set with Lebesgue measure zero and

$$\gamma = \operatorname*{esssup}_{\omega \in [-\frac{1}{2}, \frac{1}{2}]^d} \|(I_m - A^*(AA^*)^{-1} A) G_\Phi(\omega) G_\Phi^\dagger(\omega)\|.$$

Fix $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d \setminus Z$ and suppose there exists $x \in Ker(A) \cap Im(G_\Phi(\omega))$ with $\|x\| = 1$. Then, $\|P_{Ker(A)} P_{Im(G_\Phi(\omega))} x\| = \|x\| = 1$ which is a contradiction. Therefore,

$$Ker(A) \cap Im(G_\Phi(\omega)) = \{0\} \quad \text{for all } \omega \in [-\frac{1}{2}, \frac{1}{2}]^d \setminus Z,$$

and this gives that (1) in Theorem 2.5.3 is satisfied. Having $Ker(A) \cap Im(G_\Phi(\omega)) = \{0\}$ for a.e. $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d$, condition (2) of Theorem 2.5.3 can be obtained with similar arguments as in the first part of this proof.

□

Now, we consider the case of Riesz bases. We obtain necessary and sufficient conditions on A in order to preserve Riesz bases of translates. Different from the frame case, the conditions on A do not depend on the shift invariant space.

First, observe that, if $\Phi = \{\phi_1, \ldots, \phi_m\} \subseteq L^2(\mathbb{R}^d)$ is such that $E(\Phi)$ is a Riesz (orthonormal) basis for $V = S(\Phi)$, by Theorem 1.5.3, $G_\Phi(\omega)$ is an $m \times m$ invertible matrix for a.e. $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d$. Thus, using that

$$\ell(V) = \operatorname*{esssup}_{\omega \in [-\frac{1}{2}, \frac{1}{2}]^d} \operatorname{rk}(G_\Phi(\omega)),$$

we have that $\ell(V) = m$. Therefore, in order to preserve Riesz (orthonormal) bases, we need to consider squared matrices $A \in \mathbb{C}^{m \times m}$.

Proposition 2.5.7. *Let $\Phi = \{\phi_1, \ldots, \phi_m\} \subseteq L^2(\mathbb{R}^d)$ such that $E(\Phi)$ is a Riesz basis for $V = S(\Phi)$. Let $A \in \mathbb{C}^{m\times m}$ be a matrix and consider $\Psi = \{\psi_1, \ldots, \psi_m\}$ with $\Psi = A\Phi$. Then, $E(\Psi)$ is a Riesz basis for V if and only if A is an invertible matrix.*

Proof. Let $0 < \alpha \le \beta$ be the Riesz bounds for $E(\Phi)$. Suppose first that A is invertible. Then, for a.e. $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d$,

$$\begin{aligned} \|G_\Psi(\omega)\| &= \|AG_\Phi(\omega)A^*\| \le \|A\|^2\|G_\Phi(\omega)\| \le \|A\|^2\beta, \\ \|(G_\Psi(\omega))^{-1}\| &\le \|A^{-1}\|^2\|(G_\Phi(\omega))^{-1}\| \le \|A^{-1}\|^2\frac{1}{\alpha}. \end{aligned}$$

Therefore, by Theorem 1.5.3, $E(\Psi)$ is a Riesz basis for V.

Conversely, if $E(\Psi)$ is a Riesz basis for V, it follows from Theorem 1.5.3 that $G_\Psi(\omega)$ is invertible for a.e. $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d$. Since $G_\Phi(\omega)$ is invertible as well for a.e. $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d$ we have that A is invertible. □

Remark 2.5.8. The above result shows that every invertible matrix preserves Riesz bases of translates. On the other side, it is known that the set $\{A \in \mathbb{C}^{m\times m}\colon \det(A) = 0\}$ has Lebesgue measure zero. Thus, we have that almost every matrix (exactly those that are invertible) preserves Riesz bases of translates. We can then connect this with Bownik and Kaiblinger's result as follows. If in addition to the hypothesis of Theorem 2.3.1 we ask the Riesz basis condition on $E(\Phi)$, we have that the set $\mathcal{R}$ is equal to $\{A \in \mathbb{C}^{m\times m}\colon \det(A) \neq 0\}$.

It is worth to mention that Proposition 2.5.7 can be proven as a corollary of Theorem 2.5.3 or independently from this result using Ostrowski's Theorem:

Theorem 2.5.9. *[HJ90, Theorem 4.5.9] Let $A, S \in \mathbb{C}^{m\times m}$ with A Hermitian and S non-singular. Let the eigenvalues of A and SS^* be arranged in increasing order. For each $k = 1, 2, \ldots,$ there exists a positive real number θ_k such that $\lambda_1(SS^*) \le \theta_k \le \lambda_n(SS^*)$ and*

$$\lambda_k(SAS^*) = \theta_k\lambda_k(A).$$

Applying this result to $G = G_\Phi(\omega)$ for a.e. $\omega \in [-\frac{1}{2}, \frac{1}{2}]^d$, uniform bounds for the eigenvalues of $AG_\Phi(\omega)A^*$ can be found.

For frames, in the special case when the initial set of generators has exactly $\ell(V)$ elements, we have that every invertible matrix yields a set of generators that is a frame for V. This is stated in the next result and its proof is analogous to the proof of Proposition 2.5.7. It can be also viewed as a corollary of Theorem 2.5.6.

Theorem 2.5.10. *Let $\Phi = \{\phi_1, \ldots, \phi_\ell\} \subseteq L^2(\mathbb{R}^d)$ such that $E(\Phi)$ is a frame for $V = S(\Phi)$ and suppose that $\ell(V) = \ell$. Let $A \in \mathbb{C}^{\ell\times\ell}$ be a matrix and consider $\Psi = \{\psi_1, \ldots, \psi_\ell\}$ where $\Psi = A\Phi$. Then, $E(\Psi)$ is a frame for V if and only if A is an invertible matrix.*

Finally, in case of orthonormal bases, we have:

Proposition 2.5.11. *Let $\Phi = \{\phi_1, \ldots, \phi_m\} \subseteq L^2(\mathbb{R}^d)$ such that $E(\Phi)$ is an orthonormal basis for $V = S(\Phi)$ and let $A \in \mathbb{C}^{m\times m}$ be a matrix. Consider $\Psi = A\Phi$. Then, $E(\Psi)$ is an orthonormal basis for V if and only if A is an unitary matrix.*

Proof. Note that if $A = \{a_{ij}\}_{i,j}$ then,

$$\langle T_k\psi_i, T_{k'}\psi_{i'}\rangle = \sum_{j,j'}^{m} a_{ij}\overline{a_{i'j'}}\langle T_k\phi_j, T_{k'}\phi_{j'}\rangle = (AA^*)_{ii'}\delta(k-k'),$$

and from here it follows that $E(\Psi)$ is an orthonormal set if and only if A is unitary.

For the completeness of $E(\Psi)$ on V we use that, since A is unitary and $\Psi = A\Phi$, we can write $\Phi = A^*\Psi$. Then $T_k\phi_j = \sum_{i=1}^m \overline{a_{ij}} T_k\psi_i, \quad k \in \mathbb{Z}, j = 1,\dots,m$ and the result follows. □

Theorem 2.5.7 shows that almost every square matrix maps Riesz bases generators in Riesz bases generators. For the case of frames it might happen that condition (2) of Theorem 2.5.3 is not satisfied for any matrix A. That is exactly what we show in the following example on which we present a finitely generated SIS for which any linear combination of its generators yields to a minimal set of generators that is not a frame of translates.

Example 2.5.12. Let $\phi_1, \phi_2 \in L^2(\mathbb{R}^2)$ defined by

$$\widehat{\phi_1}(\omega_1,\omega_2) = -\sin(2\pi\omega_1)\chi_{[-\frac{1}{2},\frac{1}{2}]^2}(\omega_1,\omega_2)$$

and

$$\widehat{\phi_2}(\omega_1,\omega_2) = e^{2\pi i\omega_2}\cos(2\pi\omega_1)\chi_{[-\frac{1}{2},\frac{1}{2}]^2}(\omega_1,\omega_2).$$

Consider the shift invariant space generated by $\Phi = \{\phi_1,\phi_2\}$, $V = S(\phi_1,\phi_2)$. We will see that $E(\phi_1,\phi_2)$ is a frame for V, that V is a principal SIS and that $E(A\Phi)$ is not a frame for V for any matrix $A \in \mathbb{C}^{1\times 2}$.

We first compute $G_\Phi(\omega_1,\omega_2)$. Let $(\omega_1,\omega_2) \in [-\frac{1}{2},\frac{1}{2}]^2$. Then, we have

$$G_\Phi(\omega_1,\omega_2) = \begin{pmatrix} \sin^2(2\pi\omega_1) & -e^{-2\pi i\omega_2}\sin(2\pi\omega_1)\cos(2\pi\omega_1) \\ -e^{2\pi i\omega_2}\sin(2\pi\omega_1)\cos(2\pi\omega_1) & \cos^2(2\pi\omega_1) \end{pmatrix}.$$

Note that $G_\Phi(\omega_1,\omega_2) = G^2_\Phi(\omega_1,\omega_2)$ for all $(\omega_1,\omega_2) \in [-\frac{1}{2},\frac{1}{2}]^2$. Thus, by Theorem 1.5.3, $E(\phi_1,\phi_2)$ is a frame for V. Further, it can be seen that $\mathrm{rk}(G_\Phi(\omega_1,\omega_2)) = 1$ for *a.e.* $(\omega_1,\omega_2) \in [-\frac{1}{2},\frac{1}{2}]^2$ and then V has length 1. Moreover, V is the Paley-Wiener space

$$PW = \{f \in L^2(\mathbb{R}^2) : \; supp(\widehat{f}) \subseteq [-1/2,1/2]^2\}.$$

Let $A \in \mathbb{C}^{1\times 2}$. Without loss of generality we can suppose that $A = (a_1 \; a_2)$ with $|a_1|^2 + |a_2|^2 = 1$. Then, A can be written as $A = (\cos(\theta)e^{2\pi i\beta} \quad \sin(\theta)e^{2\pi i\beta'})$ for $\theta \in [0,\frac{\pi}{2}]$ and $\beta,\beta' \in \mathbb{R}$.

Therefore, the Gramian associated to $A\Phi$ is

$$\begin{aligned} AG_\Phi(\omega_1,\omega_2)A^* &= \sin^2(2\pi\omega_1)\cos^2(\theta) + \cos^2(2\pi\omega_1)\sin^2(\theta) \\ &\quad -2\cos(2\pi(w_2-\beta'+\beta))\sin(2\pi\omega_1)\cos(2\pi\omega_1)\sin(\theta)\cos(\theta). \end{aligned}$$

Observe that for each θ, β and β' fixed,

$$AG_\Phi(\omega_1, \omega_2)A^* \neq 0 \quad \text{for a.e. } (\omega_1, \omega_2) \in [-1/2, 1/2]^2.$$

In particular,

$$\operatorname{rk}(G_\Phi(\omega_1, \omega_2)) = \operatorname{rk}(AG_\Phi(\omega_1, \omega_2)A^*) \quad \text{for a.e. } (\omega_1, \omega_2) \in [-1/2, 1/2]^2$$

and then, every matrix A preserves generators.

Let $\tilde{\omega}_2 \in [-1/2, 1/2]$ such that $\tilde{\omega}_2 = \beta' - \beta + k$ for some $k \in \mathbb{Z}$. Then,

$$\begin{aligned} AG_\Phi(\omega_1, \tilde{\omega}_2)A^* &= \sin^2(2\pi\omega_1)\cos^2(\theta) + \cos^2(2\pi\omega_1)\sin^2(\theta) \\ &\quad -2\sin(2\pi\omega_1)\cos(2\pi\omega_1)\sin(\theta)\cos(\theta) \\ &= \sin^2(2\pi\omega_1 - \theta). \end{aligned}$$

Now, taking $\tilde{\omega}_1 = \frac{\theta}{2\pi}$, we get $AG_\Phi(\tilde{\omega}_1, \tilde{\omega}_2)A^* = 0$. Then, since the Gramian associated to $A\Phi$ is a continuos function with a zero, condition (b) in item (3) of Theorem 1.5.3 can never be fullfiled. Thus, $E(A\Phi)$ can not be a frame for V.

■

Capítulo 3: Extra invariancia en espacios invariantes por traslaciones enteras

(Resumen)

En este capítulo repasaremos resultados ya conocidos acerca de la caracterización de los espacios invariantes por traslaciones enteras que no solo son invariantes bajo estas traslaciones si no que presentan además invariancia sobre un conjunto particular de traslaciones en $\mathbb{R}^d$.

Para un espacio invariante por traslaciones enteras (EITE) $V \subseteq L^2(\mathbb{R}^d)$, definimos su conjunto de invariancia de la siguiente manera

$$M_V = \{x \in \mathbb{R}^d \colon T_x f \in V, \forall f \in V\}.$$

Diremos que un subespacio cerrado $V \subseteq L^2(\mathbb{R}^d)$ es un espacio $M-$invariante si $T_m f \in V$ para todo $m \in M$ y para todo $f \in V$.

Un caso límite resulta cuando $M = \mathbb{R}^d$. En este caso diremos que V es un espacio invariante por cualquier traslación real. Un ejemplo en $\mathbb{R}$ de este tipo de espacios es el espacio de Paley-Wiener de funciones que son de banda limitada en $[-1/2, 1/2]$ definido de la siguiente forma

$$PW = \{f \in L^2(\mathbb{R}) \colon \text{ sop } (\widehat{f}) \subseteq [-1/2, 1/2]\}.$$

Este espacio resulta invariante bajo cualquier traslación real pues si $f \in PW$ y $x \in \mathbb{R}$, tenemos que

$$\text{sop } (\widehat{t_x f}) = \text{sop } (e^{-2\pi i x}\widehat{f}) = \text{sop } (\widehat{f}).$$

Luego, $t_x f \in PW$ para todo $x \in \mathbb{R}$.

Es fácil probar que, para un espacio medible $\Omega \subseteq \mathbb{R}^d$, los espacios

$$V_\Omega := \{f \in L^2(\mathbb{R}^d) \colon \text{sop}(\widehat{f}) \subseteq \Omega\} \tag{2.4}$$

resultan invariantes bajo cualquier traslación real. Más aún, el Teorema de Wiener (ver [Hel64]) prueba que cualquier espacio cerrado invariante bajo cualquier traslación real de $L^2(\mathbb{R}^d)$ es de la forma (2.4).

Por otro lado, existen espacios invariantes por traslaciones enteras que sólo son invariantes bajo traslaciones enteras. Por ejemplo, es fácil ver que el siguiente espacio principal invariante por traslaciones enteras es sólo invariante por dichas traslaciones:

$$V = S(\chi_{[0,1)}) = \overline{\operatorname{span}}\{t_k\chi_{[0,1)} \colon k \in \mathbb{Z}\}.$$

Sea V un espacio invariante por traslaciones enteras de longitud ℓ y sea M un subgrupo aditivo de $\mathbb{R}^d$ que contiene a $\mathbb{Z}^d$. Diremos que V tiene extra invariancia M si V es $M-$invariante. Notemos que en este caso, si Φ es un conjunto de generadores de V, i.e. $V = S(\Phi)$, entonces

$$S(\Phi) = \overline{\operatorname{span}}\{T_k\phi \colon \phi \in \Phi, k \in \mathbb{Z}^d\} = \overline{\operatorname{span}}\{T_\alpha\phi \colon \phi \in \Phi, \alpha \in M\}.$$

Es interesante analizar cómo es la estructura del conjunto de invariancia M_V para un EITE dado $V \subseteq L^2(\mathbb{R}^d)$. Siguiendo esta dirección, en [ACHKM10] los autores caracterizan aquellos espacios invariantes por traslaciones enteras $V \subseteq L^2(\mathbb{R})$ que tienen extra invariancia. Ellos muestran que si V es un espacio invariante por traslaciones enteras, entonces su conjunto de invariancia es un subgrupo cerrado de $\mathbb{R}$ que contiene a $\mathbb{Z}$. Como consecuencia de este resultado, como cualquier subgrupo aditivo de $\mathbb{R}$ es o bien discreto o denso, entonces se tiene que o V resulta invariante bajo cualquier traslación real o existe un máximo número entero n tal que V es $\frac{1}{n}\mathbb{Z}-$invariante. Además, en [ACHKM10], los autores establecen distintas caracterizaciones acerca de cuándo un espacio invariante por traslaciones enteras resulta además $\frac{1}{n}\mathbb{Z}-$invariante.

Generalizando los resultados anteriores, en [ACP11] los autores consideran el caso d-dimensional. Ellos caracterizan la extra invariancia del EITE V para el caso donde M_V no es todo $\mathbb{R}^d$. La mayor diferencia con el caso uno dimensional es que existen subgrupos en $\mathbb{R}^d$ que no son ni discretos ni densos. Luego, no se obtiene en forma directa que todas las caracterizaciones dadas para $\mathbb{R}$ en [ACHKM10] sigan siendo válidas en varias variables.

Por otro lado, en [ACP10], los autores consideran otra generalización del problema anterior. Más precisamente, ellos consideran el mismo problema pero en el contexto de espacios invariantes por traslaciones en grupos localmente compactos y abelianos (LCA).

El capítulo está organizado de la siguiente manera. En la Section 3.2 repasamos la estructura del conjunto de invariancia para un espacio invariante por traslaciones enteras y además incluimos algunos resultados acerca de los subgrupos cerrados aditivos en $\mathbb{R}^d$. Por otro lado, presentamos resultados concernientes a la estructura de los espacios $M-$invariantes principales. Repasamos algunas caracterizaciones realizadas por [ACP11], acerca de la extra invariancia de espacios invariantes bajo traslaciones enteras de $L^2(\mathbb{R}^d)$ en la Section 3.3.

3

Extra invariance of shift invariant spaces

3.1 Introduction

In this chapter we review known results about the characterization of shift invariant spaces that are not only invariant under integer translations, but are also invariant under some particular set of translations of $\mathbb{R}^d$.

For a shift invariant space, $V \subseteq L^2(\mathbb{R}^d)$ we can define its invariance set as follows

$$M_V = \{x \in \mathbb{R}^d : T_x f \in V, \forall f \in V\}.$$

We will say that a closed subspace $V \subseteq L^2(\mathbb{R}^d)$ is $M-$invariant if $T_m f \in V$ for all $m \in M$ and for all $f \in V$.

A limit case is when $M = \mathbb{R}^d$. In this case we will say that V is a translation invariant subspace. One example of a translation invariant space in $\mathbb{R}$ is the Paley-Wiener space of functions that are bandlimited to $[-1/2, 1/2]$ defined by

$$PW := \{f \in L^2(\mathbb{R}) : \text{ supp } (\widehat{f}) \subseteq [-1/2, 1/2]\}.$$

This space is a translation invariant space since if $f \in PW$ and $x \in \mathbb{R}$, we have that supp $(\widehat{T_x f}) =$ supp $(\widehat{e^{-2\pi i x} f}) =$ supp $(\widehat{f})$, thus $T_x f \in PW$ for all $x \in \mathbb{R}$.

It is easy to prove that for a measurable set $\Omega \subseteq \mathbb{R}^d$, the space

$$V_\Omega := \{f \in L^2(\mathbb{R}^d) : \text{supp}(\widehat{f}) \subseteq \Omega\} \tag{3.1}$$

is translation invariant. Moreover, Wiener's theorem (see [Hel64]) proves that any closed translation invariant subspace of $L^2(\mathbb{R}^d)$ is of the form (3.1).

On the other hand, there exist shift invariant spaces that are only invariant under integer translates. For example, it is easy to see that the following principal shift invariant space is only invariant under integer translates:

$$V = S(\chi_{[0,1)}) = \overline{\text{span}}\{T_k \chi_{[0,1)} : k \in \mathbb{Z}\}.$$

If V is a shift invariant space of length ℓ and M is an additive subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$, we will say that V has extra invariance M if V is $M-$invariant. Note that in this case, if Φ is a set of generators of V, i.e. $V = S(\Phi)$, then

$$S(\Phi) = \overline{\operatorname{span}}\{T_k\phi \colon \phi \in \Phi, k \in \mathbb{Z}^d\} = \overline{\operatorname{span}}\{T_\alpha\phi \colon \phi \in \Phi, \alpha \in M\}.$$

One interesting question is how is the structure of the invariance set M_V for a SIS $V \subseteq L^2(\mathbb{R}^d)$ given. In this direction, in [ACHKM10] the authors characterize those shift invariant spaces $V \subseteq L^2(\mathbb{R})$ that have extra invariance. They showed that if V is a shift invariant space then the invariance set is a closed additive subgroup of $\mathbb{R}$ containing $\mathbb{Z}$. As a consequence, since every additive subgroup of $\mathbb{R}$ is either discrete or dense, then either V is translation invariant, or there exists a maximum positive integer n such that V is $\frac{1}{n}\mathbb{Z}-$invariant. They established different characterizations of when a shift invariant space is $\frac{1}{n}\mathbb{Z}-$ invariant.

In [ACP11] the authors consider the d-dimensional case. They characterize the extra invariance of V when M is not all $\mathbb{R}^d$. The main difference with the one dimensional case is that there are subgroups of $\mathbb{R}^d$ that are neither discrete or dense. So, it is not straightforward that all the characterizations given in [ACHKM10] are still valid in several variables.

The chapter is organized as follows. In Section 3.2 we resume the structure of the invariance set and the closed additive subgroups in $\mathbb{R}^d$. Also, we present a result about the structure of principal $M-$invariant spaces. We present characterizations for the extra invariance of shift invariant spaces in $L^2(\mathbb{R}^d)$ in Section 3.3.

3.2 Preliminaries

In this section we present some known results concerning the structure of the invariance set of a shift invariant space in $\mathbb{R}^d$ and the structure of the additive closed subgroups of $\mathbb{R}^d$ containing $\mathbb{Z}^d$. We will not go into details. We refer the reader to [Bou74, ACP11] and the references therein. Also, we present results about the structure of principal $M-$invariant spaces (see [ACP11] for details).

Definition 3.2.1. Let $V \subseteq L^2(\mathbb{R}^d)$ be a shift invariant space. We define the invariance set of V by

$$M_V = \{x \in \mathbb{R}^d \colon T_x f \in V, \forall f \in V\}. \tag{3.2}$$

If Φ is a set of generators for V, it is easy to check that

$$M_V = \{x \in \mathbb{R}^d \colon T_x\varphi \in V, \forall \varphi \in \Phi\}.$$

Proposition 3.2.2. *Let V be a SIS of $L^2(\mathbb{R}^d)$ and let M_V be defined as in (3.2). Then M_V is an additive closed subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$.*

Then, we are interested in closed subgroups of $\mathbb{R}^d$ containing $\mathbb{Z}^d$. For their understanding, the following definition is important.

Definition 3.2.3. Let M be a subgroup of $\mathbb{R}^d$. Consider the set

$$M^* = \{x \in \mathbb{R}^d : \langle x, m\rangle \in \mathbb{Z},\ \forall m \in M\}.$$

Then M^* is a subgroup of $\mathbb{R}^d$. In particular, $(\mathbb{Z}^d)^* = \mathbb{Z}^d$.

Also, we give the definition of range and dimension of a closed subgroup of $\mathbb{R}^d$.

Definition 3.2.4. Given M a subgroup of $\mathbb{R}^d$, the range of M, denoted by $r(M)$, is the dimension of the subspace generated by M as a real vector space.

Definition 3.2.5. Given M a closed subgroup of $\mathbb{R}^d$, there exists a subspace whose dimension is the largest of the dimensions of all the subspaces contained in M. We will denote by $d(M)$ the dimension of V. Note that $d(M)$ can be zero.

Remark 3.2.6. Given M a subgroup of $\mathbb{R}^d$,

$$0 \leq d(M) \leq r(M) \leq d.$$

Now we list some properties.

Proposition 3.2.7. *Let M, N be subgroups of $\mathbb{R}^d$.*

(a) M^* is a closed subgroup of $\mathbb{R}^d$.

(b) If $N \subseteq M$, then $M^* \subseteq N^*$.

(c) If M is closed, then $r(M^*) = d - d(M)$ and $d(M^*) = d - r(M)$.

(d) $(M^*)^* = \overline{M}$.

Let H be a subgroup of $\mathbb{Z}^d$ with $r(H) = q$, we will say that a set $\{v_1, \ldots, v_q\} \subseteq H$ is a basis for H if for every $x \in H$ there exist unique $k_1, \ldots, k_q \in \mathbb{Z}$ such that

$$x = \sum_{i=1}^{q} k_i v_i.$$

Observe that $\{v_1, \ldots, v_d\} \subseteq \mathbb{Z}^d$ is a basis for $\mathbb{Z}^d$ if and only if the determinant of the matrix A which has $\{v_1, \ldots, v_q\}$ as columns is 1 or -1.

Given $B = \{v_1, \ldots, v_d\}$ a basis for $\mathbb{Z}^d$, we call $\widetilde{B} = \{w_1, \ldots, w_d\}$ a dual basis for B if $\langle v_i, w_j\rangle$ for all $1 \leq i, j \leq d$.

The following result gives a characterization of closed subgroups of $\mathbb{R}^d$ containing $\mathbb{Z}^d$. For the proof see [Bou74].

Theorem 3.2.8. *Let $M \subseteq \mathbb{R}^d$. The following conditions are equivalent:*

(a) M is a closed subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$ and $d(M) = d - q$.

(b) There exist a basis $\{v_1, \dots, v_d\}$ for $\mathbb{Z}^d$ and integers $a_1, \dots, a_q$ satisfying $a_{i+1} \equiv 0 \,(mod\ a_i)$ for all $1 \le i \le q-1$, such that

$$M = \left\{ \sum_{i=1}^{q} k_i \frac{1}{a_i} v_i + \sum_{j=q+1}^{d} t_j v_j : \, k_i \in \mathbb{Z}, t_j \in \mathbb{R} \right\}.$$

Furthermore, the integers q and $a_1, \dots, a_q$ are uniquely determined by M.

Remark 3.2.9. If $\{v_1, \dots, v_d\}$ and $a_1, \dots, a_q$ are as in Theorem 3.2.8, let us define the linear transformation T as

$$T : \mathbb{R}^d \to \mathbb{R}^d, \quad T(e_i) = v_i, \quad \forall\, 1 \le i \le d.$$

Then T is an invertible transformation that satisfies

$$M = T\left(\frac{1}{a_1}\mathbb{Z} \times \cdots \times \frac{1}{a_q}\mathbb{Z} \times \mathbb{R}^{d-q} \right).$$

If $\{w_1, \dots, w_d\}$ is the dual basis for $\{v_1, \dots, v_d\}$, the inverse of the adjoint of T is defined by

$$(T^*)^{-1} : \mathbb{R}^d \to \mathbb{R}^d, \quad (T^*)^{-1}(e_i) = w_i, \quad \forall\, 1 \le i \le d.$$

Therefore

$$M^* = (T^*)^{-1}(a_1\mathbb{Z} \times \cdots \times a_q\mathbb{Z} \times \{0\}^{d-q}).$$

From now on, M will be a closed subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$ and M^* the group defined as in Definition 3.2.3.

We will denote for a set of functions $\Phi \subseteq L^2(\mathbb{R}^d)$, $\widehat{\Phi} = \{\widehat{\varphi} : \varphi \in \Phi\}$.

Definition 3.2.10. We will say that a closed subspace $V \subseteq L^2(\mathbb{R}^d)$ is $M-$invariant if $t_m f \in V$ for all $m \in M$ and $f \in V$.

Given $\Phi \subseteq L^2(\mathbb{R}^d)$, the $M-$invariant space generated by Φ is

$$S_M(\Phi) = \overline{\text{span}}\,\{t_m\varphi : \, m \in M, \varphi \in \Phi\}.$$

If $\Phi = \{\varphi\}$ we write $S_M(\varphi)$ instead of $S_M(\{\varphi\})$ and we say that $S_M(\varphi)$ is a principal $M-$invariant space. When $M = \mathbb{Z}^d$, we will simply write $S(\varphi)$ instead of $S_{\mathbb{Z}^d}(\varphi)$.

Principal shift invariant spaces have been completely characterized by [dBDR94, RS95] as follows.

Theorem 3.2.11. *Let $f \in L^2(\mathbb{R}^d)$ be given. If $g \in S(f)$, then there exists a $\mathbb{Z}^d$−periodic function η such that $\widehat{g} = \eta \widehat{f}$.*

Conversely, if η is a $\mathbb{Z}^d$−periodic function such that $\eta\widehat{f} \in L^2(\mathbb{R}^d)$, then the function g defined by $\widehat{g} = \eta\widehat{f}$ belong to $S(f)$.

In [ACP11], the authors generalize the previous theorem to the M−invariant case. In case that M is discrete, Theorem 3.2.11 follows easily by rescaling. The difficulty arises when M is not discrete. Theorem 3.2.11 was proved in [dBDR94] for the lattice case. In [ACP11], the authors adapt their arguments to prove the following result.

Theorem 3.2.12. *Let $f \in L^2(\mathbb{R}^d)$ and M a closed subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$. If $g \in S_M(f)$, then there exists an M^*−periodic function η such that $\widehat{g} = \eta\widehat{f}$.*

Conversely, if η is an M^*−periodic function such that $\eta\widehat{f} \in L^2(\mathbb{R}^d)$, then the function g defined by $\widehat{g} = \eta\widehat{f}$ belong to $S_M(f)$.

3.3 Characterization of extra invariance of SIS in $L^2(\mathbb{R}^d)$

Given M a closed subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$, in this section we will present results proved in [ACP11] about characterizations of M−invariance of SIS V. We refer to this work for more details and proofs.

In order to give these characterizations, the authors construct a partition $\{B_\sigma\}_{\sigma\in\mathcal{N}}$ of $\mathbb{R}^d$, where each B_σ will be an M^*−periodic set and the index set $\mathcal{N}$ will be properly chosen later. Using this partition, for each $\sigma \in \mathcal{N}$, they define the subspaces

$$V_\sigma = \{f \in L^2(\mathbb{R}^d) : \widehat{f} = \chi_{B_\sigma}\widehat{g}, \ \text{whit } g \in V\}. \tag{3.3}$$

Using this partition, the authors characterize the M−invariance of V in terms of the subspaces V_σ.

Theorem 3.3.1. *If $V \subseteq L^2(\mathbb{R}^d)$ is a SIS and M is a closed subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$, then the following are equivalent.*

(i) V is M−invariant.

(ii) $V_\sigma \subseteq V$ for all $\sigma \in \mathcal{N}$.

Moreover, in case of any of the above holds, we have that V is the orthogonal direct sum

$$V = \dot{\bigoplus}_{\sigma\in\mathcal{N}} V_\sigma.$$

Remark 3.3.2. From the above theorem we deduce that a SIS V is M−invariant if and only if

$$V = \dot{\bigoplus}_{\sigma\in\mathcal{N}} V_\sigma.$$

The following lemma is used by the authors in [ACP11] to prove the above theorem. We will not give here the proof of this theorem but we will include the lemma because it results interesting by itself.

Lemma 3.3.3. *Let V be a SIS and $\sigma \in \mathcal{N}$. Suppose that the subspace V_σ defined in (3.3) satisfies $V_\sigma \subseteq V$. Then, V_σ is a closed subspace which is $M-$invariant and in particular is a SIS.*

Now we define the partition $\{B_\sigma\}_{\sigma\in\mathcal{N}}$, in such a way that each B_σ is an M^*-periodic set.

Let Ω be a section of the quotient $\mathbb{R}^d/\mathbb{Z}^d$ given by

$$\Omega = (T^*)^{-1}([0,1)^d),$$

where T is as in Remark 3.2.9. Then Ω tiles $\mathbb{R}^d$ by $\mathbb{Z}^d$ translations, that is,

$$\mathbb{R}^d = \bigcup_{k\in\mathbb{Z}^d} \Omega + k.$$

We observe that, although the sets $\{(\Omega + k) + M^*\}_{k\in\mathbb{Z}^d}$ are M^*-periodic, they are not a partition of $\mathbb{R}^d$. So, we need to choose a subset $\mathcal{N} \subseteq \mathbb{Z}^d$ such that if $\sigma, \sigma' \in \mathcal{N}$ and $\sigma + M^* = \sigma' + M^*$, then $\sigma = \sigma'$. That implies that $\mathcal{N}$ is a section of the quotient $\mathbb{Z}^d/M^*$. For example, we can define $\mathcal{N}$ by

$$\mathcal{N} = (T^*)^{-1}(\{0,\dots,a_1-1\}\times\cdots\times\{0,\dots,a_q-1\}\times\mathbb{Z}^{d-q}),$$

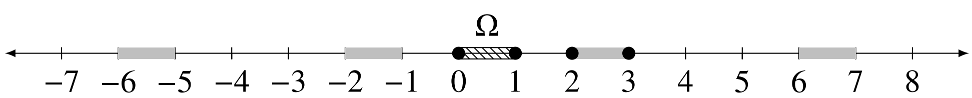
where $\{a_1,\dots,a_q\}$ are as in Therem 3.2.8. Then, given $\sigma \in \mathcal{N}$, we define

$$B_\sigma := \Omega + \sigma + M^* = \bigcup_{m^*\in M^*} (\Omega + \sigma) + m^*. \tag{3.4}$$

Example 3.3.4. Let $M = \frac{1}{n}\mathbb{Z} \subseteq \mathbb{R}$. Then $M^* = n\mathbb{Z}$, $\Omega = [0,1)$ and $\mathcal{N} = \{0,\dots,n-1\}$. Therefore,

$$B_\sigma = \bigcup_{m^*\in n\mathbb{Z}} ([0,1)+\sigma) + m^* = \bigcup_{j\in\mathbb{Z}} [\sigma,\sigma+1) + nj, \qquad \text{for each } \sigma \in \{0,\dots,n-1\}.$$

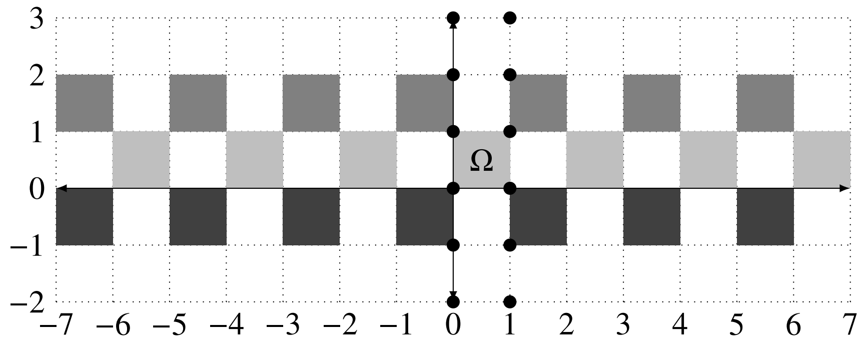
The following figure illustrates the partition for $n = 4$. In this, the black dots represent the set $\mathcal{N}$ and the set B_2 is the one which appears in gray.



Example 3.3.5. Let $M = \frac{1}{2}\mathbb{Z} \times \mathbb{R}$. Then $M^* = 2\mathbb{Z} \times \{0\}$, $\Omega = [0,1)^2$ and $\mathcal{N} = \{0,1\} \times \mathbb{Z}$. Then,

$$B_{(i,j)} = \bigcup_{k\in\mathbb{Z}} ([0,1)^2 + (i,j)) + (2k,0), \qquad \text{for each } (i,j) \in \mathcal{N}.$$

In the following figure we have the sets $B_{(0,0)}$, $B_{(1,1)}$ and $B_{(1,-1)}$ which are represented by the squares painted in light gray, gray and dark gray respectively. The set $\mathcal{N}$ is represented by the black dots.



On the other hand, in [ACP11], the authors give a characterization of the $M-$invariance for a finitely shift invariant spaces in terms of the fibers and also in terms of the Gramian.

For $f \in L^2(\mathbb{R}^d)$ and $\sigma \in \mathcal{N}$, we will denote f^σ as the function defined by

$$\widehat{f^\sigma} = \widehat{f}\chi_{B_\sigma}.$$

Let P_σ be the orthogonal projection onto S_σ, where

$$S_\sigma = \{f \in L^2(\mathbb{R}^d) : \operatorname{supp}(\widehat{f}) \subseteq B_\sigma\}.$$

Then, we have that

$$f^\sigma = P_\sigma f \quad \text{ and } \quad V_\sigma = P_\sigma(V) = \{f^\sigma : f \in V\}.$$

Let V be a SIS and V_σ be defined as in (3.3). If Φ is a countable subset of $L^2(\mathbb{R}^d)$ such that $V = S(\Phi)$, then for $\omega \in \Omega$, we define the subspace

$$\overline{\operatorname{span}}\{\tau\varphi^\sigma(\omega) : \varphi \in \Phi\}. \tag{3.5}$$

We observe that in case of V_σ is closed, it is a SIS and this subspace is the fiber space $J_{V_\sigma}(\omega)$ defined in Proposition 1.4.3.

Remark 3.3.6. We observe that $\tau\varphi^\sigma(\omega) = \{\chi_{B_\sigma}(\omega+k)\widehat{\varphi}(\omega+k)\}_{k\in\mathbb{Z}^d}$. Then, since $\chi_{B_\sigma}(\omega+k) \neq 0$ if and only if $k \in \sigma + M^*$,

$$\chi_{B_\sigma}(\omega+k)\widehat{\varphi}(\omega+k) = \begin{cases} \widehat{\varphi}(\omega+k) & \text{if } k \in \sigma + M^* \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $\sigma \neq \sigma'$, $J_{V_\sigma}(\omega)$ is orthogonal to $J_{V_{\sigma'}}(\omega)$ for a.e. $\omega \in \Omega$.

In [ACP11], the authors prove the following result that characterizes the $M-$invariance of a SIS in terms of fibers.

Theorem 3.3.7. *Let V be a SIS generated by a countable set $\Phi \subseteq L^2(\mathbb{R}^d)$. The following statements are equivalent.*

(i) *V is $M-$invariant.*

(ii) *$\tau\varphi^{\sigma}(\omega) \in J_V(\omega)$ a.e. $\omega \in \Omega$ for all $\varphi \in \Phi$ and $\sigma \in \mathcal{N}$.*

The authors also obtain a slightly simpler characterization of $M-$invariance for the finitely generated case, in terms of the Gramian and the dimension function.

Theorem 3.3.8. *If V is a finitely generated SIS generated by Φ, then the following statements are equivalent.*

(i) *V is $M-$invariant.*

(ii) *For almost every $\omega \in \Omega$, $dim_V(\omega) = \sum_{\sigma\in\mathcal{N}} dim_{V_\sigma}(\omega)$.*

(iii) *For almost every $\omega \in \Omega$, $rank(G_\Phi(\omega)) = \sum_{\sigma\in\mathcal{N}} rank(G_{\Phi^\sigma}(\omega))$, where $\Phi^\sigma = \{\varphi^\sigma : \varphi \in \Phi\}$.*

Capítulo 4: Aproximación de datos con subespacios con extra invariancia

(Resumen)

Sea H un espacio de Hilbert, $\mathcal{S}$ una clase de subespacios cerrados de H y $\mathcal{F} = \{f_1, \ldots, f_m\}$ un conjunto finito de elementos en H.

En este capítulo estudiamos la existencia y mostramos cómo construir un subespacio óptimo S en la clase $\mathcal{S}$ que maximice la distancia a un conjunto de datos dado $\mathcal{F}$, en el sentido que S minimiza el funcional $\mathcal{E}(\mathcal{F}, S)$ en $\mathcal{S}$. El funcional está definido de la siguiente manera

$$\mathcal{E}(\mathcal{F}, S) = \sum_{j=1}^{m} \|f_j - P_S f_j\|^2, \tag{3.6}$$

donde P_S denota la proyección ortogonal sobre el subespacio S.

La motivación para buscar un subespacio óptimo en $\mathcal{S}$ es que, en muchas situaciones se quiere modelar un cierto conjunto de datos. En lugar de imponer ciertas condiciones sobre los datos para que estos se adapten a un cierto modelo conocido, la idea es definir una clase amplia conveniente de subespacios y encontrar un elemento de esa clase que " mejor aproxime " al conjunto de datos.

Las señales a ser modeladas forman tipicamente un conjunto de dimensión pequeña pero dentro de un espacio de dimensión más grande. Sin embargo, como en las aplicaciones las señales son observadas con ruido, se convierten en un conjunto de dimensión grande pero cercano a un subespacio de dimensión pequeña que es el que se busca.

Cuando el espacio de Hilbert que se considera es $L^2(\mathbb{R}^d)$ resulta natural considerar como modelo para nuestro conjunto de datos la clase de los espacios invariantes por traslaciones enteras. Estos espacios se han usado en teoría de aproximación, análisis armónico, teoría de wavelets, teoría de muestreo y procesamiento de imágenes (ver, e.g., [AG01, Gro01, HW96, Mal89] y las referencias que en ellos se encuentran).

En las aplicaciones a menudo se supone que las señales a estudiar pertenecen a cierto espacio invariante por traslaciones enteras V generado por las traslaciones enteras de un conjunto finito de funciones $\Phi = \{\varphi_1, \ldots, \varphi_m\}$, i.e., $V = S(\Phi) = \overline{\text{span}}\{T_k\varphi_i : k \in \mathbb{Z}^d, i = 1, \ldots, m\}$.

La elección del espacio invariante por traslaciones enteras finitamente generado particular tipicamente no se deduce del conjunto de señales dado. Por ejemplo, en teoría de muestreo, una suposición clásica es que el conjunto de señales a ser muestreadas sean de banda limitada, es decir, las mismas pertenezcan a un espacio invariante por traslaciones enteras V generado por $\varphi(x) = \text{sinc}(x)$. Sin embargo, la suposición de ser de banda limitada no es muy aplicable en muchas aplicaciones. Luego, resulta natural buscar espacios invariantes por traslaciones enteras finitamente generados más cercanos a un conjunto de datos dado.

En este capítulo estudiaremos este problema para el caso donde $H = L^2(\mathbb{R}^d)$. Para este caso, consideraremos como clase de subespacios aproximantes, la clase de los espacios invariantes por traslaciones enteras con extra-invariancia, es decir: si M es un subgrupo de $\mathbb{R}^d$ que contiene a $\mathbb{Z}^d$, consideraremos subespacios $S(\varphi_1, \ldots, \varphi_m)$ que sean M extra-invariantes. Luego, el espacio $S(\varphi_1, \ldots, \varphi_m)$ resulta invariante no sólo bajo traslaciones enteras, a pesar de estar generado por las traslaciones enteras de un conjunto finito de funciones.

Primero vamos a considerar el caso donde el subgrupo M está contenido estrictamente en $\mathbb{R}^d$ y contiene a $\mathbb{Z}^d$. Para este caso probamos que para cualquier conjunto finito de datos $\mathcal{F} = \{f_1, \ldots, f_m\} \subseteq L^2(\mathbb{R}^d)$, para cualquier subgrupo propio M que contiene a $\mathbb{Z}^d$ y para cualquier $\ell \in \mathbb{N}$, siempre existe un espacio invariante por traslaciones enteras V de longitud a lo sumo ℓ con extra-invariancia M cuya distancia (en el sentido de (3.6)) al conjunto de datos $\mathcal{F}$ es lo más chica posible, si minimizamos sobre todos los espacios invariantes por traslaciones enteras de longitud menor o igual que ℓ que son M extra-invariantes.

Mostramos cómo construir una solución V para este problema de aproximación y exhibimos para este espacio solución un conjunto de generadores cuyas traslaciones enteras forman un marco ajustado de V. Además damos una expresión para el error $\mathcal{E}(\mathcal{F}, V)$ entre el conjunto de datos y el espacio óptimo, en función de los autovalores de una matriz especial.

Si en vez de tomar M como un subgrupo propio de $\mathbb{R}^d$, consideramos el caso donde $M = \mathbb{R}^d$, estamos en el caso en que la extra-invariancia es *total*. Es decir, espacios generados por las traslaciones enteras de un número finito de funciones que son $\mathbb{R}^d$–invariantes. En este caso, un teorema de Wiener afirma que el espacio resulta isométrico a $L^2(\Omega)$ para algún subconjunto medible $\Omega \subseteq \mathbb{R}^d$. Estos son espacios de Paley-Wiener generalizados. Para estos espacios probamos que, si las traslaciones enteras de los generadores forman una base de Riesz, entonces el subconjunto Ω asociado es un "multi-tile" de $\mathbb{R}^d$. Estudiamos nuestro problema de aproximación para estos espacios de Paley-Wiener generalizados. Exhibimos para este caso cómo construir un conjunto de generadores del espacio óptimo y además mostramos una conexión interesante con resultados recientes acerca de bases de exponenciales.

Finalmente consideramos un problema similar a los dos anteriores pero para el caso donde el espacio de Hilbert es $\ell^2(\mathbb{Z}^d)$ y la $\mathcal{S}$ es una clase de subespacios elegida convenientemente. Obtenemos para este caso resultados equivalentes a los hallados para el caso de $L^2(\mathbb{R}^d)$. Resulta importante observar que el problema de aproximación para el caso discreto está relacionado con el caso continuo como describiremos en Subsection 4.3.3.

El problema de aproximar un conjunto de datos dado por espacios invariantes por traslaciones enteras (sin imponer ninguna restricción acerca de la extra-invariancia) comenzó en [ACHM07] donde los autores probaron la existencia de un subespacio minimizante para (3.6) sobre la clase de subespacios de dimensión pequeña en un espacio de Hilbert $\mathcal{H}$ y además sobre la clase de los espacios invariantes por traslaciones enteras en $L^2(\mathbb{R}^d)$.

En [ACM08] se considera el caso de múltiples subespacios para el caso finito dimensional. Es decir, los autores construyen una unión de subespacios de dimensión pequeña que mejor aproxima a un conjunto de datos dado en $\mathbb{R}^d$ y además presentan un algoritmo para hallar dicha unión. En el 2011, usando técnicas de reducción dimensional, este algoritmo es mejorado (ver [AACM11]).

Además, en [AT11] los autores encuentran condiciones necesarias y suficientes para la existencia de subespacios óptimos en el contexto general de espacios de Hilbert. Sin embargo, los autores no exhiben cómo puede llevarse a cabo la construcción de los mismos.

El primer resultado sobre aproximación de un conjunto finito de datos usando espacios invariantes por traslaciones enteras con extra-invariancia aparece en [AKTW12], donde los autores consideran espacios invariantes por traslaciones enteras principales en una variable y suponiendo además que los espacios poseen un generador con traslaciones enteras ortogonales.

El capítulo está organizado de la siguiente forma. En Section 4.2 repasamos algunos resultados previos referentes al problema de encontrar un subespacio más cercano a un conjunto de datos dado. Además, incluimos otros resultados que necesitaremos para las pruebas. Los principales resultados del capítulo son enunciados y probados en Section 4.3. En Subsection 4.3.1 presentamos el caso de espacios invariantes por traslaciones enteras que son M extra-invariantes, en Subsection 4.3.2 consideramos el caso de espacios de Paley-Wiener generalizados y en Subsection 4.3.3 estudiamos el caso discreto.

4

Subspaces with extra invariance nearest to observed data

4.1 Introduction

Let H be a Hilbert space, $\mathcal{S}$ a class of closed subspaces of H and $\mathcal{F} = \{f_1, \dots, f_m\}$ a finite set of elements in H.

In this chapter we study the existence and show how to construct an optimal subspace S in the class $\mathcal{S}$ that minimizes the distance to the given data $\mathcal{F}$, in the sense that S minimizes the functional $\mathcal{E}(\mathcal{F}, S)$ over $\mathcal{S}$. The functional is defined as

$$\mathcal{E}(\mathcal{F}, S) := \sum_{j=1}^{m} \|f_j - P_S f_j\|^2, \tag{4.1}$$

where P_S denotes the orthogonal projection on the subspace S.

The motivation to find an optimal subspace in $\mathcal{S}$ is, that in many situations one wants to choose a model for a certain class of data. Instead of imposing some conditions on the data to fit some known model, the idea is to define a large class of subspaces convenient for the application at hand, and find from there the one that "best fits" the data under study.

The signals that need to be modelled are ideally low dimensional but living in a high dimensional space. However, since in applications they are often corrupted by noise, they become high dimensional, however they are close to a low dimensional subspace, which is the space one seeks.

When the Hilbert space is $L^2(\mathbb{R}^d)$ it is natural to consider as a model for our data the class of shift invariant spaces. These spaces have been used in approximation theory, harmonic analysis, wavelet theory, sampling theory and signal processing (see, e.g., [AG01, Gro01, HW96, Mal89] and references therein). Often, in applications, it is assumed that the signals under study belong to some shift invariant space V generated by the translations of a finite set of functions $\Phi = \{\varphi_1, \dots, \varphi_m\}$, i.e., $V = S(\Phi) = \overline{\operatorname{span}}\{T_k\varphi_i : k \in \mathbb{Z}^d, i = 1, \dots, m\}$.

The choice of the particular finitely generated shift invariant space typically is not deduced from a set of signals. For example in sampling theory, a classical assumption is that the signals to be sampled are band-limited, that is, they belong to the shift invariant space V generated by $\varphi(x) = \text{sinc}(x)$. However, the band-limited assumption is not very realistic in many applications. Thus, it is natural to search for a finitely generated shift invariant space that is nearest to a set of some observed data.

In this chapter we study the case when $H = L^2(\mathbb{R}^d)$. For this case, we restrict the class of approximating subspaces to be shift invariant spaces that have extra-invariance, that is: If M is a subgroup of $\mathbb{R}^d$ such that $\mathbb{Z}^d \subseteq M$ we consider subspaces $S(\varphi_1, \ldots, \varphi_m)$ M extra-invariant. Therefore, the space $S(\varphi_1, \ldots, \varphi_m)$ is invariant under translates other than the integers, even though it is generated by the integer translates of a finite set of functions. Such spaces with extra-invariance are important in applications specially in those where the jitter error is an issue.

We first consider the case when the subgroup M is a proper subgroup of $\mathbb{R}^d$ that contains $\mathbb{Z}^d$. For that case we obtain one of the main contribution of this chapter. We prove that for any finite set of data $\mathcal{F} = \{f_1, \ldots, f_m\} \subseteq L^2(\mathbb{R}^d)$, for any proper subgroup M containing $\mathbb{Z}^d$ and for any $\ell \in \mathbb{N}$ there always exists a SIS V of length at most ℓ with extra invariance M whose distance (in the sense of (4.1)) to the data $\mathcal{F}$ is the smallest possible among all the SIS of length smaller or equal than ℓ that are M extra-invariant.

We construct a solution V and provide a set of generators whose integer translates form a tight frame of V. An expression for the exact value of the error $\mathcal{E}(\mathcal{F}, V)$ between the data and the optimal subspace is also obtained using the eigenvalues of some special matrix.

If instead of M being a proper subgroup of $\mathbb{R}^d$ we consider $M = \mathbb{R}^d$, we are in the case of *total* extra-invariance. That is, spaces generated by integer translates of a finite number of functions that are $\mathbb{R}^d$ invariant. In this case, a theorem of Wiener says that the space is isometric to $L^2(\Omega)$ for some measurable subset $\Omega \subseteq \mathbb{R}^d$. These are generalized Paley-Wiener spaces. We prove that for these spaces, if the integer translates of the generators form a Riesz basis then the associated set Ω multi-tiles $\mathbb{R}^d$ by translates along $\mathbb{Z}^d$. We study our approximation problem for those generalized Paley-Wiener spaces. We describe for this case how to construct a set of generators and show an interesting connection with recent results about bases of exponentials.

Finally we consider a similar problem when the Hilbert space is $\ell^2(\mathbb{Z}^d)$ and $\mathcal{S}$ is a conveniently chosen class of subspaces. We obtain for this case equivalent results to the ones for $L^2(\mathbb{R}^d)$. The approximation problem for the discrete case is related with the continuous case in a very interesting way that is described in Subsection 4.3.3.

The problem of approximation of a set of data by shift invariant spaces (without the extra invariance restriction) started in [ACHM07] where the authors proved the existence of a minimizer for (4.1) over the class of low dimensional subspaces in a Hilbert space H and also over the class of shift invariant spaces in $L^2(\mathbb{R}^d)$.

In [ACM08] the case of multiple subspaces was considered in the finitely dimensional case. That is, the authors found a union of low dimensional subspaces that best fits a given set of data in $\mathbb{R}^d$ and provided an algorithm to find it. In 2011, using dimensional reduction techniques, this algorithm was improved (see [AACM11]).

Further, in [AT11] the authors found necessary and sufficient conditions for the existence of optimal subspaces in the general context of Hilbert spaces. However, they did not provide a way to construct them.

The first result for approximation of a finite set of data using shift invariant spaces with extra-invariance constrains appears in [AKTW12], where the authors consider principal shift invariant spaces in one variable and they assume that the space has a generator with orthogonal integer translates.

The rest of this chapter is organized as follows. In Section 4.2 first we review some previous work about finding a subspace that is closest to a given data set. Also, we set some other results that we need. The main results of this chapter are stated and proved in Section 4.3. In Subsection 4.3.1 we present the M extra-invariant case for shift invariant spaces, in Subsection 4.3.2 we consider the case of Paley-Wiener spaces and in Subsection 4.3.3 we study a discrete case.

4.2 Preliminaries

We begin with a review about previous related work. The results will be established without proofs, but we will provide references where the proofs can be found.

Given a Hilbert space H, $\mathcal{F} = \{f_1, \dots, f_m\}$ a set of vectors in H, and a class of closed subspaces $\mathcal{S}$ of H, the problem of finding a subspace $S^* \in \mathcal{S}$ that best approximates the data $\mathcal{F}$ has several applications in mathematics, engineering and computer science.

Definition 4.2.1. We will say that a set of closed subspaces $\mathcal{S}$ of a separable Hilbert space H has the Minimum Subspace Approximation Property (MSAP) if for every subset $\mathcal{F} \subseteq H$ there exists an element $S^* \in \mathcal{S}$ such that minimizes the expression

$$\mathcal{E}(\mathcal{F}, S) = \sum_{f \in \mathcal{S}} d^2(f, S), \quad \text{over all } S \in \mathcal{S}. \tag{4.2}$$

Any subspace $S^* \in \mathcal{S}$ satisfying (4.2) will be called an optimal subspace for $\mathcal{F}$.

In [AT11], the authors provide necessary and sufficient conditions on the class $\mathcal{S}$ of closed subspaces in a separable Hilbert space H so that $\mathcal{S}$ satisfies the MSAP.

First we begin studying the problem of finding optimal subspaces in the case $H = \mathbb{R}^d$ and $\mathcal{S}$ being the class of all subspaces of dimension at most ℓ (with $\ell << d$). Then, we want to find a subspace S^* of dimension at most ℓ that is close to a given finte data $\mathcal{F} = \{f_1, \dots, f_m\} \subseteq \mathbb{R}^d$ in the sense that S^* satisfies

$$\sum_{j=1}^{m} \|f_j - P_{S^*} f_j\|^2 \leq \sum_{j=1}^{m} \|f_j - P_S f_j\|^2, \quad \forall\, S \in \mathcal{S}_\ell,$$

where $\mathcal{S}_\ell$ is the class of all subspaces of dimension at most ℓ.

This problem is solved by Eckart-Young's Theorem (see [EY36, Sch07]) which uses the Singular Value Decomposition (SVD) of a matrix and its relation to finite dimensional least squares problems. Before stating the theorem, we will review the construction of the SVD of a matrix (for a detailed treatment see [Bha97, HJ85]).

Let $A = [f_1, \dots, f_m] \in \mathbb{R}^{d\times m}$ and $r = \text{rank}(A)$. Consider the matrix $A^*A \in \mathbb{R}^{m\times m}$. Since A^*A is self-adjoint and positive semi-definite, its eigenvalues $\lambda_1 \geq \cdots \geq \lambda_m$ are nonnegative and the associated eigenvectors $y_1, \dots, y_m$ can be chosen to form an orthonormal basis of $\mathbb{R}^m$. We observe that the rank r of A corresponds to the largest index i such that $\lambda_i > 0$, then we have that $\lambda_1 \geq \cdots \geq \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_m$. The left singular vectors $u_1, \dots, u_r$ can then be obtained from

$$\sqrt{\lambda_i} u_i = A y_i, \quad \text{that is} \quad u_i = \lambda_i^{-1/2} \sum_{j=1}^{m} y_{ij} a_{ij}, \quad 1 \leq i \leq r.$$

Here $y_i = (y_{i1}, \dots, y_{im})^t$. The remaining left singular vectors $u_{r+1}, \dots, u_m$ can be chosen to be any orthonormal collection of $m - r$ vectors in $\mathbb{R}^d$ that are perpendicular to span $\{f_1, \dots, f_m\}$. Then we can obtain the following SVD of A

$$A = U \Lambda^{1/2} Y^*,$$

where $U \in \mathbb{R}^{d\times m}$ is the matrix with columns $\{u_1, \dots, u_m\}$, $\Lambda^{1/2} = \text{diag}\,(\lambda_1^{1/2}, \dots, \lambda_m^{1/2})$ and $Y \in \mathbb{R}^{m\times m}$ is the matrix with columns $\{y_1, \dots, y_m\}$, with $U^*U = I_m = YY^*$.

Then, we state the Eckart-Young's Theorem.

Theorem 4.2.2. *Let $\mathcal{F} = \{f_1, \dots, f_m\}$ be a set of vectors in $\mathbb{R}^d$, $A = [f_1, \dots, f_m] \in \mathbb{R}^{d\times m}$, with $r = rank(A)$. Suppose that A has SVD $A = U\Lambda^{1/2}Y^*$ and that $0 < \ell \leq r$. If $S^* = span\,\{u_1, \dots, u_\ell\}$, then*

$$\sum_{j=1}^{m} \|f_j - P_{S^*} f_j\|^2 \leq \sum_{j=1}^{m} \|f_j - P_S f_j\|^2, \quad \forall\, S \in \mathcal{S}_\ell.$$

Moreover,

$$\mathcal{E}(\mathcal{F}, \mathcal{S}_\ell) = \sum_{j=\ell+1}^{r} \lambda_j,$$

where $\lambda_1 \geq \cdots \geq \lambda_r > 0$ are the positive eigenvalues of AA^*.

Then, we have that in the Hilbert space $H = \mathbb{R}^d$, the class $\mathcal{S}_\ell$ of all subspaces of dimension at most ℓ has the MSAP. Therefore, given a finite data $\mathcal{F} = \{f_1, \dots, f_m\} \subseteq \mathbb{R}^d$, there exists $S^* \in \mathcal{S}_\ell$ such that approximates the data $\mathcal{F}$. Also, the above theorem shows how to construct the generators of the optimal space S^* and gives an expression for the error in this approximation.

In [ACHM07], the authors consider the above problem but in a general context. More precisely, given a finite data $\mathcal{F}$ in an arbitrary Hilbert space H, they proved that there always exists a subspace S^* with dimension at most ℓ that approximates the data $\mathcal{F}$. They state the following theorem which is an extension of the Eckart-Young's theorem.

Theorem 4.2.3 (Theorem 4.1, [ACHM07]). *Let H be a Hilbert space, $\mathcal{F} = \{f_1, \dots, f_m\} \subseteq H$, and $\ell \in \mathbb{N}$. Define $X =$ span $\{f_1, \dots, f_m\}$, and let $\lambda_1 \geq \cdots \geq \lambda_m$ be the eigenvalues of the matrix $\mathcal{B}(\mathcal{F})$ defined as $[\mathcal{B}(\mathcal{F})]_{i,j} = \langle f_i, f_j \rangle_H$ and $y_1, \dots, y_m \in \mathbb{C}^d$, with $y_i = (y_{i_1}, \dots, y_{i_m})^t$ orthonormal left eigenvectors associated to the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_m$. Let $r = \dim X =$ rank $\mathcal{B}(\mathcal{F})$. Define the vectors $q_1, \dots, q_\ell \in H$ by*

$$q_i = \theta_i \sum_{j=1}^{m} y_{i_j} f_j, \quad i = 1, \dots, \ell$$

where $\theta_i = \lambda_i^{-1/2}$ if $\lambda_i \neq 0$ and $\theta_i = 0$ otherwise. Then $\{q_1, \dots, q_\ell\}$ is a Parseval frame of $S^* =$ span $\{q_1, \dots, q_\ell\}$ and the subspace S^* is optimal in the sense that

$$\sum_{i=1}^{m} \|f_i - P_{S^*} f_i\|^2 \leq \sum_{i=1}^{m} \|f_i - P_S f_i\|^2, \quad \forall\, S \in \mathcal{S}_\ell.$$

Furthermore we have the following formula for the error

$$\mathcal{E}(\mathcal{F}, \mathcal{S}_\ell) = \sum_{i=\ell+1}^{m} \lambda_i.$$

We observe that if r is small, that is, $r \leq \ell$, then all the vectors $q_{r+1}, \dots, q_\ell$ are null and $\{q_1, \dots, q_r\}$ is an orthonormal set.

Moreover, one could choose $q_{r+1}, \dots, q_\ell$ to be any orthonormal set in the orthogonal complement of X and so obtain an orthonormal set of ℓ elements and the formula for the error in the above theorem would still hold.

Then, using Theorem 4.2.3, we obtain that the class $\mathcal{S}_\ell$ of all subspaces of dimension at most ℓ has the MSAP in an arbitrary Hilbert space H. Also, the authors give a way to construct the generators of an optimal subspace and present a formula for the error in this approximation.

Now, we consider the problem of finding optimal subspaces in the case $H = L^2(\mathbb{R}^d)$ and $\mathcal{S}$ being the class of all shift invariant spaces of length at most ℓ. We will denote this class $\mathcal{V}^\ell$. Then, we want to find a SIS $V^* \in \mathcal{V}^\ell$ that is closed to a given finite data $\mathcal{F} = \{f_1, \dots, f_m\} \subseteq L^2(\mathbb{R}^d)$, in the sense that V^* satisfies

$$\sum_{j=1}^{m} \|f_j - P_{V^*} f_j\|^2 \leq \sum_{j=1}^{m} \|f_j - P_V f_j\|^2, \quad \forall\, V \in \mathcal{V}^\ell, \tag{4.3}$$

and estimate the error of this approximation.

Observe that in the two above cases, we take the minimum over all subspaces of dimension at most ℓ, while in this problem the minimization is taken over a particular class of infinite dimensional subspaces, so the problems are essentially different, so the SVD cannot be applied directly. However, due to the special structure of shift invariant spaces, the Fourier transform converts this problem into finite dimensional least square problems at each frequency. The following theorem shows that the above problem has a solution, that is, there exists a SIS of length at most ℓ which satisfies (4.3). Recall from Definition 1.5.2 that for a given set $\mathcal{F} = \{f_1, \dots, f_m\} \subseteq L^2(\mathbb{R}^d)$, the Gramian $G_{\mathcal{F}} \in \mathbb{C}^{m\times m}$ is defined as $[G_{\mathcal{F}}(\omega)]_{ij} = \langle \tau f_j(\omega), \tau f_i(\omega)\rangle_{\ell^2(\mathbb{Z}^d)}$ for all $1 \le i, j \le m$, where $\tau f(\omega) = \{\widehat{f}(\omega + k)\}_{k\in\mathbb{Z}^d}$.

Theorem 4.2.4 (Theorem 2.3, [ACHM07]). *Let $\mathcal{F} = \{f_1, \dots, f_m\}$ be a set of functions in $L^2(\mathbb{R}^d)$. Let $\lambda_1(\omega) \ge \cdots \ge \lambda_m(\omega)$ be the eigenvalues of the Gramian $G_{\mathcal{F}}(\omega)$. Then, there exists $V^* \in \mathcal{V}^\ell$ such that*

$$\sum_{i=1}^{m} \|f_i - P_{V^*} f_i\|^2 \le \sum_{i=1}^{m} \|f_i - P_V f_i\|^2, \quad \forall\, V \in \mathcal{V}^\ell.$$

Moreover, we have that

(1) The eigenvalues $\lambda_i(\omega)$, $1 \le i \le m$ are $\mathbb{Z}^d$–periodic, measurable functions in $L^2([0,1]^d)$ and

$$\mathcal{E}(\mathcal{F}, \ell) = \sum_{i=\ell+1}^{m} \int_{[0,1]^d} \lambda_i(\omega)\, d\omega.$$

(2) Let $\theta_i(\omega) = \lambda_i^{-1}(\omega)$ if $\lambda_i(\omega)$ is different from zero, and zero otherwise. Then, there exists a choice of measurable left eigenvectors $Y^1(\omega), \dots, Y^\ell(\omega)$ with $Y^i = (y_1^i, \dots, y_m^i)^t$, $i = 1, \dots, \ell$, associated with the first ℓ largest eigenvalues of $G_{\mathcal{F}}(\omega)$ such that the functions defined by

$$\widehat{\varphi}_i(\omega) = \theta_i(\omega) \sum_{j=1}^{m} y_j^i(\omega)\widehat{f}_j(\omega), \quad i = 1, \dots, \ell,\ \omega \in \mathbb{R}^d$$

are in $L^2(\mathbb{R}^d)$. Furthermore, the corresponding set of functions $\Phi = \{\varphi_1, \dots, \varphi_\ell\}$ is a generator set for the optimal subspace V^* and the set $\{\varphi_i(\cdot - k), k \in \mathbb{Z}^d, i = 1, \dots, \ell\}$ is a Parseval frame for V^*.

Then, we have that the class $\mathcal{V}^\ell$ of finitely generated SIS of $L^2(\mathbb{R}^d)$ with length at most ℓ has the MSAP.

As we mention above, in order to find a solution of this new problem, the authors reduce it into an uncountable set of finite dimensional problems in the Hilbert space $H = \ell^2(\mathbb{Z}^d)$, which can be solved using Theorem 4.2.3. Finally, they construct the generators of the optimal space patching together the solutions of the reduced problems to obtain a solution to the original problem.

For the proof they also use the following technical lemma concerning the measurability of the eigenvalues and the existence of measurable eigenvectors of a non-negative matrix with measurable entries. We will include here this result since we will need it later for prove our results.

Lemma 4.2.5 (Lemma 2.3.5, [RS95]). *Let $G(\omega)$ be an $m \times m$ self-adjoint matrix of measurable functions defined on a measurable subset $E \subseteq \mathbb{R}^d$ with eigenvalues $\lambda_1(\omega) \geq \cdots \geq \lambda_m(\omega)$. Then the eigenvalues λ_i, $i = 1, \ldots, m$, are measurable functions on E and there exists an $m \times m$ matrix of measurable functions $U(\omega)$ on E such that $U(\omega)U^*(\omega) = I$ a.e. $\omega \in E$ and such that*

$$G(\omega) = U(\omega)\Lambda(\omega)U^*(\omega), \qquad a.e.\ \omega \in E,$$

where $\Lambda(\omega) := diag(\lambda_1(\omega), \ldots, \lambda_m(\omega))$.

Now, we consider the problem of finding a union of low dimensional subspaces that best fits a given set of data in a Hilbert space H.

Since we are interested in models that are union of subspaces, we will arrange the subspaces in finite bundles that will be our main objects, and define the error between a bundle and a set of vectors.

Let us fix $m, n \in \mathbb{N}$ with $1 \leq n \leq m$, let $\mathcal{F} = \{f_1, \ldots, f_m\} \subseteq H$ and let $\mathcal{S}$ be a family of closed subspaces of H containing the zero subspace. Define the collection of bundles $\mathcal{B}$ to be the set of sequences of elements in $\mathcal{S}$ of length n, that is,

$$\mathcal{B} = \{V = \{V_1, \ldots, V_n\} \colon V_i \in \mathcal{S}, 1 \leq i \leq n\}.$$

For $V \in \mathcal{B}$, with $V = \{V_1, \ldots, V_n\}$, we define the error between a bundle V and $\mathcal{F}$ as follows

$$\mathbf{e}(\mathcal{F}, V) = \sum_{j=1}^{m} \min_{1 \leq i \leq n} d^2(f_j, V_i),$$

where d is the distance in H. Note that $\mathbf{e}(\mathcal{F}, V)$ is a non-linear function of $\mathcal{F}$. Also, we observe that, in the case $n = 1$, this error agrees with the error given in the above problems, that is,

$$\mathcal{E}(\mathcal{F}, \{V\}) = \mathbf{e}(\mathcal{F}, V) = \sum_{j=1}^{m} d^2(f_j, V).$$

The following theorem shows that the problem of finding an optimal union of subspaces has solution for every finite set of data $\mathcal{F} \subseteq H$ and every $n \geq 1$ if and only if the class $\mathcal{S}$ has the MSAP.

Theorem 4.2.6 ([ACM08]). *Let $\mathcal{F} = \{f_1, \ldots, f_m\}$ be a vectors in H and let n be given (with $n < m$). If $\mathcal{S}$ satisfies the MSAP, then there exists a bundle $V^* = \{V_1^*, \ldots, V_n^*\} \in \mathcal{B}$ such that*

$$\mathbf{e}(\mathcal{F}, V^*) = \inf\{\mathbf{e}(\mathcal{F}, V) \colon V \in \mathcal{B}\}. \tag{4.4}$$

Any $V^* \in \mathcal{B}$ that satisfies (4.4) will be called an optimal bundle for $\mathcal{F}$.

As we mentioned above, the class $\mathcal{V}^\ell$ of finitely generated shift invariant spaces of $H = L^2(\mathbb{R}^d)$ with length at most ℓ has the MSAP. Then, by Theorem 4.2.6, there exists a solution for the problem of optimal union of finitely generated SISs.

On the other hand, by Theorem 4.2.3, we obtain that the class $\mathcal{S}_\ell$ of all subspaces of dimension at most ℓ has the MSAP in an arbitrary Hilbert space H. Then using Theorem 4.2.6 we have that there exists a union of $\ell-$dimensional subspaces which is closest to a given data set.

Note that $\mathbf{e}(\mathcal{F}, \mathbf{V})$ in Theorem 4.2.6 is computed as follows: For each $f \in \mathcal{F}$ find the space $V_{j(f)}$ in $\mathbf{V}$ closest to f, compute $d^2(f, V_{j(f)})$, and the sum over all values found by letting f run through $\mathcal{F}$. This procedure is not feasible in practice and a search algorithm is needed. So, in [ACM08], the authors provided an algorithm to find the solution. In 2011, using dimension reduction techniques, this algorithm was improved (see [AACM11]). We will not include here all the results concerning these algorithms since we would need a lot of notation and technical aspects which are not congruent with the general line of this thesis. For a complete treatment, we refer the reader to [AACM11, ACM08].

Finally, we consider optimal models with extra invariance. We observe that, although Theorem 4.2.4 shows the existence of a finitely generated shift invariant space which is closest to a finite set of data, it does not provide an answer if we require the shift invariant space to have some extra invariance. In this chapter we prove that given an arbitrary finite set of data it always exists a finitely shift invariant space with extra invariance that minimize the distance to the data. Also we will consider a similar problem for the case of total extra invariance.

The first result about approximation using shift invariant spaces with extra invariance constrains appear in [AKTW12], where the authors consider principal shift invariant spaces in $L^2(\mathbb{R})$ and they suppose that the space has an *orthonormal generator* φ, that is, $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for the space $S(\varphi)$.

As we mentioned in Chapter 3, principal shift invariant spaces can also be invariant under a set of translates that strictly contains $\mathbb{Z}$. Also, we know that, in the real line, the invariance set M_V for a shift invariant space V is $\mathbb{R}$ or $\frac{1}{n}\mathbb{Z}$ for some $n \in \mathbb{N}$. In [AKTW12], the authors denote by $\mathcal{V}$ the class of all principal SIS with an orthonormal generator which are also translation invariant (that is, $M_V = \mathbb{R}$). Also, they consider $\mathcal{V}_n$ being the class of all principal SIS with an orthonormal generator which are also $\frac{1}{n}\mathbb{Z}-$invariant. With this notations they prove the following result.

Theorem 4.2.7. *Let $n \geq 2$ and let $\mathcal{F} = \{f_1, \ldots, f_m\} \subseteq L^2(\mathbb{R})$ be a set of functions. Then*

(i) There exists $V^* = S(\varphi) \in \mathcal{V}$ such that

$$\sum_{j=1}^{m} \|f_j - P_{V^*} f_j\|^2 \leq \sum_{j=1}^{m} \|f_j - P_V f_j\|^2, \text{ for all } V \in \mathcal{V}.$$

(ii) There exists $V^* = S(\varphi) \in \mathcal{V}_n$ such that

$$\sum_{j=1}^{m} \|f_j - P_{V^*} f_j\|^2 \leq \sum_{j=1}^{m} \|f_j - P_V f_j\|^2, \text{ for all } V \in \mathcal{V}_n.$$

We want to remark here that a key element in their proof is that the space is principal and the generator has orthogonal integer translates. So theirs techniques do not apply to our general case.

4.3 Main results

4.3.1 Optimality for the class of SIS with extra-invariance

In this subsection we consider the approximation problem for the class of finitely generated SIS with extra invariance under a given *proper* subgroup M of $\mathbb{R}^d$.

Let us start introducing some notation. Let $m, \ell \in \mathbb{N}$, M be a closed proper subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$ and $\mathcal{F} = \{f_1, \ldots, f_m\} \subseteq L^2(\mathbb{R}^d)$. Define

$$\mathcal{V}_M^\ell := \{V : V \text{ is a SIS of length at most } \ell \text{ and } V \text{ is } M\text{-invariant}\}. \tag{4.5}$$

Let $\mathcal{N} = \{\sigma_1, \ldots, \sigma_\kappa\}$ be a section of the quotient $\mathbb{Z}^d/M^*$ and $\{B_\sigma : \sigma \in \mathcal{N}\}$ the partition defined in (3.4).

For each $\sigma \in \mathcal{N}$, we consider $\mathcal{F}^\sigma = \{f_1^\sigma, \ldots, f_m^\sigma\} \subseteq L^2(\mathbb{R}^d)$ where, f_j^σ is such that $\widehat{f_j^\sigma} = \widehat{f_j}\chi_{B_\sigma}$ for $j = 1, \ldots, m$. Also, let

$$\widetilde{\mathcal{F}} = \{f_1^{\sigma_1}, \ldots, f_m^{\sigma_1}, \ldots\ldots, f_1^{\sigma_\kappa}, \ldots, f_m^{\sigma_\kappa}\}.$$

For each $\omega \in \mathcal{U} \subseteq \mathbb{R}^d$ (a measurable set of representatives of the quotient $\mathbb{R}^d/\mathbb{Z}^d$) let $G_{\widetilde{\mathcal{F}}}(\omega)$ be the associated Gramian matrix of the vectors in $\widetilde{\mathcal{F}}$ with eigenvalues

$$\lambda_1(\omega) \geq \cdots \geq \lambda_{m\kappa}(\omega) \geq 0.$$

Using Lemma 4.2.5, we have that these eigenvalues are measurable functions.

Since $f_i^{\sigma_s}$ is orthogonal to $f_i^{\sigma_t}$ if $s \neq t$, the Gramian $G_{\widetilde{\mathcal{F}}}(\omega)$ is a diagonal block matrix with blocks $G_\sigma(\omega)$, $\sigma \in \mathcal{N}$. Here $G_\sigma(\omega)$ denotes the $m \times m$ Gramian associated to the data $\mathcal{F}^\sigma$.

On the other hand, using Lemma 4.2.5 again, we have that

$$G_\sigma(\omega) = U_\sigma(\omega)\Lambda_\sigma(\omega)U_\sigma^*(\omega) \quad a.e.\ \omega \in \mathcal{U}$$

where U_σ are unitary and $\Lambda_\sigma(\omega) := \operatorname{diag}(\lambda_1^\sigma(\omega), \ldots, \lambda_m^\sigma(\omega)) \in \mathbb{C}^{m\times m}$ and they are also measurable matrices as in Lemma 4.2.5. We also have $\lambda_1^\sigma(\omega) \geq \cdots \geq \lambda_m^\sigma(\omega)$ for each $\sigma \in \mathcal{N}$.

Using the decompositions of the blocks G_σ we have that

$$G_{\widetilde{\mathcal{F}}}(\omega) = U(\omega)\Lambda(\omega)U^*(\omega) \tag{4.6}$$

where U has blocks U_σ in the diagonal, and Λ is diagonal with blocks Λ_σ.

We want to recall here that for almost each $\omega \in \mathcal{U}$ the matrix $\Lambda(\omega)$ collects all the eigenvalues of the Gramian $G_{\widetilde{\mathcal{F}}}(\omega)$ and the columns of the matrix $U(\omega)$ are the associated left eigenvectors. Note that an eigenvector associated to the eigenvalue $\lambda_j^\sigma(\omega)$ has all the components not corresponding to the block σ equal to zero.

Now for each fixed $\omega \in \mathcal{U}$, we consider $\{(i_1(\omega), j_1(\omega)), \ldots, (i_n(\omega), j_n(\omega))\}$ with $i_s(\omega) \in \mathcal{N}$ and $j_s(\omega) \in \{1, \ldots, m\}$ and $n = m\kappa$ such that

$$\lambda_{j_1(\omega)}^{i_1(\omega)} \geq \cdots \geq \lambda_{j_n(\omega)}^{i_n(\omega)} \geq 0$$

are the ordered eigenvalues of $G_{\widetilde{\mathcal{F}}}(\omega)$, with corresponding left eigenvectors $Y^{(i_s(\omega), j_s(\omega))} \in \mathbb{C}^n$, for $s = 1, \cdots, n$.

Here $i_s(\omega)$ indicates the block of the matrix $G_{\widetilde{\mathcal{F}}}(\omega)$ in which the eigenvalue $\lambda_{j_s(\omega)}^{i_s(\omega)}(\omega)$ is found and $j_s(\omega)$ indicates the displacement in this block of the matrix $G_{\widetilde{\mathcal{F}}}(\omega)$. More precisely, we have that $\lambda_{j_s(\omega)}^{i_s(\omega)}(\omega)$ coincides with $\lambda_{(i_s(\omega)-1)m+j_s(\omega)}(\omega)$, the $((i_s(\omega) - 1)m + j_s(\omega))$–th eigenvalue of $G_{\widetilde{\mathcal{F}}}(\omega)$. When $\omega \in \mathcal{U}$ is fixed, we will write i_s instead of $i_s(\omega)$ and j_s instead of $j_s(\omega)$.

We will prove now that $\gamma_s(\omega) := \lambda_{j_s(\omega)}^{i_s(\omega)}(\omega)$ is measurable as a function on ω for each $s = 1, \ldots, n$.

Let $s \in \{1, \ldots, n\}$ fixed. Let $i_s(\omega) \in \mathcal{N}$ and $j_s(\omega) \in \{1, \ldots, m\}$. We have that $\gamma_s(\omega) = \lambda_j^\sigma(\omega)$ for all $\omega \in E_{\sigma j} := \{\omega \in \mathcal{U} \colon i_s(\omega) = \sigma, j_s(\omega) = j\}$.

We observe that

$$E_{\sigma j} = \{\omega \in \mathcal{U} \colon i_s(\omega) = \sigma, j_s(\omega) = j\} = \{\omega \in \mathcal{U} \colon \lambda_s(\omega) = \lambda_j^\sigma(\omega)\}.$$

Using Lemma 4.2.5 applied to $G_{\widetilde{\mathcal{F}}}(\omega)$ and $G_\sigma(\omega)$, we have that λ_s and λ_j^σ are measurable functions of ω. Therefore $E_{\sigma j}$ are measurable sets for all $\sigma \in \mathcal{N}$ and for all $j \in \{1, \ldots, m\}$.

We further observe that $\gamma_s(\omega) = \lambda_j^\sigma(\omega)$, for $\omega \in E_{\sigma j}$. So $\gamma_s(\omega)$ is a measurable function. A similar argument shows that the eigenvectors are measurable.

Finally we define $h_s \colon \mathbb{R}^d \to \mathbb{C}$, for $s = 1, \ldots, \ell$

$$h_s(\omega) := \theta_{j_s}^{i_s}(\omega) \sum_{k=1}^{m} y_{(i_s-1)m+k}^{(i_s, j_s)}(\omega) \widehat{f_k^{i_s}}(\omega), \tag{4.7}$$

where $\theta_{j_s}^{i_s}(\omega) = (\lambda_{j_s}^{i_s}(\omega))^{-1/2}$ if $\lambda_{j_s}^{i_s}(\omega) \neq 0$ and $\theta_{j_s}^{i_s}(\omega) = 0$ otherwise.

Now we are ready to state the main result of this section.

Theorem 4.3.1. *Let $m, \ell \in \mathbb{N}$, and M be a closed proper subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$. Assume that $\mathcal{F} = \{f_1, \ldots, f_m\} \subseteq L^2(\mathbb{R}^d)$ is given data and let $\mathcal{V}_M^\ell$ be the class defined in* (4.5)*. Then, there exists a shift invariant space $V^* \in \mathcal{V}_M^\ell$ such that*

$$V^* = \underset{V \in \mathcal{V}_M^\ell}{argmin} \sum_{j=1}^{m} \|f_j - P_V f_j\|^2. \tag{4.8}$$

Furthermore, with the above notation,

(1) *The eigenvalues* $\{\lambda_j^\sigma(\omega) : \sigma \in \mathcal{N}, j = 1,\dots,m\}$, *are* $\mathbb{Z}^d$*−periodic, measurable functions in* $L^2(\mathcal{U})$ *and the error of approximation is*

$$\mathcal{E}(\mathcal{F}, M, \ell) := \sum_{j=1}^{m} \|f_j - P_{V^*} f_j\|^2 = \int_{\mathcal{U}} \sum_{s=\ell+1}^{m\kappa} \lambda_{j_s}^{i_s}(\omega)\, d\omega.$$

(2) *The functions* $\{h_1,\dots,h_\ell\}$ *defined in* (4.7) *are in* $L^2(\mathbb{R}^d)$ *and if* $\varphi_1,\dots,\varphi_\ell$ *are defined by* $\widehat{\varphi_j} = h_j$*, then* $\Phi = \{\varphi_1,\cdots,\varphi_\ell\}$ *is a generator set for the optimal subspace* V^* *and the set* $\{\varphi_i(\cdot - k), k \in \mathbb{Z}^d, i = 1,\dots,\ell\}$ *is a Parseval frame for* V^*.

Proof. Let $\mathcal{V}^\ell$ be the class as in Theorem 4.2.4, that is $\mathcal{V}^\ell$ is the set of all shift invariant spaces V that can be generated by ℓ or less generators. (Note that we do not ask the elements of the class $\mathcal{V}^\ell$ to have extra invariance.)

Define $V^* \in \mathcal{V}^\ell$ to be the optimal space given by Theorem 4.2.4 for the data $\widetilde{\mathcal{F}}$. That is, V^* satisfies

$$\sum_{\sigma\in\mathcal{N}} \sum_{j=1}^{m} \|f_j^\sigma - P_{V^*} f_j^\sigma\|^2 \le \sum_{\sigma\in\mathcal{N}} \sum_{j=1}^{m} \|f_j^\sigma - P_V f_j^\sigma\|^2 \qquad \forall\, V \in \mathcal{V}^\ell. \tag{4.9}$$

We claim that $V^* \in \mathcal{V}_M^\ell$ (in particular V^* is M extra-invariant) and it is optimal in this class for the data $\mathcal{F}$, i.e.

$$\sum_{j=1}^{m} \|f_j - P_{V^*} f_j\|^2 \le \sum_{j=1}^{m} \|f_j - P_V f_j\|^2 \qquad \forall\, V \in \mathcal{V}_M^\ell. \tag{4.10}$$

Let us prove first that V^* is M extra-invariant. For this we will check that the generators of V^* satisfy condition (ii) in Theorem 3.3.7. We have from (4.6) that the Gramian $G_{\widetilde{\mathcal{F}}}(\omega)$ can be decomposed as $G_{\widetilde{\mathcal{F}}}(\omega) = U(\omega)\Lambda(\omega)U^*(\omega)$ with eigenvalues $\{\lambda_j^\sigma(\omega) : \sigma \in \mathcal{N}, j = 1,\dots,m\}$.

By Theorem 4.2.4, the ℓ generators of V^* have the form defined in (4.7),

$$\widehat{\varphi}_s(\omega) = \theta_{j_s}^{i_s}(\omega) \sum_{k=1}^{m} y_{(i_s-1)m+k}^{(i_s,j_s)}(\omega) \widehat{f}_k^{i_s}(\omega), \qquad \text{for } s = 1,\dots,\ell. \tag{4.11}$$

From (4.11) it is clear that $\widehat{\varphi}_s$ is supported in B_{i_s}, since each $\widehat{f}_k^{i_s}$ is supported in B_{i_s}. Then if we apply the cut off operator to these generators we obtain

$$\widehat{\varphi_s^\sigma}(\omega) = \begin{cases} \widehat{\varphi}_s(\omega) & \text{if } \sigma = i_s(\omega), \\ 0 & \text{otherwise.} \end{cases}$$

The above implies that

$$\tau\varphi_s^\sigma(\omega) = \begin{cases} \tau\varphi_s(\omega) & \text{if } \sigma = i_s(\omega), \\ 0 & \text{otherwise.} \end{cases}$$

So, in any case $\tau\varphi_s^\sigma(\omega) \in J_{V^*}(\omega)$ for a.e. $\omega \in \mathcal{U}$ and for all $\sigma \in \mathcal{N}$, $s = 1, \ldots, \ell$ which proves the M-invariance of V^*.

What is left now is to prove that V^* is optimal over the class $\mathcal{V}_M^\ell$, that is V^* satisfies equation (4.10). For this note that if $V \in \mathcal{V}_M^\ell$ then $V = \bigoplus_{\sigma\in\mathcal{N}} V_\sigma$. So we have for any $f \in L^2(\mathbb{R}^d)$,

$$\|P_V f\|^2 = \|P_V \sum_{\sigma\in\mathcal{N}} f^\sigma\|^2 = \|\sum_{\sigma\in\mathcal{N}} P_V f^\sigma\|^2 = \|\sum_{\sigma\in\mathcal{N}} P_{V^\sigma} f^\sigma\|^2 = \sum_{\sigma\in\mathcal{N}} \|P_{V^\sigma} f^\sigma\|^2 = \sum_{\sigma\in\mathcal{N}} \|P_V f^\sigma\|^2,$$

which implies together with (4.9) that

$$\sum_{j=1}^m \|P_{V^*} f_j\|^2 \geq \sum_{j=1}^m \|P_V f_j\|^2, \qquad \forall\, V \in \mathcal{V}_M^\ell.$$

The others claims of the theorem are a direct consequence of Theorem 4.2.4.

□

4.3.2 Approximation with Paley-Wiener spaces

Preliminaries

In this section the class of approximation subspaces will be finitely generated SIS with total translation invariance. That is translation invariant spaces that are generated by the integer translates of a finite number of functions.

More precisely, given $\ell \in \mathbb{N}$ define $\mathcal{T}^\ell$ to be the set of all shift invariant spaces $V = S(\varphi_1, \ldots, \varphi_\ell)$ for some functions $\varphi_1, \ldots, \varphi_\ell$ in $L^2(\mathbb{R}^d)$, and such that V is translation invariant and the integer translates of $\{\varphi_1, \ldots, \varphi_\ell\}$ form a Riesz basis of V.

Given a set $\mathcal{F} = \{f_1, \ldots, f_m\} \subseteq L^2(\mathbb{R}^d)$, we want to find $V^* \in \mathcal{T}^\ell$ such that

$$V^* = \underset{V\in\mathcal{T}^\ell}{argmin} \sum_{j=1}^m \|f_j - P_V f_j\|^2. \tag{4.12}$$

Here P_V denotes the orthogonal projection on V.

Before going to the approximation problem, we will obtain a characterization of the class $\mathcal{T}^\ell$.

Using Wiener's theorem, we have that V is a translation invariant space in $L^2(\mathbb{R}^d)$ if and only if there exists a measurable set $\Omega \subseteq \mathbb{R}^d$ such that

$$V = \{f \in L^2(\mathbb{R}^d) : \widehat{f}(\omega) = 0 \text{ a.e. } \omega \in \mathbb{R}^d \setminus \Omega\}.$$

Since Ω is unique up to measure zero, we will write $V = V_\Omega$.

Definition 4.3.2. Let $\Omega \subseteq \mathbb{R}^d$ be measurable and $L \subseteq \mathbb{R}^d$ be a countable set. We say that Ω *tiles* $\mathbb{R}^d$ *when translated by* L *at level* $\ell \in \mathbb{N}$ if

$$\sum_{t \in L} \chi_\Omega(\omega - t) = \ell \quad \text{for a.e. } \omega \in \mathbb{R}^d.$$

In case of $L = \mathbb{Z}^d$ we will say that Ω is an ℓ *multi-tile*.

It is known (see for example [Ko15],) that Ω is an ℓ multi-tile of $\mathbb{R}^d$, if and only if, up to measure zero, Ω is the union of ℓ measurable and disjoint 1 tile sets. i.e. Ω is a quasi-disjoint union of ℓ sets of representatives of $\mathbb{R}^d/\mathbb{Z}^d$.

Lemma 4.3.3. *A measurable set* $\Omega \subseteq \mathbb{R}^d$*,* ℓ *multi-tiles* $\mathbb{R}^d$ *if and only if*

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_\ell \cup N,$$

where N *is a zero measure set, and the sets* Ω_j*,* $1 \le j \le \ell$ *are measurable, disjoint and each of them tiles* $\mathbb{R}^d$ *by translations on* $\mathbb{Z}^d$*.*

The following proposition characterizes the set Ω for the elements in $\mathcal{T}^\ell$.

Proposition 4.3.4. *A subspace* V *is in* $\mathcal{T}^\ell$ *if and only if* $V = V_\Omega$ *with* Ω *a measurable* ℓ *multi-tile of* $\mathbb{R}^d$*.*

Proof. Assume first that $V \in \mathcal{T}^\ell$, so $V = V_\Omega$ for some measurable $\Omega \subseteq \mathbb{R}^d$. Also, as a consequence of Wiener's theorem, for almost all $\omega \in \mathcal{U}$ we have $J_V(\omega) \cong \ell^2(O_\omega)$ with

$$O_\omega = \{k \in \mathbb{Z}^d : w + k \in \Omega\}.$$

To see this, we note that $J_V(\omega) \subseteq \ell^2(O_\omega)$. For the other inclusion, fix $\omega \in \mathcal{U}$. Using that

$$\Omega = \bigcup_{k \in \mathbb{Z}^d} E_k \quad \text{where} \quad E_k = (\mathcal{U} + k) \cap \Omega,$$

we have that $k \in O_\omega$, if and only if $\omega + k \in E_k$. Hence, if $a \in \ell^2(O_\omega)$ consider the function

$$G_\omega(\xi) := \sum_{k \in O_\omega} a_k \chi_{E_k}(\xi).$$

Since G_ω is in $L^2(\Omega)$, the function g defined by $\widehat{g} = G_\omega$ is in V, and $\widehat{g}(\omega + k) = a_k$ if $k \in O_\omega$. Therefore, $g \in V$ and $a = \tau g(\omega) \in J_V(\omega)$.

Now, since $V = S(\varphi_1, \ldots, \varphi_\ell)$, and the integer translates of $\varphi_1, \ldots, \varphi_\ell$ form a Riesz basis of V, using Theorem 1.5.1 we obtain that $\{\tau\varphi_1(\omega), \ldots, \tau\varphi_\ell(\omega)\}$ form a Riesz basis of $J_V(\omega)$ with the same Riesz bounds for a.e. $\omega \in \mathcal{U}$. We conclude that $\dim(J_V(\omega)) = \ell$ a. e. $\omega \in \mathcal{U}$.

Since V is translation invariant, by the observation above $\dim(J_V(\omega)) = \#O_w$. Then $\#O_\omega = \ell$ for almost all $\omega \in \mathcal{U}$, which implies that Ω is an ℓ multi-tile. (Here $\#A$ denote the cardinal of the set A).

For the converse, assume that Ω is a measurable ℓ multi-tile of $\mathbb{R}^d$. Define $V = V_\Omega$. So, V is translation invariant.

By Lemma 4.3.3 we have that $\Omega = \Omega_1 \cup \cdots \cup \Omega_\ell$ up to a measure zero set, where each Ω_j is a set of representatives or $\mathbb{R}^d/\mathbb{Z}^d$. We define φ_j by its Fourier transform:

$$\widehat{\varphi}_j = \chi_{\Omega_j}, \quad j = 1, \ldots, \ell.$$

Since $\{e^{2\pi i \omega k}\widehat{\varphi}_j : k \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(\Omega_j)$, we have that $\{e^{2\pi i \omega k}\widehat{\varphi}_j : k \in \mathbb{Z}^d, j = 1, \ldots, \ell\}$ is an orthonormal basis of $L^2(\Omega)$, and so, $\{t_k\varphi_j : k \in \mathbb{Z}^d, j = 1, \ldots, \ell\}$ is an orthonormal basis of V, in particular a Riesz basis. □

The approximation problem for Paley-Wiener Spaces

Now we come back to our approximation problem. In order to find an optimal subspace in the class $\mathcal{T}^\ell$ for a set of data $\mathcal{F} = \{f_1, \ldots, f_m\}$, it is enough to find the associated ℓ multi-tile Ω in $\mathbb{R}^d$.

It is not difficult to see that if we allow Ω to be *any* ℓ multi-tile the minimum in (4.12) may not exist. So we will restrict Ω to be inside a cube that could be arbitrarily large. Let us fix $N \in \mathbb{N}$. Define

$$\begin{aligned} C_N &:= [-(N+1/2), N+1/2]^d, \\ M_N^\ell &:= \{\Omega \subseteq C_N : \Omega \text{ is measurable and } \ell \text{ multi-tiles } \mathbb{R}^d\} \text{ and} \\ \mathcal{T}_N^\ell &:= \{V \in \mathcal{T}^\ell : V = V_\Omega \text{ with } \Omega \in M_N^\ell\}. \end{aligned}$$

With this notation we can state the main result of this section.

Theorem 4.3.5. *Assume that $m, \ell \in \mathbb{N}$ and a set $\mathcal{F} = \{f_1, \ldots, f_m\} \subseteq L^2(\mathbb{R}^d)$, are given. Then for each $N \geq \ell$ there exists a Paley-Wiener space $V^* \in \mathcal{T}_N^\ell$ that satisfies*

$$V^* = \underset{V \in \mathcal{T}_N^\ell}{argmin} \sum_{j=1}^m \|f_j - P_V f_j\|^2, \tag{4.13}$$

where $\mathcal{T}_N^\ell$ is the class defined above.

Proof. First we observe that if a solution space V^* exists then

$$V^* = \underset{V \in \mathcal{T}_N^\ell}{argmin} \sum_{j=1}^m \|f_j - P_V f_j\|^2 = \underset{V \in \mathcal{T}_N^\ell}{argmax} \sum_{j=1}^m \|P_V f_j\|^2, \tag{4.14}$$

and using the definition of $\mathcal{T}_N^\ell$, we have that

$$\max_{V \in \mathcal{T}_N^\ell} \sum_{j=1}^m \|P_V f_j\|^2 = \max_{\Omega \in M_N^\ell} \sum_{j=1}^m \|P_{V_\Omega} f_j\|^2. \tag{4.15}$$

So, we need to find $\Omega \in M_N^\ell$ that yields the maximum in (4.15).

Using Lemma 1.4.4 we see that for each $\Omega \in M_N^\ell$,

$$\begin{aligned}\sum_{j=1}^{m} \|P_{V_\Omega} f_j\|^2 &= \sum_{j=1}^{m} \|P_{\widehat{V_\Omega}} \widehat{f_j}\|^2 \\ &= \sum_{j=1}^{m} \int_{\mathcal{U}} \|P_{J_{V_\Omega}}(\omega)(\tau f_j(\omega))\|^2_{\ell^2(\mathbb{Z}^d)}\, d\omega \\ &= \int_{\mathcal{U}} \sum_{j=1}^{m} \|P_{J_{V_\Omega}}(\omega)(\tau f_j(\omega))\|^2_{\ell^2(\mathbb{Z}^d)}\, d\omega.\end{aligned} \tag{4.16}$$

Recall that $P_{J_{V_\Omega}}(\omega)$ denotes the orthogonal projection onto the closed subspace $J_{V_\Omega}(\omega)$ of $\ell^2(\mathbb{Z}^d)$.

Furthermore, if $\Omega \in M_N^\ell$, we know from the proof of Proposition 4.3.4 that $\dim(J_{V_\Omega}(\omega)) = \ell$ for a.e. $\omega \in \mathcal{U}$. Note that $J_{V_\Omega}(\omega)$ agrees with the subspace of $\ell^2(\mathbb{Z}^d)$ of the sequences supported in O_ω. Then there exists a unique set of ℓ integer vectors $\mathbf{k}^\Omega(\omega) := \{k_1^\Omega(\omega), \dots, k_\ell^\Omega(\omega)\} \subseteq \mathbb{Z}^d$ such that

$$\operatorname{span}\{\delta_{k_j^\Omega(\omega)} : j = 1, \dots, \ell\} = J_{V_\Omega}(\omega) \quad \text{for a.e. } \omega \in \mathcal{U}.$$

Here δ_j denotes the canonical vector in $\ell^2(\mathbb{Z}^d)$. i.e. $\delta_j(s) = 0$ if $s \neq j$ and 1 otherwise. Note that, since $\Omega \subseteq C_N$ necessarely $\|k_j^\Omega(\omega)\|_\infty \leq N$, for each j and ω. Combining this observation with (4.16) we obtain,

$$\sum_{j=1}^{m} \|P_{V_\Omega} f_j\|^2 = \int_{\mathcal{U}} \sum_{j=1}^{m} \sum_{s=1}^{\ell} |\widehat{f_j}(\omega + k_s^\Omega(\omega))|^2\, d\omega. \tag{4.17}$$

So, now we need to maximize the left hand side in (4.17) over all the sets $\Omega \in M_N^\ell$.

Note that given $\Omega \in M_N^\ell$, for almost each $\omega \in \mathcal{U}$, the set Ω contains exactly ℓ elements from the sequence $\{\omega + k, k \in \mathbb{Z}^d\}$. Then we can pick for each $\omega \in \mathcal{U}$ (up to a set of zero measure) ℓ translations $k_s^*(\omega)$ such that $\sum_{j=1}^m \sum_{s=1}^\ell |\widehat{f_j}(\omega + k_s^*(\omega))|^2$ is maximum over all sets of ℓ translations $\mathbf{k} = \{k_1, \dots, k_\ell\} \subseteq \mathbb{Z}^d$, with $\|k_j\|_\infty \leq N$. The maximum exists since the fibers of f_j are $\ell^2(\mathbb{Z}^d)$-sequences and the number of translations considered is finite.

Call $\mathcal{K}$ the set of admisibles translations i.e. $\mathcal{K} := \{\mathbf{k} = \{k_1, \dots, k_\ell\} \subseteq \mathbb{Z}^d : \|k_j\|_\infty \leq N\}$ and for $\mathbf{k} \in \mathcal{K}$ set

$$H_{\mathbf{k}}(\omega) := \sum_{j=1}^{m} \sum_{s=1}^{\ell} |\widehat{f_j}(\omega + k_s(\omega))|^2.$$

Our goal is to construct a set Ω such that the associated space V_Ω is optimal. So the idea is to construct the optimal set Ω^* considering for each $\omega \in \mathcal{U}$ the optimal translations $\{\omega + k_s^*(\omega) : s = 1, \dots, \ell\}$, and then taking the union over almost all $\omega \in \mathcal{U}$.

For this we define for each $\mathbf{k} = \{k_1, \ldots, k_\ell\} \in \mathcal{K}$ the following subset of $\mathcal{U}$,

$$E_{\mathbf{k}} := \{\omega \in \mathcal{U} \colon H_{\mathbf{k}}(\omega) \geq H_{\mathbf{r}}(\omega),\ \forall\ \mathbf{r} = \{r_1, \ldots, r_\ell\} \in \mathcal{K}\},$$

i.e., $E_{\mathbf{k}}$ is the set of $\omega \in \mathcal{U}$ for which the maximum is attained for $\mathbf{k} = \{k_1, \ldots, k_\ell\}$. Note that $E_{\mathbf{k}}$ could be the empty set for some $\mathbf{k} = \{k_1, \ldots, k_\ell\}$ and the sets $E_{\mathbf{k}}$ may not be disjoint.

Finally we define our optimal set as,

$$\Omega^* := \bigcup_{\mathbf{k} \in \mathcal{K}} \bigcup_{j=1}^{\ell} E_{\mathbf{k}} + k_j.$$

We will now prove that Ω^* is measurable. First we note that $E_{\mathbf{k}}$ is a measurable set for each $\mathbf{k} \in \mathcal{K}$ since,

$$E_{\mathbf{k}} = \bigcap_{\mathbf{r} \in \mathcal{K}} F_{\mathbf{r}}^{\mathbf{k}},$$

where,

$$F_{\mathbf{r}}^{\mathbf{k}} := \{\omega \in \mathcal{U} \colon H_{\mathbf{k}}(\omega) \geq H_{\mathbf{r}}(\omega)\}.$$

Now, since $F_{\mathbf{r}}^{\mathbf{k}}$ is measurable for all $\mathbf{r} \in \mathcal{K}$, we obtain that $E_{\mathbf{k}}$ is measurable and so is Ω^*.

Furthermore, by construction, Ω^* is in M_N^ℓ. Since for all $\Omega \in M_N^\ell$ we have that,

$$\sum_{j=1}^{m} \sum_{s=1}^{\ell} |\widehat{f_j}(\omega + k_s^{\Omega}(\omega))|^2 \leq \sum_{j=1}^{m} \sum_{s=1}^{\ell} |\widehat{f_j}(\omega + k_s^{*}(\omega))|^2 \ \text{ for almost all } \omega \in \mathcal{U},$$

taking the integral over $\mathcal{U}$ we get

$$\begin{aligned}
\sum_{j=1}^{m} \|P_{V_\Omega} f_j\|^2 &= \int_{\mathcal{U}} \sum_{j=1}^{m} \sum_{s=1}^{\ell} |\widehat{f_j}(\omega + k_s^{\Omega}(\omega))|^2 \, d\omega \\
&\leq \int_{\mathcal{U}} \sum_{j=1}^{m} \sum_{s=1}^{\ell} |\widehat{f_j}(\omega + k_s^{*}(\omega))|^2 \, d\omega \\
&= \sum_{j=1}^{m} \|P_{V_{\Omega^*}} f_j\|^2.
\end{aligned}$$

This shows that $\Omega^* \in M_N^\ell$ is optimal over all $\Omega \in M_N^\ell$. We conclude that $V_{\Omega^*} \in \mathcal{T}_N^\ell$ is a solution for the data $\mathcal{F}$. □

Remark 4.3.6. Notice that if Ω_N^* is the optimal multi-tile set for the class $\mathcal{T}_N^\ell$ for some data $\mathcal{F}$, then the approximation error $\mathcal{E}_N(\mathcal{F}, \ell)$ is given by

$$\begin{aligned}
\mathcal{E}_N(\mathcal{F}, \ell) &= \int_{\mathbb{R}^d \setminus \Omega_N^*} \sum_{j=1}^{m} |\widehat{f_j}(\omega)|^2 d\omega \\
&= \int_{C_N \setminus \Omega_N^*} \sum_{j=1}^{m} |\widehat{f_j}(\omega)|^2 d\omega + \int_{\mathbb{R}^d \setminus C_N} \sum_{j=1}^{m} |\widehat{f_j}(\omega)|^2 d\omega.
\end{aligned}$$

Clearly $\mathcal{E}_N(\mathcal{F}, \ell) \geq \mathcal{E}_{N+1}(\mathcal{F}, \ell)$. So $\mathcal{E}(\mathcal{F}, \ell) := \lim_{N\to\infty} \mathcal{E}_N(\mathcal{F}, \ell)$ is somehow the optimal error. Since $\mathcal{F} \subseteq L^2(\mathbb{R}^d)$ then the second integral goes to zero when N goes to infinite, for functions with good decay at infinite we will be close to the optimal error for conveniently large N.

Remark 4.3.7. In Proposition 4.3.4 we show, for an element of $\mathcal{T}^\ell$, how to construct a set of generators that gives a Riesz basis of translates in $\mathbb{Z}^d$. There are many ways to construct other sets of generators that give Riesz basis of integer translates. Recently Grepstad-Lev in [GL14] constructed a basis of exponentials for $L^2(\Omega)$ when $\Omega \subseteq \mathbb{R}^d$ is a multi-tile. Later on, Kolountzakis [Ko15] gave a simpler proof of this result in a slightly more general form. Precisely they prove the following result.

Theorem 4.3.8 (Theorem 1,[Ko15]). *Suppose $\Omega \subseteq \mathbb{R}^d$ is bounded, measurable and multi-tiles $\mathbb{R}^d$ when translated by $\mathbb{Z}^d$ at level ℓ. Then there exist vectors $a_1, \ldots, a_\ell \in \mathbb{R}^d$ such that the exponentials*

$$e^{-2\pi i(a_j+k)\omega} \quad j = 1, \ldots, \ell, \quad k \in \mathbb{Z}^d$$

form a Riesz basis for $L^2(\Omega)$.

From Theorem 4.3.8, we can obtain immediately a set of generators for V_Ω. Let φ be such that $\widehat{\varphi} = \chi_\Omega$. If $a_1, \ldots, a_\ell \in \mathbb{R}^d$ are as in Theorem 4.3.8, then

$$V_\Omega = S(\varphi_1, \ldots, \varphi_\ell), \quad \text{with} \quad \varphi_j = t_{a_j}\varphi, \ j = 1, \ldots, \ell,$$

and the integer translates of $\varphi_1, \ldots, \varphi_\ell$ form a Riesz basis of V_Ω.

In general, all the Riesz basis for V_Ω can be described in the following way:

Let $A = \{a_{js}\}_{j,s} \in [L^2(\mathcal{U})]^{\ell\times\ell}$ be a measurable matrix, such that $0 < c_1 \leq \lambda(\omega) \leq c_2$ for every eigenvalue $\lambda(\omega)$ of A and for almost each $\omega \in \mathcal{U}$. Set $k(\omega) = (k_1(\omega), \ldots, k_\ell(\omega))$ such that $w + k_s(\omega) \in \Omega$. Define φ_j such that $\widehat{\varphi}_j(\omega + k_s(w)) = a_{js}(\omega)$.

It is not difficult to see that $\varphi_1, \ldots, \varphi_\ell$ are measurable, $V_\Omega = S(\varphi_1, \ldots, \varphi_\ell)$ and the integer translates of $\varphi_1, \ldots, \varphi_\ell$ form a Riesz basis of V_Ω.

4.3.3 The discrete case

Consequently the Hilbert space we consider in this section is $\ell^2(\mathbb{Z}^d)$. We define the class of approximating subspaces in the following way:

Let $\mathcal{N}$ be an arbitrary finite set and $\{D_\sigma : \sigma \in \mathcal{N}\}$ be a partition of $\mathbb{Z}^d$, that is,

$$\mathbb{Z}^d = \bigcup_{\sigma\in\mathcal{N}} D_\sigma,$$

where the union is disjoint.

For $a \in \ell^2(\mathbb{Z}^d)$, we denote $a^\sigma = \mathbf{1}_{\mathbf{D}_\sigma} a$, where $\mathbf{1}_{\mathbf{D}_\sigma}$ denotes the indicator of D_σ. Given $S \subseteq \ell^2(\mathbb{Z}^d)$ a closed subspace we define

$$S_\sigma := \{a^\sigma : a \in S\}, \quad \text{for each } \sigma \in \mathcal{N}.$$

We define the class of approximating subspaces by

$$\mathcal{D}^{\ell}_{\mathcal{N}} := \{S \subseteq \ell^2(\mathbb{Z}^d) \colon S \text{ is a subspace, } \dim(S) \leq \ell \text{ and } S_\sigma \subseteq S, \ \ \forall \sigma \in \mathcal{N}\}. \tag{4.18}$$

Note that $S \in \mathcal{D}^{\ell}_{\mathcal{N}}$ if and only if $\dim(S) \leq \ell$ and S is the orthogonal sum of the subspaces S_σ i. e.,

$$S = \oplus_{\sigma \in \mathcal{N}} S_\sigma.$$

For a given set $\mathcal{A} = \{a_1, \ldots, a_m\} \subseteq \ell^2(\mathbb{Z}^d)$ consider for each $\sigma \in \mathcal{N}$, the Gramian matrix $G_\sigma \in \mathbb{C}^{m\times m}$ of the data $\mathcal{A}_\sigma = \{a_1^\sigma, \ldots, a_m^\sigma\}$, that is $(G_\sigma)_{k,l} = \langle a_k^\sigma, a_l^\sigma, \rangle$, $k, l = 1, \ldots, m$. Let $\lambda_1^\sigma \geq \cdots \geq \lambda_m^\sigma$ be the eigenvalues of the matrix G_σ and $y_1^\sigma, \ldots, y_m^\sigma$ the orthonormal corresponding left eigenvectors.

Now set

$$\Lambda := \{\lambda_j^\sigma : j = 1, \ldots, m, \ \sigma \in \mathcal{N}\}$$

and collect in Λ_ℓ the ℓ first biggest eigenvalues of Λ, that is if $\lambda \in \Lambda_\ell$ then $\lambda \geq \mu$ for all $\mu \in \Lambda \setminus \Lambda_\ell$. Write $\Lambda_\ell = \{\lambda_1, \ldots, \lambda_\ell\}$.

Now, we will proceed to define the sequence $\{q_s\}_{s=1}^{\ell} \in \ell^2(\mathbb{Z}^d)$:

Since $\lambda_s = \lambda_{j_s}^{\sigma_s}$ for some $\sigma_s \in \mathcal{N}$ and some $j_s \in \{1, \ldots, m\}$, then λ_s is an eigenvalue of G_{σ_s}. Let $y_{j_s}^{\sigma_s}$ be the corresponding left eigenvector, $y_{j_s}^{\sigma_s} = (y_{j_s}^{\sigma_s}(1), \ldots, y_{j_s}^{\sigma_s}(m)) \in \mathbb{C}^m$.

Then define, if $\lambda_s \in \Lambda_\ell, \lambda_s \neq 0$

$$q_s := (\lambda_s)^{-1/2} \sum_{k=1}^{m} y_{j_s}^{\sigma_s}(k) a_k^{\sigma_s}. \tag{4.19}$$

If $\lambda_s = 0$ we define q_s to be the zero sequence.

With this notation we can state the main theorem of this section:

Theorem 4.3.9. *Let $m, \ell \in \mathbb{N}$ and $\mathcal{N}$ a finite set. Assume that a set $\mathcal{A} = \{a_1, \ldots, a_m\} \subseteq \ell^2(\mathbb{Z}^d)$ is given.*

Then there exists $S^* \in \mathcal{D}^{\ell}_{\mathcal{N}}$ that satisfies

$$\sum_{j=1}^{m} \|a_j - P_{S^*} a_j\|^2 \leq \sum_{j=1}^{m} \|a_j - P_S a_j\|^2, \quad \forall\, S \in \mathcal{D}^{\ell}_{\mathcal{N}}. \tag{4.20}$$

Moreover, we have that

(1) $S^* = span\{q_1, \ldots, q_\ell\}$ where $q_1, \ldots, q_\ell$ are defined in (4.19). Also, the vectors $\{q_1, \ldots, q_\ell\}$ form a Parseval frame for S^*.

(2) The error in the approximation is

$$\mathcal{E}(\mathcal{A}, \mathcal{N}, \ell) = \sum_{\lambda \in \Lambda \setminus \Lambda_\ell} \lambda.$$

Proof. First, we observe that (4.20) is equivalently to,

$$\sum_{j=1}^{m} \|P_{S^*} a_j\|^2 \geq \sum_{j=1}^{m} \|P_S a_j\|^2, \quad \forall\, S \in \mathcal{D}_{\mathcal{N}}^{\ell}.$$

Furthermore, if $S \in \mathcal{D}_{\mathcal{N}}^{\ell}$ then

$$\sum_{j=1}^{m} \|P_S a_j\|^2 = \sum_{j=1}^{m} \|\sum_{\sigma\in\mathcal{N}} P_{S_\sigma} a_j\|^2 = \sum_{j=1}^{m} \|\sum_{\sigma\in\mathcal{N}} P_{S_\sigma} a_j^\sigma\|^2 = \sum_{\sigma\in\mathcal{N}} \sum_{j=1}^{m} \|P_{S_\sigma} a_j^\sigma\|^2,$$

where $a_j = \sum_{\sigma\in\mathcal{N}} a_j^\sigma$.

In order to construct an optimal subspace $S^* \in \mathcal{D}_{\mathcal{N}}^{\ell}$ for the data $\mathcal{A}$, we will find an optimal subspace S_σ for each $\sigma \in \mathcal{N}$, of dimension at most $\alpha_\sigma \in \{1, \ldots, \ell\}$ for the data $\mathcal{A}_\sigma = \{a_1^\sigma, \ldots, a_m^\sigma\}$. The existence of the optimal subspaces is provided by Theorem 4.2.3. We need

$$\sum_{\sigma\in\mathcal{N}} \dim(S_\sigma) = \sum_{\sigma\in\mathcal{N}} \alpha_\sigma = \dim(S^*) \leq \ell.$$

Thus if

$$Q := \left\{ \alpha = \{\alpha_\sigma\} : 0 \leq \alpha_\sigma \leq \ell \text{ and } \sum_{\sigma\in\mathcal{N}} \alpha_\sigma \leq \ell \right\},$$

then for each choice of $\alpha \in Q$ we will find optimal subspaces $\{S_\sigma^\alpha : \sigma \in \mathcal{N}\}$ and define

$$S^\alpha := \oplus_{\sigma\in\mathcal{N}} S_\sigma^\alpha.$$

The candidate for S^* is the space S^α which minimizes the expression (4.20) over all $\alpha \in Q$.

Let $\beta \in Q$ be the minimizer. Hence β satisfies,

$$\sum_{\sigma\in\mathcal{N}} \sum_{j=1}^{m} \|P_{S_\sigma^\alpha} a_j^\sigma\|^2 \leq \sum_{\sigma\in\mathcal{N}} \sum_{j=1}^{m} \|P_{S_\sigma^\beta} a_j^\sigma\|^2 \quad \forall\, \alpha \in Q. \tag{4.21}$$

Therefore, the subspace $S^* := S^\beta = \oplus_{\sigma\in\mathcal{N}} S_\sigma^\beta$ is the optimal subspace we need. It is straightforward to see that $S^* \in \mathcal{D}_{\mathcal{N}}^{\ell}$ and that S^* is optimal.

Using Theorem 4.2.3, we obtain that for each $\alpha \in Q$, the error of approximation for the data $\mathcal{A}_\sigma$ and the class of subspaces of dimension at most α_σ is given by

$$\mathcal{E}(\mathcal{A}_\sigma, \alpha_\sigma) = \sum_{s=\alpha_\sigma+1}^{m} \lambda_s^\sigma.$$

So the distance between the α-optimal subspace S^α and the data $\mathcal{A}$ is,

$$E(\alpha) := \sum_{\sigma\in\mathcal{N}} \mathcal{E}(\mathcal{A}_\sigma, \alpha_\sigma) = \sum_{\sigma\in\mathcal{N}} \sum_{s=\alpha_\sigma+1}^{m} \lambda_s^\sigma. \tag{4.22}$$

Let κ be the number of elements in $\mathcal{N}$. We see that $E(\alpha)$ is minimum when the $m\kappa - \ell$ eigenvalues used in (4.22) are the smallest from the set $\Lambda = \{\lambda_j^\sigma : j = 1, \dots, m, \ \sigma \in \mathcal{N}\}$. Therefore if we set $\Lambda_\ell \subseteq \Lambda$ the set of the ℓ biggest eigenvalues from Λ, the optimal $\beta = \{\beta_\sigma\} \in \mathcal{Q}$ satisfies that

$$\bigcup_{\sigma \in \mathcal{N}} \{\lambda_1^\sigma, \dots, \lambda_{\beta_\sigma}^\sigma\} = \Lambda_\ell.$$

Therefore,

$$\mathcal{E}(\mathcal{A}, \mathcal{N}, \ell) = \sum_{\sigma \in \mathcal{N}} \mathcal{E}(\mathcal{A}_\sigma, \beta_\sigma) = \sum_{\sigma \in \mathcal{N}} \sum_{j=1}^{m} \|a_j^\sigma - P_{S_\sigma^\beta} a_j^\sigma\|^2 = \sum_{\sigma \in \mathcal{N}} \sum_{s=\beta_\sigma+1}^{m} \lambda_s^\sigma = \sum_{\lambda \in \Lambda \setminus \Lambda_\ell} \lambda.$$

In order to construct the generators of S^β it is enough to construct the generators of each S_σ^β. Since S_σ^β are optimal subspaces for the data $\mathcal{A}_\sigma$ according with Theorem 4.2.3 the generators of S_σ^β are given by (4.19). That is the set $\{q_s : \sigma_s = \sigma\}$ is a Parseval frame of S_σ^β.

Since the subspaces S_σ^β are mutually orthogonal, $\{q_1, \dots, q_\ell\}$ is a set of Parseval frame generators for the optimal space $S^* = S^\beta$. □

Remark 4.3.10. Even though the approximation problem for this discrete case is interesting by itself, that is related with the continuous case in $L^2(\mathbb{R}^d)$ in a very interesting way. In fact, there is a reason to consider this particular class of subspaces for the discrete case. If a SIS V is M extra-invariant for some proper subgroup M of $\mathbb{R}^d$ containing $\mathbb{Z}^d$, then its fiber spaces $J_V(\omega)$ satisfies

$$J_V(\omega) = \bigoplus_{\sigma \in \mathcal{N}} J_{V_\sigma}(\omega),$$

where $J_{V_\sigma}(\omega) = J_V(\omega) \cap \ell^2(B_\sigma \cap \mathbb{Z}^d)$. That is, they satisfies exactly the conditions that we imposed on the class $\mathcal{D}_\mathcal{N}^\ell$ where the partition of $\mathbb{Z}^d$ is $\{B_\sigma \cap \mathbb{Z}^d : \sigma \in \mathcal{N}\}$ and B_σ and $\mathcal{N}$ are defined as in (3.4).

The optimal subspace V^* in Theorem 4.3.1 is the closest to the data $\mathcal{F}$ over all subspaces V in the class $\mathcal{V}_M^\ell$. Using Lemma 1.4.4 and Lemma 1.4.5, it is not difficult to see that almost each fiber space $J_{V^*}(\omega) \subseteq \ell^2(\mathbb{Z}^d)$ of V^* is the closest to the fibers of our data $\tau(\mathcal{F})(\omega) := \{\tau f_1(\omega), \dots, \tau f_m(\omega)\}$ over the particular class $\mathcal{D}_\mathcal{N}^\ell$ defined as above. (Clearly this class is determine by the class $\mathcal{V}_M^\ell$). So, the discrete result (Theorem 4.3.9) provides a different proof of Theorem 4.3.1 using properties of range functions, without the need of Theorem 4.2.4.

4.4 Approximations in the context of LCA groups

We want to mention here, that the notion of shift invariant spaces still make sense in the context of locally compact abelian groups. In that setting, the additive group $\mathbb{R}^d$ is replaced with a general LCA group G and the additive subgroup $\mathbb{Z}^d$ with a uniform lattice $H \subseteq G$, (see [CP10].)

Most of the result for the Euclidean case can be adapted, with certain difficulties, to this abstract setting. In particular the notion of extra invariance ([ACP10]). In particular, after the completion of this thesis, the new results in chapter Chapter 2, were extended by Paternostro [Pat14].

We are working now in the extension of the approximation's results to the setting of LCA groups and its generalizations. This work is in progress and it is not going to be part of this thesis.

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