An Elliptic Singular System with Nonlocal Boundary Conditions

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Abstract

We study the existence of solutions for the nonlinear elliptic system $\Delta u + g(u) = f(x)$, where $g \in C(\mathbb{R}^N \backslash S, \mathbb{R}^N)$ and S is a bounded set of singularities. Using topological degree methods, we prove existence results. We analyze in particular the case in which $S = \{0\}$ and the isolated singularity is of a repulsive nature, by approximating problems and prove that if an appropriate Nirenberg type condition holds then the problem has a solution.

Keywords: singularities; elliptic system; nonlocal conditions; topological degree.

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1 Introduction

Let $\Omega \subset \mathbb{R}^d$ a smooth bounded domain. We consider the following elliptic system:

$$\begin{cases} \Delta u + g(u) & = f(x) \quad \text{in } \Omega \\ u & = C \quad \text{on } \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS & = 0 \end{cases} \tag{1}$$

with $C \in \mathbb{R}^N$ a yet to be determined constant vector, $f : \overline{\Omega} \to \mathbb{R}^N$ continuous and $g : \mathbb{R}^N \backslash S \to R^N$ continuous, with $S \subset \mathbb{R}^N$ bounded. Without loss of generality we may assume that $\overline{f} := \frac{1}{|\Omega|} \int_\Omega f(x)dx = 0$

The particular case $S = \{0\}$ was extensively studied in the literature: for example, several results when $d = 1$ can be found in [5], [6] and [11], among other works.

The nonlocal boundary conditions in (1) have been studied by Berestycki and Brézis in [4] and also by Ortega in [9]. They arise from certain models in plasma physics: specifically, a model describing the equilibrium of a plasma confined in a toroidal cavity, called a Tokamak machine. A detailed description of this problem can be found in the appendix of [12].

Note that when $d = 1$ and $\Omega = (a, b)$, the system reads:

$$u'' + g(u) = p(t), \qquad t \in (a, b).$$

In this framework, the boundary conditions can be interpreted as follows:

$$u = C \text{ on } \partial\Omega \quad \Rightarrow \quad u(a) = u(b); \quad \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS = 0 \quad \Rightarrow \quad u'(a) = u'(b).$$

Hence, for $d > 1$ the nonlocal boundary condition in (1) can be seen as a generalization of the well known periodic conditions.

The case $d = 1$ has been studied by the authors in [3]. Using topological degree methods it was proved that if the nonlinearity $g : \mathbb{R}^N \backslash \{0\} \to \mathbb{R}^N$ is continuous, repulsive at the origin and bounded at infinity, and an appropriate Nirenberg type condition [8] holds, then either the problem has a classical solution, or else there exists a family of solutions of perturbed problems that converges uniformly and weakly in H^1 to some limit function u. Furthermore, if the singularity is *strong* (in a sense that will be explained below), then u is nontrivial and it can be shown, under extra assumptions, that the problem has always a classical solution.

In this work, we shall consider two different problems. In the next section we shall allow the (bounded) set S of singularities to be arbitrary and focus our attention on the behavior of the nonlinear term g over the boundary of an appropriate domain $D \subset \mathbb{R}^N \backslash S$. More precisely, we shall assume the boundedness condition

(B) $\qquad \limsup_{|u|\to\infty} |g(u)| < \infty$

and introduce a condition of geometric nature that involves the geodesic distance on Ω, namely:

$$d(x, y) := \inf\{lenght(\gamma) : \gamma \in C^1([0, 1], \Omega) : \gamma(0) = x, \gamma(1) = y\}.$$

Indeed, we shall fix a compact neighborhood $\mathcal{C}$ of S and a number

$$r := k\, diam_d(\Omega)(\|f\|_\infty + \sup_{u \notin \mathcal{C}} |g(u)|), \tag{2}$$

where k is a constant such that

$$\|\nabla u\|_\infty \leq k\|\Delta u\|_\infty$$

for all $u \in C^2(\overline{\Omega}, \mathbb{R}^N)$ satisfying the nonlocal boundary conditions of (1). Then we shall assume, for a certain $D \subset \mathbb{R}^N \backslash (\mathcal{C} + \overline{B_r}(0))$:

(D_1) For all $v \in \partial D$, $0 \notin co(g(B_r(v)))$, where '$co(X)$' stands for the convex hull of a set $X \subset \mathbb{R}^N$.

(D_2) $\deg(g, D, 0) \neq 0$.

Condition (D_1) was introduced by Ruiz and Ward in [10] and extended in [2] by the first author and Clapp. It generalizes a classical condition given by Nirenberg in [8] which, in particular, implies that g cannot rotate around the origin when $|u|$ is large. Condition (D_1) is weaker: it allows g to rotate, although not too fast since r cannot be arbitrarily small.

The main result in Section 2 reads as follows:

Theorem 1.1 *Let $g \in C(\mathbb{R}^N \backslash S, \mathbb{R}^N)$ satisfying (B) and $f \in C(\overline{\Omega}, \mathbb{R}^N)$ such that $\overline{f} = 0$. Let $\mathcal{C}$ be a compact neighborhood of S and let r be as in (2). If there exists a domain $D \subset \mathbb{R}^N \backslash (\mathcal{C} + \overline{B_r}(0))$ such that (D_1) and (D_2) hold, then (1) has at least one solution u with $\overline{u} \in D$ and $\|u - \overline{u}\|_\infty < r$.*

In Section 3 we study the case in which S consists in a single point; without loss of generality, it may be assumed $S = \{0\}$. We shall focus our attention on the way g behaves near the singular point. In first place, we shall assume that g is repulsive, namely:

(Rep) There exists $c > 0$ such that $\langle g(u), u \rangle < 0$ for $0 < |u| < c$.

Furthermore, it will be assumed that g is *sequentially strongly repulsive*, in the following sense:

(Seq) There exists a sequence $r_n \searrow 0$ such that.

$$\sup_{|u|=r_n} \left\langle g(u), \frac{u}{|u|} \right\rangle \to -\infty \qquad \text{as } n \to \infty.$$

We shall proceed as follows: firstly, we shall prove existence of at least one solution of an approximated problem. Next, we shall obtain accurate estimates and deduce the existence of a convergent sequence of these solutions.

In order to define the approximated problems, fix a sequence $\varepsilon_n \to 0$ and consider the problem

$$\Delta u + g_n(u) = f(x) \qquad \text{in } \Omega \tag{3}$$

together with the nonlocal boundary conditions of (1). Although more general perturbations are admitted, for convenience we shall define g_n by

$$g_n(u) = \begin{cases} g(u) & |u| \geq \varepsilon_n \\ \rho_n(|u|)g\Big(\varepsilon_n \frac{u}{|u|}\Big) & 0 < |u| < \varepsilon_n \\ 0 & u = 0, \end{cases} \tag{4}$$

with $\rho_n : [0, \varepsilon_n] \to [0, +\infty)$ continuous such that $\rho_n(0) = 0, \rho_n(\varepsilon_n) = 1$.

The conditions on g shall be, as before, of geometric nature. However, a stronger assumption is needed in order to obtain uniform estimates. A similar condition has been introduced by one of the authors and De Nápoli in [1] and has been employed also in [3] for a system of singular periodic ordinary differential equations:

(P_1) There exists a family $\mathcal{F} = \{(U_j, w_j)\}_{j=1,\ldots,J}$, where $\{U_j\}_{j=1,\ldots,J}$ is an open cover of S^{N-1}, constants $c_j > 0$ and $w_j \in S^{N-1}$, such that for $j = 1, \ldots, K$:

$$\limsup_{r\to+\infty} \langle g(ru), w_j\rangle \leq -c_j$$

uniformly for $u \in U_j$.

On the other hand, we shall take advantage of the repulsiveness condition (Seq), which ensures that the degree over certain small balls centered at the origin is $(-1)^N$. Thus, (D_2) shall be replaced by

(P_2) There exists a $R_0 > 0$ such that $\deg(g, B_R, 0) \neq (-1)^N$ for $r \geq R_0$.

The preceding conditions will allow us to construct a sequence $\{u_n\}$ of solutions of the approximated problems that converges weakly in H^1 to some function u. It is easy to see that if u does not vanish on Ω, then u is a classical solution of the problem. If $u \not\equiv 0$ but possibly vanishes in Ω, then we shall call it a *generalized solution*. With this idea in mind, let us introduce a stronger repulsiveness condition:

(SR) $\quad \lim_{u\to 0} \langle g(u), u\rangle = -\infty.$

We now state the main result of Section 3:

Theorem 1.2 *Let $g : \mathbb{R}^N\backslash\{0\} \to \mathbb{R}^N$ be continuous satisfying (B), (Rep), (Seq) and let $f \in C(\overline{\Omega}, \mathbb{R}^N)$ with $\overline{f} = 0$. Suppose that (P_1) and (P_2) hold and let $\{g_n\}$ be as in (4). Then there exist $\{u_n\}_n$ solutions of (3), a positive constant $\tilde{r}$ such that $\|u_n\|_\infty \geq \tilde{r}$ and a subsequence of $\{u_n\}$ that converges weakly in H^1 to some function u. If furthermore (SR) is assumed, then u is a generalized solution of the problem.*

Remark 1.3 *All the preceding results can be reproduced similarly for the Neumann boundary conditions.*

2 The general case. Proof of Theorem 1.1

Let $U = \{u \in C(\overline{\Omega}, \mathbb{R}^N) : \|u - \overline{u}\|_\infty < r, \, \overline{u} \in D\}$ and consider, for $\lambda \in (0,1]$, the problem

$$\left\{ \begin{array}{lll} \Delta u + \lambda \hat{g}(u) & = \lambda f(x) & \text{in } \Omega \\ u & = C & \text{on } \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS & = 0, & \end{array} \right. \tag{5}$$

where $\hat{g} : \mathbb{R}^N \to \mathbb{R}^N$ is continuous and bounded with $\hat{g} = g$ over $\overline{D + B_r(0)}$. It is clear that if $u \in \overline{U}$ solves (5) for $\lambda = 1$ then u is a solution of (1). Thus, from the standard continuation methods [7] it suffices to prove that (5) has no solutions on ∂U for $0 < \lambda < 1$.

Indeed, if $u \in \partial U$ is a solution of (5), then $\overline{u} \in \overline{D}$ and $\|u - \overline{u}\|_\infty \leq r$, so $\hat{g} \circ u = g \circ u$. As $dist(\overline{u}, \mathcal{C}) \geq r$, we deduce that $u(x) \in \overline{\mathbb{R}^N - \mathcal{C}}$ and hence $|g(u(x))| \leq sup_{z \notin \mathcal{C}} |g(z)|$ for all x. This implies

$$\|\nabla u\|_\infty \leq k \|\Delta u\|_\infty < k(\|f\|_\infty + sup_{z \notin \mathcal{C}} |g(z)|),$$

and thus

$$\|u - \overline{u}\|_\infty \leq diam_d(\Omega) \|\nabla u\|_\infty < r.$$

Hence, $\overline{u} \in \partial D$. Moreover, it follows from the mean value theorem for vector integrals that

$$\frac{1}{|\Omega|} \int_\Omega g(u(x))\, dx \in co(g(u(\overline{\Omega}))) \subset co(g(B_r(\overline{u}))).$$

On the other hand, simple integration shows that

$$\int_\Omega g(u(x))\, dx = 0,$$

so $0 \in co(g(B_r(\overline{u})))$, a contradiction. ■

Remark 2.1 *In this framework, taking $S = \emptyset$ we obtain the main result in [10] for the non-singular case, conveniently adapted to our problem.*

Remark 2.2 *After a more accurate computation of the* a priori *estimates, the preceding theorem can be extended for g sublinear, namely, for g satisfying:*

$$\lim_{|u| \to \infty} \frac{g(u)}{|u|} = 0.$$

Let us show an example that illustrates the possibility of obtaining multiple solutions. For convenience, let us call $B_\rho := B_\rho(0) = \{u \in \mathbb{R}^N : |u| < \rho\}$.

Example 2.3 *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be continuous and bounded, $a = \|A\|_\infty$ and $b > 0$. Define $g(u) = \frac{A(u)}{|u|(b-|u|)}$, so $S = \{0\} \cup \partial B_b$. Let $\eta > 0$ and consider the following compact set:*

$$\mathcal{C} = \overline{B_\eta} \cup \left(\overline{B_{b+\eta}} \backslash B_{b-\eta}\right).$$

Hence, $\mathbb{R}^N \backslash \mathcal{C} = (B_{b-\eta} \backslash \overline{B_\eta}) \cup (\mathbb{R}^N \backslash \overline{B_{b+\eta}})$. From the previous computations, the following estimate holds:

$$\|\nabla u\|_\infty \leq K := k\left(\|f\|_\infty + \frac{a}{\eta(b+\eta)}\right)$$

Thus,

$$r = diam_d(\Omega)k\left(\|f\|_\infty + \frac{a}{\eta(b+\eta)}\right).$$

If also $b > 2(r+\eta)$, then we might be able to obtain two disjoint sets $D^1, D^2 \subset \mathbb{R}^N \backslash (\mathcal{C} + B_r)$ such that:

$$D^1 \subset B_{b-\eta-r} \backslash B_{\eta+r}, \qquad D^2 \subset \mathbb{R}^N \backslash B_{b+\eta+r}$$

leading to two different solutions u_1, u_2 with $\overline{u_1} \in D^1$ and $\overline{u_2} \in D^2$ respectively.

In order to apply our previous result, observe that condition (D_1) requires $\eta + 2r < b - \eta - 2r$, that is: $b > 4r + 2\eta$.

For example, let $T > 0$ be large enough and define $g : B_{b+T} \backslash S \to \mathbb{R}^N$ by

$$g(u) := \frac{(|u| - x_1)(|u| - x_2)\, u}{|u|(|u| - b)}$$

for some numbers $x_1, x_2 > 0$. The numerator of this function can be extended continuously to $\mathbb{R}^N \backslash S$ in such a way that $a \leq (b+T)^3$. Taking $diam(\Omega)$ small enough, the preceding inequalities for r are satisfied, so we may fix $x_1 \in (\eta + 2r, b - \eta - 2r)$ and $x_2 \in (b + \eta + 2r, b + T - 2r)$.

Thus, all the assumptions are satisfied for D^1 and D^2; hence, by Theorem 1.1 we deduce the existence of classical solutions $u^1 \neq u^2$ of problem (1) such that $\overline{u^i} \in D^i$, for $i = 1, 2$.

Remark 2.4 *This example shows that the if the assumptions of Theorem 1.1 are verified, then the distance between different conected components of S cannot be too small.*

3 The case $S = \{0\}$

Before giving a proof of Theorem 1.2, let us make some comments on the concept of generalized solution. Let u_n be a weak solution of (3) such that $u_n \to u$ weakly in H^1. From the equality

$$\int_\Omega \Delta u_n \varphi + \int_\Omega g_n(u_n)\varphi = \int_\Omega f\varphi \qquad \forall \varphi \in H$$

we deduce that the operator $A : H \to \mathbb{R}^N$ given by

$$A\varphi = \lim_{n\to\infty} \int_\Omega g_n(u_n)\varphi$$

is well defined and continuous, that is: $A \in H^{-1}$. In fact,

$$A\varphi = \int_\Omega f\varphi dx + \sum_{j=1} \nabla u^j \nabla \varphi^j dx$$

so we may regard it as a pair $(f, \nabla u) \in H^{-1}$, namely

$$A\varphi := (f, \nabla u)[\varphi].$$

Thus, we are able to define the operator $\mathcal{G} : H \to H^{-1}$ by

$$\mathcal{G}(u) := (f, \nabla u); \qquad i.e. \qquad \mathcal{G}(u)[\varphi] = A\varphi. \tag{6}$$

As shown in [3], it is always possible to find approximations in such a way that $u \equiv 0$, this is why we need to exclude this case in the definition of generalized solution.

Also, observe that if u does not vanish in Ω then for any $\varphi \in H$ then

$$\mathcal{G}(u)[\varphi] = A\varphi = \lim_{n\to\infty} \int_\Omega g_n(u_n)\varphi dx = \int_\Omega g(u)\varphi dx$$

So a generalized solution can be regarded as a nontrivial distributional solution of the equation

$$\Delta u + \mathcal{G}(u) = f.$$

In order to prove Theorem 1.2, firstly let us state an existence result for the approximated problems.

Proposition 3.1 *Let $\Omega \subset \mathbb{R}^d$ a bounded C^2 domain. Let $g : \mathbb{R}^N \backslash \{0\} \to \mathbb{R}^N$ be continuous satisfying (B), (Rep), (Seq) and let $f \in C(\overline{\Omega}, \mathbb{R}^N)$ with $\overline{f} = 0$. Suppose that (P_1) and (P_2) hold and let $\{g_n\}$ be as in (4). Then there exist $\{u_n\}_n$ solutions of (3) and a constant $\tilde{r} > 0$ such that $\|u_n\|_\infty \geq \tilde{r}$.*

Proof:
Fix $\tilde{r} > 0$ such that

$$\left\langle g(u), \frac{u}{|u|} \right\rangle + \|f\|_{L^\infty} < 0 \text{ for } |u| = \tilde{r}. \tag{7}$$

As before, we shall apply the continuation method, now over the set

$$U := \{u \in C(\overline{\Omega}, \mathbb{R}^N) : \tilde{r} < \|u\|_\infty < R\}$$

for some $R > \tilde{r}$ to be specified.

Suppose that for some $\lambda \in (0, 1)$ there exists $u \in \partial U$ a solution of (5).

If $\|u\|_\infty = \tilde{r}$, then we may fix x_0 such that $\|u\|_\infty = |u(x_0)| = \tilde{r}$ and define $\phi(x) := \frac{|u(x)|^2}{2}$.

For $x_0 \in \Omega$, it is seen that

$$\Delta\phi(x_0) = |\nabla u(x_0)|^2 + \langle u(x_0), \Delta u(x_0)\rangle \geq \langle u(x_0), f(x_0) - g(u(x_0))\rangle =$$

$$= \lambda \left[\langle u(x_0), f(x_0)\rangle - |u(x_0)| \langle g(u(x_0)), \frac{u(x_0)}{|u(x_0)|} \rangle \right] \geq$$

$$\geq \tilde{r} \left[-\|f\|_\infty - \left\langle g(u(x_0)), \frac{u(x_0)}{|u(x_0)|} \right\rangle \right] > 0,$$

a contradiction.

If $x_0 \in \partial\Omega$, then $\tilde{r} = |C|$. Moreover,

$$\int_{\partial\Omega} \frac{\partial \phi}{\partial \nu} dS = \int_{\partial\Omega} \left\langle u, \frac{\partial u}{\partial \nu} \right\rangle dS = \langle C, \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS \rangle = 0. \tag{8}$$

From the continuity of ϕ, arguing as before we deduce that, $\Delta\phi > 0$ in $B_{2\delta}(x_0) \cap \Omega$ for some $\delta > 0$.

From the standard regularity theory, it follows that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Moreover, we may consider a C^2 domain $\Omega_0 \subset \Omega$ such that $B_\delta \cap \Omega \subset \Omega_0$ and $\Omega_0 \subset B_{2\delta} \cap \Omega$; then $\phi(x_0) > \phi(x)$ for every $x \in \Omega_0$, and from Hopf's Lemma we obtain

$$\frac{\partial \phi}{\partial \nu}(x_0) > 0.$$

As $u \equiv C$ on the boundary, then $|u(x)| \equiv \tilde{r}$ and so $\frac{\partial \phi}{\partial \nu}(x) > 0$ for each $x \in \partial\Omega$. This contradicts (8) and thus $\|u\|_\infty = R$.

For n large, it follows that $\|u - \overline{u}\|_\infty < r$ and from condition (P_1) we deduce (D_1) for $D = B_R(0)$ when R is sufficiently large. As in Theorem 1.1, a contradiction yields.

Finally, observe that the repulsiveness condition implies that the degree $\deg(g_n, B_{\tilde{r}}, 0) = (-1)^N$ so, by the excision property of the degree, condition (P_2) ensures that $deg(g_n, U \cap \mathbb{R}^N, 0) \neq 0$ and so completes the proof.

■

The following Lemma shows that the solutions of the perturbed problems are also bounded for the H^1 norm.

Lemma 3.2 *In the situation of Proposition 3.1, there exists a constant $\mathfrak{C}$ independent of n such that $\|u_n\|_{H^1} \leq \mathfrak{C}$ for all n.*

Proof:
As $\Delta u_n + g_n(u_n) = f(x)$ in Ω and $u_n \equiv C_n$ on $\partial\Omega$, we may multiply by $u_n - C_n$ and integrate to obtain:

$$\int_\Omega \langle \Delta u_n + g_n(u_n), u_n - C_n \rangle \, dx = \int_\Omega \langle p, u_n - C_n \rangle \, dx.$$

Integrating by parts, the left hand side is equal to:

$$-\int_\Omega |\nabla u_n|^2 \, dx + \int_{\partial\Omega} \left\langle \frac{\partial u_n}{\partial \nu}, u_n - C_n \right\rangle dS + \int_\Omega \langle g_n(u_n), u_n - C_n \rangle \, dx$$

As $u_n \equiv C_n$ on $\partial\Omega$, it follows that

$$\|\nabla u_n\|_{L^2}^2 = \int_\Omega \langle g_n(u_n), u_n - C_n \rangle \, dx - \int_\Omega \langle p, u_n - C_n \rangle \, dx.$$

Now, taking absolute value and using the Cauchy-Schwarz inequality, we get

$$\|\nabla u_n\|_{L^2}^2 \leq \left| \int_\Omega \langle g_n(u_n), u_n - C_n \rangle \, dx \right| + \|p\|_{L^2} \|u_n - C_n\|_{L^2}.$$

Let c be the constant in condition (Rep) and write:

$$\begin{aligned} \left| \int_\Omega \langle g_n(u_n), u_n - C_n \rangle \, dx \right| \leq & \left| \int_{\{|u_n|<c\}} \langle g_n(u_n), u_n - C_n \rangle \, dx \right| \\ & + \left| \int_{\{|u_n|\geq c\}} \langle g_n(u_n), u_n - C_n \rangle \, dx \right|. \end{aligned}$$

Fix $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < c$ for every $n \geq n_0$, then $g_n(u_n(x)) = g(u_n(x))$ if $|u_n(x)| > c > \frac{1}{n}$ and hence on the one hand

$$\left| \int_{\{|u_n|\geq c\}} \langle g_n(u_n), u_n - C_n \rangle \, dx \right| \leq |\Omega|^{1/2} \gamma_c \|u_n - C_n\|_{L^2},$$

where $\gamma_c := \sup_{|u|>c} |g(u)|$ and, on the other hand:

$$\int_{\{|u_n|<c\}} \langle g_n(u_n), u_n - C_n \rangle \, dx \leq - \int_{\{|u_n|<c\}} \langle g_n(u_n), C_n \rangle \, dx.$$

Moreover, as $\int_\Omega g_n(u_n)\, dx = 0$, we deduce that

$$\int_{\{|u_n|<c\}} \langle g_n(u_n), u_n - C_n \rangle \, dx \leq \left\langle C_n, \int_{\{|u|\geq c\}} g_n(u_n) \right\rangle \, dx \leq |\Omega|^{1/2} \gamma_c |C_n|.$$

Gathering all together,

$$\left| \int_\Omega \langle g_n(u_n), u_n - C_n \rangle \, dx \right| \leq |\Omega|^{1/2} \gamma_c \left(\|u_n - C_n\|_{L^2} + |C_n| \right).$$

Thus,

$$\|\nabla u_n\|_{L^2}^2 \leq \mathfrak{C}_1 \|u_n - C_n\|_{L^2} + \mathfrak{C}_2 |C_n|$$

for some constants $\mathfrak{C}_1, \mathfrak{C}_2$. Using Poincaré inequality, we deduce the existence of a constant $\mathfrak{C}$ such that

$$\|\nabla u_n\|_{L^2}^2 \leq \mathfrak{C} |C_n|$$

and hence

$$\|u_n - C_n\|_{H^1}^2 \leq A + B|C_n| \qquad \text{for some } A, B > 0.$$

Suppose that $|C_n|$ is unbounded, then taking a subsequence (still denoted C_n) we may assume that $|C_n| \to +\infty$, $\frac{C_n}{|C_n|} \to \eta \in S^{N-1}$. From the inequality

$$\left\| \frac{u_n - C_n}{\sqrt{|C_n|}} \right\|_{H^1}^2 \leq \frac{A}{|C_n|} + B \qquad \forall n \geq n_0,$$

we may take again a subsequence and thus assume that $\frac{u_n - C_n}{\sqrt{|C_n|}}$ converges almost everywhere and weakly in H^1 to some $w \in H^1$.

Let $\varepsilon > 0$ and fix M large enough so that $|\Omega \backslash \Omega_M| < \varepsilon$, where

$$\Omega_M := \{x \in \Omega : |w(x)| \leq M\}.$$

Then $\frac{u_n - C_n}{|C_n|} \to 0$ and $\frac{u_n}{|u_n|} \to \eta$ almost everywhere in Ω_M.

Fix $U_k \subset S^{N-1}$ as in (P1) such that $\eta \in U_k$, then writing

$$\langle g(u_n(x)), w_k \rangle = \left\langle g\left(|u_n(x)| \frac{u_n(x))}{|u_n(x)|} \right), w_k \right\rangle$$

we deduce that

$$\limsup_{n\to\infty}\langle g(u_n(x)), w_k\rangle \leq -c_k$$

a.e. in Ω_M. Thus we obtain, from Fatou's Lemma:

$$\limsup_{n\to\infty}\int_{\Omega_M}\langle g(u_n(x)), w_k\rangle\, dx \leq \int_{\Omega_M}\limsup_{n\to\infty}\langle g(u_n(x)), w_k\rangle\, dx \leq -c_k|\Omega_M|.$$

We may assume that $M \geq c$, then taking $\varepsilon < \frac{c_k|\Omega|}{\gamma_c}$ we conclude:

$$\limsup_{n\to\infty}\int_{\Omega}\langle g(u_n(x)), w_k\rangle\, dx \leq -c_k|\Omega_M| + \limsup_{n\to\infty}\int_{\Omega\setminus\Omega_M}\langle g(u_n(x)), w_k\rangle\, dx$$

$$\leq -c_k|\Omega_M| + \gamma_c|\Omega\setminus\Omega_M| < 0,$$

which contradicts the fact that $\int_\Omega g(u_n(x))\, dx = 0$.

■

Proof of Theorem 1.2:
From the preceding results, there exists a sequence (still denoted $\{u_n\}$) of solutions of the approximated problems converging a.e. and weakly in H^1 to some function u, and also such that $\|u_n\|_\infty \geq \tilde{r}$. It remains to prove that if (SR) holds then $u \not\equiv 0$.

Suppose that $u \equiv 0$, then from (3) we obtain

$$\int_\Omega \langle \Delta u_n(x), u_n(x)\rangle + \langle g(u_n(x)), u_n(x)\rangle\, dx = \int_\Omega \langle p(x), u_n(x)\rangle\, dx \to 0$$

as $n \to \infty$. Moreover,

$$\int_\Omega \langle \Delta u_n(x), u_n(x)\rangle\, dx = -\int_\Omega |\nabla u_n(x)|^2\, dx$$

is bounded, and from (SR) an Fatou's Lemma we obtain

$$\limsup_{n\to\infty}\int_\Omega \langle g(u_n(x)), u_n(x)\rangle\, dx \leq \int_\Omega \limsup_{n\to\infty}\langle g(u_n(x)), u_n(x)\rangle\, dx = -\infty$$

a contradiction.

■

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