Convexity and eigenvalues of the Hessian.

Julio D. Rossi

U. Buenos Aires (Argentina)

jrossi@dm.uba.ar
www.dm.uba.ar/∼jrossi

Joint works with P. Blanc, C. Esteve, L. Del Pezzo and N. Frevenza

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Convex envelopes

A function $u : \Omega \subset \mathbb{R}^n \mapsto \mathbb{R}$ is convex if

$$u(\lambda x + (1-\lambda)y) \leq \lambda u(x) + (1-\lambda)u(y).$$

Given $F : \partial\Omega \mapsto \mathbb{R}$ the convex envelope of F in Ω is

$$u^*(x) = \sup_{u \text{ convex}, \, u|_{\partial\Omega} \leq F} u(x).$$

That is, u^* is the largest convex function that is below F on $\partial\Omega$.

If $u \in C^2$ is convex then $D^2u(x)$ must be positive semidefinite,

$$\langle D^2u(x)v, v\rangle \geq 0.$$

Convex envelopes

In terms of the eigenvalues of D^2u this can be written as

$$\lambda_1(D^2u(x)) = \min_{\lambda \text{ eigenvalue of } D^2u(x)} \{\lambda\} = \inf_{|v|=1} \langle D^2u(x)v, v\rangle \geq 0.$$

Moreover, the convex envelope of F in Ω, u^*, is the largest viscosity solution to

$$\begin{cases} \lambda_1(D^2u) = 0, & \text{in } \Omega, \\ u \leq F, & \text{on } \partial\Omega. \end{cases}$$

For larger eigenvalues of D^2u we have

$$\lambda_j(D^2u(x)) = \inf_{dim(S)=j} \sup_{v\in S, |v|=1} \langle D^2u(x)v, v\rangle.$$

Theorem *When* Ω *strictly convex and* F *continuous, there is a unique viscosity solution to*

$$\begin{cases} \lambda_1(D^2u) = 0, & \Omega, \\ u = F, & \partial\Omega. \end{cases}$$

Moreover, if F *is* $C^{1,\alpha}$*, solutions are* $C^{1,\alpha}$ *(Oberman–Silvestre).*

Let $\lambda_1(D^2u) \le \lambda_2(D^2u) \le \cdots \le \lambda_N(D^2u)$ be the ordered eigenvalues od the Hessian.

Theorem (Blanc-R.) *When* Ω *strictly convex and* F *continuous, there is a unique viscosity solution to*

$$\begin{cases} \lambda_j(D^2u) = 0, & \Omega, \\ u = F, & \partial\Omega. \end{cases}$$

Question: if F is $C^{1,\alpha}$, solutions are $C^{1,\alpha}$??.

A parabolic version

Consider

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \lambda_j(D^2 u(x,t)) = 0, & \Omega \times (0,+\infty), \\ u(x,t) = F(x), & \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & \Omega, \end{cases}$$

This problem is the evolution version of our previous elliptic problem

$$\begin{cases} \lambda_j(D^2 z(x)) = 0, & \Omega, \\ z(x) = F(x), & \partial\Omega. \end{cases}$$

A parabolic version

Theorem (Blanc–Esteve–R.)

For Ω strictly convex and u_0 compatible with F ($u_0|_{\partial\Omega} = F$), existence and uniqueness for the parabolic problem holds.

Asymptotic behaviour

There exist positive constants C (depending on the initial condition u_0) and $\mu > 0$ (depending only on Ω), such that

$$\|u(\cdot, t) - z(\cdot)\|_\infty \leq Ce^{-\mu t}.$$

A parabolic version

Let $1 < j < N$,

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \lambda_j(D^2 u(x,t)) = 0, & \Omega \times (0,+\infty), \\ u(x,t) = 0, & \partial\Omega \times [0,+\infty), \\ u(x,0) = u_0(x), & \Omega, \end{cases}$$

with u_0 any continuous function.

Theorem *There exists $T > 0$ depending only on Ω, such that the solution u satisfies*

$$u(x,t) \equiv 0, \qquad \textit{for any } t > T.$$

For any affine function F the same phenomenon also holds (just apply the same argument to $\tilde{u} = u - z$).

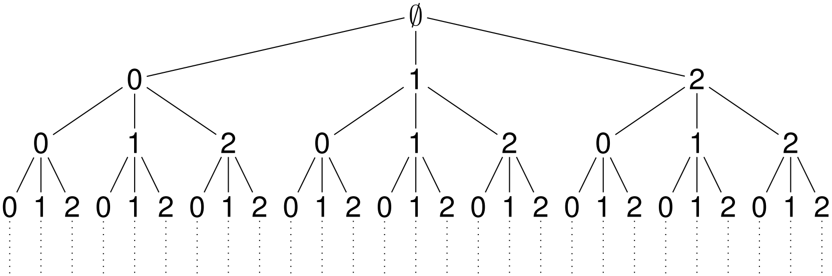
We have two proofs of this result

1. A pure PDE proof, based on a comparison argument constructing sub a super solutions.

2. A game-theory proof, based on the fact that there is a game theoretical approximation of the solutions.

Convexity on Trees

Given $m \in \mathbb{N}_{\geq 2}$, a tree $\mathbb{T}_m$ with regular $m-$branching is an infinite graph that consists of the root $\emptyset$ and all finite sequences $(a_1, a_2, \ldots, a_k)$ with $k \in \mathbb{N}$, whose coordinates a_i are chosen from $\{0, 1, \ldots, m-1\}$.



A tree with $3-$branching.

Boundary of the tree

A branch of $\mathbb{T}_m$ is an infinite sequence of vertices, where each one of them is followed by one of its immediate successors.

The collection of all branches forms the boundary of $\mathbb{T}_m$, denoted by $\partial\mathbb{T}_m$.

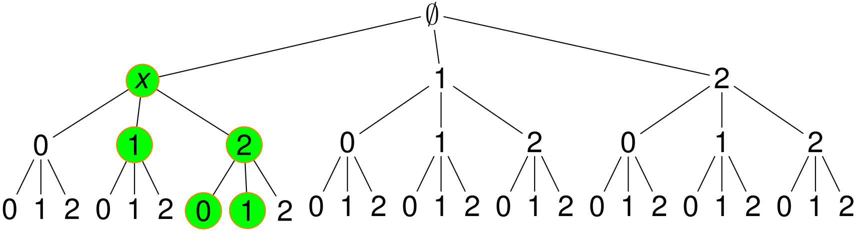
Observe that the mapping $\psi : \partial\mathbb{T}_m \to [0, 1]$ defined as

$$\psi(\pi) := \sum_{k=1}^{+\infty} \frac{a_k}{m^k}$$

is surjective, where $\pi = (a_1, \ldots, a_k, \ldots) \in \partial\mathbb{T}_m$ and $a_k \in \{0, 1, \ldots, m-1\}$ for all $k \in \mathbb{N}$.

Convex envelopes on trees

Given $x \in \mathbb{T}_m$, $\mathbb{B}_x$ denotes the set of all binary subgraphs $\mathbb{B}$ such that $x \in \mathbb{B}$ is the root.



An element of $\mathbb{B}_x$. Root: x. Endpoints: $(x, 1)$, $(x, 2, 0)$, and $(x, 2, 1)$.

Convex envelopes on trees

Definition $u : \mathbb{T}_m \to \mathbb{R}$ is called binary convex if for any $x \in \mathbb{T}_m$

$$u(x) \leq \sum_{y \in \mathcal{E}(\mathbb{B})} \frac{1}{2^{|y|-|x|}} u(y) \quad \forall \mathbb{B} \in \mathbb{B}_x .$$

In this notion of convexity, a segment is $\mathbb{B}$, a finite binary subgraph of $\mathbb{T}_m$; a midpoint is the root of this subgraph $\mathbb{B}$. T

The convexity property just says that the value of the function at the midpoint is less or equal than the mean value of the values at the endpoints.

Convex envelopes on trees

Associated to this version of convexity we have a convex envelope

$$\tilde{u}_F(x) = \sup\left\{ u(x) \colon u \in \mathcal{B}(F) \right\},$$

where

$$\mathcal{B}(F) = \left\{ u \text{ is binary convex and } \limsup_{x \to \pi \in \partial \mathbb{T}_m} u(x) \leq F(\psi(\pi)) \right\}.$$

Convex envelopes on trees

For this notion of binary convex envelope, we also have an equation.

Theorem (Del Pezzo–Frevenza–R.) *The binary convex envelope of a bounded boundary datum F is the largest solution to*

$$u(x) = \min_{\substack{y,z \in \mathcal{S}(x) \\ y \neq z}} \left\{ \frac{u(y) + u(z)}{2} \right\} \qquad x \in \mathbb{T}_m,$$

that verifies

$$\limsup_{x \to \pi \in \partial \mathbb{T}_m} u(x) \leq F(\psi(\pi)).$$

Given a continuous boundary datum F, there exists a unique solution to the equation that verifies

$$\lim_{x \to \pi \in \partial \mathbb{T}_m} u(x) = F(\psi(\pi)).$$

The Laplacian

In this case, we have

$$0 = \min_{\substack{y,z \in \mathcal{S}(x) \\ y \neq z}} \left\{ \frac{1}{2} u(y) + \frac{1}{2} u(z) - u(x) \right\},$$

and we can identify the analogous to the eigenvalues of the Hessian,

$$\left\{ \frac{1}{2} u(y) + \frac{1}{2} u(z) - u(x) \right\}_{i<j}.$$

Then, adding the eigenvalues, we obtain

$$0 = \frac{1}{m} \sum_{y \in S(x)} u(y) - u(x).$$

Notice that this is the usual Laplacian in the directed tree.

Quasiconvexity in $\mathbb{R}^N$

Definition u is quasiconvex if for all $x, y \in \mathbb{R}^N$ and any $\lambda \in [0,1]$ we have

$$u(\lambda x + (1-\lambda)y) \leq \max\Big\{u(x), u(y)\Big\}.$$

An alternative and more geometrical way of defining a quasiconvex function u is to require that each sublevel set

$$S_\alpha(u) = \{x \mid u(x) \leq \alpha\}$$

is a convex set.

Quasiconvexity

Based in the previously mentioned idea of a segment and a midpoint in the tree $\mathbb{T}_m$, we introduce a definition of a quasiconvex function on a tree.

Our definition is based on thinking on segments as subtrees $\mathbb{B} \in \mathbb{T}_k^x$ with the root x as the midpoint of the segment and the terminal nodes $y \in \mathcal{E}(\mathbb{B})$ as the endpoints.

Quasiconvexity

A convex set in the tree $C \subset \mathbb{T}_m$ is then a subset such that it contains every midpoint of every segment with terminal nodes in the set. That is, $C \subset \mathbb{T}_m$ is convex if for every $\mathbb{B} \in \mathbb{T}_k^x$ with $\mathcal{E}(\mathbb{B}) \subset C$ we have that $x \in C$.

Then, the natural definition for quasiconvexity runs as follows: a function on the tree u is quasiconvex if it is such that every sublevel set

$$S_\alpha(u) = \{x \mid u(x) \leq \alpha\}$$

is a convex set in $\mathbb{T}_m$.

Quasiconvexity

We prove that a function u is quasiconvex on the tree if and only if for every vertex $x \in \mathbb{T}_m$ it holds that

$$u(x) \leq \min_{y_1,\dots y_k \in S(x),\, y_i \neq y_j} \max_{i=1,\dots,k} \Big\{ u(y_i) \Big\}.$$

Notice that the right side is the $k-$th smallest value among all the values of u at the successors of x, $S(x)$.

Given a boundary datum, f, the Dirichlet problem for

$$u(x) = \min_{y_1,\dots,y_k \in S(x),\, y_i \neq y_j} \max_{i=1,\dots,k} \Big\{ u(y_i) \Big\},$$

has a unique solution.

Quasiconvexity

It turns out that this equation is a mean value property that involves the k median value among values of the function on the successors of a given vertex. In the particular case of the m-branch directed tree with m odd and $k = \frac{m-1}{2}$, we obtain the median operator, that is, the quasiconvex envelope is the solution to

$$u(x) = \text{median}\left\{ u(y) \colon y \in S(x) \right\} \quad \text{for } x \in \mathbb{T}_m.$$

Quasiconvexity

The pure second derivative in the direction of $y \in S(x)$ is given by

$$u(y) - u(x)$$

Adding these pure second derivatives in every direction (we add with respect to y) and dividing by m we obtain again the usual Laplacian (we find the Laplacian as the sum of the pure second derivatives).

One the other hand, the infinity Laplacian in the tree is given by

$$u(x) = \frac{1}{2} \max_{y \in S(x)} u(y) + \frac{1}{2} \min_{y \in S(x)} u(y).$$

Then, the ”direction of the gradient” in the tree are the two directions given by the two successors at which the $\max_{y \in S(x)} u(y)$ and the $\min_{y \in S(x)} u(y)$ are attained.

Quasiconvexity

In the Euclidean space the equation for quasiconvexity is

$$\min_{v:|v|=1,\langle v,\nabla u(x)\rangle=0} \langle D^2u(x)v,v\rangle = 0.$$

Now, for the quasiconvex envelope with $k=2$ if we consider pure second derivatives of u in directions ”orthogonal to the direction of the gradient” and compute the minimum, we get

$$\min_{y_i\in S(x),\, y_i\neq\tilde{y}_M, y_i\neq\tilde{y}_m} \Big\{ u(y_i)-u(x) \Big\},$$

with $\tilde{y}_M, \tilde{y}_m$ the two successors at which the $\max_{y\in S(x)} u(y)$ and the $\min_{y\in S(x)} u(y)$ are attained. Then, we get

$$\min_{y_i\in S(x),\, y_i\neq\tilde{y}_M, y_i\neq\tilde{y}_m} \Big\{ u(y_i)-u(x) \Big\} = u_2(x)-u(x)$$
$$= \min_{y_1,y_2\in S(x),\, y_1\neq y_2} \max_{i=1,2} \Big\{ u(y_i)-u(x) \Big\} = 0.$$

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Thanks !!! Gracias !!!