UNIVERSIDAD DE BUENOS AIRES

Facultad de Ciencias Exactas y Naturales

Departamento de Matemática

Medidas de Gibbs sobre permutaciones de procesos puntuales de baja densidad

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

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Fecha de defensa: 29 de marzo de 2017.

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Resumen

En esta tesis se estudia un modelo probabilístico sobre el espacio de permutaciones de un conjunto discreto e infinito de puntos siguiendo el enfoque de la mecánica estadística. Concretamente, una permutación σ es sorteada de forma proporcional al peso

$$\exp\{-\alpha \sum_x V(\sigma(x) - x)\}, \tag{1}$$

donde $\alpha > 0$ representa la temperatura y V es un potencial no negativo y continuo. Desde el punto de vista físico el caso más relevante es $V(x) = \|x\|^2$, ya que está relacionado con una representación del fenómeno de la condensación de Bose-Einstein introducida por Feynman en los 50'. Los pesos (1) definen una medida de probabilidad cuando el conjunto de puntos es finito, pero obtener una construcción consistente cuando el conjunto de puntos es infinito no es trivial y requiere de hipótesis adecuadas. El primer problema de este modelo, es encontrar condiciones bajo las qué se puede obtener una medida de probabilidad cuando el conjunto de puntos es infinito. Establecida la existencia, interesa saber si es única y cómo es la estructura de ciclos de una permutación típica bajo esta medida.

Las preguntas anteriores se analizan en el régimen de alta temperatura cuando el conjunto de puntos viene dado por un proceso puntual de Poisson en $\mathbb{Z}^d$ con intensidad $\rho \in (0, 1/2)$, y el potencial V verifica algunas condiciones de regularidad. En particular, se prueba que si α es suficientemente grande, para casi toda realización del proceso puntual, existe y es única la medida de Gibbs asociada a las distribuciones finito dimensionales de peso (1). A su vez se demuestra que bajo la medida de Gibbs anterior, una permutación típica contiene solamente ciclos finitos.

Los resultados anteriores se extienden al contexto continuo, es decir, cuando el conjunto de puntos es una realización de un proceso puntual de $\mathbb{R}^d$ con baja densidad.

Palabras clave: Medidas de Gibbs, permutaciones, ciclos finitos, procesos puntuales de Poisson.

Gibbs measures over permutations of point processes with low density

Abstract

In this thesis we study a model of spatial random permutations over a discrete set of points. Formally, a permutation σ is sampled proportionally to

$$\exp\{-\alpha \sum_x V(\sigma(x) - x)\}, \qquad (2)$$

where $\alpha > 0$ is the temperature and V is a non negative and continuous potential. The most relevant case for physics is when $V(x) = \|x\|^2$, since it is related to Bose-Einstein condensation through a representation introduced by Feynman in the '50s. In the context of statistical mechanics, the weights in (2) define a probability when the set of points is finite, but the construction associated to an infinite set is not trivial and may fail without appropriate hypotheses. The first problem is to establish conditions for the existence of such a measure at infinite volume when the set of points is infinite. Once existence derived, we are interested in establishing it uniqueness and the cycle structure of a typical permutation.

The previous questions are analyzed in the large temperature regime, when the set of points is given by a Poisson point process on $\mathbb{Z}^d$ with intensity $\rho \in (0, 1/2)$, and the potential verifies some regularity conditions. In particular, we prove that if α is large enough, for almost every realization of the point process, there exists a unique Gibbs measure that concentrates on finite cycle permutations.

We then extend these results to the continuous setting, when the set of points is given by a Poisson point process in $\mathbb{R}^d$ with sufficient low intensity.

Key words: Gibbs measures, permutations, finite cycles, Poisson point process.

Agradecimientos

Al CONICET y al Departamento de Matemática de la UBA, por la beca y el lugar de trabajo.

A Inés y Pablo por la confianza, las charlas, las ideas, y las horas de cuentas que salieron buenas y las que no.

A todo el grupo de Probabilidad por recibirme y pasarla bien. A Juli, Anita, Sergio, Santi, Nahuel, Clara y Aurelia, por las horas de trabajo compartidas, las consultas y los tecnicismos, las ideas, y todo ese tiempo donde la probabilidad era sólo una excusa.

A la 2046, con los que están ahora y los que pasaron en estos 5 años y a los hermanos de la igualmente gloriosa 2038, por tantos almuerzos llenos de anécdotas, charlas generosas de café, reuniones y excusas para festejar. Agradecimiento especial a aquellos que ponen y ponían su fervor para discutir sobre el alGOLitmo y que hacían de los jueves un gran día. Y las disculpas a todos los que aburrimos con semejante pasión.

A Lea, Manuel, Miji, y Vicky, que son los hermanos que me dio la militancia para toda la vida.

A los reformistas, porque con la razón y el corazón, vamos a volver.

A mi familia por su fuerza y su paciencia.

A Luciana, por acompañarme en esta aventura al otro lado del río con todo su amor.

Todo lo resume La Mojigata 2017: "El que no puede soñar piensa que todo es mentira".

Contents

Chapter 1

Preliminaries

We consider a model of spatial random permutations. The interest in these permutations was initially driven by their connection to Bose-Einstein condensation. In the 50s Richard Feynman introduced a representation of the Bose gas through trajectories of interacting Brownian motions that evolve over a fixed time interval, starting and finishing at the points of a spatial point process. Several simplifications have been proposed over the years to reduce this representation to spatial random permutations. Sütő showed that macroscopic cycles are present in the ideal Bose gas ([Süt93,Süt02]). Tóth [Tót93] relates a particular model of spatial random permutations, the interchange process, to the quantum Heisenberg ferromagnet model, proving that the existence of macroscopic cycles in the former is equivalent to the presence of spontaneous magnetization for the latter. Aizenman and Nachtergaele [AN94] introduce a probabilistic representation for the Heisenberg anti-ferromagnetic model on a space of permutations.

Let $\Omega = (x_i)_{i \in \mathbb{N}} \subset \mathbb{R}^d$ be a locally finite set, that is set of points such that its intersection with any compact subset of $\mathbb{R}^d$ is finite. We focus on the cases when Ω equals the integer lattice or is given by a realization of a spatial point process. Consider the space state S_Ω given by the set of permutations or bijections $\sigma \colon \Omega \to \Omega$. We want to consider a probability measure μ formally given by

$$\mu(\sigma) = \frac{e^{-\alpha H(\sigma)}}{Z}, \qquad \sigma \in S_\Omega,\, \alpha > 0, \tag{1.1}$$

where Z is a normalization factor and H the Hamiltonian, formally defined by

$$H(\sigma) = \sum_i \|\sigma(x_i) - x_i\|^2, \qquad \sigma \in S_\Omega. \tag{1.2}$$

The Hamiltonian discourages the appearance of large jumps, whose probability decays

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exponentially. The parameter α is physically interpreted as the temperature of the system, we can also understand it as a degree of penalization of large jumps. Note that if Ω is finite the measure (1.1) is well defined. In general, for infinite Ω, the definition (1.1) does not make sense. Statistical mechanics provides a standard approach to extending finite volume probabilities to infinite volume. The method consists of specifying what the conditional probabilities of the infinite volume measure given boundary conditions outside a compact set should look like, these so-called specifications are given by (1.1) plus consistency with the boundary conditions, and then proving that the specifications have weak limits, called Gibbs measures. One of the fundamental problems is to determine whether there exists more than one Gibbs measure.

The main questions of the model are related to the cycle structure of a typical permutation, and how this depends on parameters such as point density and temperature α. In particular it is interesting to know if infinitely long cycles appear with positive probability, and in this case, whether they are macroscopic, i.e., if their intersection with a large finite box contains a positive density of the points in the box.

For the model determined by the quadratic Hamiltonian (1.2) it is conjectured that, at all temperatures, a typical permutation will decompose into finite cycles if $d = 1$, 2, whereas for $d \geq 3$, there exists a critical value α_c below which a typical permutation contains an infinite cycle with positive probability. This conjecture can be explained heuristically and it is supported by numerical simulations (see for instance [GRU07, GLU12, GUW11]).

The first rigourous result are given by Gandolfo, Ruiz and Ueltschi [GRU07], they show that for high enough temperature all cycles are finite, in any dimension. The case $d = 1$ was then settled by Biskup and Richthammer in [BR15]. They prove uniqueness of the Gibbs measure associated to identity boundary conditions, at any temperature, and show that is is supported on finite cycle permutations. They also establish a bijection between ground states (local minima) of the Hamiltonian and extremal Gibbs measures. This one-to-one correspondence is expected to fail in dimensions higher than 1. Armendáriz, Ferrari, Groisman and Leonardi [AFGL15] consider the large temperature regime for general strictly convex potentials in $d \geq 2$. They derive the existence and uniqueness of Gibbs measures concentrating on finite cycle permutations for α large determined by the potential.

We are interested in the case when the set of points Ω is random. In the 1-dimensional case, Biskup and Richthammer [BR15] show that if Ω is a realization of a shift-invariant point process then, again, cycles are almost surely finite at all temperatures, a quenched result.

In the annealed case, when points and permutations are jointly sampled, Betz and Uelts-

chi proved in [BU09, BU11b] that if $d \geq 3$, there exists a critical density of points ρ_c below which a typical permutation has only finite cycles and while above ρ_c it contains macroscopic cycles. The asymptotic behavior of the lengths of cycles is also proved in [BU11a]. Precisely, the sorted lengths of the cycles, scaled down by N converges to the Poisson-Dirichlet distribution, as was previously proved to be the case for uniformly distributed permutations [Sch05].

We here consider the set of points given by a realization of a Poisson process on $\mathbb{Z}^d$ with intensity ρ, that is, for each $x \in \mathbb{Z}^d$ we place a Poisson(ρ) number of points $\theta(x)$ at x, independently among locations, and consider the set $\Omega_\theta = \{(x,i)\colon 1 \leq i \leq \theta(x),\, x \in \mathbb{Z}^d\}$ hence determined. This is a simpler version of a regular Poisson point process on $\mathbb{R}^d$ with intensity ρ, obtained by collecting all points in the unit box $x + [0,1)^d$ and placing them at x. We study the permutation group of Ω_θ under the probability induced by (1.2), where the distance between two points in Ω_θ is defined as the distance between their projections on $\mathbb{Z}^d$. Our first result proves the existence of Gibbs measures compatible with finite volume probabilities for almost every realization of the environment $\{\theta(x)\}_{x \in \mathbb{Z}^d}$ in the large temperature regime and with fixed density $\rho \in (0, 1/2)$. We next show that any permutation sampled with respect to this Gibbs measure has finite cycles, almost surely with respect to the Poisson point process. We finally derive uniqueness for the Gibbs measures supported on the set of finite cycle permutations of Ω_θ, for almost all $\{\theta(x)\}_{x \in \mathbb{Z}^d}$.

The results for the quenched discrete case can be extended to results in the continuous setting when the set of points is a realization of a Poisson point process on $\mathbb{R}^d$ with low density. Concretely, we show that if the density ρ is small enough, in the large temperature regime, there exists a Gibbs measure for almost every realization of the point process. The Gibbs measure is unique among the Gibbs measures that concentrates over permutations for whose decomposition has only finite cycles.

To prove these results we follow an approach introduced by Fernández, Ferrari and Garcia in [FFG01] that would realize the Gibbs measure as the stationary distribution of a suitable Markov process on the space of finite cycles using the coupling from the past algorithm. In the particular case considered here, however, the infinite volume stationary distribution can not be obtained using the algorithm. Instead, we apply this method to bounded regions, where the corresponding Markov process is well defined, and realize the specifications on these regions as the invariant distribution. We prove that the specifications are dominated by a Poisson point process on the space of finite cycles, and apply renewal arguments from [GRU07, BR15] to obtain tightness of the family of specifications, show that all weak limits are supported on finite cycle permutations, and conclude uniqueness.

1.1 Setting and results

Let $\theta = \{\theta(x)\}_{x \in \mathbb{Z}^d}$ be an i.i.d. sequence of Poisson random variables with mean ρ. We say that $\theta(x)$ is the multiplicity of the site x, that is, $\theta(x)$ is the number of points located at the same site x. We denote by $\mathbb{P}$ and $\mathbb{E}$ the probability and expectation with respect to product measure for which each marginal is Poisson(ρ).

Fix a realization of θ. With this realization we will define the set of points and the setup of Gibbs measures, so, it will be fixed for the rest of article. The set of points Ω_θ is defined by

$$\Omega_\theta = \{(x, i) \in \mathbb{Z}^d \times \mathbb{N} \colon i = 1, \dots, \theta(x)\}.$$

For example, if $\theta(x) = 0$, there is not point at x, and if $\theta(x) = 2$, there are two points located at x but with different label. For $s \in \Omega_\theta$ we write $X(s)$ for the projection on $\mathbb{Z}^d$. If $\Lambda \subset \mathbb{Z}^d$ we say that $s \in \Lambda$ when $X(s) \in \Lambda$. We write $\Lambda \Subset \mathbb{Z}^d$ for denote that Λ is a finite set. In general, we will use point to refer a point of Ω_θ and use site for its location in $\mathbb{Z}^d$.

Denote by S_θ the set of bijections of Ω_θ equipped with the topology generated by the sets

$$\{\sigma \in S_\theta \colon \sigma(s) = r\} \quad s, r \in \Omega_\theta.$$

With this topology S_θ is metrizable. For check it, consider the metric d defined by:

$$d(\sigma, \sigma') = 2^{-\min\{\|X(s)\|_2 \colon \sigma(s) \neq \sigma'(s)\}},$$

with the conventions $\min \emptyset = -\infty$ and $2^{-\infty} = 0$. The metric space (S_θ, d) is a complete and separable. We associate the Borel sigma-algebra $\mathcal{F}_\theta$. We write $\mathcal{F}_{\theta, \Lambda}$ for the sigma-algebra generated by sets $\{\sigma \in S_\theta \colon \sigma(s) = r\}$, $s \in \Lambda$, $r \in \Omega_\theta$.

A function $f \colon S_\theta \to \mathbb{R}$ is local when exists $\Lambda \Subset \mathbb{Z}^d$ such that f is $\mathcal{F}_{\theta, \Lambda}$-measurable.

Let $V \colon [0, +\infty) \to [0, +\infty)$ a continuous potential. We define the Hamiltonian associated to V restricted to the set Λ as

$$H_{\theta, \Lambda}(\sigma) = \sum_{s \in \Lambda} V(\|X(\sigma(s)) - X(s)\|). \tag{1.3}$$

We will focus in the case $V(t) = t^2$ but we indicate when a results needs a different proof for a more general potential.

Let $\xi \in S_\theta$ and consider $\Lambda \Subset \mathbb{Z}^d$. The permutations that are compatible with boundary

condition ξ at volume Λ are given by

$$S_{\theta,\Lambda}^{\xi} = \{\sigma \in S_\theta : \sigma^n(s) = \xi^n(s) \text{ for all } s \in \Lambda^c, \, n \in \mathbb{Z}\}\,,$$

where σ^n means the n-fold composition σ with itself. When $\Lambda \Subset \mathbb{Z}^d$ the set $S_{\theta,\Lambda}^{\xi}$ is also finite. In Figure 1.1 we illustrate these definitions for a particular choice of Λ, θ and boundary condition ξ.

The specification at volume Λ corresponding to temperature $\alpha > 0$ and boundary condition ξ is given by:

$$G_{\theta,\Lambda}^{\xi}(\sigma) = \frac{e^{-\alpha H_{\theta,\Lambda}(\sigma)}}{Z_{\theta,\Lambda}^{\xi}} \mathbf{1}\{\sigma \in S_{\theta,\Lambda}^{\xi}\}\,, \tag{1.4}$$

where $Z_{\theta,\Lambda}^{\xi}$ is a normalizing constant that depends on α, ξ, θ and Λ. We will refer to $e^{-\alpha H_\Lambda(\sigma)}$ as the weight of σ and for be short we write $w(\sigma)$.

Definition 1.1.1. A probability measure μ on $(S_\theta, \mathcal{F}_\theta)$ is a Gibbs measure with respect to the Hamiltonian (1.3) when for any $\Lambda \Subset \mathbb{Z}^d$ and $A \in \mathcal{F}_\theta$ we have

$$\mu(A) = \int G_{\theta,\Lambda}^{\xi}(A)\,\mathrm{d}\mu(\xi)\,.$$

A cycle γ associated to $(s_1, \ldots, s_n) \in \Omega_\theta{}^n$, $n \in \mathbb{N} \cup \{\infty\}$, is a permutation $\gamma \in S_\theta$ such that $\gamma(s_i) = s_{i+1}$, $i = 1, \ldots, n$, $\gamma(s_n) = s_1$ if $n < \infty$, and $\gamma(s) = s$ otherwise. We write $\gamma = (s_1, \ldots, s_n)$ for short. Due to cyclic structure of the cycle, the choice of starting point in this representation is arbitrary. Observe that we could have multiplicities in the cycle, i.e., a site $x \in \mathbb{Z}^d$ might appear as many times is allowed for its multiplicity $\theta(x)$.

Any permutation $\sigma \neq \mathrm{id}$ can be written as a finite or countable composition of disjoint cycles. A finite cycle permutation is a permutation σ for which its decomposition has only finite cycles.

The main results are summarized in the next theorems.

Theorem 1.1.2. *Consider the model with quadratic potential and let $\rho \in (0, 1/2)$. We can choose $\alpha > 0$ large enough, such that for almost every realization of $\{\theta(x)\}_{x \in \mathbb{Z}^d}$ there exists a Gibbs measure μ_θ. It can be obtained as a subsequential weak limit of specifications with identity boundary condition.*

Furthermore, μ_θ concentrates on permutations whose decomposition has only finite cycles and it is the unique Gibbs measure with this property.
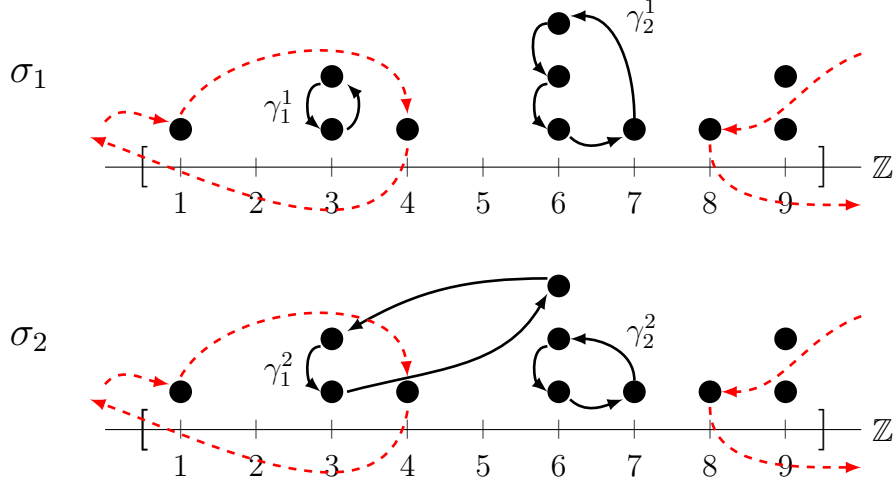
Figure 1.1: Each figure above depicts a permutation in $S_{\theta,\Lambda}^{\xi}$, $\Lambda = \{1, \ldots, 9\} \in \mathbb{Z}$. Cycles in the decomposition of the boundary condition ξ are drawn using dashed lines, and cycles contained in Λ that are compatible with ξ are drawn in solid lines.

Remark 1.1.3. Define α_*

$$\alpha_* = \frac{\pi}{\left[\left(\frac{r_0}{C_\rho} + 1\right)^{\frac{1}{d}} - 1\right]^2},$$

where $C_\rho = \frac{\rho e^{-\rho + \frac{1}{2}}}{1 - 2\rho}$ and r_0 is the unique solution in $[0, 1]$ of $\frac{r}{(1-r)^2} - r = \frac{1}{2}$ (solving it one gets $r_0 \approx 0.35542$). For the theorem is sufficient to choose $\alpha > \alpha_*$.

Remark 1.1.4. The restriction for the density ρ comes from the control of the expectation of a combinatorial factor. Essentially, we need to dominated the number of cycles that have the same ordered support. With good bounds for this number, the range for density can be extended.

Remark 1.1.5. The proof also works when the environment is not necessarily generated by Poisson random variables. If the environment is given by a i.i.d. family of random variables $\theta(x)$ such that $\mathbb{E}(\theta(x)! 2^{\theta(x)}) < \infty$, then the theorem holds.

An analogous theorem also holds if the Hamiltonian is given by a non negative potential $V : [0, +\infty) \to \mathbb{R}$ with the growing condition: $\exists \, \alpha_V > 0$ such that for $\alpha \geq \alpha_V$ we have

$$\varphi_V(\alpha) = \sum_{\substack{x \in \mathbb{Z}^d \\ x \neq 0_d}} e^{-\alpha V(\|x\|)} < \infty. \tag{1.5}$$

Note that this condition implies that $\lim_{t \to +\infty} V(t) = +\infty$.

To enunciate the theorem we need to solve the following problem with the potential. If V is not strictly positive on $[1, +\infty)$, perhaps, some lengths of jump contribute zero to the Hamiltonian even if the points are located at different sites. For example, if $V(t) = (t^2 - 2t)^+$ the jumps of length 0, 1 and 2 pay zero. So, we can have an infinite cycle with finite energy. To allow more general potentials we need to restrict the density. Set $L_V = \sup\{t \geq 0 \colon V(t) = 0\}$ and note that by the growing condition for V it is finite. If $L_V \geq 1$ we need to choice the density ρ small enough to ensure that the event that there exists an infinite sequence of points $(s_i)_{i \in \mathbb{N}} \subset \Omega_\theta$ such that $\|X(s_i) - X(s_{i+1})\| \geq L_V$, has zero probability with respect to the probability of the environment.

Remark 1.1.6. Fix $\rho > 0$ and $L \geq 1$. We say that $x \in \mathbb{Z}^d$ is retained when $\theta(x) \neq 0$, so, $\mathbb{P}(x \text{ is retained}) = 1 - e^{-\rho}$. A site $z \in \mathbb{Z}^d$ is open if there exists a retained site $x \in \mathbb{Z}^d$ such that $\|x - z\| \leq L$. We can ask about site percolation in this model with respect to $\mathbb{P}$. This model is a particular case of the Bernoulli Boolean model on $\mathbb{Z}^d$ and it is also close to the standard Poisson Boolean model on $\mathbb{R}^d$. See for instance [MR96, Gou09, CG14]. It is easy to see that for fixed L there exists a critical parameter ρ_c such that if $\rho > \rho_c$ there is site percolation and if $\rho < \rho_c$ all clusters are finite. To check it, first note that by standard coupling we have monotonicity in ρ for the model. Then observe that if we take $\rho < \rho_*$, where ρ_* is the critical density for the standard Poisson Boolean model on $\mathbb{R}^d$ with fix radii $L + \sqrt{d}$, there is not percolation in the model. However, if ρ is such that $1 - e^{-\rho} > p_{\text{site}}$, where p_{site} is the site percolation threshold on $\mathbb{Z}^d$, we have percolation in the model.

Definition 1.1.7. If the potential V is such that $L_V < 1$ we say that any density ρ is good. If $L_V \geq 1$ we say that ρ is good for V when the percolation model described before with parameters ρ and L_V is in the subcritical regime.

Remark 1.1.8. If the density is good for V, an infinite cycle in the model has infinite energy. Suppose that γ is an infinite cycle. Then for any $s \in \gamma$ we have $\gamma^j(s) \neq s$ for all $j \in \mathbb{Z}$. If the cycle has finite energy, there exists $M > 0$ such that

$$\sum_{j \in \mathbb{Z}} V(\|X(\gamma^j(s)) - X(\gamma^{j-1}(s))\|) < M \,.$$

So, as V is non negative, there exists j_0 such that $V(\|X(\gamma^j(s)) - X(\gamma^{j-1}(s))\|) = 0$ if $j > j_0$. By definition of L_V we have $\|X(\gamma^j(s)) - X(\gamma^{j-1}(s))\| \leq L_V$ for all $j > j_0$, but this is equivalent to have percolation in the Bernoulli Boolean model of parameter ρ and radii L_V introduced in the Remark (1.1.6). So, if ρ is good for V, an infinite cycle has infinite energy.

Theorem 1.1.9. *Consider the model with a non negative potential V and such* (1.5)

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holds. Let $\rho \in (0, 1/2)$ a good density for V and $\alpha > 0$ such that

$$C_\rho \varphi_V(\alpha) < r_0 \,, \tag{1.6}$$

where $C_\rho = \frac{\rho e^{-\rho+\frac{1}{2}}}{1-2\rho}$ is the same constant that in the previous theorem, φ_V is defined in (1.5), and r_0 is the unique solution in $[0,1]$ of the equation $\frac{r}{(1-r)^2} - r = \frac{1}{2}$.

Then for almost every realization of $\{\theta(x)\}_{x \in \mathbb{Z}^d}$ there exists a Gibbs measure μ_θ and it is a subsequential weak limit of specifications with identity boundary condition. It concentrates also the finite cycle permutations and it is the unique Gibbs measure with this property.

The theorem include some cases that appear in the literature of random permutations, see for instance [BU11a, BU09]. For example, we can consider a model of random permutations related to the formal Hamiltonian

$$H(\sigma) = \sum_{s \in \Omega_\theta} \|X(\sigma(s)) - X(s)\|^2 + \sum_{j \geq 2} \frac{\epsilon}{\alpha} r_j(\sigma) \,,$$

where $r_j(\sigma)$ is the number of cyles in σ with length j and $\epsilon > 0$ is a constant. To do this, we can use the potential $V(t) = t^2 + \epsilon/\alpha$ if $t \geq 1$ and $V(t) = 0$ if $t \in (0, 1)$. If we want to replace ϵ by a bounded non negative sequence $(\epsilon_j)_{j \geq 2}$ we can prove a similar result to (1.1.9) using coupling arguments. We are also able to consider the case with $\epsilon \in (-1, 0)$ taking the same V as before.

In the next theorem we extend the results when the set of points is given by a realization of an homogeneous Poisson point process.

Theorem 1.1.10. *Let $V(t) = (t^2 - 2\sqrt{d}t)^+$. Choose $\rho \in (0, 1/2)$ a good density for V and $\alpha > 0$ such that $\varphi_V < \infty$. Suppose that ρ and α also satisfy the condition (1.6), i.e.,*

$$C_\rho \varphi_V(\alpha) < r_0 \,.$$

Consider the model of random permutations with quadratic potential when the set of points $\Omega \subset \mathbb{R}^d$ is a realization of a Poisson process with intensity ρ.

Then for almost every realization of Ω, the model has a unique Gibbs measure μ_Ω that concentrates on finite cycle permutations. It can be obtained as subsequential weak limit of specifications with identity boundary condition.

For the next we need some additional definitions. For $s \in \Omega_\theta$ and $\sigma \in S_\theta$ we say $s \in \sigma$ when $\sigma(s) \neq s$. The set $\{s \in \Omega_\theta : s \in \sigma\}$ is called the support of σ and it is denoted by $\{\sigma\}$. Also, for $x \in \mathbb{Z}^d$ we say that $x \in \sigma$ if exists $s \in \theta$ such that $X(s) = x$ and $s \in \sigma$.

We say that a cycle γ is a trivial cycle if only uses points that projects onto the same site in $\mathbb{Z}^d$. For a cycle γ that is present in the decomposition of σ we write $\gamma \in \sigma$. The set of finite cycles on S_θ is denoted by Γ_θ and we write $\Gamma_{\theta, \Lambda}$ for the set of finite cycles with support included in Λ.

Recall that σ is a finite cycle if its decomposition has only finite cycles. Write S_θ^F for the set of finite cycle permutations and $S_{\theta, \Lambda}^F$ for the finite cycle permutations supported in Λ. Note that S_θ^F and $S_{\theta, \Lambda}^F$ are groups. When Λ is finite, $S_{\theta, \Lambda}^{\mathrm{id}}$, the set of permutations compatibles with identity boundary conditions, coincides with $S_{\theta, \Lambda}^F$.

The ordered support $\bar{\gamma}$ of a cycle $\gamma = (s_1, s_2, \ldots, s_n)$ is the vector in $(\mathbb{Z}^d)^m$, $m \leq n$, given by $\bar{\gamma} = (x_1, x_2, \cdots, x_m)$ with $x_i = X(s_{\pi(i)})$, where $\pi(1) = 1$, and inductively,

$$\pi(i) = \inf\{k > \pi(i-1), X(s_k) \neq X(s_{\pi(i-1)})\}, \ i > 1.$$

That is, the ordered support is obtained by considering the projection of γ on $\mathbb{Z}^d$, and erasing consecutive repetitions of points. Note that both in the representation of γ as a vector and in the definition of its ordered support $\bar{\gamma}$ due to the cyclic property of γ, the choice of initial point is arbitrary. Starting from any other point $s \in \{\gamma\}$ for the former, or of its spatial coordinate $X(s)$ for the latter, lead to alternative representations of the cycle and its ordered support.

The weight of a finite cycle γ is

$$w(\gamma) := e^{-\alpha H_\Lambda(\gamma)}, \quad \Lambda = \{\gamma\}.$$

Note that the weight of a cycle is a function of its ordered support.

The following refers to Figure 1.1. The supports of the cycles in σ_1 are $\{\gamma_1^1\} = \{(3,1); (3,2)\}$ and $\{\gamma_2^1\} = \{(6,1); (6,2); (6,3); (7,1)\}$, and the ordered supports are $\bar{\gamma}_1^1 = (3)$ and $\bar{\gamma}_2^1 = (6,7)$. Also, $\{\gamma_2^2\} = \{(6,1); (6,2); (7,1)\} \neq \{\gamma_2^1\}$, but they share the ordered support, $\bar{\gamma}_2^1 = \bar{\gamma}_2^2$.

1.2 Previous results

We begin by recalling the expected behaivor for the model with quadratic Hamiltonian based in heuristic arguments and numerical simulations (see [GRU07, GLU12, GUW11]). On $\mathbb{Z}^d$, if $d = 1, 2$, it is expected that a typical permutations will have only finite cycles on its decomposition. For $d \geq 3$, the conjecture says that there exists a critical value α_c below which a typical permutation contains an infinite cycle with positive probability.

The same conjectures are maintained in the continuum case, when the set of points is given by a realization of a Poisson process or other ergodic translation invariant point process.

1.2.1 The 1-dimensional case

We begin the study describing the results of Biskup and Richthammer [BR15] for the 1-dimensional lattice. First we present the notion of flow and then we introduce the specifications and Gibbs measures to enunciate their theorems.

Denote by $S_{\mathbb{Z}}$ the symmetric group over $\mathbb{Z}$. Given $a \in \mathbb{Z}^*$, the dual lattice of $\mathbb{Z}$, we define

$$F_a^+(\sigma) = \#\{x \in \mathbb{Z} \colon x < a < \sigma(x)\} \text{ and } F_a^-(\sigma) = \#\{x \in \mathbb{Z} \colon \sigma(x) < a < x\}.$$

If both quantities are finite, we define the flow trough $a \in \mathbb{Z}^*$ by $F_a(\sigma) = F_a^+(\sigma) - F_a^-(\sigma)$, otherwise we set $F_a(\sigma) = \infty$.

Remark 1.2.1. Note that $F_a(\sigma)$ does not depends on $a \in \mathbb{Z}^*$. Indeed, let $a = x - 1/2$ and $b = x + 1/2$ and suppose that $F_a(\sigma)$ is finite. A jump from σ that do not contribute in the same way to F_a and F_b is a jump to or from x. So, $F_b(\sigma)$ is also finite. To show that $F_b(\sigma) = F_a(\sigma)$ we analyze each case. If $\sigma^{-1}(x) < x < \sigma(x)$ both jumps contribute $+1$ both flows. With the case $\sigma(x) < x < \sigma^{-1}(x)$ is the same but contribute -1 to both flows. If $\sigma^{-1}(x), \sigma(x) < x$ these jumps do not contribute to the flow F_b but the jump from $\sigma^{-1}(x)$ to x contribute $+1$ to $F_a(\sigma)$ and the jump from x to $\sigma(x)$ contribute -1 to $F_a(\sigma)$, so, $F_a(\sigma) = F_b(\sigma)$. The last case, $\sigma^{-1}(x), \sigma(x) > x$, is analogous to the previous. Then $F_a(\sigma)$ has the same value for all $a \in \mathbb{Z}^*$. Therefore, we define $F \colon S_{\mathbb{Z}} \to \mathbb{Z} \cup \{\infty\}$ to be this value.

To define the Gibbs setting we need to endow $S_{\mathbb{Z}}$ with a topology, and introduce the Hamiltonian and specifications. We use the smallest topology for which the projections $\sigma \mapsto P_x^+(\sigma) := \sigma(x)$ and $\sigma \mapsto P_x^-(\sigma) := \sigma^{-1}(x)$ are continuous. For $\Lambda \subset \mathbb{Z}$ we write $\mathcal{F}_\Lambda$ for the σ-algebra generated by $\{P_x^+, P_x^-\}_{x \in \Lambda}$ and $\mathcal{T}$ for the tail σ-algebra.

The Hamiltonian at volume Λ is given by

$$H_\Lambda(\sigma) = \sum_{x \in \sigma^{-1}(\Lambda) \cap \Lambda} V(\sigma(x) - x),$$

where V is symmetric and strictly convex satisfying the following growth condition: for all $s > 0$

$$\lim_{t \to +\infty} \frac{V(t) + V(0) - 2V\left(\frac{t+s}{2}\right)}{t \log t} = +\infty.$$

Note that $V(t) = |t|^p$ for $p > 1$ satisfies these conditions. For the physics point of view, the relevant choice of the one-body potential is $V(t) = t^2$ as was presented on [Fey53].

The bijections σ, ξ are compatible with the relation $\sim_\Lambda$ if $\sigma(x) = \xi(x)$ and $\sigma^{-1}(x) = \xi^{-1}(x)$ for all $x \in \Lambda^c$. So, as is usual, the specifications at volume Λ corresponding to boundary conditions ξ and temperature $\alpha > 0$ is given by

$$ G_\Lambda^\xi(\sigma) = \frac{1}{Z_\Lambda(\xi)} e^{-\alpha H_\Lambda(\sigma)} \mathbf{1}\{\sigma \sim_\Lambda \xi\}, $$

where $Z_\Lambda(\xi)$ is the normalizing constant. A probability measure μ on $(S_\mathbb{Z}, \mathcal{F}_\mathbb{Z})$ is a Gibbs measure if for all $A \in \mathcal{F}_\mathbb{Z}$ we have $\mu(A) = \int G_\Lambda^\xi(A) \, d\mu(\xi)$.

A ground state for the Hamiltonian is a permutation σ such that $H_\Lambda(\tau) \leq H_\Lambda(\sigma)$ for all finite Λ and all $\sigma \sim_\Lambda \tau$. Since the potential V is strictly convex the ground states are the translations, that is the permutations $\{\tau_n\}_{n \in \mathbb{Z}}$ where for each $n \in \mathbb{Z}$ are defined by $\tau_n(x) = x + n$ for all $x \in \mathbb{Z}$. Note that τ_n has flow n.

Theorem 1.2.2. *Let μ a Gibbs measure. The flow F is a σ-tail measurable function and it is finite μ-almost surely.*

A consequence of this theorem is that if μ is a extremal Gibbs measure, then F is constant μ-almost surely. The next theorem gives a complete description for the set of extremal Gibbs measures.

Theorem 1.2.3. *Let $n \in \mathbb{Z}$. Then:*

1. *There exists a unique extremal Gibbs measure μ_n that has flow n.*

2. *μ_n-almost surely a permutation has exactly $|n|$ infinite cycles.*

3. *For any $\xi \in S_\mathbb{Z}$ with $F(\xi) = n$ and any increasing sequence of finite sets $\Lambda_n \uparrow \mathbb{Z}$, the specifications with boundary conditions ξ converges weakly to μ_n.*

The last theorem says that the extremal Gibbs measures are in one-to-one correspondence with the ground states of the Hamiltonian $\{\tau_n\}_{n \in \mathbb{Z}}$.

To prove these theorems they used estimations for the probability to have large jumps. In particular, they used that if there is a jump from x to $\sigma(x)$ larger than $|F(\sigma)|$, i.e. $|\sigma(x) - x| > |F(\sigma)|$, there exists $y \in \mathbb{Z}$ such that the jump from y is the reverse flow direction. So, the permutation $\sigma \circ (xy)$ where $\circ$ is the composition, has lower energy than σ and they apply it to control the probability to have a large jump. These arguments cannot be extended to dimensions $d > 1$.

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The theorems hold when the set of points is an infinite discrete subset of $\mathbb{R}$ with certain regularity conditions. For instance they consider point sets produced by a Poisson process on $\mathbb{R}$. Denote the set of points by Ω with the convention that $x_i < x_{i+1}$ for all i. and write Ω^* for $\{\frac{x_1+x_2}{2}: x_1, x_2 \in \Omega\}$. For $a \in \Omega^*$ and $k \in \mathbb{N}$ we write $\Lambda(a, k) = \{a_{-k-1}, \ldots, a_{-1}, a, a_1, \ldots, a_{k+1}\}$ where a_j denotes the j-th point of Ω^* in the positive direction if $j > 0$ and in the negative direction if $j < 0$. Now, define the separation constant and the growth rate from a as:

$$c_s(a, k) = \min\{c \geq 1: \text{ consecutive points of } \Lambda(a, k) \text{ keep distances } \in [c^{-1}, c]\},$$

$$c_g(a) = \inf\{c \geq 0: \#\{x \in \Omega: 0 \leq |x - a| \leq t\} \leq ct \text{ for all } t\}.$$

The regularity conditions are the following:

- Ω is locally finite and bi-infinite, that is, $\inf \Omega = -\infty$ and $\sup \Omega = +\infty$.

- For some $c_k < \infty$ such that bi-infinitely many points $a \in \Omega^*$ we have $c_g(n) \leq c_k$ and $c_s(a, k) \leq c_k$.

In the case that $\Omega = \mathbb{Z}$ we have $c_s(a, k) = 1$ and $c_g(a) = 2$ for all $a \in \mathbb{Z}^*$.

1.2.2 The d-dimensional case

For the d-dimensional lattice, Armendáriz, Ferrari, Groisman and Leonardi prove in [AFGL15] the existence of Gibbs measures when $\alpha > 0$ is large enough. We introduce the basic setup of this paper to present their results.

They consider on $S_{\mathbb{Z}^d}$ an analogous topology from $S_{\mathbb{Z}}$, that is, the smallest topology which the projections are continuous. The Hamiltonian and the specifications are slighty different that in the previous case. Let $V: \mathbb{R} \to [0, +\infty]$ a strictly convex potential with $V(0_d) = 0$ and define

$$H_\Lambda(\sigma) = \sum_{x \in \Lambda} V(\|\sigma(x) - x\|) \quad \text{for all } \sigma \in S_{\mathbb{Z}^d}.$$

For Hamiltonian the difference is in sum range, it is Λ instead $\sigma^{-1}(\Lambda) \cap \Lambda$. The permutations σ and ξ are compatible on Λ if $\sigma(x) = \xi(x)$ for all $x \in \Lambda^c$. The specifications and the Gibbs measures are defined as in the 1-dimensional case.

Recall that if $\gamma \in S_{\mathbb{Z}^d}$ is a finite cycle its weight is defined by $w(\gamma) = \exp\{-\alpha H_{\Lambda_0}(\gamma)\}$

where Λ_0 is the support of γ, i.e., the set of points for which $\gamma(x) \neq x$. Define

$$\beta(V, \alpha) = \sum_{\gamma \ni 0_d} |\gamma| w(\gamma),$$

where the sum is over finite cycles and $|\gamma|$ denote its length. If $\beta(V, \alpha)$ is finite for some α it is a decreasing function of α and we can define $\alpha_*(V) = \inf\{\alpha \colon \beta(V, \alpha) < 1\}$. If $\beta(V, \alpha) = \infty$ for all α we set $\alpha_*(V) = \infty$.

For $v \in \mathbb{Z}^d$ denote by τ_v the shift permutation given by $\tau_v(x) = x + v$. Given a measure μ on $S_{\mathbb{Z}^d}$, the v-shifted measure is given by $\mu^v(f) = \int f(\tau_v \sigma) \, d\mu^v(\sigma)$ where f is a continuous function.

We focus on the case $V(t) = t^2$ but a similar theorem holds for a strictly convex potential. For the quadratic potential $\alpha_*(V) \leq (1.44504^{1/d} - 1)^{-2}$.

Theorem 1.2.4. *Let $v \in \mathbb{Z}^d$ and $V(t) = t^2$. For each $\alpha > \alpha_*(V)$ there exists a random process $(\sigma_\Lambda \colon \Lambda \subset \mathbb{Z}^d)$ on $S_{\mathbb{Z}^d}$ such that:*

1. *for $\Lambda \Subset \mathbb{Z}^d$, $\tau_v \sigma_\Lambda$ is distributed according to $G_\Lambda^{\tau_v}$.*

2. *$\lim_\Lambda \tau_v \sigma_\Lambda(x) = \tau_v \sigma_{\mathbb{Z}^d}(x)$ almost surely when Λ increase to $\mathbb{Z}^d$, for each x.*

3. *Call μ the distribution of $\sigma_{\mathbb{Z}^d}$. Then, $G_\Lambda^{\tau_v}$ converges weakly to $\mu\tau_v$.*

4. *$\mu\tau_v$ is an ergodic Gibbs measure corresponding to temperature α with mean jump v.*

5. *μ is the unique Gibbs measure for the family of specifications that concentrates on the finite cycle permutations.*

For the proof the authors write a finite cycle permutation as a gas of cycles and use a coupling from the past algorithm from [FFG01] to sample the specifications and the Gibbs measure as their thermodynamic limit. The conditions to apply the algorithm restrict the temperature values.

As in the 1-dimensional case, the translations $\{\tau_v\}_{v \in \mathbb{Z}^d}$ are gound states for the Hamiltonian but it has other ground states. For instance, if $x = (x_1 \ldots, x_d)$ the permutation ξ given by

$$\xi(x) = \begin{cases} x_1 + e_1 & \text{if } x_2 = \cdots = x_d = 0, \\ x & \text{otherwise,} \end{cases}$$

where $e_1 = (1, 0 \ldots, 0)$, is a ground state. When $d = 1$, for $\alpha > 0$ the extremal Gibbs measure are in one-to-one correspondence with the ground states of the Hamiltonian. It is expected that this correspondence fails if $d > 1$. For example, it is conjectured that

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the specifications with boundary conditions id and ξ have the same limit distribution. Moreover, it is believed that for $d = 2$ and α large enough, in the box $[-n, n]^2$ the (forced) cycle from $(-n - 1, 0)$ to $(n + 1, 0)$ properly rescaled converges to a Brownian bridge. The situation is similar to line separation for the Ising model with boundary condition $\eta(x) = \mathbf{1}\{x_2 \geq 0\} - \mathbf{1}\{x_2 < 0\}$. In that case, Gallavotti and Higuchi proved in [Gal72, Hig79] that for large β (the inverse of the temperature in the Ising model) the phase separation line, when it is suitably normalized, converges almost surely to 1-dimensional Brownian bridge. There is a recent progress in this direction but is not proved (see [BT17]).

If in our model set $\theta(x) = 1$ for all $x \in \mathbb{Z}^d$, we recover the d-lattice structure. So it is natural to ask the relation or the range of our approach with respect to [AFGL15]. For simplicity, we focus in the case $V(t) = t^2$ and the question related to the identity boundary conditions (the shift boundary conditions are not defined in the model with multiplicities since the lattice is not translation invariant). In [AFGL15], for the existence and uniqueness they need α large enough such that $\varphi(\alpha) < r_1$ where r_1 is the unique solution in $[0, 1]$ of the equation $\frac{r}{(1-r)^2} - r = 1$. The solution r_1 is approximately 0.44504. For prove the existence in this case we essentially need α large enough such that $\varphi(\alpha) < 1$. To obtain also the the uniqueness we need α large enough that satisfies $\varphi(\alpha) < r_0$, where r_0 is the unique solution in $[0, 1]$ of the equation $\frac{r}{(1-r)^2} - r = \frac{1}{2}$ (solving one gets 0.35542). This extends a bit the range of α for the existence but it is more restrictive for the uniqueness.

1.2.3   On a regular lattice embedded in $\mathbb{R}^d$

A lattice $\Omega \subset \mathbb{R}^d$ is regular when exists a basis $\{v_i \colon i = 1, \ldots, d\}$ of $\mathbb{R}^d$ such that

$$\Omega = \left\{ \sum_{i=1}^d a_i v_i \colon a_i \in \mathbb{Z} \right\}.$$

For $L \in \mathbb{N}$, Ω_L denotes the subset of Ω with side lenght L and periodic boundary conditions. For each L, it is defined a probability P_L on S_{Ω_L} such that

$$P_L(\sigma) = \frac{1}{Z_L} e^{-\alpha H_L(\sigma)} \quad \text{for all } \sigma \in S_{\Omega_L},$$

where $H_L(\sigma) = \sum_{x \in \Omega_L} V(\sigma(x) - x)$ is a Hamiltonian associated to the potential $V \colon \mathbb{R}^d \to \mathbb{R}$. Under some conditions on V, Betz proved in [Bet14] the existence of a limit measure μ supported on Ω for the family $(P_L)_{L \in \mathbb{N}}$ and corresponding to temperature α. Note that he did not use the setting of Gibbs measure theory. When one tries to view the model as

a spin system, the condition to be permutation is hard core condition of infinite range. So, the classical theory of Gibbs measures does not work. The condition is the following: for $\alpha > 0$ assume that there exists $\varepsilon > 0$ such that

$$\sum_{x \in \Omega} e^{-(\alpha - \varepsilon) V(x)} < \infty.$$

In our setting, the condition is translated to there exists $\varepsilon > 0$ such that $\varphi_V(\alpha - \varepsilon) < \infty$. Observe that V is non necessarily convex, symmetric or non-negative. The proof uses the translation invariance of Ω and extension theorems of probability theory. In particular, the proof cannot be adapted to models where the points set Ω is a realization of a point process on $\mathbb{R}^d$. The result implies the existence of a limit measure for the case $\Omega = \mathbb{Z}^d$, $V(x) = \|x\|^2$ and $\alpha > 0$ but it does not say anything the cycle structure of a typical permutation.

1.3   Resumen del capítulo

El interés por las permutaciones aleatorias comenzó con los estudios de Feynmann en [Fey53] relacionados a la condensación del gas $_2^4$He. En este artículo se introduce una representación probabilística que consiste en un conjunto de trayectorias de movimientos Brownianos en un intervalo de tiempo fijo que comienzan y terminan en puntos aleatoriamente distribuídos en el espacio y que interactúan entre sí. Varias simplificaciones de este modelo se propusieron para llegar a algunos modelos sobre espacios de permutaciones. Por ejemplo, Sütő mostró que en la versión de gas ideal de Bose existen ciclos macroscópicos (ver [Süt93, Süt02]). A su vez, en [Tót93] Tóth relacionó un modelo particular de permutaciones entre puntos de un retículo, el *interchange process*, con el modelo cuántico ferromagnético de Heisenberg de forma tal que, la existencia de ciclos ínfinitos en el primero es equivalente a la magnetización espontanea del segundo. En la misma época, Aizenman y Nachtergaele [AN94] introdujeron otra representación probabilística sobre un espacio de permutaciones para el modelo anti-ferromagnético de Heisenberg.

Sea $\Omega = (x_i)_{i \in \mathbb{N}} \subset \mathbb{R}^d$ un conjunto localmente finito, esto es, que la intersección con cualquier subconjunto compacto de $\mathbb{R}^d$ es finita. Nos concentramos en los casos donde Ω es el retículo $\mathbb{Z}^d$ o es la realización de un proceso puntual en $\mathbb{R}^d$, como por ejemplo, un proceso puntual de Poisson. Para el conjunto de biyecciones (permutaciones) de Ω se escribirá S_Ω. En este espacio consideramos la medida formal dada por

$$\mu(\sigma) = \frac{e^{-\alpha H(\sigma)}}{Z}, \qquad \sigma \in S_\Omega, \alpha > 0, \tag{1.7}$$

donde Z es la constante de normalización y H es el Hamiltoniano definido formalmente por

$$H(\sigma) = \sum_i \|\sigma(x_i) - x_i\|^2, \qquad \sigma \in S_\Omega.$$ (1.8)

Observar que el Hamiltoniano favorece saltos cortos por sobre saltos largos en virtud de que el peso decae con el cuadrado de la magnitud del salto. El parámetro α se intrepreta físicamente como la temperatura y puede entenderse también como una medida de la penalización a los saltos largos. Cuando Ω es finito la medida (1.7) está bien definida porque todas las sumas involucradas son finitas. Si Ω es un conjunto infinito es necesario dar un sentido a definición. La mecánica estadística permite definir una medida sobre el conjunto infinito manteniendo la forma (1.7) para cuando observamos se condiciona a eventos que suceden dentro de un conjunto finito. Más precisamente, el método consiste en describir las probabilidades condicionales de la medida a volumen infinito dada una condición de borde fuera de un conjunto compacto. Estas probabilidades condicionales, llamadas especificaciones, están dadas por (1.7) y por cierta consistencia entre las condiciones de borde. Un problema fundamental de la mecánica estadística es determinar cuándo estas especificaciones dan lugar a una medida límite, llamada medida de Gibbs, y en caso de que éstas existan, ver si existe más de una medida de Gibbs.

Las preguntas fundamentales en los modelos de permutaciones son sobre la estructura de ciclos de una permutación típica y su relación con los parámetros del modelo de acuerdo a la dimensión del espacio. Los parámetros son esencialmente dos: la densidad de puntos y la temperatura α. En particular, interesa saber si en la permutación típica aparecen ciclos infinitos con probabilidad positiva, y más aún, si estos ciclos infinitos son macroscópicos o no, es decir, si la intersección de uno de estos ciclos con cajas arbitrariamente grandes contiene una fracción positiva de los puntos de la caja.

En el modelo con Hamiltoniano cuadrático (1.8) se conjetura que si $d = 1, 2$, una pemutación típica tiene solamente ciclos finitos en su descomposición, mientras que si $d \geq 3$, existe una temperatura crítica α_c, por debajo de la cual una permutación típica contiene un ciclo infinito con probabilidad positiva. Esta conjetura está soportada por evidencia numérica e heurística que puede encontrarse en los artículos [GRU07, GLU12, GUW11].

Los primeros resultados rigurosos de este modelo fueron presentados por Gandolfo, Ruiz y Ueltschi, en [GRU07], siguiendo un enfoque clásico de extensión de medidas sin usar el formalismo termodinámico. En este artículo se prueba que para cualquier dimensión si la temperatura es suficientemente alta (dependiendo de la dimensión), todos los ciclos son finitos. El caso $d = 1$ fue resuelto por Biskup and Richthammer en [BR15], donde se probó que para cada $\alpha > 0$ existe una única medida de Gibbs asociada a las condiciones de borde identidad que está soportada sobre permutaciones con ciclos finitos.

Más aún, se demostró que las medidas de Gibbs extremales están en correspondencia biunívoca con los *ground states* (mínimos locales) del Hamiltoniano. Se cree que esta correspondencia, que permite describir totalmente el conjunto de medidas extremales, falla en dimensiones mayores. Existen actualmente algunos indicios en esa dirección (ver discusión en [AFGL15] y [BT17]). En [AFGL15], Armendáriz, Ferrari, Groisman y Leonardi, consideraron en $d \geq 2$ el modelo con potencial estrictamente convexo y probaron que si la temperatura es suficientemente alta (dependiendo de d y del potencial) existe una única medida de Gibbs que concentra su masa en permutaciones con ciclos finitos.

Estamos interesados en el caso donde Ω es un conjunto aleatorio de puntos. En el caso $d = 1$ el problema de la existencia de medidas de Gibbs y sus propiedades, fue resuelto también en [BR15], para cuando el conjunto de puntos coincide con la realización de un proceso puntual en $\mathbb{R}$ invariante por traslaciones. Concretamente se prueba que para cualquier valor de la temperatura y para casi todas realización del proceso puntual, la medida de Gibbs existe y concentra sobre permutaciones con ciclos finitos. A su vez se caracteriza el conjunto de medidas de Gibbs extremales.

Cuando se considera un modelo *annealead*, es decir, cuando puntos y permutaciones se sortean conjuntamente, Betz y Ueltschi mostraron en [BU09, BU11b] que si $d \geq 3$, existe una densidad crítica de puntos ρ_c por encima de la cual el modelo presenta ciclos macroscópicos y debajo de la que una permutación tiene solamente ciclos finitos. Ellos también estudiaron el comportamiento asintótico de los ciclos macroscópicos. Precisamente, los largos ordenados de estos ciclos, reescalados apropiadamente, convergen a la distribución Poisson-Dirichlet al igual de lo que sucede cuando las permutaciones son distribuídas uniformemente (ver [Sch05]).

En esta tesis consideramos el caso donde el conjunto de puntos es un proceso de Poisson de intensidad ρ en $\mathbb{Z}^d$. Esto es que, en cada sitio $x \in \mathbb{Z}^d$ se coloca una cantidad de puntos $\theta(x)$, donde $\theta(x)$ se distribuye Poisson(ρ) y las variables aleatorias de cada sitio son independientes del resto; luego se considera el siguiente conjunto determinado por una realización: $\Omega_\theta = \{(x,i) : 1 \leq i \leq \theta(x), x \in \mathbb{Z}^d\}$. Observar que este proceso es una versión simplificada de un proceso puntual de Poisson estándar en $\mathbb{R}^d$ de intensidad ρ, donde los puntos de éste que están en la caja $x + [0,1)^d$ se colapsan a una misma posición x. Se estudiarán las permutaciones de puntos de Ω_θ bajo la probabilidad inducida por el Hamiltoniano (1.8) considerando que la distancia entre dos puntos de Ω_θ es la distancia entre sus proyecciones a $\mathbb{Z}^d$. Nuestro primer resultado muestra la existencia de medidas de Gibbs consistentes con las especificaciones para casi toda realización del medio $\{\theta(x)\}_{x \in \mathbb{Z}^d}$, cuando la temperatura es suficientemente alta y la densidad ρ está fija en $(0, 1/2)$. Posteriormente se prueba que para casi toda realización del medio, la descomposición de una permutación típica usa solo ciclos finitos y que la medida de

Gibbs es la única entre aquellas que concentran su masa en ciclos finitos.

Los resultados para el modelo discreto puede extenderse a cuando el conjunto de puntos es la realización de un proceso de Poisson en $\mathbb{R}^d$ con baja intensidad. Concretamente, si la densidad es pequeña en el régimen de alta temperatura existe una medida de Gibbs que concentra sobre permutaciones con ciclos finitos y es la única medida de Gibbs entre ellas.

Los resultados se basan en una técnica introducida por Fernández, Ferrari y García en [FFG01], donde la medida de Gibbs es la medida invariante de cierto proceso de Markov sobre el espacio de ciclos finitos y se construye usando acoplamientos y el algoritmo de simulación perfecta. En el caso del modelo considerado en esta tesis, el proceso de Markov asociado no está bien definido, y la medida a volumen infinito no puede ser construída usando el algoritmo de simulación perfecta. Sin embargo, se puede aplicar la técnica a las especificaciones y establecer que éstas están dominadas por un proceso de Poisson en el espacio de ciclos finitos. Luego, utilizando esta dominación se demuestra que la familia de especificaciones es *tight* y por tanto existe un límite débil y este es Gibbs. Ello muestra la existencia. Con argumentos de renovación se muestra que la medida de Gibbs está soportada sobre permutaciones con ciclos finitos y la unicidad entre estas.

En la primer sección de este capítulo se introduce la notación y las definiciones básicas del modelo. También se enuncian los principales resultados de la tesis que aquí repetimos por simplicidad y se presentan algunas posibles variantes de estos. En los que sigue los teoremas se enuncian para el caso de potencial cuadrático pero pueden extenderse a potenciales más generales.

Teorema. *Sea $\rho \in (0, 1/2)$. Se puede elegir $\alpha > 0$ suficientemente grande tal que para casi toda realización del medio $\{\theta(x)\}_{x \in \mathbb{Z}^d}$, existe una medida de Gibbs. Esta medida se obtiene como límite débil por subsucesiones de las especificaciones con condición de borde identidad.*

Más aún, esta medida está soportada en las permutaciones cuya descomposición usa solamente ciclos finitos y es la única medida de Gibbs con esa propiedad.

La definición de que una densidad ρ sea buena para un potencial V se encuentra en (1.1.7). En el caso $V(t) = (t^2 - 2\sqrt{d}t)^+$, se puede pensar que $\rho < \rho_*$, siendo ρ_* la densidad crítica del modelo de percolación Booleano estándar con radio $3\sqrt{d}$.

Teorema. *Sean $\rho \in (0, 1/2)$ una densidad buena para el potencial $V(t) = (t^2 - 2\sqrt{d}t)^+$ y $\alpha > 0$ tales que se satisface*

$$C_\rho \varphi_V(\alpha) < r_0,$$

donde $C_\rho = \frac{\rho e^{-\rho + \frac{1}{2}}}{1 - 2\rho}$, φ_V es la función definida en (1.5) y r_0 es la única solución en $[0, 1]$

de la ecuación $\frac{r}{(1-r)^2} - r = \frac{1}{2}$.

Se considera el modelo de permutaciones sobre el conjunto $\Omega \subset \mathbb{R}^d$ dado por una realización de un proceso puntual de Poisson de intensidad ρ y con potencial cuadrático.

Entonces, para casi toda realización de Ω, el modelo tiene una única medida de Gibbs que concentra sobre permutaciones con ciclos finitos.

En la siguiente sección de este capítulo se presentan sumariamente los resultados previos sobre este modelo o similares. En particular se incluyen descripciones de los principales resultados de [BR15, AFGL15, Bet14].

Chapter 2

Permutations over $\mathbb{Z}^d$ with random multiplicities

In this chapter we present the results for the $\mathbb{Z}^d$ lattice with random multiplicities. In order to do this, we first compute an upper bound for the number of cycles that have the same order support. Then, we show that a finite cycle permutation can be viewed as a gas of cycles in the space of finite cycles. In this new space, we establish a domination of the specificactions with finite cycle boundary condition by a certain Poisson process on the cycles space. This stochastic domination is the key for prove of existence and uniqueness of the Gibbs measure.

2.1   A combinatorial bound

Fix $\bar{y} = (y_1, \ldots, y_m)$ with $y_i \in \mathbb{Z}^d$ such that $y_i \neq y_{i+1}$. It will represent a fix ordered support. Set $N_\theta(\bar{y})$ the number of cycles γ that have ordered support $\bar{y}$. We want to compute an upper bound for $N_\theta(\bar{y})$.

Set $z_1 = y_1$. Given z_j, we define $z_{j+1} = y_{j*}$ where $j*$ is the first integer such that $y_{j*} \neq z_k$ for all $k = 1, \ldots, j$. Write $\{\bar{y}\}$ for the set $\{z_1, \ldots, z_n\}$. Obviously, $n \leq m$. For a cycle γ with ordered support $\bar{y}$, we know that all of points used by the cycle γ are located at sites included $\{\bar{y}\}$, so, for this reason we call $\{\bar{y}\}$ the support of $\bar{y}$. This is a different notion from the support of a cycle, since now, the support is a subset of $\mathbb{Z}^d$ instead a subset of Ω_θ as in the case of cycles.

For each $z \in \{\bar{y}\}$ let $k(z, \bar{y}) = \#\{i \colon y_i = z\}$ be the number of times that z appears in the ordered support $\bar{y}$, i.e., the effective uses of site z, in the sense that computes non

trivially for the Hamiltonian. For be short, we omit the dependence on $\bar{y}$ in $k(z, \bar{y})$. A cycle γ such that $\bar{\gamma} = \bar{y}$, can use the site z at least $k(z)$ times and at most $\theta(z)$ times. Hence, if $k(z) > \theta(z)$ for some $z \in \{\bar{y}\}$ we have $N_\theta(\bar{y}) = 0$. Also, if for any $z \in \{\bar{y}\}$ we have $\theta(z) = 0$, then $N_\theta(\bar{y}) = 0$. Now suppose that $0 < k(z) \le \theta(z)$ for all $z \in \{\bar{y}\}$.

For each difference between the real use of z and its effective use, there exists consecutive points in the cycle that are located in the same spatial site but with different label. We need to estimate in how many ways we can put z in a cycle to obtain $k(z)$ effective uses.

Suppose that we have $k(z_j)$ urns (coordinates of $\bar{y}$ associated to z_j), a_j balls to place on them (the real use of z_j) and the a_j balls are chosen from a set of $\theta(z_j)$ labelled balls.

For choice the a_j balls we have $\binom{\theta(z_j)}{a_j} a_j!$ options, where the first factor corresponds to how many subsets of a_j elements are in a set of $\theta(z_j)$ elements and the second factor is for the different ways to order the labels. Fixed the a_j balls, we need to distribute them into $k(z_j)$ urns putting at least one ball in each urn. It is easy to show that there are $\binom{a_j-1}{k(z_j)-1}$ ways to do it. So that:

$$
\begin{aligned}
N_\theta(\bar{y}) \;\le\; & \sum_{\substack{k(z_1)\le a_1 \le \theta(z_1) \\ \vdots \\ k(z_n)\le a_n \le \theta(z_n)}} \prod_{j=1}^{n} \binom{\theta(z_j)}{a_j} a_j! \binom{a_j-1}{k(z_j)-1} \\[2ex]
=\; & \sum_{\substack{k(z_1)\le a_1 \le \theta(z_1) \\ \vdots \\ k(z_n)\le a_n \le \theta(z_n)}} \prod_{j=1}^{n} \frac{\theta(z_j)!}{(\theta(z_j)-a_j)!} \binom{a_j-1}{k(z_j)-1} \\[2ex]
\le\; & \sum_{\substack{k(z_1)\le a_1 \le \theta(z_1) \\ \vdots \\ k(z_n)\le a_n \le \theta(z_n)}} \prod_{j=1}^{n} \frac{\theta(z_j)!}{(\theta(z_j)-a_j)!} \, 2^{a_j-1} \\[2ex]
\le\; & \prod_{j=1}^{n} \left(\frac{e^{\frac{1}{2}}}{2} \theta(z_j)! \, 2^{\theta(z_j)} \right).
\end{aligned}
\tag{2.1}
$$

For the last inequality we used that:

$$
\sum_{k(z_1)\leq a_1 \leq \theta(z_1)} \frac{\theta(z_1)!}{(\theta(z_1)-a_1)!} 2^{a_1-1} = \theta(z_1)! \sum_{0\leq b_1 \leq \theta(z_1)-k(z_1)} \frac{2^{\theta(z_1)-b_1-1}}{b_1!}
$$
$$
\leq \frac{1}{2}\theta(z_1)!\, 2^{\theta(z_1)} \sum_{b_1\geq 0} \frac{2^{-b_1}}{b_1!}
$$
$$
\leq \frac{e^{\frac{1}{2}}}{2}\theta(z_1)!\, 2^{\theta(z_1)}.
$$

We define the upper bound for $N_\theta(\bar{y})$ as

$$
M_\theta(\bar{y}) = \prod_{j=1}^{n} \left(\frac{e^{\frac{1}{2}}}{2}\theta(z_j)!\, 2^{\theta(z_j)} \mathbf{1}\{\theta(z_j)\neq 0\} \right).
$$

Remark 2.1.1. This upper bound does not depend on the relative order of the coordinates of $\bar{y}$, that is $M_\theta(\bar{y}^\pi) = M_\theta(\bar{y})$ where $\bar{y}^\pi = (y_{\pi(1)}, \ldots, y_{\pi(m)})$ for any permutation π in the symmetric group of m elements.

This observation will be useful for the following case. Suppose that $\bar{y}, \bar{y}'$ are ordered supports that share a site, that is, $\{\bar{y}\} \cap \{\bar{y}'\} \neq \emptyset$. If we need an upper bound for the number of pairs of cycles (γ, γ') such that $\bar{\gamma} = \bar{y}$ and $\bar{\gamma}' = \bar{y}'$, we can use the upper bound $M_\theta(\bar{y}\bar{y}')$, where $\bar{y}\bar{y}'$ is a concatenation of vectors such that $(\bar{y}, \bar{y}')$ is an ordered support. If both ordered supports have the common site at end, the concatenation $(\bar{y}, \bar{y}')$ is also an ordered support. Otherwise, we can use a suitable permutation of coordinates of $\bar{y}$ and $\bar{y}'$ to rearrange their and obtain an ordered support.

Remark 2.1.2. Fix $\bar{y} = (y_1, \ldots y_m)$ an ordered support and consider the bound $M_\theta(\bar{y})$. We need to compute the expectation under $\mathbb{P}$ of this upper bound. Denote by $n = \#\{\bar{y}\}$. Note that $m \geq n$. As the multiplicities associated to different sites are independent and identically distributed, we have:

$$
\mathbb{E}[M_\theta(\bar{y})] = \left(\frac{e^{\frac{1}{2}}}{2}\right)^n \mathbb{E}\left[\theta(z_1)!\, 2^{\theta(z_1)} \mathbf{1}_{\{\theta(z_1)\neq 0\}}\right]^n
$$
$$
= \left(\frac{e^{\frac{1}{2}}}{2}\right)^n \left(\sum_{l\geq 1} l!\, 2^l\, e^{-\rho}\frac{\rho^l}{l!}\right)^n
$$
$$
\leq \left(\frac{\rho e^{-\rho+\frac{1}{2}}}{1-2\rho}\right)^m, \tag{2.2}
$$

if $\rho < 1/2$. Note that $\mathbb{E}[M_\theta(\bar{y})]$ is increasing in ρ.

Remark 2.1.3. Denote by $w(\bar{y}) = \exp\{-\alpha \sum\limits_{i=1}^{m} \|y_{i+1} - y_i\|^2\}$ the weight of the ordered support $\bar{y}$ (we assume that $y_{m+1} = y_1$). For a cycle γ is clear that $w(\gamma) = w(\bar{\gamma})$. We want to compute the sum of weights of ordered supports that contain some site x and have length m. So, we have:

$$\sum_{\substack{\bar{y}:\, x \in \bar{y} \\ |\bar{y}| = m \\ y_i \neq y_{i+1}}} w(\bar{y}) = \sum_{\substack{\bar{y}:\, x \in \bar{y} \\ |\bar{y}| = m \\ y_i \neq y_{i+1}}} \prod_{i=1}^{m} e^{-\alpha \|y_i - y_{i+1}\|^2} = \sum_{\substack{t_1 \ldots t_m \\ t_i \neq 0_d}} \prod_{i=1}^{m} e^{-\alpha \|t_i\|^2} = \prod_{i=1}^{m} \sum_{\substack{t_i \in \mathbb{Z}^d \\ t_i \neq 0_d}} e^{-\alpha \|t_i\|^2} = \varphi(\alpha)^m,$$

where $\varphi(\alpha) = \sum\limits_{\substack{t \in \mathbb{Z}^d \\ t \neq 0_d}} e^{-\alpha \|t\|^2}$. To complete the computation we need an upper bound for $\varphi(\alpha)$. So,

$$\varphi(\alpha) = \sum_{\substack{l_1, \ldots, l_d \\ l_i \neq 0}} \prod_{j=1}^{d} e^{-\alpha l_i^2} = \left(\sum_{l \in \mathbb{Z}} e^{-\alpha l^2} \right)^d - 1 < \left(1 + \int_{-\infty}^{+\infty} e^{-\alpha x^2}\, \mathrm{d}x \right)^d - 1 = \left(1 + \sqrt{\frac{\pi}{\alpha}} \right)^d - 1.$$

$$(2.3)$$

Note that φ is a decreasing function of α and $\varphi(\alpha) \to 0$ when $\alpha \to +\infty$.

Remark 2.1.4. If the Hamiltonian is given by a potential V in the conditions of (1.1.9), the new calculation has the same steps that above. Indeed, for all $\alpha \geq \alpha_V$ we have that

$$\sum_{\substack{\bar{y}:\, x \in \bar{y} \\ |\bar{y}| = m \\ y_i \neq y_{i+1}}} w_V(\bar{y}) = \left(\sum_{\substack{t \in \mathbb{Z}^d \\ t \neq 0_d}} e^{-\alpha V(\|t\|)} \right)^m = \varphi_V(\alpha)^m,$$

where φ_V is the function defined in (1.5). Note that in the general case φ_V is also a decreasing function of α but it does not necessary hold $\varphi_V(\alpha) \to 0$ when $\alpha \to +\infty$. For example, if $V(t) = (t^2 - 2t)^+$ we have $\varphi_V(\alpha) > \#\{x \in \mathbb{Z}^d \colon \|x\| \leq 2\}$.

2.2 Domination by a Poisson process

In this section we will prove that the specifications at finite volume are dominated by a Poisson measure on S_θ^F. The approach follows ideas from [FFG01, FFG02, AFGL15] but only in the finite volume case. In these articles, the Gibbs measure is realized as the stationary distribution of a suitable Markov process on the space of finite cycles using perfect simulation algorithm. At infinite volume, the conditions for apply the coupling from the past algorithm are not satisfied in our case. However, at finite volume the algorithm works.

For $s \in \Omega_\theta$ define $\beta(\alpha, s) = \sum_{\gamma \in \Gamma_\theta} |\gamma| w(\gamma) \mathbf{1}\{s \in \gamma\}$. To apply the coupling from the past algorithm at infinite volume as in [AFGL15], we need to choice $\alpha > 0$ such that $\beta(\alpha) = \sup_{s \in \Omega_\theta} \beta(\alpha, s) < 1$. It does not hold in our case. Indeed, given $s \in \Omega_\theta$, if $X(s) = x$ we have at least $(\theta(x) - 1)!$ cycles with zero Hamiltonian. Hence, $\beta(\alpha, s) \geq (\theta(X(s)) - 1)!$ for all $\alpha > 0$ and taking the supremum we have $\beta(\alpha) = +\infty$ for any α and for almost every realization of the environment θ. In the lattice without multiplicities, $\beta(\alpha, s)$ does not depend on s, so, the supremum is not necessary and if $\beta(\alpha, 0) < \infty$ it is possible to choice α large enough such that $\beta(\alpha) < 1$.

We first attempt to the identity boundary conditions case since the explanation is simpler and main ideas are present.

For this section does not matter the properties of the potential in the Hamiltonian. The only assumption that we need is $w(\gamma) \leq 1$ for all cycle γ.

2.2.1 A particular case: the identity boundary conditions

Each $\sigma \in S_\theta^F$ can be represented as a configuration $\eta \in \{0, 1\}^{\Gamma_\theta}$. Indeed, if we can define $\eta(\gamma) = \mathbf{1}\{\gamma \in \sigma\}$. We say that η is the gas of cycles of σ. For any Λ the set $S_{\theta,\Lambda}^{\mathrm{id}}$ can be described as:

$$S_{\theta,\Lambda}^{\mathrm{id}} = \{\eta \in \{0, 1\}^{\Gamma_\theta} : \eta(\gamma)\eta(\gamma') = 0 \text{ if } \{\gamma\} \cap \{\gamma'\} \neq \emptyset \text{ for all } \gamma, \gamma' \in \Gamma_{\theta,\Lambda}\}.$$

The specification at finite volume Λ with boundary condition id, can be written in terms of η as:

$$G_{\theta,\Lambda}^{\mathrm{id}}(\eta) = \frac{1}{Z_{\theta,\Lambda}^{\mathrm{id}}} \prod_{\gamma \in \Gamma_{\theta,\Lambda}} w(\gamma)^{\eta(\gamma)} \mathbf{1}\{\eta \in S_{\theta,\Lambda}^{\mathrm{id}}\}. \tag{2.4}$$

From this perspective, we understand the specification $G_{\theta,\Lambda}^{\mathrm{id}}$ as a distribution over gas of cycles space with interactions between cycles. The interaction is by exclusion, if a cycle is present in the gas, any other cycle that use a point of it can not be in the gas. Here and subsequently we use indistinctly σ and its associated gas of cycle η.

On $\mathbb{N}_0^{\Gamma_\theta}$ we consider the product measure $\nu_\theta \otimes_{\gamma \in}$ with marginals Poisson with parameter $w(\gamma)$ for each $\gamma \in \Gamma_\theta$. Note that ν_θ assigns positive probability to the set $S_{\theta,\Lambda}^{\mathrm{id}}$ when Λ is finite. In that case, the conditional measure $\nu_\theta(\,\cdot\,|S_{\theta,\Lambda}^{\mathrm{id}})$ is well defined and it is easy to check that $G_{\theta,\Lambda}^{\mathrm{id}}(\,\cdot\,) = \nu_\theta(\,\cdot\,|S_{\theta,\Lambda}^{\mathrm{id}})$. For each finite Λ we want to construct a continuous time Markov process supported on $S_{\theta,\Lambda}^{\mathrm{id}}$ such that $G_{\theta,\Lambda}^{\mathrm{id}}$ is the unique invariant distribution for it.

Let $\mathcal{N}$ be a Poisson process on $\Gamma_\theta \times \mathbb{R} \times \mathbb{R}_+$ with rate measure $w(\gamma) \times dt \times e^{-r} dr$. Now,

define the process $(\eta_t^o \colon t \in \mathbb{R})$ on $\mathbb{N}_0^{\Gamma_\theta}$, called the free process, by its marginal

$$\eta_t^o(\gamma) = \sum_{(\gamma,t',r') \in \mathcal{N}} \mathbf{1}\{t' \le t < t' + r'\}. \tag{2.5}$$

For a mark $(\gamma, t', r') \in \mathcal{N}$ we say that a cycle γ is born at time t' and lives until time $t' + r'$. For each γ the process $(\eta_t^o(\gamma) \colon t \in \mathbb{R})$ is a birth and death process of individuals of type γ. So η_t^o is a product of independent birth and death process for where a cycle of type γ borns at rate $w(\gamma)$ and dies at rate 1. The process has generator given by

$$\mathcal{L}^o f(\eta) = \sum_{\gamma \in \Gamma_\theta} w(\gamma) \left[f(\eta + \delta_\gamma) - f(\eta) \right] + \sum_{\gamma \in \Gamma_\theta} \eta(\gamma) \left[f(\eta - \delta_\gamma) - f(\eta) \right],$$

where f is a local test function and δ_γ is the configuration that has only one cycle and it has type γ.

Lemma 2.2.1. *The measure ν_θ is reversible for $\mathcal{L}^o$.*

Proof. For each $\gamma \in \Gamma_\theta$ we denote by $\nu_{\theta,\gamma}$ the Poisson distribution with parameter $w(\gamma)$. Note that for each marginal dynamics, the distribution $\nu_{\theta,\gamma}$ is reversible since the detail balance equations holds. It also implies that fro any finite Λ the measure $\otimes_{\gamma \in \Gamma_{\theta,\Lambda}} \nu_{\theta,\gamma}$ is reversible for the dynamics of the generator $\mathcal{L}^o$ restricted to $\Gamma_{\theta,\Lambda}$.

For prove the ν_θ is reversible for $\mathcal{L}^o$ it is sufficient to show that for f, g local test fuctions we have $\int g \mathcal{L}^o f \, d\nu_\theta = \int \mathcal{L}^o g f \, d\nu_\theta$. Let $\Lambda \Subset \mathbb{Z}^d$ such that f and g are measurable with respect to the cycle coordinates in $\Gamma_{\theta,\Lambda}$. Then,

$$\int g \mathcal{L}^o f \, d\nu_\theta = \int g \mathcal{L}^o f \left(\otimes_{\gamma \in \Gamma_{\theta,\Lambda}} d\nu_{\theta,\gamma} \right) = \int \mathcal{L}^o g f \left(\otimes_{\gamma \in \Gamma_{\theta,\Lambda}} d\nu_{\theta,\gamma} \right) = \int \mathcal{L}^o g f \, d\nu_\theta,$$

where in the second equality we use that $\Gamma_{\theta,\Lambda}$ is finite and the reversibility holds in such case (there is a slight abuse of notation since we are identifiying $\mathcal{L}^o$ with its restriction to $\Gamma_{\theta,\Lambda}$).

$\square$

Let $\Lambda \Subset \mathbb{Z}^d$. We are interested in a continuous time Markov process on S_θ with invariant distribution $G_{\theta,\Lambda}^{\mathrm{id}}$. So, the process must be supported on $S_{\theta,\Lambda}^{\mathrm{id}}$. This process is called the loss network of cycles at volume Λ. Our aim is construct it as a function of the Poisson process $\mathcal{N}$. If we do it in appropriate way, we obtain a coupling between the birth and death process $(\eta_t^o \colon t \in \mathbb{R})$ and the loss network of cycles.

We say that the cycles γ and γ' are compatible if $\{\gamma\} \cap \{\gamma'\} = \emptyset$. Otherwise they are incompatible. Note that the intersection is among subsets of Ω_θ. A cycle γ is compatible

with the gas of cycles $\eta \in \{0,1\}^{\Gamma_\theta}$ and we write $\gamma \sim \eta$ when γ is compatible with all cycles γ' such that $\eta(\gamma') = 1$. In particular, if γ is compatible with η, we have $\eta(\gamma) = 0$. When γ is incompatible with η we write $\gamma \nsim \eta$. Informally, the loss network has the following dynamic: we try to add a cycle γ to η at rate $w(\gamma)$ but the attempt is effective only when γ is compatible with the cycles already present in η, and we remove each cycle at rate 1, independently from others. The formal generator is given by

$$\mathcal{L}^\Lambda f(\eta) = \sum_{\gamma \in \Gamma_{\theta,\Lambda}} w(\gamma) \mathbf{1}\{\gamma \sim \eta\} \left[f(\eta + \delta_\gamma) - f(\eta) \right] + \sum_{\gamma \in \Gamma_{\theta,\Lambda}} \eta(\gamma) \left[f(\eta - \delta_\gamma) - f(\eta) \right], \quad (2.6)$$

where $f \colon \{0,1\}^{\Gamma_{\theta,\Lambda}} \mapsto \mathbb{R}$ is a test function. Note that if Λ is finite, the state space is also finite, so, the loss network of cycles is well defined and it is an irreducible Markov process with a unique invariant measure. When Λ is infinite, the space state is uncountable, so, we do not know if the loss network process is well defined.

Lemma 2.2.2. *Let $\Lambda \Subset \mathbb{Z}^d$. The measure $G^{id}_{\theta,\Lambda}$ defined in (2.4) is the unique invariant distribution for the generator $\mathcal{L}^\Lambda$. Furthermore, $G^{id}_{\theta,\Lambda}$ can be obtained as the weak limit distribution of the process starting from any initial permutation.*

Proof. The space state is finite, so, the loss network of cycles is a continuous-time Markov chain. Denote by $q(\eta, \xi)$ the rate to jump from η to ξ. We want to prove that

$$G^{id}_{\theta,\Lambda}(\eta) q(\eta, \xi) = G^{id}_{\theta,\Lambda}(\xi) q(\xi, \eta) \qquad \forall \eta, \xi \in S_{\theta,\Lambda}.$$

The only non trivial transitions from $\eta \in S_{\theta,\Lambda}$ are jumps to $\eta \pm \delta_\gamma$ depending on γ and η.

Let $\gamma \in \Gamma_{\theta,\Lambda}$. If $\gamma \in \eta$ we have $q(\eta, \eta - \delta_\gamma) = 1$, otherwise the rate is zero. It also implies that the cycle γ is compatible with $\eta - \delta_\gamma$, so, $q(\eta - \delta_\gamma, \eta) = w(\gamma)$. Finally, the equality follows from

$$G^{id}_{\theta,\Lambda}(\eta - \delta_\gamma) = \frac{G^{id}_{\theta,\Lambda}(\eta)}{w(\gamma)}.$$

If $\eta(g) = 0$ and γ is compatible with η, the jump from η to $\eta + \delta_\gamma$ is allowed and ocurrs with rate $w(\gamma)$. So, as $q(\eta, \eta + \delta_\gamma) = w(\gamma)$, $q(\eta + \delta_\gamma, \eta) = 1$ and

$$G^{id}_{\theta,\Lambda}(\eta + \delta_\gamma) = w(\gamma) G^{id}_{\theta,\Lambda}(\eta),$$

the reversibility is proved.

$\square$

Fix Λ finite. Denote by η_t^Λ the loss network process at time t. We want to construct η_t^Λ using a convenient thinning of η_t^o to obtain $\eta_t^\Lambda \leq \eta_t^o$ for all t. The algorithm to delete

cycles needs to know if the birth attempt is allowed or not. So, we consider the clan of ancestors of a mark $\zeta = (\gamma, t, r) \in \Gamma_{\theta,\Lambda} \times \mathbb{R} \times \mathbb{R}^+$ as follows. The first generation of ancestors supported on Λ is defined by:

$$A_1^{\zeta,\Lambda} = \{(\gamma', t', r') \in \mathcal{N} \colon \gamma' \in \Gamma_{\theta,\Lambda},\ \{\gamma'\} \cap \{\gamma\} \neq \emptyset,\ t' < t < t' + r'\}.$$

Inductively, if $A_{n-1}^{\zeta,\Lambda}$ is determined, for the n-th generation we set:

$$A_n^{\zeta,\Lambda} = \bigcup_{v \in A_{n-1}^{\zeta,\Lambda}} A_1^{v,\Lambda}.$$

The clan of ancestors of the mark ζ supported in Λ is defined by $A^{\zeta,\Lambda} = \bigcup_{n \geq 1} A_n^{\zeta,\Lambda}$.

Note that for $\Lambda' \subset \Lambda$ the clan of ancestors of ζ corresponding to Λ' and Λ might be different. We do not compare $\eta_t^{\Lambda'}$ with η_t^{Λ}, but we will compare both with η_t^o restricted to Λ.

Suppose that $A^{\zeta,\Lambda}$ is finite for all $\zeta \in \Gamma_{\theta,\Lambda} \times \mathbb{R} \times \mathbb{R}^+$ and for almost all realizations of $\mathcal{N}$. Let $\mathcal{D}_0^\Lambda = \emptyset$ and for $n \geq 1$ we define

$$\mathcal{K}_n^\Lambda = \{\zeta \in \mathcal{N} \colon A_1^{\zeta,\Lambda} \setminus \mathcal{D}_{n-1}^\Lambda = \emptyset\}, \qquad \mathcal{D}_n^\Lambda = \{\zeta \in \mathcal{N} \colon A_1^{\zeta,\Lambda} \cap \mathcal{K}_n^\Lambda \neq \emptyset\}.$$

Let $\mathcal{K}^\Lambda = \cup_{n \geq 1} \mathcal{K}_n^\Lambda$ and $\mathcal{D}^\Lambda = \cup_{n \geq 1} \mathcal{D}_n^\Lambda$. By our assumption, all of clans in Λ are finite, so, every mark $\zeta \in \Gamma_{\theta,\Lambda} \times \mathbb{R} \times \mathbb{R}^+$ is in $\mathcal{K}^\Lambda$ or $\mathcal{D}^\Lambda$. Hence, under the assumption, we define the loss network of cycles at volume Λ by

$$\eta_t^\Lambda(\gamma) = \sum_{(\gamma, t', r') \in \mathcal{N}} \mathbf{1}\{t' \leq t < t' + r'\}\, \mathbf{1}\{(\gamma, t', r') \in \mathcal{K}^\Lambda\}\, \mathbf{1}\{\gamma \in \Gamma_{\theta,\Lambda}\}. \tag{2.7}$$

If the clan of ancestors of any mark is finite almost surely, then the loss network is well defined by (2.7). The next Lemma shows that if Λ is finite, the existence of the loss network of cycles is guaranteed. Unfortunately the argument does not work if Λ is infinite as we seen at the beginning of the section.

Lemma 2.2.3. *Let $\Lambda \Subset \mathbb{Z}^d$. Then, the process $(\eta_t^\Lambda \colon t \in \mathbb{R})$ is well defined and it is a Markov process with generator given by (2.6).*

Further, it is stationary by construction, so, η_t^Λ is distributed according to $G_{\theta,\Lambda}$ for all t.

Proof. To check that is well defined we need to prove that for all $\zeta \in \Gamma_{\theta,\Lambda} \times \mathbb{R} \times \mathbb{R}^+$ the clan $A^{\zeta,\Lambda}$ is finite. First, observe that if $\zeta' \in A^{\zeta,\Lambda}$, its associated cycle (the first coordinate of ζ') is supported on Λ. If the process $(\eta_t^o \colon t \in \mathbb{R})$ is restricted to cycles supported on

Λ, the marks $\{(\gamma, t', s') \in \mathcal{N} : \gamma \in \Gamma_{\theta, \Lambda}\}$ can be ordered by their birth time (the second coordinate of the mark). Since $\Gamma_{\theta, \Lambda}$ is finite, for almost every realization of $\mathcal{N}$ there exists a sequence of times $\{t_j : j \in \mathbb{Z}\}$ with $t_j \to \pm\infty$ as $j \to \pm\infty$ such that $\eta_{t_j}^o(\gamma) = 0$ for all $\gamma \in \Gamma_{\theta, \Lambda}$. Therefore, $A^{\zeta, \Lambda}$ must be finite almost surely.

The stationary construction and Lemma (2.2.2) implies that η_t^Λ is distributed according to the specification at volume Λ for all t.

$\square$

Lemma 2.2.4. *For almost every realization of θ, we have $G_{\Lambda, \theta} \preceq \nu_\theta$ for all $\Lambda \Subset \mathbb{Z}^d$.*

Proof. As Λ is finite, Lemma (2.2.3) says that $(\eta_t^\Lambda : t \in \mathbb{R})$ is well defined. By (2.5) and (2.7), the loss network of cycles is constructed by a thinning of the birth and death process restricted to Λ, then we have that $\eta_t^\Lambda \leq \eta_t^o$ for all t. Hence, for all finite Λ the specification $G_{\Lambda, \theta}^{\mathrm{id}}$ is dominated by ν_θ.

$\square$

2.2.2 The general case

Fix $\xi \in S_\theta^F$. We will dominate the specification related to ξ at volume Λ by a birth and death process on Γ_θ conditioned to be always non empty in same particular coordinates. As in the identity boundary condition case, our goal is construct a realization of the $G_{\theta, \Lambda}^\xi$ as function of a Poisson process on $\Gamma_\theta \times \mathbb{R} \times \mathbb{R}_+$.

Given $\Lambda \Subset \mathbb{Z}^d$, we consider the cycles of ξ that use locations in Λ and Λ^c simultaneously. Concretely, set $B(\xi, \Lambda) = \{\gamma \in \xi : \{\bar{\gamma}\} \cap \Lambda \neq \emptyset, \{\bar{\gamma}\} \cap \Lambda^c \neq \emptyset\}$. There are a finite number of cycles in $B(\xi, \Lambda)$ since ξ is a finite cycle permutation and Λ is finite.

The set of permutations that are compatible with ξ at volume Λ, can be written in terms of the gas of cycles representation as:

$$S_{\theta, \Lambda}^\xi = \Big\{ \eta \in \{0, 1\}^{\Gamma_\theta} \quad : \quad \eta(\gamma) = 1 \text{ for all } \gamma \in B(\xi, \Lambda),$$
$$\eta(\gamma) = 0 \text{ if exists } \gamma' \in B(\xi, \Lambda) \text{ with } \{\gamma\} \cap \{\gamma'\} \neq \emptyset,\ \gamma \in \Gamma_{\theta, \Lambda}$$
$$\eta(\gamma)\eta(\gamma') = 0 \text{ if } \{\gamma\} \cap \{\gamma'\} \neq \emptyset \text{ for all } \gamma, \gamma' \in \Gamma_{\theta, \Lambda} \Big\}$$

As in the previous section, we use indistinctly the permutation σ and its associated gas of cycles η. When $\Lambda \Subset \mathbb{Z}^d$, the specification with boundary condition ξ at Λ, can be

expressed as:

$$G_{\theta,\Lambda}^{\xi}(\eta) = \frac{1}{Z_{\theta,\Lambda}^{\xi}} \prod_{\gamma \in \Gamma_{\theta,\Lambda}} w(\gamma)^{\eta(\gamma)} \mathbf{1}\{\eta \in S_{\theta,\Lambda}^{\xi}\}. \tag{2.8}$$

On $\mathbb{N}_0^{\Gamma_\theta}$ we consider the product measure $\nu_{\theta,\Lambda}^{\xi}$ with marginals Poisson of mean $w(\gamma)$ conditioned to be non zero when $\gamma \in B(\xi,\Lambda)$ and with marginals Poisson of parameter $w(\gamma)$ in other case. Observe that $\nu_{\theta,\Lambda}^{\xi}$ assigns positive probability to $S_{\theta,\Lambda}^{\xi}$ when Λ is finite. So, the conditional measure $\nu_{\theta,\Lambda}^{\xi}(\,\cdot\,|S_{\theta,\Lambda}^{\xi})$ is well defined and $G_{\theta,\Lambda}^{\xi}(\,\cdot\,) = \nu_{\theta,\Lambda}^{\xi}(\,\cdot\,|S_{\theta,\Lambda}^{\xi})$. Again, when Λ is finite we want to construct a continuous time Markov process supported on $S_{\theta,\Lambda}^{\xi}$ such that $G_{\theta,\Lambda}^{\xi}$ is the unique invariant distribution for this process.

Let $\mathcal{N}$ be a Poisson process on $\Gamma_\theta \times \mathbb{R} \times \mathbb{R}_+$ with rate measure $w(\gamma) \times dt \times e^{-r}dr$. Now, define the process $(\eta_t^{o,\xi,\Lambda} \colon t \in \mathbb{R})$ on $\mathbb{N}_0^{\Gamma_\theta}$ and called the free process related to ξ and Λ, by

$$\eta_t^{o,\xi,\Lambda}(\gamma) = \mathbf{1}\{\gamma \in B(\xi,\Lambda)\} + \sum_{(\gamma,t',r') \in \mathcal{N}} \mathbf{1}\{t' \le t < t' + r'\}. \tag{2.9}$$

The first summand is the effect of the boundary condition. For a mark $(\gamma, t', r') \in \mathcal{N}$ we say that a cycle γ is born at time t' and lives until time $t' + r'$. The marginal process $(\eta_t^{o,\xi,\Lambda}(\gamma) \colon t \in \mathbb{R})$ is a birth and death process of individuals of type γ, shifted by 1 when $\gamma \in B(\xi,\Lambda)$. The generator of this process evaluated on a local test fuction f is given by

$$\mathcal{L}^{o,\xi,\Lambda} f(\eta) = \sum_{\gamma \in \Gamma_\theta} w(\gamma)\left[f(\eta + \delta_\gamma) - f(\eta)\right] + \sum_{\gamma \in B(\xi,\Lambda)} \eta(\gamma)\mathbf{1}\{\eta(\gamma) \ge 2\}\left[f(\eta - \delta_\gamma) - f(\eta)\right]$$
$$+ \sum_{\gamma \notin B(\xi,\Lambda)} \eta(\gamma)\left[f(\eta - \delta_\gamma) - f(\eta)\right].$$

Lemma 2.2.5. *Let $\Lambda \Subset \mathbb{Z}^d$. The measure $\nu_{\theta,\Lambda}^{\xi}$ is reversible for $\mathcal{L}^{o,\xi,\Lambda}$.*

Proof. It is the same proof that in the case with identity boundary conditions, Lemma (2.2.1) of the previous sub-section.

$\square$

As in the previous case, the goal is define a loss network of cycles related to ξ at volume Λ, where $\Lambda \Subset \mathbb{Z}^d$, using a convenient thinning of $(\eta_t^{o,\xi,\Lambda} \colon t \in \mathbb{R})$. The loss network associated to ξ works as follows: cycles in $B(\xi,\Lambda)$ are always present in the process. A cycle $\gamma \in \Gamma_{\theta,\Lambda}$ tries to be added at rate $w(\gamma)$ but the attempt is effective only when γ is compatible with the cycles already present, and it is removed at rate 1 independently from others. The

formal generator is given by

$$\mathcal{L}^{\xi,\Lambda} f(\eta) = \sum_{\gamma \in \Gamma_{\theta,\Lambda}} w(\gamma) \mathbf{1}\{\gamma \sim \eta\} \left[f(\eta + \delta_\gamma) - f(\eta) \right] + \sum_{\gamma \in \Gamma_{\theta,\Lambda}} \eta(\gamma) \left[f(\eta - \delta_\gamma) - f(\eta) \right] ,$$

(2.10)

where $f \colon S_{\theta,\Lambda}^\xi \mapsto \mathbb{R}$ is a test function. At first sight, $\mathcal{L}^{\xi,\Lambda}$ does not depend on ξ, but observe that if a cycle is incompatible with any cycle from $B(\xi,\Lambda)$, then it is not allowed to be added, since cycles from $B(\xi,\Lambda)$ are always present.

Lemma 2.2.6. *Let $\Lambda \Subset \mathbb{Z}^d$ and ξ a finite cycle permutation. The measure $G_{\theta,\Lambda}^\xi$ defined in (2.8) is the unique invariant distribution for the generator $\mathcal{L}^{\xi,\Lambda}$. Furthermore, $G_{\theta,\Lambda}^\xi$ can be obtained as the weak limit distribution of the process starting from any initial permutation.*

Proof. The same proof of (2.2.2) works.

$\square$

By Lemma (2.2.3) we know that the clan of ancestors restricted to Λ, $A^{\zeta,\Lambda}$, is finite for all $\zeta \in \Gamma_{\theta,\Lambda} \times \mathbb{R} \times \mathbb{R}^+$ for almost every realization of $\mathcal{N}$. So, we only need to redefine the algorithm appropriately to obtain a stationary version of the loss network related to ξ at volume Λ.

Let $\mathcal{D}_0^{\xi,\Lambda} = \{(\gamma,t,r) \in \mathcal{N} \colon \gamma \nsim \gamma' \text{ for some } \gamma' \in B(\xi,\Lambda)\}$ and for $n \geq 1$ set

$$\mathcal{K}_n^{\xi,\Lambda} = \{\zeta \in \mathcal{N} \colon A_1^{\zeta,\Lambda} \setminus \mathcal{D}_{n-1}^{\xi,\Lambda} = \emptyset\}, \qquad \mathcal{D}_n^{\xi,\Lambda} = \{\zeta \in \mathcal{N} \colon A_1^{\zeta,\Lambda} \cap \mathcal{K}_n^{\xi,\Lambda} \neq \emptyset\} .$$

Let $\mathcal{K}^{\xi,\Lambda} = \cup_{n \geq 1} \mathcal{K}_n^{\xi,\Lambda}$ and $\mathcal{D}^{\xi,\Lambda} = \cup_{n \geq 1} \mathcal{D}_n^{\xi,\Lambda}$. Since all of clans of ancestors supported in Λ are finite, every mark $\zeta \in \Gamma_{\theta,\Lambda} \times \mathbb{R} \times \mathbb{R}^+$ is in $\mathcal{K}^{\xi,\Lambda}$ or $\mathcal{D}^{\xi,\Lambda}$. We define the loss network of cycles at volume Λ by

$$\eta_t^{\xi,\Lambda}(\gamma) = \sum_{(\gamma,t',r') \in \mathcal{N}} \mathbf{1}\{t' \leq t < t' + r'\} \mathbf{1}\{(\gamma,t',r') \in \mathcal{K}^{\xi,\Lambda}\} \mathbf{1}\{\gamma \in \Gamma_{\theta,\Lambda}\} .$$

(2.11)

Note that the algorithm works essentially as the identity case. The differences are in the first two steps. While in the identity case any ζ with $A^{\zeta,\Lambda} = \emptyset$ is kept in the second step, in the general case, the same ζ will be kept in the second step if and only if its cycle coordinate is compatible with all cycles of $B(\xi,\Lambda)$.

Lemma 2.2.7. *If $\Lambda \Subset \mathbb{Z}^d$, the process $(\eta_t^{\xi,\Lambda} \colon t \in \mathbb{R})$ is well defined. It is a Markov process with generator given by (2.10).*

Further, the construction (2.11) is stationary, so, $\eta_t^{\xi,\Lambda}$ is distributed according to $G_{\theta,\Lambda}^\xi$ for all t.

Proof. The assertions follows from Lemmas (2.2.3) and (2.2.6).

$\square$

Lemma 2.2.8. *For almost every realization of θ, we have $G_{\Lambda,\theta}^{\xi} \preceq \nu_{\theta,\theta}^{\xi}$ for all finite Λ.*

Proof. By (2.9) and (2.11), both process are constructed as a function of a Poisson process over $\Gamma_{\theta,\Lambda} \times \mathbb{R} \times \mathbb{R}^+$ such that $\eta_t^{\xi,\Lambda} \leq \eta_t^{\xi,\Lambda,o}$ for all $t \in \mathbb{R}$. Then as the constructions are stationary, so, both process are sampled according to their invariant distribution, and lemma follows.

$\square$

2.3   Existence of Gibbs measures

In this section we prove that if $\rho \in (0, 1/2)$, the family of specifications with identity boundary condition is tight in the large temperature regime for almost every realization of θ. The result also holds considering the specifications related to ξ boundary condition when ξ is a finite cycle permutation. In the next section we prove that in the large temperature regime there exists a unique Gibbs measure that concentrates over finite cycle permutations, so, the weak limits from specifications with a general finite cycle boundary condition are the same. For this reason, we focus on the identity boundary condition case.

Most of proofs are done in the case of quadratic potential but works in the general case with slight modifications. When is not the case, we present an alternative proof for the more general case.

The family $\{G_{\theta,\Lambda}^{\mathrm{id}}\}_{\Lambda \in \mathbb{Z}^d}$ is tight when given $\epsilon > 0$ there exists a compact set $K \subset S_{\theta}$ such that $\sup_{\Lambda \in \mathbb{Z}^d} G_{\theta,\Lambda}^{\mathrm{id}}(K^c) < \epsilon$.

Let $x \in \mathbb{Z}^d$ and $f : \mathbb{Z}^d \mapsto \mathbb{N}$. Define the set $\widehat{K}_f = \bigcap_{x \in \mathbb{Z}^d} \widehat{K}_f(x)$, where

$$\widehat{K}_f(x) = \{\eta \in \mathbb{N}_0^{\Gamma_\theta} : \forall \gamma \in \eta \text{ such that } x \in \gamma \text{ we have } H(\gamma) \leq f(x)\}. \tag{2.12}$$

Denote by $\widehat{K}_f^c(x)$ the complement of $\widehat{K}_f(x)$.

Lemma 2.3.1. *Let $\rho \in (0, 1/2)$ and $\alpha > 0$ such that*

$$C_\rho \, \varphi(\alpha/2) < 1\,, \tag{2.13}$$

where φ is the function defined in (2.3) and $C_\rho = \frac{\rho e^{-\rho+\frac{1}{2}}}{1-2\rho}$. Then,

$$\mathbb{E}\left[\nu_\theta(\widehat{K_f^c}(x))\right] \leq C(\rho,\alpha)e^{-\frac{\alpha}{2}f(x)}. \tag{2.14}$$

In the quadratic potential case we know that $\varphi(\alpha/2) \to 0$ when $\alpha \to +\infty$, so, for any $\rho \in (0, 1/2)$ we can choose α large enough such that $C_\rho \varphi(\alpha/2) < 1$. For general potential the conditions (2.13) becomes to $C_\rho \varphi_V(\alpha/2) < 1$ where φ_V is defined in (1.5). The problem is that $\varphi_V(\alpha)$ does not tend to 0 necessarily. For example, it is the case of $V(t) = (t^2 - 2t)^+$. So, perhaps we need to choice the density ρ small enough such that the condition $C_\rho \varphi_V(\alpha/2) < 1$ holds.

Proof of Lemma (2.3.1). Using the cycle gas representation and the marginal distributions, we have that

$$\mathbb{E}\left[\nu_\theta\left(\widehat{K_f^c}(x)\right)\right] \leq \mathbb{E}\left[\sum_{\substack{\gamma:\,\gamma\ni x \\ H(\gamma)>f(x)}} \nu_\theta(\mathbf{1}\{\gamma \in \eta\})\right] = \mathbb{E}\left[\sum_{\substack{\gamma:\,\gamma\ni x \\ H(\gamma)>f(x)}} (1 - e^{-w(\gamma)})\right].$$

Now, we sum over ordered supports instead cycles. For any cycle, we write its weight as a function of the ordered support. Recall that for any ordered support $\bar{y}$, the number of cycles that have ordered support $\bar{y}$ is bounded above by $M_\theta(\bar{y})$, where $M_\theta(\bar{y})$ was defined in (2.1). Then:

$$
\begin{aligned}
\mathbb{E}\left[\nu_\theta(\widehat{K_f^c}(x))\right] &\leq \mathbb{E}\left[\sum_{m\geq 2}\sum_{\substack{\bar{y}:\,\bar{y}\ni x \\ \bar{y}_i\neq\bar{y}_{i+1} \\ |\bar{y}|=m \\ H(\bar{y})>f(x)}} M_\theta(\bar{y})(1 - e^{-w(\bar{y})})\right] \\
&= \sum_{m\geq 2}\sum_{\substack{\bar{y}:\,\bar{y}\ni x \\ \bar{y}_i\neq\bar{y}_{i+1} \\ |\bar{y}|=m \\ H(\bar{y})>f(x)}} \mathbb{E}\left[M_\theta(\bar{y})\right](1 - e^{-w(\bar{y})}) \\
&\leq \sum_{m\geq 2}\sum_{\substack{\bar{y}:\,\bar{y}\ni x \\ \bar{y}_i\neq\bar{y}_{i+1} \\ |\bar{y}|=m \\ H(\bar{y})>f(x)}} C_\rho^m(1 - e^{-w(\bar{y})}),
\end{aligned}
$$

where in the last line we use the bound (2.2) for the expectation of $M_\theta(\bar{y})$. Then, using

that $1 - e^{-w(\bar{y})} \le w(\bar{y})$ and $H(\bar{y}) > f(x)$ we have:

$$
\begin{aligned}
\mathbb{E}\Big[\nu_\theta\big(\widehat{K}_f^c(x)\big)\Big] &\le \sum_{m \ge 2} \sum_{\substack{\bar{y}\,:\,\bar{y}\ni x \\ \bar{y}_i \ne \bar{y}_{i+1} \\ |\bar{y}|=m \\ H(\bar{y})>f(x)}} C_\rho^m w(\bar{y}) \\
&\le e^{-\frac{\alpha}{2}f(x)} \sum_{m \ge 2} \sum_{\substack{\bar{y}\,:\,\bar{y}\ni x \\ \bar{y}_i \ne \bar{y}_{i+1} \\ |\bar{y}|=m \\ H(\bar{y})>f(x)}} C_\rho^m e^{-\frac{\alpha}{2}H(\bar{y})} \\
&\le e^{-\frac{\alpha}{2}f(x)} \sum_{m \ge 2} C_\rho^m \sum_{\substack{\bar{y}\,:\,\bar{y}\ni x \\ \bar{y}_i \ne \bar{y}_{i+1} \\ |\bar{y}|=m}} e^{-\frac{\alpha}{2}H(\bar{y})} \\
&= e^{-\frac{\alpha}{2}f(x)} \sum_{m \ge 2} C_\rho^m \, \varphi\left(\alpha/2\right)^m \\
&= C(\rho,\alpha) e^{-\frac{\alpha}{2}f(x)} .
\end{aligned}
$$

$\square$

Lemma 2.3.2. *Let ρ and α as in the conditions of Lemma (2.3.1). Given $\epsilon > 0$, for almost every realization of θ there exists a function f that depends on θ, such that $\nu_\theta\big(\widehat{K}_f^c\big) < \epsilon$.*

Proof. Using (2.14), for each $n > 0$ we can pick $f_n(x)$ large enough such that:

$$
\mathbb{E}\Big[\nu_\theta\big(\widehat{K}_{f_n}^c(x)\big)\Big] \le \frac{1}{n^2} \frac{1}{2^{\|x\|_1}} .
$$

If we define $f_n : \mathbb{Z}^d \mapsto \mathbb{N}$ is the obvious way, we have:

$$
\mathbb{E}\Big[\nu_\theta\big(\widehat{K}_{f_n}^c\big)\Big] \le \mathbb{E}\bigg[\sum_{x \in \mathbb{Z}^d} \nu_\theta\big(\widehat{K}_{f_n}^c(x)\big)\bigg] \le \sum_{x \in \mathbb{Z}^d} \frac{1}{n^2} \frac{1}{2^{\|x\|_1}} = \frac{2^d}{n^2} .
$$

So, we have a sequence of functions $\{f_n\}_{n \ge 1}$ such that

$$
\mathbb{E}\bigg[\sum_{n \ge 1} \nu_\theta\big(\widehat{K}_{f_n}^c\big)\bigg] = \sum_{n \ge 1} \mathbb{E}\Big[\nu_\theta\big(\widehat{K}_{f_n}^c\big)\Big] < \infty .
$$

Therefore $\sum_{n \ge 1} \nu_\theta\big(\widehat{K}_{f_n}^c\big) < \infty$ for almost every realization of θ. So, for almost every realization of θ there exists $n_0(\theta,\epsilon)$ such that $\nu_\theta\big(\widehat{K}_{f_n}^c\big) < \epsilon$ for all $n \ge n_0$

$\square$

The following lemma works in the quadratic potential case. A lemma for the general potential case is present below it.

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Lemma 2.3.3. *Consider a function $f : \mathbb{Z}^d \mapsto \mathbb{N}$ and let $K_f := \widehat{K}_f \cap S_\theta^F$, where S_θ^F is the finite cycle permutation space. Then, K_f is a non empty compact set.*

Proof. The gas of cycles of identity is the null configuration, that is $\eta(\gamma) = 0$ for all γ. Hence, $\mathrm{id} \in \widehat{K}_f(x)$ for all x and all choices of $f(x)$. Thus, $\mathrm{id} \in K_f$.

Let $\{\sigma_n\}_{n \geq 1} \subset K_f$. Recall that $\{\sigma_n\}_{n \geq 1}$ converges to σ if and only if $\sigma_n(s) \to \sigma(s)$ for all $s \in \Omega_\theta$. Fix $s \in \Omega_\theta$. By definition of K_f we have $\|X(\sigma_n(s)) - X(s)\|^2 \leq f(X(s))$ for all n. Let $R(s) = \max\{\theta(x) \colon \|x - X(s)\|^2 \leq f(X(s))\}$. Then, $\{\sigma_n(s)\}_{n \geq 1}$ is bounded, since $\|\sigma_n(s) - s\|^2 \leq f(X(s)) + R(s)$, and it has a convergent subsequence. As Ω_θ is countable and S_θ is a complete metric space, a Cantor's diagonal argument follows to show that exists a convergent subsequence. Write σ for the limit permutation of this subsequence.

We claim that $\sigma \in S_\theta^F$. On the contrary, suppose that exists $s \in \Omega_\theta$ such that it is supported in an infinite cycle of σ, i.e., $\sigma^j(s) \neq s$ for all $j \in \mathbb{Z}$. We can choose a subsequence $(j_k)_{k \in \mathbb{Z}}$ such that $X(\sigma^{j_k}(s)) \neq X(\sigma^{j_{k-1}}(s))$ for all $k \in \mathbb{Z}$. Since $\sigma_n \to \sigma$, we can pick n such that $\sigma^j(s) = \sigma_n^j(s)$ for all $j = 1, \ldots, j_{k_0}$, where $k_0 = f(X(s)) + 1$. Then, if $\gamma' \in \sigma_n$ is the cycle that contain s, we have:

$$H(\gamma') \geq \sum_{j=1}^{k_0} \|X(\sigma_n^j(s)) - X(\sigma_n^{j-1}(s))\|^2 \geq \sum_{j=1}^{k_0} \|X(\sigma^j(s)) - X(\sigma^{j-1}(s))\|^2 \geq f(X(s)) + 1 \,,$$

which contradicts that $\sigma_n \in K_f$. Then σ is a finite cycle permutation.

Let $\gamma \in \sigma$ a cycle and suppose that exists $x \in \gamma$ such that $H(\gamma) > f(x)$. As γ is a finite cycle, there exist n_0 such that $\sigma_n(s) = \sigma(s)$ for all $s \in \{\gamma\}$ and $n \geq n_0$. If γ_n' denote the cycle of σ_n that contains x, we have $H(\gamma_n') \geq H(\gamma) > f(x)$ that contradicts $\sigma_n \in K_f$.

Then, K_f is a non empty compact set.

$\square$

The statement of the previous Lemma is different in the general potential case. In some cases we can only proof that K_f is compact for almost every realization of θ with some restrictions for the density ρ. The proof is essentially the same, but we need some extra conditions for the potential and the density. In order to prove the existence of a subsequential limit we need the continuity of V and the growing condition $V(t) \to +\infty$ as $t \to +\infty$.

For show that the permutation limit is a finite cycle permutation we have the problem that, depending on the potential, sometimes an infinite cycle can have finite energy. For example, when the potential $V(t) = (t^2 - 2t)^+$ this is the case of the cycle that contains

the origin in the permutation σ given by, $\sigma(x) = x$ if the first coordinate is non-zero and $\sigma(x) = x + e_1$ in other case. We cannot allow this type of situations.

If V is strictly positive on $[1, +\infty)$ the previous proof works exactly the same, since each jump among points that project to different location contributes to the Hamiltonian. Otherwise, if $L_V = \sup\{t \geq 0 : V(t) = 0\} \geq 1$, we need to choice the density ρ small enough to ensure that the event that there exists an infinite sequence of points $(s_i)_{i \in \mathbb{N}} \subset \Omega_\theta$ such that $\|X(s_i) - X(s_{i+1})\| \geq L_V$, has zero probability with respect to the probability of the environment. This is exactly our definition that ρ is good for the potential V given in (1.1.7). Now, we present the proof in this second case.

Lemma 2.3.4. *Let V is a non negative continuous potential such that $V(t) \to +\infty$ as $t \to +\infty$. Let the density ρ such that it is good for V.*

Then K_f is a non empty compact set for almost every realization of θ.

Proof. The gas of cycles of identity is the null configuration, that is $\eta(\gamma) = 0$ for all γ. Hence, $\mathrm{id} \in \widehat{K}_f(x)$ for all x and all choices of $f(x)$. Thus, $\mathrm{id} \in K_f$.

Let $\{\sigma_n\}_{n \geq 1} \subset K_f$. Fix $s \in \Omega_\theta$. By definition of K_f we have

$$0 \leq V(\|X(\sigma_n(s)) - X(s)\|) \leq f(X(s)) \quad \text{for all } n.$$

As the potential V is continuous and tends to $+\infty$ when the argument goes to $+\infty$, we have that $\|X(\sigma_n(s)) - X(s)\|^2 \leq M_{f,s}$ for all n. Let $R(s) = \max\{\theta(x) : \|x - X(s)\|^2 \leq M_{f,s}\}$. Then, $\{\sigma_n(s)\}_{n \geq 1}$ is bounded because $\|\sigma_n(s) - s\|^2 \leq M_{f,s} + R(s)$, and it has a convergent subsequence. As Ω_θ is countable and S_θ is a complete metric space, a Cantor's diagonal argument follows to show that exists a convergent subsequence. Write σ for the limit permutation of this subsequence.

We claim that $\sigma \in S_\theta^F$ for almost every realization of θ. On the contrary, suppose that there exists an infinite cycle $\gamma \in \sigma$. We can think the cycle as the orbit of a point s, that is, $\gamma = \{\sigma^j(s)\}_{j \in \mathbb{Z}}$ with $\sigma^j(s) \neq s$ for all j. Since ρ is good for V, by the Remark (1.1.8) we know that γ has infinite energy. In particular, $H(\gamma) > f(X(s))$, so there exists a finite set $A \subset \mathbb{Z}$ such that

$$\sum_{j \in A} V(\|X(\sigma^j(s)) - X(\sigma^{j-1}(s))\|) > f(X(s)).$$

Now, we can choose n large enough such that $\sigma^j(s) = \sigma_n^j(s)$ for all $j \in A$. If γ' is the cycle in σ_n that contain s, we have that $H(\gamma') > f(X(s))$ that contradicts the fact $\sigma_n \in K_f$. So, γ must be finite and for any cycle γ we have $H(\gamma) \leq f(X(s))$ if $s \in \gamma$. Hence, σ is a finite cycle permutation and $\sigma \in K_f$. So, K_f is compact for almost every realization of θ.

$\square$

S_θ^F has a natural partial order, i.e., $\eta \leq \eta'$ if $\eta(\gamma) \leq \eta'(\gamma)$ for all $\gamma \in \Gamma_\theta$. So, we say that an event $A \subset S_\theta^F$ is increasing when $\mathbf{1}_A$ is an increasing function with respect to the partial order.

Lemma 2.3.5. *Set $K_f^c = \widehat{K_f^c} \cap S_\theta^F$. Then K_f^c is an increasing event.*

Proof. It is sufficient to show that $\eta \in K_f^c$ implies $\eta' \in K_f^c$ when $\eta \leq \eta'$. Let $\eta \in \widehat{K_f^c}(x)$ for some x. Then there exists $\gamma \in \eta$ such $H(\gamma) > f(x)$ but also $\gamma \in \eta'$, so $\eta' \in \widehat{K_f^c}(x) \subset K_f^c$.

$\square$

The following results are general facts from Gibbs measures theory. The proofs can be found in [BR15, Section 4.3].

Lemma 2.3.6.

1. *Let $\Lambda \Subset \mathbb{Z}^d$ and f a local function. Then $G_{\theta,\Lambda}^\xi(f)$ is also a local and continuous function of the boundary condition ξ.*

2. *Let $\{\Lambda_n\}_{n\geq 1} \Subset \mathbb{Z}^d$ an increasing sequence such that $\Lambda_n \uparrow \mathbb{Z}^d$ and $\{G_{\theta,\Lambda_n}^{id}\}_{n\geq 1}$ converges weakly to a probability measure μ. Then, μ is a Gibbs measure.*

Lemma 2.3.7. *Let ρ and α satisfying the conditions of Lemma (2.3.1). Then, for almost every realization of θ, there exists a Gibbs measure μ_θ associated to temperature α, Hamiltonian given in (1.3) and specifications defined in (1.4).*

Proof. First we prove that the family of specifications $\{G_{\theta,\Lambda}^{id}\}_{\Lambda \in \mathbb{Z}^d}$ is tight. By Lemma (2.2.4) we know that $G_{\theta,\Lambda}^{id}$ is stochastic dominated by ν_θ for all $\Lambda \Subset \mathbb{Z}^d$ and K_f^c is an increasing event for any function f, so, we obtain:

$$\sup_{\Lambda \in \mathbb{Z}^d} G_{\theta,\Lambda}^{id}(K_f^c) \leq \nu_\theta(K_f^c), \qquad \theta \text{ a.s.}.$$

Given $\epsilon > 0$, by Lemma (2.3.2) for almost every realization of θ there exists a function f, such that $\nu_\theta(K_f^c) < \epsilon$. Using that K_f is a compact for almost every realization of the multiplicities, the tightness follows.

Now, we have a subsequential limit μ_θ and Lemma (2.3.6) proves that μ_θ is a Gibbs measure.

$\square$

The additional restrictions for the general potential case becomes from the compactness of K_f, that is, we need the hypothesis of Lemma (2.3.4). With this assumption the proof is the same.

Remark 2.3.8. As the Gibbs measure μ_θ is a subsequential weak limit of specifications with identity boundary conditions, we have that for any increasing and continuous function f, we have

$$\mu_\theta(f) = \lim_{n \to \infty} G_{\theta, \Lambda_n}(f) \le \nu_\theta(f).$$

2.4 Uniqueness of Gibbs measures

We we want to prove that if μ and μ' are Gibbs measures supported on the finite cycle permutations, then $\mu = \mu'$. So, μ_θ obtained in (2.3.7) is the unique Gibbs measure. For establish it, we need to work with the product measure $\mu \otimes \mu'$. It is also a Gibbs measure with respect to the product specifications $G_{\theta, \Lambda_1}^{\xi_1} \otimes G_{\theta, \Lambda_2}^{\xi_2}$ with ξ_1, ξ_2 and $\Lambda_1, \Lambda_2 \Subset \mathbb{Z}^d$.

As in the previous section we focus in the quadratic potential case. When the Hamiltonian is given by a potential V the only change is to use φ_V instead φ in the conditions.

In this section we denote by $\Lambda_l = [-l, l]^d \cap \mathbb{Z}^d$.

Definition 2.4.1. Given $\eta \in \mathbb{N}_0^{\Gamma_\theta}$, we say $\Delta \subset \mathbb{Z}^d$ separates η when η has not cycles for which its support intersect Δ and Δ^c simultaneously, i.e, for all $\gamma \in \eta$ we have $\Delta \cap \{\bar{\gamma}\} = \emptyset$ or $\Delta^c \cap \{\bar{\gamma}\} = \emptyset$. For a pair (η, η') we say that the pair is separated by $\Delta \subset \mathbb{Z}^d$ when both are separated by Δ.

Note that in any case Δ is not necessarily a cubic box, it can be any finite subset of $\mathbb{Z}^d$.

The separating set property is closed by unions. Indeed, let Δ_1 and Δ_2 such that both are separating sets for (η, η'). If $\gamma \in \eta$ is such that $(\Delta_1 \cup \Delta_2) \cap \{\bar{\gamma}\} = \emptyset$, as Δ_1 and Δ_2 separates η, we have that $\{\bar{\gamma}\} \subset \Delta_1^c$ and $\{\bar{\gamma}\} \subset \Delta_2^c$. So, $\{\bar{\gamma}\} \subset (\Delta_1 \cup \Delta_2)^c$. The same holds for $\gamma \in \eta'$.

Let A_n be the event that exists $\Delta \Subset \mathbb{Z}^d$ such that Δ separates (η, η') and $\Delta \supset \Lambda_n$. We also say that A_n ocurrs in Λ_l when $\Delta \subset \Lambda_l$. Note that $A_{n+1} \subset A_n$ and define $A = \cap_{n \ge 1} A_n$. Observe also that A and A_n are decreasing events.

We want to prove that if μ and μ' are Gibbs measures supported on the finite cycle permutations the event A has full measure with respect to the product measure $\mu \otimes \mu'$. Then we use the existence of an arbitrary large separating set to prove that $\mu = \mu'$.

Lemma 2.4.2. *Let μ a Gibbs measure that concentrates on the finite cycle permutations. For $s \in \Omega_\theta$ and $l \in \mathbb{N}$ define the event B_l^s that there exists a cycle γ that contains the*

point s and $H(\gamma) > l$. Then

$$\mu(B_l^s) \leq \nu_\theta(B_l^s), \tag{2.15}$$

where ν_θ is the invariant measure for the loss network of cycles associated to identity boundary conditions.

Remark 2.4.3. Note that $\nu_\theta(B_l^s) \leq \nu_\theta(\widehat{K}_l^c(x))$ where $\widehat{K}_l^c(x)$ is the complement of the event defined in (2.12), with $x = X(s)$ the projection of s onto $\mathbb{Z}^d$.

Proof of Lemma 2.4.2. First note that B_l^s is an increasing event.

Fix j large enough such that $x + \Lambda_l \subset \Lambda_j$. Denote by C_j the event that there exists a cycle γ such that $s \in \gamma$, and $\{\bar{\gamma}\} \cap \Lambda_j^c$. By definition of Gibbs measure and using the domination of the specification G_{θ,Λ_j}^ξ by $\nu_{\theta,\Lambda_j}^{\xi,}$ we have

$$
\begin{aligned}
\mu(B_k^s) = \int G_{\theta,\Lambda_j}^\xi(B_l^s)\mathrm{d}\mu(\xi) &\leq \int \nu_{\theta,\Lambda_j}^\xi(B_l^s)\mathrm{d}\mu(\xi) \\
&= \int \nu_{\theta,\Lambda_j}^\xi(B_l^s)\mathbf{1}\{\xi \in C_j\}\mathrm{d}\mu(\xi) \\
&\quad + \int \nu_{\theta,\Lambda_j}^\xi(B_l^s)\mathbf{1}\{\xi \notin C_j\}\mathrm{d}\mu(\xi) \\
&\leq \int \mathbf{1}\{\xi \in C_j\}\mathrm{d}\mu(\xi) + \int \nu_\theta(B_l^s)\mathbf{1}\{\xi \notin C_j\}\mathrm{d}\mu(\xi) \\
&\leq \int \mathbf{1}\{\xi \in C_j\}\mathrm{d}\mu(\xi) + \nu_\theta(B_l^s) \int \mathbf{1}\{\xi \notin C_j\}\mathrm{d}\mu(\xi).
\end{aligned}
$$

In the third line we use that if $\xi \notin C_j$ the cycle that contains x can be sampled with ν_θ instead $\nu_{\theta,\Lambda_j}^\xi$.

Now, observe that if ξ is a finite cycle permutation $\mathbf{1}\{\xi \in C_j\} \to 0$ as j goes to $+\infty$, so, taking the limit in j and using the dominate convergence theorem we obtain (2.15).

$\square$

We say that the cycles $\gamma, \gamma' \in \Gamma_\theta$ are neighbours and we write $\gamma \bowtie \gamma'$, if exists $s, s' \in \Omega_\theta$ such that $s \in \gamma$, $s' \in \gamma'$ and $X(s) = X(s')$. In terms of ordered supports, γ and γ' are neighbours if $\{\bar{\gamma}\} \cap \{\bar{\gamma}'\} \neq \emptyset$. A path of cycles of length n is a sequence of n different cycles $\gamma_1, \ldots, \gamma_n$ such that $\gamma_i \bowtie \gamma_{i+1}$ for $i = 1, \ldots, n-1$. The idea is to consider a random subgraph of $(\Gamma_\theta, \bowtie)$ and ask about percolation on it, i.e., the existence of an infinite path of cycles with positive probability.

Fix $\eta, \eta' \in \mathbb{N}^{\Gamma_\theta}$. We declare a cycle γ open when $\eta(\gamma) + \eta'(\gamma) \geq 1$, that is, when γ is in η or η'. A path $\gamma_1, \ldots, \gamma_n$ is open when every γ_i is open. The path $\gamma_1, \ldots, \gamma_n$ starts at site x if $x \in \gamma_1$.

Recall that a trivial cycle is a cycle that only uses points located at the same site. Fix

a site $x_0 \in \mathbb{Z}^d$ with $\theta(x_0) \neq 0$. Let $D(n)$ be the event that there exists an open path of length n that starts at site x_0 and it is formed only by non trivial cycles. Obviously, we have $D(n+1) \subset D(n)$ for all n. Clearly, there are paths of cycles of length n that are not included in $D(n)$. However, an infinite path of cycles exists if and only if there exists an infinite path using only non trivial cycles.

Remark 2.4.4. Let r_0 be the unique solution in $[0, 1]$ of the equation $\frac{r}{(1-r)^2} - r = \frac{1}{2}$. Solving the equation we get $r_0 \approx 0.35542$. Note that $\sum_{m \geq 2} m r^m = \frac{r}{(1-r)^2} - r < \frac{1}{2}$ when $r < r_0$.

Lemma 2.4.5. *Suppose that $\rho \in (0, 1/2)$ and $\alpha > 0$ satisfy $C_\rho \varphi(\alpha) < r_0$, where C_ρ is the constant that appears in Lemma (2.3.1) and r_0 is defined in the Remark (2.4.4). Consider η, η' independently sampled according to ν_θ and the graph structure induced by them.*

Then, for almost every realization of θ, the event that exists an infinite open path of cycles has zero probability with respect to $\nu_\theta \otimes \nu_\theta$.

Proof. It is sufficient to show that $\lim_{n \to +\infty} \nu_\theta \otimes \nu_\theta(D(n)) = 0$.

We first compute the annealed expectation of $D(n)$ with respect to $\nu_\theta \otimes \nu_\theta$. Using the marginals distribution and independence, we obtain

$$\mathbb{E}[\nu_\theta \otimes \nu_\theta(D(n))] = \mathbb{E}[\sum_{\gamma_1, \dots, \gamma_n} \prod_{i=1}^n (1 - e^{-2w(\gamma_i)})] \leq \mathbb{E}[\sum_{\gamma_1, \dots, \gamma_n} \prod_{i=1}^n 2w(\gamma_i)],$$

where the sum denoted by $\sum_{\gamma_1, \dots, \gamma_n}$ is over all paths of cycles of length n that start at x_0 and are formed only by non trivial cycles. As each non trivial cycle has an ordered support, we sum over sequence of ordered supports instead cycles. Concretely, we sum over all sequences of n ordered supports $\bar{y}_1, \dots, \bar{y}_n$ such that $x_0 \in \{\bar{y}_1\}$ and $\bar{y}_i$ shares a site with $\bar{y}_{i+1}$ for all i, that is, $\{\bar{y}_i\} \cap \{\bar{y}_{i+1}\}$ for all i. This sum is denoted by $\sum_{\bar{y}_1, \dots, \bar{y}_n}$. Recall that $M_\theta(\bar{y}_1 \dots \bar{y}_n)$ is an upper bound for the number of cycles $\gamma_1, \dots, \gamma_n$ that have ordered supports $\bar{y}_1, \dots, \bar{y}_n$ respectively when $\bar{y}_1, \dots, \bar{y}_n$ is a sequence of ordered supports that share sites as above (see Remark (2.1.1)). So,

$$\mathbb{E}[\nu_\theta \otimes \nu_\theta(D(n))] \leq \mathbb{E}[\sum_{\bar{y}_1, \dots, \bar{y}_n} M_\theta(\bar{y}_1 \dots \bar{y}_n) \prod_{i=1}^n 2w(\bar{y}_i)] = \sum_{\bar{y}_1, \dots, \bar{y}_n} \prod_{i=1}^n 2C_\rho^{|\bar{y}_i|} w(\bar{y}_i).$$

To estimate the last sum we need to consider any possibility for that $\bar{y}_n$ shares a site with $\bar{y}_{n-1}$, and after this, all of possibilities such that $\bar{y}_{n-1}$ shares a site with $\bar{y}_{n-2}$ and so on. Using that $C_\rho \varphi(\alpha) < 1$ we compute it to obtain

$$\mathbb{E}[\nu_\theta \otimes \nu_\theta(D(n))] \leq \sum_{\bar{y}_1, \dots, \bar{y}_n} \prod_{i=1}^n 2C_\rho^{|\bar{y}_i|} w(\bar{y}_i) \leq 2^n \left[\sum_{m \geq 2} (C_\rho \varphi(\alpha))^m\right] \left[\sum_{m \geq 2} m(C_\rho \varphi(\alpha))^m\right]^{n-1}.$$

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The choice of ρ and α implies that $\sum_{m \geq 2} m(C_\rho \varphi(\alpha))^m < 1/2$. So, it follows that

$$\mathbb{E}[\sum_{n \geq 1} \nu_\theta \otimes \nu_\theta(D(n))] = \sum_{n \geq 1} \mathbb{E}[\nu_\theta \otimes \nu_\theta(D(n))] < \infty \,,$$

and then for almost every realization of θ, $\nu_\theta \otimes \nu_\theta(D(n)) \to 0$ when $n \to +\infty$.

$\square$

Lemma 2.4.6. *Let ρ and α in the same conditions that in the previous Lemma. Recall that A_n be the event that exists a separating set Δ that contains the box Λ_n and $A = \cap_{n \geq 1} A_n$.*

For almost every realization of θ we have that $\nu_\theta \otimes \nu_\theta(A) = 1$.

Proof. Consider $\Delta_0(\eta, \eta')$ defined by

$$\Delta_0(\eta, \eta') = \sup\{\Delta \Subset \mathbb{Z}^d : 0_d \in \Delta, \ \Delta \text{ separates } (\eta, \eta')\} \,.$$

Such Δ_0 exists because the separating set property is closed by finite unions. Suppose that exists $x \in \Delta_0^c$ with $\theta(x) \neq 0$ (it is equivalent to have $\Delta_0 \neq \mathbb{Z}^d$). This also implies that the event A does not hold. In such case, there exists $s \in \Omega_\theta$ with $X(s) = x$ such that $(\eta(s), \eta'(s)) \neq (s, s)$. Otherwise $\Delta_0 \cup \{x\}$ contains 0_d and it is a separating set for (η, η') larger than Δ_0. Call γ_1 to the cycle of η or η' such that $\gamma_1(s) \neq s$. Clearly, γ_1 is an open cycle and since Δ_0 is a separating set, $\{\bar{\gamma}_1\} \subset \Delta_0^c$.

Consider $\Delta_1 = \Delta_0 \cup \{\bar{\gamma}_1\}$. It cannot be a separating set for (η, η') by maximality assumption. Then there exists γ_2 in η or η' such that $\Delta_1 \cap \{\bar{\gamma}_2\} \neq \emptyset$ and $\Delta_1^c \cap \{\bar{\gamma}_2\} \neq \emptyset$. From the second intersection, we deduce that $\Delta_0^c \cap \{\bar{\gamma}_2\} \neq \emptyset$, but as Δ_0 is a separating set, $\{\bar{\gamma}_2\} \subset \Delta_0^c$. We have also $\{\bar{\gamma}_2\} \cap \{\bar{\gamma}_1\}^c \neq \emptyset$, so $\gamma_2 \neq \gamma_1$. From the first intersection we know that $\{\bar{\gamma}_2\} \cap \{\bar{\gamma}_1\} \neq \emptyset$, and as $\gamma_2 \neq \gamma_1$, we have $\gamma_2 \bowtie \gamma_1$. Note that γ_2 is open.

Now, suppose that we have n different open cycles, $\gamma_1, \ldots, \gamma_n$, such that $\{\bar{\gamma}_i\} \subset \Delta_0^c$ for all $i = 1, \ldots, n$, and $\gamma_i \bowtie \gamma_j$ for some $j = 1, \ldots, i-1$. Denote by Δ_n the set $\Delta_0 \cup (\cup_{i=1}^n \{\bar{\gamma}_i\})$. Since Δ_n cannot separate (η, η') there exists a cycle γ_{n+1} in η or η' (so, γ_{n+1} is open) such that $\Delta_n \cap \{\bar{\gamma}_{n+1}\} \neq \emptyset$ and $\Delta_n^c \cap \{\bar{\gamma}_{n+1}\} \neq \emptyset$. As Δ_0 is a separating set, the second condition implies that $\{\bar{\gamma}_{n+1}\} \subset \Delta_0^c$ and $\gamma_{n+1} \neq \gamma_j$ for all $j = 1, \ldots, n$. Then the condition $\Delta_n \cap \{\bar{\gamma}_{n+1}\} \neq \emptyset$ says that $\gamma_{n+1} \bowtie \gamma_j$ for some $j = 1, \ldots, n$.

By induction, we have a sequence of different open cycles $\{\gamma_i\}_{i \in \mathbb{N}}$ with the following property: for each γ_n there exists γ_j with $j < n$, such that $\gamma_n \bowtie \gamma_j$. Then all cycles are

in the the same connected component, so, it is infinite. Hence, we have proved that

$$A^c \subset \{\text{exists an infinite open path of cycles}\},$$

and by Lemma (2.4.5) it follows that $\nu_{\theta,\Lambda}^{\xi} \otimes \nu_{\theta,\Lambda}^{\xi'}(A^c) = 0$ for almost every realization of θ.

$\square$

Lemma 2.4.7. *Consider ρ and α in the conditions of Lemma (2.4.5). Let μ, μ' be a Gibbs measures that concentrate on finite cycle permutations. Then $\mu \otimes \mu'(A) = 1$.*

Proof. As $A_{n+1} \subset A_n$, it is sufficient to show that $\lim_{n \to \infty} \mu \otimes \mu'(A_n) = 1$.

We say that a permutation ξ is bad for Λ_l if exists $\gamma \in \xi$ such that $\{\bar{\gamma}\} \cap \Lambda_{2l} \neq \emptyset$ and $\{\bar{\gamma}\} \cap \Lambda_l \neq \emptyset$. Otherwise, we say that ξ is good for Λ_l. Let $l \gg n$. By definition of Gibbs measures we obtain

$$
\begin{aligned}
\mu \otimes \mu'(A_n) &= \int G_{\theta,\Lambda_{2l}}^{\xi} \otimes G_{\theta,\Lambda_{2l}}^{\xi'}(A_n) \mathrm{d}\mu \otimes \mu'(\xi,\xi') \\
&\geq \int G_{\theta,\Lambda_{2l}}^{\xi} \otimes G_{\theta,\Lambda_{2l}}^{\xi'}(A_n \text{ occurs in } \Lambda_l) \mathrm{d}\mu \otimes \mu'(\xi,\xi') \\
&\geq \int G_{\theta,\Lambda_{2l}}^{\xi} \otimes G_{\theta,\Lambda_{2l}}^{\xi'}(A_n \text{ occurs in } \Lambda_l) \mathbf{1}\{\xi, \xi' \text{ are good for } \Lambda_l\} \mathrm{d}\mu \otimes \mu'(\xi,\xi').
\end{aligned}
$$

The events A_n and A_n occurs in Λ_l are decreasing events, so, using the stochastic domination by $\nu_{\theta,\Lambda_l}^{\xi} \otimes \nu_{\theta,\Lambda_l}^{\xi'}$ we have:

$$
\mu \otimes \mu'(A_n) \geq \int \nu_{\theta,\Lambda_l}^{\xi} \otimes \nu_{\theta,\Lambda_l}^{\xi'}(A_n \text{ occurs in } \Lambda_l) \mathbf{1}\{\xi, \xi' \text{ good for } \Lambda_l\} \mathrm{d}\mu \otimes \mu'(\xi,\xi').
$$

If ξ and ξ' are good for Λ_l, the boundary conditions does not affect the events that depends on the configuration inside Λ_l, so,

$$
\nu_{\theta,\Lambda_l}^{\xi} \otimes \nu_{\theta,\Lambda_l}^{\xi'}(A_n \text{ occurs in } \Lambda_l) = \nu_\theta \otimes \nu_\theta(A_n \text{ occurs in } \Lambda_l),
$$

where $\nu_\theta \otimes \nu_\theta$ is the product of two independent copies of the invariant measure for the free process with identity boundary condition. Then,

$$
\begin{aligned}
\mu \otimes \mu'(A_n) &\geq \int \nu_\theta \otimes \nu_\theta(A_n \text{ occurs in } \Lambda_l) \mathbf{1}\{\xi, \xi' \text{ good for } \Lambda_l\} \mathrm{d}\mu \otimes \mu'(\xi,\xi') \\
&\geq \nu_\theta \otimes \nu_\theta(A_n \text{ occurs in } \Lambda_l) \\
&\quad - \int \mathbf{1}\{\xi \text{ bad for } \Lambda_l\} + \mathbf{1}\{\xi' \text{ bad for } \Lambda_l\} \mathrm{d}\mu \otimes \mu'(\xi,\xi') \\
&\geq \nu_\theta \otimes \nu_\theta(A_n \text{ occurs in } \Lambda_l) - \mu(\xi \text{ bad for } \Lambda_l) - \mu'(\xi' \text{ bad for } \Lambda_l). \quad (2.16)
\end{aligned}
$$

If l goes to $+\infty$ the first term tends to $\nu_\theta \otimes \nu_\theta(A_n)$, and the other terms converge to 0. Indeed, if ξ es bad for Λ_l there exists γ such that its ordered support intersects Λ_{2l}^c and Λ_l, so, $H(\gamma) \geq l$. Hence $\{\xi$ bad for $\Lambda_l\} \subset \cup_{s \in \Lambda_l} B_l^s$ and as μ concentrates on finite cycle permutations by (2.15) and the later remark we have

$$\mu(\xi \text{ bad for } \Lambda_l) \leq \sum_{s \in \Lambda_l} \nu_\theta(B_l^s) \leq \sum_{x \in \Lambda_l} \nu_\theta(\widehat{K_l^c}(x)).$$

Now, we use the estimations of Lemma (2.3.1) to obtain

$$\mathbb{E}[\sum_{x \in \Lambda_l} \nu_\theta(\widehat{K_l^c}(x))] \leq C(2l+1)^d e^{-\frac{\alpha}{2}l},$$

and deduce that $\lim_{l \to +\infty} \mathbb{E}[\sum_{x \in \Lambda_l} \nu_\theta(\widehat{K_l^c}(x))] = 0$. Since the sum inside of the expectation is non-negative, we prove that $\sum_{x \in \Lambda_l} \nu_\theta(\widehat{K_l^c}(x))$ tends to 0 as l tends to ∞ and $\mu(\xi$ bad for $\Lambda_l)$ also converges to 0. So, taking limit in l for the inequality (2.16), we have $\mu \otimes \mu'(A_n) \geq \nu \otimes \nu(A_n)$ and the results follows by Lemma (2.4.6).

$\square$

Lemma 2.4.8. *Let $\rho \in (0, 1/2)$ and $\alpha > 0$ such that $C_\rho \varphi(\alpha) < r_0$, where r_0 is the constant defined in the Remark (2.4.4). If μ and μ' are Gibbs measures supported on finite cycle permutations, then $\mu = \mu'$.*

Proof. It is sufficient to prove that $\mu(B) = \mu'(B)$ holds for a local event B. Fix $B \in \mathcal{F}_{\theta,\Lambda}$. Let $J_\Lambda(\Delta)$ be the event that $\Delta \Subset \mathbb{Z}^d$ is the first separating set that contains Λ. The existence of Δ is guaranteed by (2.4.7). It also proves that

$$\sum_{\substack{\Delta \supset \Lambda \\ \Delta \Subset \mathbb{Z}^d}} \mu \otimes \mu'(J_\Lambda(\Delta)) = 1.$$

Now, suppose that for any finite cycle boundary conditions ξ, ξ', we have

$$G_{\theta,\Delta}^\xi \otimes G_{\theta,\Delta}^{\xi'} \left((B \times S_\theta^F) \cap J_\Lambda(\Delta)\right) = G_{\theta,\Delta}^\xi \otimes G_{\theta,\Delta}^{\xi'} \left((S_\theta^F \times B) \cap J_\Lambda(\Delta)\right). \qquad (2.17)$$

Then, if we integrate with respect to $\mu \otimes \mu'$ and use that μ and μ' are supported in the finite cycle permutations, we obtain

$$\mu \otimes \mu' \left((B \times S_\theta^F) \cap J_\Lambda(\Delta)\right) = \mu \otimes \mu' \left((S_\theta^F \times B) \cap J_\Lambda(\Delta)\right),$$

and summing over all choices of Δ, we obtain $\mu(B) = \mu'(B)$.

It remains to prove that (2.17). To show it, observe that if Δ does not separate (ξ, ξ'), both sides of (2.17) are 0. If Δ separates, the boundary conditions (ξ, ξ') have the same effect as the identity boundary conditions. Hence, we can replace ξ and ξ' by id. Since the event $J_\Lambda(\Delta)$ and $G^{id}_{\theta,\Delta} \otimes G^{id}_{\theta,\Delta}$ are invariant under the map $(\sigma, \sigma') \mapsto (\sigma', \sigma)$, the equation (2.17) holds.

$\square$

2.5 Resumen del capítulo

En este capítulo se prueba la existencia y unicidad de la medida de Gibbs en el régimen de baja densidad y alta temperatura para el retículo discreto con multiplicidades. En la primer sección se presenta un resultado combinatorio que permite acotar en media el número de ciclos que puede provenir de un mismo soporte ordenado.

En la segunda sección se presenta la dinámica de [FFG01] a volumen finito que permite dominar a las especificaciones por un proceso de Poisson en el espacio de los ciclos finitos. Para ello se definen dos procesos de Markov como sigue:

- $(\eta^o_t(\gamma)\colon \gamma \in \Gamma_\theta)_{t\in\mathbb{R}}$ donde cada ciclo finito γ nace a tasa $w(\gamma)$ y muere a tasa 1, independientemente del resto de los ciclos. Este proceso tiene como medida invariante (reversible) a ν_θ, la medida Poisson producto con marginales Poisson $w(\gamma)$.

- Para cada $\Lambda \Subset \mathbb{Z}^d$, se tiene un proceso $(\eta^\Lambda_t(\gamma)\colon \gamma \in \Gamma_{\theta,\Lambda})_{t\in\mathbb{R}}$ donde cada ciclo γ con soporte en Λ nace a tasa $w(\gamma)$ siempre y cuando no esté vivo otro ciclo que use puntos de su soporte, y muere a tasa 1 independiente del resto. Observar que la condición a chequear para ver si tiene permitido nacer, es una condición sobre una cantidad finita de ciclos porque Λ es un subconjunto finito. Este proceso está bien definido si Λ es finito y tiene como medida invariante a la especificación asociada a las condiciones de borde identidad $G^{id}_{\theta,\Lambda}$.

Ambos procesos pueden ser acoplados usando la misma representación gráfica y no es difícil observar que $\eta^\Lambda_t \leq \eta^o_t$. A su vez los procesos puede construirse de forma estacionaria, con lo que se obtiene que $G^{id}_{\theta,\Lambda}$ está dominada estocásticamente por ν_θ. Un resultado similar se consigue si se coloca como condición de borde una permutación que use ciclos finitos, lo que es muy útil al momento de probar la unicidad de la medida de Gibbs.

En la sección siguiente se prueba que la familia de especificaciones $\{G^{id}_{\theta,\Lambda}\}_{\Lambda\Subset\mathbb{Z}^d}$ es *tight*. Para ello se utiliza la dominación estocástica antes mencionada y se acota la esperanza respecto del medio θ de la medida ν_θ sobre ciertos eventos que involucran saltos largos.

Ello, en conjunto con el Lema de Borel-Cantelli permite extraer una subsucesión convergente en la familia de especificaciones, lo que combinado con un resultado general sobre el límite débil de especificaciones, nos da la existencia de la medida de Gibbs y más aún, que esta concentra sobre permutaciones con ciclos finitos.

Para probar la unicidad de la medida de Gibbs entre aquellas que concentran su masa en permutaciones con ciclos finitos, se usa que los procesos de Poisson en el espacio de ciclos finitos, que dominan a las especificaciones, tiene la propiedad de que para casi toda realización del medio existen regiones acotadas que separan a las configuraciones. Esta propiedad también se traslada a las medidas de Gibbs. Concretamente, se prueba que si μ y μ' son medidas de Gibbs que concentran sobre ciclos finitos, para casi toda realización (σ, σ') de $\mu \otimes \mu'$ existe $\Delta \Subset \mathbb{Z}^d$ tal que todo ciclo de σ o σ' está soportado en Δ o bien en Δ^c, es decir, no hay ciclos en σ o σ' que involucren simultáneamente puntos de Δ y Δ^c. Finalmente, si se condiciona a este conjunto Δ y se usa la consistencia de las medidas de Gibbs con sus especificaciones, se concluye la unicidad.

Chapter 3

Permutations over a Poisson process on $\mathbb{R}^d$

In this chapter we consider the problem of the existence of Gibbs measure when the set of points given by a realization of an homogeneous Poisson point process on $\mathbb{R}^d$ with low intensity. The Hamiltonian H is given by a quadratic potential as in (1.2). We understand the finite volume Λ as a compact subset of $\mathbb{R}^d$ and write $\Lambda \Subset \mathbb{R}^d$ for it. The thermodynamic formalism have analogous definitions that in the random lattice case.

Let $\Omega \subset \mathbb{R}^d$ be an infinite discrete set of points. The notation is the same that for previous sections, we write Ω instead Ω_θ or θ. So, S_Ω is the permutation space, S_Ω^F is the set of finite cycle permutation and Γ_Ω is the space of finite cycles. The definition of the support of a cycle is the same that in the previuos case. However, the notion of ordered support does not make sense in the continuous setting, since each jump contributes non-zero to the Hamiltonian.

The section 2.2 works with the same proofs in the continuum case. The birth and death process, called the free process, and the loss network of cycles have the same definitions and properties that in the discrete case. In particular, the specification at finite volume Λ corresponding to a finite cycle boundary condition is stochastically dominated by the corresponding invariant measure of the free process related to it. So, for show the existence we will prove the tightness of the family of specifications $\{G_{\Omega,\Lambda}^{\mathrm{id}}\}_{\Lambda \Subset \mathbb{R}^d}$. To prove the uniqueness we use an appropriate version of the separating set Lemma similar to (2.4.7).

We need to construct a coupling between the free process in the continuum setup with the free process of certain discrete model on $\mathbb{Z}^d$ with Poisson multiplicities in such way that the first is dominated by the second. Then we are able to apply the results of the previous section.

If $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$ we write $\lfloor z \rfloor = (\lfloor z_1 \rfloor, \ldots, \lfloor z_d \rfloor) \in \mathbb{Z}^d$. Obviously $z \in I_{\lfloor z \rfloor}$.

We want to construct a simultaneously a Poisson process on $\mathbb{R}^d$ and $\mathbb{Z}^d$. Let Ω be an homogeneous Poisson process on $\mathbb{R}^d$ with intesity ρ. For $x \in \mathbb{Z}^d$, let $\theta(x)$ be the number of points in $\Omega \cap I_x$, where $I_x = x + [0,1)^d$. Then $\theta = \{\theta(x)\}_{x \in \mathbb{Z}^d}$ is an i.i.d. sequence with Poisson distribution of mean ρ. For each x such that $\theta(x) \neq 0$ we can tag the points of I_x from 1 to $\theta(x)$ with some rule. For example we can tag by the relative order of the distance to x. So, there is a bijection among Ω and Ω_θ, where $\Omega_\theta \subset \mathbb{Z}^d \times \mathbb{N}$ is the set associated to the sequence θ, such that each $z \in \Omega$ is mapped to some $(\lfloor z \rfloor, i)$ with $i \in \{1, \ldots, \theta(\lfloor z \rfloor)\}$. This bijection induces also a bijection among the finite cycle spaces. We denote this bijection as Ψ.

Lemma 3.0.1. *The bijection $\Psi\colon S_\Omega \to S_\theta$ is an homeomorphism. Further, it induces an homeomorphism among $\mathbb{N}_0^{\Gamma_\Omega}$ and $\mathbb{N}_0^{\Gamma_\theta}$ by the relation $\eta \mapsto \varsigma$ where $\varsigma(\gamma) = \eta(\Psi^{-1}(\gamma))$ for all $\gamma \in \Gamma_\theta$.*

Lemma 3.0.2. *For $x, z \in \mathbb{R}^d$ we have that*

$$\|x - z\|^2 \geq V(\|\lfloor x \rfloor - \lfloor z \rfloor\|)$$

where $V\colon [0, +\infty) \to [0, +\infty)$ is given by $V(t) = \max\{t^2 - 2\sqrt{d}t, 0\}$.

Proof. For $x \in \mathbb{R}^d$ write $x = \lfloor x \rfloor + \tilde{x}$ with $\tilde{x} \in [0,1)^d$. So, using Cauchy-Schwarz we have

$$\begin{aligned}
\|x - z\|^2 &= \|\lfloor x \rfloor - \lfloor z \rfloor\|^2 + \|\tilde{x} - \tilde{z}\|^2 - 2\langle \lfloor x \rfloor - \lfloor z \rfloor, \tilde{x} - \tilde{z} \rangle \\
&\geq \|\lfloor x \rfloor - \lfloor z \rfloor\|^2 - 2\|\lfloor x \rfloor - \lfloor z \rfloor\|\|\tilde{x} - \tilde{z}\| \\
&\geq \|\lfloor x \rfloor - \lfloor z \rfloor\|^2 - 2\sqrt{d}\|\lfloor x \rfloor - \lfloor z \rfloor\|.
\end{aligned}$$

$\square$

In the rest of this section, V will denote the potential defined in the previous Lemma and H_V its related Hamiltonian. Note that $H(\gamma) \geq H_V(\Psi(\gamma))$ for all $\gamma \in \Gamma_\Omega$, so, the respective weights satisfies $w(\gamma) \leq w_V(\Psi(\gamma))$ for all $\gamma \in \Gamma_\Omega$.

Denote by $(\eta_t^o\colon t \in \mathbb{R})$ and $(\varsigma_t^o\colon t \in \mathbb{R})$ the stationary constructions of free processes for the continuum model with quadratic Hamiltonian and for the discrete model with Hamiltonian H_V respectively. By the relation among the weights we can give a coupling between the processes such that $\Psi(\eta_t^o) \leq \varsigma_t^o$ for all t, where Ψ acts in each coordinate. Each free process is constructed as a function of a Poisson process in the cycle spaces, so, it is sufficient to couple both processes. Denote by $\mathcal{N}$ and $\mathcal{N}_V$ the Poisson processes corresponding to $(\eta_t^o\colon t \in \mathbb{R})$ and $(\varsigma_t^o\colon t \in \mathbb{R})$ respectively. It is sufficient to construct $\mathcal{N}$

as an independent thinning of $\mathcal{N}_V$ as follows: a mark $(\Psi(\gamma), t, s) \in \mathcal{N}_V$ induces a mark (γ, t, s) in the process $\mathcal{N}$ with probability $w(\gamma)/w_V(\Psi(\gamma))$ independent from each other.

Let $K \subset S_\Omega^F \subset \{0,1\}^{\Gamma_\Omega}$ be a Borel set and suppose that K is a decreasing event. Since the Lemma (2.2.4) holds for the continuum setting, the specifications are dominated by the Poisson measure ν_Ω. Using this fact with the coupling among the free processes we obtain that for all bounded Borel set $\Lambda \subset \mathbb{R}^d$:

$$G_{\Omega, \Lambda}^{\mathrm{id}}(K^c) \leq \nu_\Omega(K^c) \leq \nu_\theta(\Psi(K^c)) \tag{3.1}$$

We recall some definitions for the random lattice model. For $f : \mathbb{Z}^d \mapsto \mathbb{N}$ define the set $\widehat{K}_f = \bigcap_{x \in \mathbb{Z}^d} \widehat{K}_f(x) \subset \mathbb{N}_0^{\Gamma_\theta}$ where

$$\widehat{K}_f(x) = \{\eta \in \mathbb{N}_0^{\Gamma_\theta} : \forall \gamma \in \eta \text{ such that } x \in \gamma \text{ we have } H_V(\gamma) \leq f(x)\}.$$

It is the same set that in the previous sections but now is associated to the Hamiltonian given by V. The set $K_f = \widehat{K}_f \cap S_\theta^F$ is a decreasing event by Lemma (2.3.5) and if the density ρ is good for V, we know that K_f is a non empty compact set for almost every realization of θ. For the rest of the section we assume that ρ is good for V.

Note that in such case, $\Psi^{-1}(K_f)$ is also a decreasing event and a non empty compact set. Hence, the equation (3.1) implies that for almost every realization of the Poisson point process

$$G_{\Omega, \Lambda}^{\mathrm{id}}(\Psi^{-1}(K_f^c)) \leq \nu_\Omega(\Psi^{-1}(K_f^c)) \leq \nu_\theta(K_f^c).$$

So, given $\epsilon > 0$, for almost every realization of it is sufficient to choose a function f such that $\nu_\theta(K_f^c) < \epsilon$ for the model over $\mathbb{Z}^d$ with multiplicities and potential V. Indeed, this can be done using the Lemma (2.3.2).

Lemma 3.0.3. *Let $\rho \in (0, 1/2)$ such that ρ is good for V. Suppose that ρ and $\alpha > 0$ satisfy $C_\rho \varphi_V(\alpha/2) < 1$.*

Then the family $\{G_{\Omega, \Lambda}^{id}\}_{\Lambda \in \mathbb{R}^d}$ is tight and there exists a Gibbs measure μ. Moreover, μ concentrates on finite cycle permutations.

We want to show an uniquennes as in the discrete case, that is, there is only one Gibbs measures in the continuous model that concentrate on the finite cycle permutations. To prove it, we will use the existence of a sequence of separating sets in the discrete model with potential V. We need to redefine the notion of separating set since the definition in the discrete setup uses the notion of ordered support that does not make sense in the continuous.

We say that a compact $\Delta \subset \mathbb{R}^d$ is a separating set for $\eta \in \mathbb{N}_0^{\Gamma_\Omega}$ if $\partial\Delta \cap \Omega = \emptyset$ and for any $\gamma \in \eta$ we have $\{\gamma\} \subset \Delta$ or $\{\gamma\} \subset \Delta^c$. We say that Δ is a separating set for the pair (η, η'), if it is a separating set for η and η'.

Given $z_1, \ldots, z_n \in \Omega$, we can pick a compact set $\Lambda \subset \mathbb{R}^d$ such that $\partial\Lambda \cap \Omega = \emptyset$ and $\{z_1, \ldots, z_n\} \subset \Omega$. There exists a lot of ways to choose Λ, but fix one of these. Now, given $\Delta \Subset \mathbb{Z}^d$ we define $\Psi^{-1}(\Delta)$ as the previous fixed compact set that contains $\cup_{x \in \Delta}(\Omega \cap I_x)$.

Now note that, if $\Delta \Subset \mathbb{Z}^d$ is a separating set for $\varsigma \in \mathbb{N}_0^{\Gamma_\theta}$, the compact set $\Psi^{-1}(\Delta) \subset \mathbb{R}^d$ is also a separating set for $\eta = \Psi^{-1}(\varsigma) \in \mathbb{N}_0^{\Gamma_\Omega}$, that is, for any $\gamma \in \eta$ we have $\{\gamma\} \subset \Psi^{-1}(\Delta)$ or $\{\gamma\} \subset \Psi^{-1}(\Delta^c)$. This fact can be extended to pair of configurations.

Let A be the event that exists a sequence of compact sets that increase to $\mathbb{R}^d$ and each set is a separating set for pairs of gases of cycles on $\Gamma_\Omega \times \Gamma_\Omega$. Using the coupling between both free processes and the Lemma (2.4.6) about the existence or arbitrary large separating sets for the discrete model, one shows that for almost every realization of the Poisson point process $\nu_\Omega \otimes \nu_\Omega(A) = 1$. Moreover, we can pick the sequence of sets as $\{\Psi^{-1}(\Delta_j)\}_{j \in \mathbb{N}}$, where $\{\Delta_j\}_{j \in \mathbb{N}}$ is the separating set sequence for the model on $\mathbb{Z}^d$.

Let μ and μ' Gibbs measures in the continuous model that concentrate on the finite cycle permutations. The Lemma (2.4.7) also holds in the continuum context, since it only uses the definition of Gibbs measures and the domination of specifications by the corresponding free process, so, we have
$$\mu \otimes \mu'(A) = \nu_\Omega \otimes \nu_\Omega(A) = 1.$$

Hence, we ensured the existence of an increasing sequence of separating sets in the continuous model.

Lemma 3.0.4. *Let $\rho \in (0, 1/2)$ such that ρ is good for V. Suppose that ρ and α satisfy $C_\rho \varphi_V(\alpha) < r_0$, where r_0 is the solution of the equation given in (2.4.4).*

Then, if μ and μ' are Gibbs measures supported on finite cycle permutations we have $\mu = \mu'$.

Proof. Observe that replacing $\{\Psi^{-1}(\Delta_j)\}_{j \in \mathbb{N}}$ instead $\{\Delta_j\}_{j \in \mathbb{N}}$ the same proof of Lemma (2.4.8) works.

$\square$

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