Frames by Iterations in Shift-invariant Spaces

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***Abstract*—In this note we solve the dynamical sampling problem for a class of shift-preserving (SP) operators, acting on a finitely generated shift-invariant space (FSIS). We find conditions on the operator and on a finite set of functions in the FSIS in order that the iterations of the operator on the functions produce a frame generator set. That is, the integer translations of the frame generator set is a frame of the FSIS. In order to obtain these results, we study the structure of SP operators and obtain a generalized finite dimensional spectral theorem.**

I. INTRODUCTION

Dynamical Sampling (DS) is a new paradigm in sampling theory, where we aim to reconstruct a signal from its time-space samples. Usually the signal we seek for is evolving with time. Assuming that the samples at initial time are insufficient for reconstruction, we sample at different instances in time, trying to compensate for the initial lost. That is, Dynamical Sampling answers the question of whether it is possible to reconstruct a signal from coarse sampling at different times.

The general question can be stated as:

Given a separable Hilbert space $\mathcal{H}$ find conditions on a bounded linear operator $T \in B(\mathcal{H})$ and functions $\mathcal{F} = \{f_1, \dots, f_m\} \subset \mathcal{H}$ such that the collection of iterations $\{T^j f_i : i = 1, \dots, m, j = 0, 1, 2, ...\}$ form a frame of the Hilbert space $\mathcal{H}$.

Recently, a number of significant results have appeared in the literature. The case where $\mathcal{H}$ is finite dimensional was completely solved in a very general setting in [6]. The DS infinite dimensional case is a challenging problem that requiere the use of techniques from different areas such as Complex Analysis, Operator theory, Spectral theory, etc. See [5], [6], [9], [12], [13], [14].

Early results in DS considered convolution operators on the spaces $l^2(\mathbb{Z}^d), l^2(\mathbb{Z})$ and $L^2(\mathbb{R})$, see [1], [2], [4], [7], [8].

In this paper we studied the dynamical sampling problem for operators acting on shift-invariant (SI) spaces. Shift-invariant spaces are the standard subspaces for sampling. In fact, the classical Sampling Theorem is a statement about the Paley-Wiener space that is a translation invariant space.

The natural operators acting in SI spaces are the shift-preserving (SP) operators, that is, operators that commute with integer shifts. These operators have been studied by Bownik in [11].

The fiberization techniques introduced by Helson in [16] through the range functions, allow to reduce the study of the SP operators acting on *finitely generated* shift-invariant (FSI) spaces, to the action of uncountable linear transformations acting on finite dimensional vector spaces. These linear transformations are the *range operators* that act in the fiber spaces.

Using this characterization of SP operators we reduce the problem of DS for SP operators on FSI spaces to the finite dimensional DS problem. Since the finite dimensional DS problem was solved, we can use the known results. However, there is a substancial obstacle in doing that. We need certain kind of uniformity in translating the finite dimensional results to the SP operator. Moreover, there are some issues of medibility involved in the operation that we have to sort. In this article we solved the DS problem for a normal and bounded SP operator acting on a FSI subspace of $L^2(\mathbb{R}^d)$. We refer the readers to [3] for a more extensive and comprehensive development of the results presented here.

II. SHIFT-INVARIANT SPACES

In this section we describe the basic theory of SI spaces that we need in what follows.

Definition II.1. We say that a closed subspace $V \subset L^2(\mathbb{R}^d)$ is *shift invariant* if for each $f \in V$ we have that $T_k f \in V$, $\forall\, k \in \mathbb{Z}^d$. Here, $T_k f(x) = f(x - k)$.

Given a countable set of functions $\Phi \subset L^2(\mathbb{R}^d)$, we will denote $S(\Phi) = \overline{\operatorname{span}}\left\{T_k \varphi : \varphi \in \Phi,\, k \in \mathbb{Z}^d\right\}$. We say Φ is a *set of generators* of V if $V = S(\Phi)$. When Φ is finite, we say that V is a *finitely generated* shift-invariant space, and if $\Phi = \{\varphi\}$ we say that V is a *principal* shift-invariant space. Moreover, we will denote $E(\Phi) = \left\{T_k \varphi : \varphi \in \Phi,\, k \in \mathbb{Z}^d\right\}$. When $E(\Phi)$ form a frame of V with bounds A and B, we will say that Φ is a *frame generator set* of V with bounds A and B.

An essential tool in the development of shift-invariant theory is the technique known as fiberization that we will introduce now.

Proposition II.1. *The fiberization map,* $\mathcal{T} : L^2(\mathbb{R}^d) \to L^2\left([0,1)^d, \ell^2(\mathbb{Z}^d)\right)$ *defined by* $\mathcal{T}f(\omega) = \{\hat{f}(\omega + k)\}_{k \in \mathbb{Z}^d}$, *is an isometric isomorphism satisfying that* $\mathcal{T}T_k f(\omega) = e^{-2\pi i \langle \omega, k \rangle}\, \mathcal{T}f(\omega)$. *We call* $\mathcal{T}f(\omega)$ *the fiber of* f *at* ω.

Here, the Hilbert space $L^2\left([0,1)^d, \ell^2(\mathbb{Z}^d)\right)$ consists of all vector valued measurable functions $\psi : [0,1)^d \to \ell^2(\mathbb{Z}^d)$ and the norm of the space is

$$\|\psi\| = \left(\int_{[0,1)^d} \|\psi(\omega)\|_{\ell^2}^2 \, d\omega\right)^{1/2} < \infty.$$

Definition II.2. A *range function* is a mapping

$$J : [0,1)^d \to \{\,\text{closed subspaces of } \ell^2(\mathbb{Z}^d)\,\}$$
$$\omega \mapsto J(\omega),$$

We say J is measurable if for every $u, v \in \ell^2(\mathbb{Z}^d)$, the scalar function $\omega \mapsto \langle P_{J(\omega)} u, v\rangle$ is measurable, where $P_{J(\omega)}$ is the orthogonal projection of $\ell^2(\mathbb{Z}^d)$ onto $J(\omega)$.

In [11] and [16], it is proved that for every shift-invariant space $V \subset L^2(\mathbb{R}^d)$ there exists an essentially unique measurable range function J_V which satisfies that

$$f \in V \text{ if and only if } \mathcal{T} f(\omega) \in J_V(\omega), \text{ for a.e. } \omega \in [0,1)^d.$$

Furthermore, if $V = S(\Phi)$ for some countable set $\Phi \subset L^2(\mathbb{R}^d)$, then for a.e. $\omega \in [0,1)^d$,

$$J_V(\omega) = \overline{\operatorname{span}}\{\mathcal{T} f(\omega) \,:\, f \in \Phi\,\}.$$

We call the subspace $J_V(\omega)$ the fiber space of V at ω. We will simply write J when it is evident that we are referring to the corresponding range function of V.

We have the following useful property.

Lemma II.1. *(Helson, [16]) Let $V \subset L^2(\mathbb{R}^d)$ be a shift-invariant space with range function J. Then, for each $f \in L^2(\mathbb{R}^d)$ one has that*

$$\mathcal{T}(P_V f)(\omega) = P_{J(\omega)}(\mathcal{T} f(\omega)).$$

A characterization of frames and Riesz bases of a shift-invariant space V can be given in terms of the fibers.

Theorem II.1. *(Bownik, [11]) Let $\Phi \subset L^2(\mathbb{R}^d)$ be a countable set. Then, the system $E(\Phi)$ is a frame (Riesz basis) of V with constants $A, B > 0$ if and only if $\{\mathcal{T} f(\omega) \,:\, f \in \Phi\,\} \subset \ell^2(\mathbb{Z}^d)$ is a frame (Riesz basis) of $J(\omega)$ with uniform constants $A, B > 0$ for a.e. $\omega \in [0,1)^d$.*

In particular, when Φ is a finite set, this allows us to translate problems in infinite dimensional shift-invariant spaces, into problems of finite dimension that can be treated with linear algebra.

Definition II.3. If V is a finitely generated shift-invariant space, its *length* denoted by $\mathcal{L}(V)$, is defined as the smallest natural number ℓ for which there exits $\varphi_1, ..., \varphi_\ell \in V$ with $V = S(\varphi_1, ..., \varphi_\ell)$.

If J is the range function associated to V, an equivalent definition is $\mathcal{L}(V) = \sup\operatorname{ess}_{\omega\in[0,1)^d} \dim J(\omega)$, where J is the range function associated to V. Moreover, the *spectrum* of V is defined by $\sigma(V) = \left\{\omega \in [0,1)^d \,:\, \dim J(\omega) > 0\right\}$.

Note that, in particular, if $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ and $E(\Phi)$ is a Riesz basis of V, by Theorem II.1 we obtain that $\dim J(\omega) = n$ almost everywhere, in which case $\sigma(V) = [0,1)^d$, up to a set of zero measure.

III. Shift-preserving operators

Definition III.1. Let $V \subset L^2(\mathbb{R}^d)$ be a shift-invariant space and $L : V \to L^2(\mathbb{R}^d)$ be a bounded operator. We say that L is *shift preserving* if $LT_k = T_k L$ for all $k \in \mathbb{Z}^d$.

Definition III.2. Let V be a shift-invariant space with range function J. A *range operator on J* is a mapping

$$R : [0,1)^d \to \left\{\begin{array}{l}\text{bounded operators defined on} \\ \text{closed subspaces of } \ell^2(\mathbb{Z}^d)\end{array}\right\}$$

such that the domain of $R(\omega)$ is $J(\omega)$ for a.e. $\omega \in [0,1)^d$.

We say that R is measurable if for every $u, v \in \ell^2(\mathbb{Z}^d)$, $\omega \mapsto \langle R(\omega) P_{J(\omega)} u, v\rangle$ is a measurable scalar function.

In [11], Bownik proved that given a bounded, shift-preserving operator $L : V \to L^2(\mathbb{R}^d)$, there exists a measurable range operator R on J such that

$$(\mathcal{T} \circ L) f(\omega) = R(\omega)\left(\mathcal{T} f(\omega)\right), \tag{1}$$

for a.e. $\omega \in [0,1)^d$ and $f \in V$. If we identify the range operators which are equal almost everywhere, then the range operator of L is unique.

Furthermore, he also proved the following formula for the operator norm of L,

$$\|L\| = \sup_{\omega\in[0,1)^d}\operatorname{ess} \|R(\omega)\|. \tag{2}$$

Let us now focus on the particular case where $L : V \to V$. Thus, we have a corresponding range operator such that for a.e. $\omega \in [0,1)^d$, $R(\omega) : J(\omega) \to J(\omega)$. We have the following result (see [11]).

Theorem III.1. *The dual operator $L^* : V \to V$ is also a shift-preserving operator and its corresponding range operator R^* satisfies that $R^*(\omega) = (R(\omega))^*$ for a.e. $\omega \in [0,1)^d$. As a consequence, if L is self-adjoint, then $R(\omega)$ is self-adjoint, for almost all $\omega \in [0,1)^d$, and if L is a normal operator, then $R(\omega)$ is a normal operator for a.e. $\omega \in [0,1)^d$.*

Now we give two examples of shift-preserving operators and their corresponding range operator. The first one was shown in [11].

Example III.1. Let I be an index set. Suppose $\Phi = \{\varphi_i : i \in I\}$ is a set of functions such that $E(\Phi)$ is a Bessel family. Then, the frame operator of $E(\Phi)$ is self-adjoint and shift-preserving with corresponding range operator $R(\omega)$, given by the frame operator of $\{\mathcal{T}\varphi_i(\omega) \,:\, i \in I\}$ for a.e. $\omega \in [0,1)^d$.

For the next example we need to introduce a new definition.

Definition III.3. We say that a sequence $a \in \ell^2(\mathbb{Z}^d)$ is of *bounded spectrum* if $\hat{a} \in L^\infty\left([0,1)^d\right)$, where $\hat{a}(\omega) = \sum_{j\in\mathbb{Z}^d} a_j e^{-2\pi i\langle\omega, j\rangle}$.

Example III.2. Let $\Phi = \{\varphi_1, \ldots, \varphi_n\} \subset L^2(\mathbb{R}^d)$ be a set of functions such that $V = S(\Phi)$ and $E(\Phi)$ is a Riesz basis of V. For every $f \in V$, there exist unique sequences $b_1, \ldots, b_n$ in $\ell^2(\mathbb{Z}^d)$ such that $f = \sum_{s=1}^n \sum_{j\in\mathbb{Z}^d} b_s(j) T_j \varphi_s$.

Now, let $a_1,\dots,a_n$ be sequences of bounded spectrum and let $L:V\to V$ be the operator defined by:

$$Lf=\sum_{s=1}^{n}\sum_{j\in\mathbb{Z}^d}(b_s*a_s)(j)T_j\varphi_s.$$

Then, L is a bounded, shift-preserving operator, and its range operator is a $n\times n$ diagonal matrix with diagonal $\hat{a}_1(\omega),\dots,\hat{a}_n(\omega)$ in the basis $\{\mathcal{T}\varphi_1(\omega),\dots,\mathcal{T}\varphi_n(\omega)\}$ for a.e. $\omega\in[0,1)^d$.

IV. SHIFT-DIAGONALIZATION

Most of the results in this section are still valid in the more general context of LCA groups. However in order to keep the exposition simpler we present it here in the Euclidean space $\mathbb{R}^d$.

Let $V\subset L^2(\mathbb{R}^d)$ be a finitely generated shift-invariant space. From now on we will asume that there exists a set $\Phi=\{\varphi_1,\dots,\varphi_n\}\subset V$ such that $E(\Phi)$ is a Riesz basis of V.

Let $L:V\to V$ be a bounded, shift-preserving operator, and $R(\omega):J(\omega)\to J(\omega)$ its corresponding range operator. Since $\dim J(\omega)=n$ for a.e. $\omega\in[0,1)^d$, one could represent $R(\omega)$ as an $n\times n$ matrix with measurable entries, relative to the basis $\{\mathcal{T}\varphi_1(\omega),\dots,\mathcal{T}\varphi_n(\omega)\}$ of $J(\omega)$. It is thus natural to study which properties inherent to the structure of the operators $R(\omega)$ could be translated into properties of L. One of these is, for instance, diagonalization.

Proposition IV.1. *Given $a\in\ell^2(\mathbb{Z}^d)$, define the operator $\Lambda_a:V\to V$ as*

$$\Lambda_a=\sum_{j\in\mathbb{Z}^d}a_jT_j. \tag{3}$$

Then Λ_a is well-defined and bounded if and only if the sequence a is of bounded spectrum. Moreover, in that case, Λ_a is a shift-preserving operator.

Definition IV.1. Let $a\in\ell^2(\mathbb{Z}^d)$ be a sequence of bounded spectrum and define the shift-invariant subspace

$$V_a:=\ker(L-\Lambda_a). \tag{4}$$

We say that Λ_a is an *s-eigenvalue* of L if $\sigma(V_a)=\sigma(V)=[0,1)^d$.

This new definition of s-eigenvalue is related with the eigenvalues of the range operator of L, as we state in the next lemma.

Lemma IV.1. *If Λ_a is an s-eigenvalue of L, then $\lambda_a(\omega):=\hat{a}(\omega)$ is an eigenvalue of $R(\omega)$ for a.e. $\omega\in[0,1)^d$.*

In what follows we will see that, in fact, the range function associated to V_a has as values the eigenspaces of $R(\omega)$ associated to the eigenvalues $\hat{a}(\omega)$ for almost every $\omega\in[0,1)^d$.

Proposition IV.2. *The function $\omega\in[0,1)^d\mapsto\ker(R(\omega)-\lambda_a(\omega)\mathcal{I})$ is the measurable range function of V_a, which we will denote J_{V_a}. Here $\mathcal{I}$ is the identity operator.*

The existence of the s-eigenvalues of L can be proved using the range operator R. As we mentioned before, $R(\omega)$ can be represented as an $n\times n$ matrix of measurable entries. Thus, for every ω one could consider the n eigenvalues (with multiplicity) of $R(\omega)$ in some order. This define n functions of ω, say $\lambda_1(\omega),\dots,\lambda_n(\omega)$. However, it is not straightforward that these functions are measurable. So we need the following proposition which can be found in [3]:

Proposition IV.3. *If $R=R(\omega)$ is an $n\times n$ matrix of measurable entries defined on $[0,1)^d$, then there exist n* measurable *functions $\lambda_i:[0,1)^d\to\mathbb{C}$, $i=1,\dots,n$, such that $\lambda_1(\omega),\dots,\lambda_n(\omega)$ are the eigenvalues of $R(\omega)$.*

Since R is the range operator of L, by (2) we get that $|\lambda_i(\omega)|\le\|L\|$ for a.e. $\omega\in[0,1)^d$, and $i=1,\dots,n$. This says that $\lambda_i\in L^\infty\left([0,1)^d\right)$ and so there exist $a_i\in\ell^2(\mathbb{Z}^d)$ such that $\hat{a}_i=\lambda_i$. On the other hand, since $\lambda_i(\omega)$ is an eigenvalue of $R(\omega)$ almost everywhere, then $\ker(R(\omega)-\lambda_i(\omega)\mathcal{I})\ne\{0\}$. This implies that $\sigma(V_{a_i})=[0,1)^d$, and therefore Λ_{a_i} is an s-eigenvalue of L.

The next proposition gives conditions on the s-eigenvalues in order that its associated subspaces are in direct sum.

Proposition IV.4. *Let a,b be two distinct sequences of bounded spectrum such that Λ_a and Λ_b are s-eigenvalues of L. The following statements are equivalent,*

1) $V_a\cap V_b=\{0\}$,
2) $\hat{a}(\omega)\ne\hat{b}(\omega)$ *for almost all ω in $[0,1)^d$.*

Proposition IV.4 is a consequence of the following lemma, whose proof can be found in [10].

Lemma IV.2. *Let V and U two shift-invariant spaces in $L^2(\mathbb{R}^d)$. Then*

$$J_{V\cap U}(\omega)=J_V(\omega)\cap J_U(\omega),$$

for a.e. $\omega\in[0,1)^d$.

Definition IV.2. We say that L is *s-diagonalizable* if there exist $r\in\mathbb{N}$ and sequences $a_1,\dots,a_r$ of bounded spectrum such that

$$V=V_{a_1}\oplus\cdots\oplus V_{a_r}. \tag{5}$$

Remark IV.1. Notice that $r\le\mathcal{L}(V)=n$ since, otherwise, by Lemma IV.1, $R(\omega)$ would have more than n eigenvalues but $\dim J(\omega)=n$ for a.e. $\omega\in[0,1)^d$.

Proposition IV.5. *If L is s-diagonalizable, then $R(\omega)$ is diagonalizable for a.e. $\omega\in[0,1)^d$.*

Proposition IV.6. *If L is normal and s-diagonalizable, then the associated decomposition in (5) is orthogonal and*

$$L=\sum_{s=1}^{r}\Lambda_{a_s}P_{V_{a_s}},$$

where $P_{V_{a_s}}$ is the orthogonal projection onto the space V_{a_s} for every $s=1,\dots r$.

V. Dynamical sampling for shift-preserving operators

In this section we show the solution of the dynamical sampling problem for a bounded, normal, s-diagonalizable shift-preserving operator L acting on a set of functions $F = \{f_1, \ldots, f_m\}$ of a FSI space V. That is, we give conditions on L and F in order that the collection $\left\{L^j f_i : j = 0, \ldots n-1, i = 1, \ldots m\right\}$ is a frame generator set of V. See [3]. The necessary condition is reminiscent of the condition for the finite dimensional case.

Theorem V.1. *Let $V = S(\Phi)$ a finitely generated shift-invariant space, with $\Phi = \{\varphi_1, ..., \varphi_n\} \subset L^2(\mathbb{R}^d)$ such that $E(\Phi)$ is a Riesz basis of V. Let $\{f_1, ..., f_m\} \subset V$ and $L : V \to V$ a bounded, normal, s-diagonalizable shift-preserving operator. Let $\Lambda_1, ..., \Lambda_r$ be r different s-eigenvalues of L^* such that V can be decomposed as an orthogonal sum $V = V_1 \oplus ... \oplus V_r$, where $V_s = \ker(L^* - \Lambda_s)$ for all $1 \leq s \leq r$.*

If $\left\{L^j f_i : i = 1, \ldots, m,\ j = 0, \ldots, n-1\right\}$ is a frame generator set of V with bounds $A, B > 0$, then $\{P_{V_s} f_i : i = 1, \ldots, m\}$ is a frame generator set of V_s for all $1 \leq s \leq r$ with bounds

$$A\left(\frac{1 - \|L\|^2}{1 - \|L\|^{2n}}\right) \quad \text{and} \quad B.$$

Different than the finite dimensional case, for the converse we have to add an assumption of uniformity.

Theorem V.2. *Under the same assumptions as in Theorem V.1, let $\lambda_1(\omega), ..., \lambda_r(\omega)$ be the eigenvalues of $R(\omega)$ a.e. $\omega \in [0,1)^d$ and suppose that*

$$\operatorname*{inf\,ess}_{\omega \in [0,1)^d} \left\{ \min_{1 \leq s \leq r} \prod_{j \neq s} |\lambda_j(\omega) - \lambda_s(\omega)|^2 \right\} \geq \alpha > 0.$$

If $\{P_{V_s} f_i : i = 1, \ldots, m\}$ is a frame generator set of V_s of all $1 \leq s \leq r$ with bounds $A, B > 0$, then $\left\{L^j f_i : i = 1, \ldots, m,\ j = 0, \ldots, n-1\right\}$ is a frame generator set of V with bounds

$$A\left(\frac{r}{\alpha} \sum_{j=1}^{r-1} \binom{r-1}{j} \|L\|^{2j}\right)^{-1} \quad \text{and} \quad B\left(r \frac{1 - \|L\|^{2n}}{1 - \|L\|^2}\right).$$

To obtain the conditions in Theorem V.1 and Theorem V.2, we applied the results for the finite dimensional case in [6], [12] to the range operator associated to L. However these results can not be applied directly, since we need uniformity in the frame bounds to translate results from the range operator to the SP operator L and in [6], [12] frame bounds estimates were not provided. In the next two theorems we obtain frame bounds estimates for the conditions in the finite dimensional dynamical sampling problem. (See [3]).

Let $R : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation and $\{v_i\}_{i \in I} \subseteq \mathbb{C}^n$. Given $\lambda \in \mathbb{C}$, an eigenvalue of R^*, we denote $E_\lambda = \ker(R^* - \lambda \mathcal{I})$ and P_{E_λ} its orthogonal projection.

Theorem V.3. *If $\left\{R^j v_i : i = 1, \ldots, m,\ j = 0, \ldots, n-1\right\}$ is a frame of $\mathbb{C}^n$ with bounds $A, B > 0$, then for every eigenvalue λ of R^*, $\{P_{E_\lambda} v_i : i = 1, \ldots, m\}$ is a frame of E_λ, with bounds A/C_λ and B/C_λ, where $C_\lambda = \sum_{j=0}^{n-1} |\lambda|^{2j}$.*

For the converse we will now assume that R is a normal operator and hence there exist $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ with $\lambda_i \neq \lambda_j$ if $i \neq j$ such that $R = \sum_{s=1}^{r} \lambda_s P_{E_{\lambda_s}}$.

Theorem V.4. *If for every $1 \leq s \leq r$, $\left\{P_{E_{\lambda_s}} v_i : i = 1, \ldots, m\right\}$ is a frame of E_{λ_s} with bounds $A, B > 0$, then $\left\{R^j v_i : i = 1, \ldots, m,\ j = 0, \ldots, n-1\right\}$ is a frame of $\mathbb{C}^n$ with bounds*

$$A\left(\frac{r}{\alpha_\Lambda} \sum_{j=0}^{r-1} \binom{r-1}{j}^2 \|R\|^{2j}\right)^{-1} \quad \text{and} \quad B\left(r \frac{1 - \|R\|^{2n}}{1 - \|R\|^2}\right),$$

where

$$\alpha_\Lambda = \min_{1 \leq s \leq r} \prod_{\substack{j=1 \\ j \neq s}}^{r} |\lambda_s - \lambda_j|^2 > 0. \tag{6}$$

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