A PRIORI BOUNDS AND EXISTENCE OF SOLUTIONS FOR SOME NONLOCAL ELLIPTIC PROBLEMS

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Abstract. In this paper we show existence of solutions for some elliptic problems with nonlocal diffusion by means of nonvariational tools. Our proof is based on the use of topological degree, which requires a priori bounds for the solutions. We obtain the a priori bounds by adapting the classical scaling method of Gidas and Spruck. We also deal with problems involving gradient terms.

1. Introduction

Nonlocal diffusion problems have received considerable attention during the last years, mainly because their appearance when modelling different situations. To name a few, let us mention anomalous diffusion and quasi-geostrophic flows, turbulence and water waves, molecular dynamics and relativistic quantum mechanics of stars (see [11, 20, 29, 57] and references therein). They also appear in mathematical finance (cf. [3, 9, 28]), elasticity problems [51], thin obstacle problem [15], phase transition [1, 13, 55], crystal dislocation [31, 58] and stratified materials [46].

A particular class of nonlocal operators which have been widely analyzed is given, up to a normalization constant, by

$$(-\Delta)^s_K u(x) = \int_{\mathbb{R}^N} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{N+2s}} K(y) dy,$$

where $s \in (0,1)$ and K is a measurable function defined in $\mathbb{R}^N$ ($N \geq 2$). A remarkable example of such operators is obtained by setting $K = 1$, when $(-\Delta)^s_K$ reduces to the well-known fractional Laplacian (see [56, Chapter 5] or [30, 39, 52] for further details). Of course, we will require the operators $(-\Delta)^s_K$ to be elliptic, which in our context means that there exist positive constants $\lambda \leq \Lambda$ such that

$$\lambda \leq K(x) \leq \Lambda \quad \text{in } \mathbb{R}^N \tag{1.1}$$

(cf. [18]). While there is a large literature dealing with this class of operators, very little is known about existence of solutions for nonlinear problems, except for cases where variational methods can be employed (see for instance [5, 6, 7, 47, 49, 50] and references therein).

But when the problem under consideration is not of variational type, for instance when gradient terms are present, as far as we know, results about existence of solutions are very scarce in the literature. Thus our objective is to find a way to show existence of solutions for some problems under this assumption. For this aim, we will resort to the use of the fruitful topological methods, in particular Leray-Schauder degree.

It is well-known that the use of these methods requires the knowledge of the so-called a priori bounds for all possible solutions. Therefore we will be mainly concerned with the obtention of these a priori bounds for a particular class of equations. A natural starting point for this program is to consider the problem:

$$\begin{cases} (-\Delta)^s_K u = u^p + g(x,u) & \text{in } \Omega, \\ \ \ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $p > 1$ and g is a perturbation term which is small in some sense. Under several expected restrictions on g and p we will show that all positive solutions of this problem are a priori bounded. The most important requirement is that p is subcritical, that is

$$1 < p < \frac{N+2s}{N-2s} \tag{1.3}$$

and that the term $g(x,u)$ is a small perturbation of u^p at infinity. By adapting the classical scaling method of Gidas and Spruck ([35]) we can show that all positive solutions of (1.2) are a priori bounded.

An important additional assumption that we will be imposing on the kernel K is that

$$\lim_{x\to 0} K(x) = 1. \tag{1.4}$$

It is important to clarify at this moment that we are always dealing with viscosity solutions $u \in C(\mathbb{R}^N)$ in the sense of [18], although in some cases the solutions will turn out to be more regular with the help of the regularity theory developed in [18, 19].

With regard to problem (1.2), our main result is the following:

Theorem 1. *Assume Ω is a C^2 bounded domain of $\mathbb{R}^N$, $N \geq 2$, $s \in (0,1)$ and p verifies* (1.3)*. Let K be a measurable kernel that satisfies* (1.1) *and* (1.4)*. If $g \in C(\overline{\Omega} \times \mathbb{R})$ verifies*

$$|g(x,z)| \leq C|z|^r \qquad x \in \overline{\Omega},\ z \in \mathbb{R},$$

where $1 < r < p$, then problem (1.2) *admits at least a positive viscosity solution.*

It is to be noted that the scaling method requires on one side of good estimates for solutions, both interior and at the boundary, and on the other side of a Liouville theorem in $\mathbb{R}^N$. In the present case interior estimates are well known (cf. [18]), but good local estimates near the boundary do not seem to be available. We overcome this problem by constructing suitable barriers which can be controlled when the scaled domains are moving. It is worthy of mention at this point that the corresponding Liouville theorems are already available (cf. [60, 25, 43, 32]).

Let us also mention that we were not aware of any work dealing with the question of a priori bounds for problem (1.2); however, when we were completing this manuscript, it has just come to our attention the very recent preprint [24], where a priori bounds for smooth solutions are obtained in problem (1.2) with $K = 1$ and $g = 0$ (but no existence is shown). On the other hand, it is important to mention the papers [12, 14, 26, 27], where a priori bounds and Liouville results have been obtained for related operators, like the "spectral" fractional laplacian. To see some diferences between this operator and $(-\Delta)^s$, obtained by setting $K = 1$ in the present work, see for instance [48]. In all the previous works dealing with the spectral fractional Laplacian, the main tool is the well-known Caffarelli-Silvestre extension obtained in [17]. This tool is not available for us here, hence we will treat the problem in a nonlocal way with a direct approach.

As we commented before, we will also be concerned with the adaptation of the previous result to some more general equations. More precisely, we will study the perturbation of equation (1.2) with the introduction of gradient terms, that is,

$$\left\{ \begin{array}{ll} (-\Delta)^s_K u = u^p + h(x,u,\nabla u) & \text{in } \Omega, \\ \quad u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{array} \right. \tag{1.5}$$

For the type of nonlocal equations that we are analyzing, a natural restriction in order that the gradient is meaningful is $s > \frac{1}{2}$. However, there seem to be few works dealing with nonlocal equations with gradient terms (see for example [2, 4, 10, 20, 22, 23, 37, 53, 54, 59]).

It is to be noted that, at least in the case $K = 1$, since solutions u are expected to behave like $\text{dist}(x, \partial\Omega)^s$ near the boundary by Hopf's principle (cf. [45]), then the gradient is expected to be singular near $\partial\Omega$. This implies that the standard scaling method has to be modified to take care of this singularity. We achieve this by introducing some suitable weighted norms which have been already used in the context of second order elliptic equations (cf. [36]).

However, the introduction of this weighted norms presents some problems since the scaling needed near the boundary is not the same one as in the interior. Therefore we need to split our study into two parts: first, we obtain "rough" universal bounds for all solutions of (1.5), by using the well-known doubling lemma in [41]. Since our problems are nonlocal in nature this forces us to strengthen the subcriticality hypothesis (1.3) and to require instead

$$1 < p < \frac{N}{N-2s} \tag{1.6}$$

(cf. Remarks 1 (b) in Section 3). After that, we reduce the obtention of the a priori bounds to an analysis near the boundary. With a suitable scaling, the lack of a priori bounds leads to a problem in a half-space which has no solutions according to the results in [43] or [32].

It is worth stressing that the main results in this paper rely in the construction of suitable barriers for equations with a singular right-hand side, which are well-behaved with respect to suitable perturbations of the domain (cf. Section 2).

Le us finally state our result for problem (1.5). In this context, a solution of (1.5) is a function $u \in C^1(\Omega) \cap C(\mathbb{R}^N)$ vanishing outside Ω and verifying the equation in the viscosity sense.

Theorem 2. *Assume Ω is a C^2 bounded domain of $\mathbb{R}^N$, $N \geq 2$, $s \in (\frac{1}{2}, 1)$ and p verifies* (1.6). *Let K be a measurable kernel that satisfies* (1.1) *and*

(1.4). *If* $h \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N)$ *is nonnegative and verifies*

$$h(x, z, \xi) \leq C(|z|^r + |\xi|^t), \quad x \in \Omega, \ z \in \mathbb{R}, \ \xi \in \mathbb{R}^N,$$

where $1 < r < p$ *and* $1 < t < \frac{2sp}{p+2s-1}$, *then problem* (1.5) *admits at least a positive solution.*

The rest of the paper is organized as follows: in Section 2 we recall some interior regularity results needed for our arguments, and we solve some linear problems by constructing suitable barriers. Section 3 is dedicated to the obtention of a priori bounds, while in Secion 4 we show the existence of solutions that is, we give the proofs of Theorems 1 and 2.

2. Interior regularity and some barriers

The aim of this section is to collect several results regarding the construction of suitable barriers and also some interior regularity for equations related to (1.2) and (1.5). We will use throughout the standard convention that the letter C denotes a positive constant, probably different from line to line.

Consider $s \in (0, 1)$, a measurable kernel K verifying (1.1) and (1.4) and a C^2 bounded domain Ω. We begin by analyzing the linear equation

$$(-\Delta)^s_K u = f \quad \text{in } \Omega, \tag{2.1}$$

where $f \in L^\infty_{\rm loc}(\Omega)$. As a consequence of Theorem 12.1 in [18] we get that if $u \in C(\Omega) \cap L^\infty(\mathbb{R}^N)$ is a viscosity solution of (2.1) then $u \in C^\alpha_{\rm loc}(\Omega)$ for some $\alpha \in (0, 1)$. Moreover, for every ball $B_R \subset\subset \Omega$ there exists a positive constant $C = C(N, s, \lambda, \Lambda, R)$ such that:

$$\|u\|_{C^\alpha(\overline{B_{R/2}})} \leq C\|f\|_{L^\infty(B_R)} + \|u\|_{L^\infty(\mathbb{R}^N)}. \tag{2.2}$$

The precise dependence of the constant C on R can be determined by means of a simple scaling, as in Lemma 5 below; however, for interior estimates this will be of no importance to us. When $s > \frac{1}{2}$, the Hölder estimate for the solution can be improved to obtain an estimate for the first derivatives. In fact, as a consequence of Theorem 1.2 in [38], we have that $u \in C^{1,\beta}_{\rm loc}(\Omega)$, for some $\beta = \beta(N, s, \lambda, \Lambda) \in (0, 1)$. Also, for every ball $B_R \subset\subset \Omega$ there exists a positive constant $C = C(N, s, \lambda, \Lambda, R)$ such that:

$$\|u\|_{C^{1,\beta}(\overline{B_{R/2}})} \leq C \left(\|f\|_{L^\infty(B_R)} + \|u\|_{L^\infty(\mathbb{R}^N)} \right). \tag{2.3}$$

Both estimates will play a prominent role in our proof of a priori bounds for positive solutions of (1.2) and (1.5).

Next we need to deal with problems with a right hand side which is possibly singular at $\partial\Omega$. For this aim, it is convenient to introduce some norms which will help us to quantify the singularity of both the right hand sides and the gradient of the solutions in case $s > \frac{1}{2}$.

Let us denote, for $x \in \Omega$, $d(x) = \text{dist}(x, \partial\Omega)$. It is well known that d is Lipschitz continuous in Ω with Lipschitz constant 1 and it is a C^2 function in a neighborhood of $\partial\Omega$. We modify it outside this neighborhood to make it a C^2 function (still with Lipschitz constant 1), and we extend it to be zero outside Ω.

Now, for $\theta \in \mathbb{R}$ and $u \in C(\Omega)$, let us denote (cf. Chapter 6 in [36]):

$$\|u\|_0^{(\theta)} = \sup_{\Omega} \ d(x)^\theta |u(x)|.$$

When $u \in C^1(\Omega)$ we also set

$$\|u\|_1^{(\theta)} = \sup_{\Omega} \ \Big(d(x)^\theta |u(x)| + d(x)^{\theta+1} |\nabla u(x)| \Big) . \tag{2.4}$$

Then we have the following existence result for the Dirichlet problem associated to (2.1).

Lemma 3. *Assume Ω is a C^2 bounded domain, $0 < s < 1$ and K is a measurable function verifying* (1.1) *and* (1.4)*. Let $f \in C(\Omega)$ be such that $\|f\|_0^{(\theta)} < +\infty$ for some $\theta \in (s, 2s)$. Then the problem*

$$\begin{cases} (-\Delta)_K^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.5}$$

admits a unique viscosity solution. Moreover, there exists a positive constant C such that

$$\|u\|_0^{(\theta-2s)} \le C \|f\|_0^{(\theta)}. \tag{2.6}$$

Finally, if $f \ge 0$ in Ω then $u \ge 0$ in Ω.

The proof of this result relies in the construction of a suitable barrier in a neighborhood of the boundary of Ω which we will undertake in the following lemma. This barrier will also turn out to be important to obtain bounds for the solutions when trying to apply the scaling method. It is worthy of mention that for quite general operators, the lemma below can be obtained provided that θ is taken close enough to $2s$ (cf. for instance Lemma 3.2 in [34]). But the precise assumptions we are imposing on K, especifically (1.4), allow us to construct the barrier in the whole range $\theta \in (s, 2s)$.

In what follows, we denote, for small positive δ,

$$\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\},$$

and $K_\mu(x) = K(\mu x)$ for $\mu > 0$.

Lemma 4. *Let Ω be a C^2 bounded domain of $\mathbb{R}^N$, $0 < s < 1$ and K be measurable and verify* (1.1) *and* (1.4)*. For every $\theta \in (s, 2s)$ and $\mu_0 > 0$, there exist $C_0, \delta > 0$ such that*

$$(-\Delta)_{K_\mu}^s d^{2s-\theta} \ge C_0 d^{-\theta} \quad \text{in } \Omega_\delta,$$

if $0 < \mu \le \mu_0$.

Proof. By contradiction, let us assume that the conclusion of the lemma is not true. Then there exist $\theta \in (s, 2s)$, $\mu_0 > 0$, sequences of points $x_n \in \Omega$ with $d(x_n) \to 0$ and numbers $\mu_n \in (0, \mu_0]$ such that

$$\lim_{n \to +\infty} d(x_n)^\theta (-\Delta)_{K_{\mu_n}}^s d^{2s-\theta}(x_n) \le 0. \tag{2.7}$$

Denoting for simplicity $d_n := d(x_n)$, and performing the change of variables $y = d_n z$ in the integral appearing in (2.7) we obtain

$$\int_{\mathbb{R}^N} \frac{2 - \left(\frac{d(x_n+d_nz)}{d_n}\right)^{2s-\theta} - \left(\frac{d(x_n-d_nz)}{d_n}\right)^{2s-\theta}}{|z|^{N+2s}} K(\mu_n d_n z) dz \le o(1). \tag{2.8}$$

Before passing to the limit in this integral, let us estimate it from below. Observe that when $x_n + d_n z \in \Omega$, we have by the Lipschitz property of d that $d(x_n + d_n z) \le d_n(1 + |z|)$. Of course, the same is true when $x_n + d_n z \notin \Omega$ and it similarly follows that $d(x_n - d_n z) \le d_n(1 + |z|)$. Thus, taking $L > 0$ we obtain for large n

$$\begin{aligned} &\int_{|z|\ge L} \frac{2 - \left(\frac{d(x_n+d_nz)}{d_n}\right)^{2s-\theta} - \left(\frac{d(x_n-d_nz)}{d_n}\right)^{2s-\theta}}{|z|^{N+2s}} K(\mu_n d_n z) dz \\ &\qquad \ge -2\Lambda \int_{|z|\ge L} \frac{(1+|z|)^{2s-\theta}}{|z|^{N+2s}} dz. \end{aligned} \tag{2.9}$$

On the other hand, since d is smooth in a neighborhood of the boundary, when $|z| \le L$ and $x_n + d_n z \in \Omega$, we obtain by Taylor's theorem

$$d(x_n + d_n z) = d_n + d_n \nabla d(x_n) z + \Theta_n(d_n, z) d_n^2 |z|^2, \tag{2.10}$$

where Θ_n is uniformly bounded. Hence

$$d(x_n + d_n z) \le d_n + d_n \nabla d(x_n) z + C d_n^2 |z|^2. \tag{2.11}$$

Now choose $\eta \in (0, 1)$ small enough. Since $d(x_n) \to 0$ and $|\nabla d| = 1$ in a neighborhood of the boundary, we can assume that

$$\nabla d(x_n) \to e \text{ as } n \to +\infty \text{ for some unit vector } e. \tag{2.12}$$

Without loss of generality, we may take $e = e_N$, the last vector of the canonical basis of $\mathbb{R}^N$. If we restrict z further to satisfy $|z| \le \eta$, we obtain $1+\nabla d(x_n)z \sim 1+z_N \ge 1-\eta > 0$ for large n, since $|z_N| \le |z| \le \eta$. Therefore, the right-hand side in (2.11) is positive for large n (depending only on η), so that the inequality (2.11) is also true when $x_n + d_n z \notin \Omega$. Moreover, by using again Taylor's theorem

$$(1 + \nabla d(x_n) z + C d_n |z|^2)^{2s-\theta} \le 1 + (2s - \theta) \nabla d(x_n) z + C|z|^2,$$

for large enough n. Thus from (2.11),

$$\left(\frac{d(x_n + d_n z)}{d_n}\right)^{2s-\theta} \le 1 + (2s - \theta) \nabla d(x_n) z + C|z|^2,$$

for large enough n. A similar inequality is obtained for the term involving $d(x_n - d_n z)$. Therefore we deduce that

$$\begin{aligned} &\int_{|z|\le \eta} \frac{2 - \left(\frac{d(x_n+d_nz)}{d_n}\right)^{2s-\theta} - \left(\frac{d(x_n-d_nz)}{d_n}\right)^{2s-\theta}}{|z|^{N+2s}} K(\mu_n d_n z) dz \\ &\qquad \ge -2\Lambda C \int_{|z|\le\eta} \frac{1}{|z|^{N-2(1-s)}} dz. \end{aligned} \tag{2.13}$$

We finally observe that it follows from the above discussion (more precisely from (2.10) and (2.12) with $e = e_N$) that for $\eta \le |z| \le L$

$$\frac{d(x_n \pm d_n z)}{d_n} \to (1 \pm z_N)_+ \qquad \text{as } n \to +\infty. \tag{2.14}$$

Therefore using (2.9), (2.13) and (2.14), and passing to the limit as $n \to +\infty$ in (2.8), by dominated convergence we arrive at

$$-2\Lambda \int_{|z|\ge L} \frac{(1+|z|)^{2s-\theta}}{|z|^{N+2s}} dz + \int_{\eta\le|z|\le L} \frac{2-(1+z_N)_+^{2s-\theta}-(1-z_N)_+^{2s-\theta}}{|z|^{N+2s}} dz$$
$$-2\Lambda C \int_{|z|\le\eta} \frac{1}{|z|^{N-2(1-s)}} dz \le 0.$$

We have also used that $\lim_{n\to+\infty} K(\mu_n d_n z) = 1$ uniformly, by (1.4) and the boundedness of $\{\mu_n\}$. Letting now $\eta \to 0$ and then $L \to +\infty$, we have

$$\int_{\mathbb{R}^N} \frac{2-(1+z_N)_+^{2s-\theta}-(1-z_N)_+^{2s-\theta}}{|z|^{N+2s}} dz \le 0.$$

It is well-known, with the use of Fubini's theorem and a change of variables, that this integral can be rewritten as a one-dimensional integral

$$\int_{\mathbb{R}} \frac{2-(1+t)_+^{2s-\theta}-(1-t)_+^{2s-\theta}}{|t|^{1+2s}} dt \le 0. \tag{2.15}$$

We will see that this is impossible because of our assumption $\theta \in (s, 2s)$. Indeed, consider the function

$$F(\tau) = \int_{\mathbb{R}} \frac{2-(1+t)_+^{\tau}-(1-t)_+^{\tau}}{|t|^{1+2s}} dt, \qquad \tau \in (0, 2s),$$

which is well-defined. We claim that $F \in C^\infty(0, 2s)$ and it is strictly concave. In fact, observe that for $k \in \mathbb{N}$, the candidate for the $k-$th derivative $F^{(k)}(\tau)$ is given by

$$-\int_{\mathbb{R}} \frac{(1+t)_+^{\tau}(\log(1+t))_+^k + (1-t)_+^{\tau}(\log(1-t))_+^k}{|t|^{1+2s}} dt.$$

It is easily seen that this integral converges for every $k \ge 1$, since by Taylor's expansion for $t \sim 0$ we deduce $(1+t)^\tau(\log(1+t))^k + (1-t)^\tau(\log(1-t))^k = O(t^2)$. Therefore it follows that F is C^∞ in $(s, 2s)$. To see that F is strictly concave, just notice that

$$F_\varepsilon''(\tau) = -\int_{\mathbb{R}} \frac{(1+t)_+^{\tau}(\log(1+t)_+)^2 + (1-t)_+^{\tau}(\log(1-t)_+)^2}{|t|^{1+2s}} dt < 0.$$

Finally, it is clear that $F(0) = 0$. Moreover, since $v(x) = (x_+)^s$, $x \in \mathbb{R}$ verifies $(-\Delta)^s v = 0$ in $\mathbb{R}_+$ (see for instance the introduction in [16] or Proposition 3.1 in [45]), we also deduce that $F(s) = 0$. By strict concavity we have $F(\tau) > 0$ for $\tau \in (0, s)$, which clearly contradicts (2.15) if $\theta \in (s, 2s)$. Therefore (2.15) is not true and this concludes the proof of the lemma. $\square$

Proof of Lemma 3. By Lemma 4 with $\mu_0 = 1$, there exist $C_0 > 0$ and $\delta > 0$ such that

$$(-\Delta)^s_K d^{2s-\theta} \geq C_0 d^{-\theta} \text{ in } \Omega_\delta. \tag{2.16}$$

Let us show that it is possible to construct a supersolution of the problem

$$\begin{cases} (-\Delta)^s_K v = C_0 d^{-\theta} & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.17}$$

vanishing outside Ω.

First of all, by Theorem 3.1 in [34], there exists a nonnegative function $w \in C(\mathbb{R}^N)$ such that $(-\Delta)^s_K w = 1$ in Ω, with $w = 0$ in $\mathbb{R}^N \setminus \Omega$. We claim that $v = d^{2s-\theta} + tw$ is a supersolution of (2.17) if $t > 0$ is large enough. For this aim, observe that $(-\Delta)^s_K d^{2s-\theta} \geq -C$ in $\Omega \setminus \Omega_\delta$, since d is a C^2 function there. Therefore,

$$(-\Delta)^s_K v \geq t - C \geq C_0 d^{-\theta} \text{ in } \Omega \setminus \Omega_\delta$$

if t is large enough. Since clearly $(-\Delta)^s_K v \geq C_0 d^{-\theta}$ in Ω_δ as well, we see that v is a supersolution of (2.17), which vanishes outside Ω.

Now choose a sequence of smooth functions $\{\psi_n\}$ verifying $0 \leq \psi_n \leq 1$, $\psi_n = 1$ in $\Omega \setminus \Omega_{2/n}$ and $\psi_n = 0$ in $\Omega_{1/n}$. Define $f_n = f\psi_n$, and consider the problem

$$\begin{cases} (-\Delta)^s_K u = f_n & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.18}$$

Since $f_n \in C(\overline{\Omega})$, we can use Theorem 3.1 in [34] which gives a viscosity solution $u_n \in C(\mathbb{R}^N)$ of (2.18).

On the other hand, $|f_n| \leq |f| \leq \|f\|_0^{(\theta)} d^{-\theta}$ in Ω, so that the functions $v_\pm = \pm C_0^{-1} \|f\|_0^{(\theta)} v$ are sub and supersolution of (2.18). By comparison (cf. Theorem 5.2 in [18]), we obtain

$$-C_0^{-1} \|f\|_0^{(\theta)} v \leq u_n \leq C_0^{-1} \|f\|_0^{(\theta)} v \qquad \text{in } \Omega.$$

Now, this bound together with (2.2), Ascoli-Arzelá's theorem and a standard diagonal argument allow us to obtain a subsequence, still denoted by $\{u_n\}$, and a function $u \in C(\Omega)$ such that $u_n \to u$ uniformly on compact sets of Ω. In addition, u verifies

$$|u| \leq C_0^{-1} \|f\|_0^{(\theta)} v \quad \text{in } \Omega. \tag{2.19}$$

By Corollary 4.7 in [18], we can pass to the limit in (2.18) to obtain that $u \in C(\mathbb{R}^N)$ is a viscosity solution of (2.5). Moreover inequality (2.19) implies that $|u| \leq C \|f\|_0^{(\theta)} d^{2s-\theta}$ in $\Omega \setminus \Omega_\delta$ for some $C > 0$, so that, by (2.5), (2.16) and the comparison principle, we obtain that

$$|u| \leq C \|f\|_0^\theta d^{2s-\theta} \quad \text{in } \Omega$$

which shows (2.6).

The uniqueness and the nonnegativity of u when $f \geq 0$ are a consequence of the maximum principle (again Theorem 5.2 in [18]). This concludes the proof. $\square$

Our next estimate concerns the gradient of the solutions of (2.5) when $s > \frac{1}{2}$. The proof is more or less standard starting from (2.3) (cf. [36]) but we include it for completeness

Lemma 5. *Assume Ω is a smooth bounded domain and $s > \frac{1}{2}$. There exists a constant C_0 which depends on N, s, λ and Λ but not on Ω such that, for every $\theta \in (s, 2s)$ and $f \in C(\Omega)$ with $\|f\|_0^{(\theta)} < +\infty$ the unique solution u of* (2.5) *verifies*

$$\|\nabla u\|_0^{(\theta-2s+1)} \leq C_0(\|f\|_0^{(\theta)} + \|u\|_0^{(\theta-2s)}). \tag{2.20}$$

Proof. By (2.3) with $R = 1$ we know that if $(-\Delta)^s_K u = f$ in B_1 then there exists a constant which depends on N, s, λ and Λ such that $\|\nabla u\|_{L^\infty(B_{1/2})} \leq C(\|f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^N)})$. By a simple scaling, it can be seen that if $(-\Delta)^s_K u = f$ in Ω and $B_R \subset\subset \Omega$ then

$$R\|\nabla u\|_{L^\infty(B_{R/2})} \leq C(R^{2s}\|f\|_{L^\infty(B_R)} + \|u\|_{L^\infty(\mathbb{R}^N)}).$$

Choose a point $x \in \Omega$. By applying the previous inequality in the ball $B = B_{d(x)/2}(x)$ and multiplying by $d(x)^{\theta-2s}$ we arrive at

$$d(x)^{\theta-2s+1}|\nabla u(x)| \leq C\left(d(x)^\theta\|f\|_{L^\infty(B)} + d(x)^{\theta-2s}\|u\|_{L^\infty(\mathbb{R}^N)}\right).$$

Finally, notice that $\frac{d(x)}{2} < d(y) < \frac{3d(x)}{2}$ for every $y \in B$, so that $d(x)^\theta|f(y)| \leq 2^\theta d(y)^\theta f(y) \leq 2^{2s}\|f\|_0^{(\theta)}$, this implying $d(x)^\theta\|f\|_{L^\infty(B)} \leq 2^{2s}\|f\|_0^{(\theta)}$. A similar inequality can be achieved for the term involving $\|u\|_{L^\infty(\mathbb{R}^N)}$. After taking supremum, (2.20) is obtained. □

Our next lemma is intended to take care of the constant in (2.6) when we consider problem (2.5) in expanding domains, since in general it depends on Ω. This is the key for the scaling method to work properly in our setting. For a C^2 bounded domain Ω, we take $\xi \in \partial\Omega$, $\mu > 0$ and let

$$\Omega^\mu := \{y \in \mathbb{R}^N : \ \xi + \mu y \in \Omega\}.$$

It is clear then that $d_\mu(y) := \mathrm{dist}(y, \partial\Omega^\mu) = \mu^{-1}d(\xi + \mu y)$. Let us explicitly remark that the constant in (2.6) for the solution of (2.5) posed in Ω^μ will depend then on the domain Ω, but not on the dilation parameter μ, as we show next.

Lemma 6. *Assume Ω is a C^2 bounded domain, $0 < s < 1$ and K is a measurable function verifying* (1.1) *and* (1.4). *For every $\theta \in (s, 2s)$ and $\mu_0 > 0$, there exist $C_0, \delta > 0$ such that*

$$(-\Delta)^s_{K_\mu} d_\mu^{2s-\theta} \geq C_0 d_\mu^{-\theta} \quad \textit{in } (\Omega^\mu)_\delta,$$

if $0 < \mu \leq \mu_0$. Moreover, if u verifies $(-\Delta)^s_{K_\mu} u \leq C_1 d_\mu^{-\theta}$ in Ω^μ for some $C_1 > 0$ with $u = 0$ in $\mathbb{R}^N \setminus \Omega^\mu$, then

$$u(x) \leq C_2(C_1 + \|u\|_{L^\infty(\Omega^\mu)})\, d_\mu^{2s-\theta} \quad \textit{for } x \in (\Omega^\mu)_\delta.$$

for some $C_2 > 0$ only depending on s, δ, θ and C_0.

Proof. The first part of the proof is similar to that of Lemma 4 but taking a little more care in the estimates. By contradiction let us assume that there exist sequences $\xi_n \in \partial\Omega$, $\mu_n \in (0,\mu_0]$ and

$$x_n \in \Omega^n := \{y \in \mathbb{R}^N : \ \xi_n + \mu_n y \in \Omega\},$$

such that $d_n(x_n) \to 0$ and

$$d_n(x_n)^\theta (-\Delta)^s_{K_{\mu_n}} d_n^{2s-\theta}(x_n) \le o(1).$$

Here we have denoted

$$d_n(y) := \operatorname{dist}(y, \partial\Omega^n) = \mu_n^{-1} d(\xi_n + \mu_n y).$$

For $L > 0$, we obtain as in Lemma 4, letting $d_n = d_n(x_n)$

$$\int_{|z|\ge L} \frac{2 - \left(\frac{d_n(x_n+d_nz)}{d_n}\right)^{2s-\theta} - \left(\frac{d_n(x_n-d_nz)}{d_n}\right)^{2s-\theta}}{|z|^{N+2s}} K(\mu_n d_n z) dz$$
$$\ge -2\Lambda \int_{|z|\ge L} \frac{(1+|z|)^{2s-\theta}}{|z|^{N+2s}} dz.$$

Moreover, we also have an equation like (2.10). In fact taking into account that $\|D^2 d_n\| = \mu_n \|D^2 d\|$ is bounded we have for $|z| \le \eta < 1$:

$$d_n(x_n \pm d_n z) \le d_n \pm d_n \nabla d_n(x_n) z + C d_n^2 |z|^2.$$

with a constant $C > 0$ independent of n. Hence

$$\int_{|z|\le \eta} \frac{2 - \left(\frac{d_n(x_n+d_nz)}{d_n}\right)^{2s-\theta} - \left(\frac{d_n(x_n-d_nz)}{d_n}\right)^{2s-\theta}}{|z|^{N+2s}} K(\mu_n d_n z) dz$$
$$\ge -2\Lambda C \int_{|z|\le\eta} \frac{1}{|z|^{N-2(1-s)}} dz.$$

Now observe that $d_n(x_n) \to 0$ implies in particular $d(\xi_n + \mu_n x_n) \to 0$, so that $|\nabla d(\xi_n + \mu_n x_n)| = 1$ for large n and then $|\nabla d_n(x_n)| = 1$. As in (2.12), passing to a subsequence we may assume that $\nabla d_n(x_n) \to e_N$. Then

$$\frac{d_n(x_n \pm d_n z)}{d_n} \to (1 \pm z_N)_+ \qquad \text{as } n \to +\infty,$$

for $\eta \le |z| \le L$ and the proof of the first part concludes as in Lemma 4.

Now let u be a viscosity solution of

$$\begin{cases} (-\Delta)^s_{K_\mu} u \le C_1 d_\mu^{-\theta} & \text{in } \Omega^\mu, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega^\mu. \end{cases}$$

Choose $R > 0$ and let $v = R d_\mu^{2s-\theta}$. Then clearly

$$(-\Delta)^s_{K_\mu} v \ge R C_0 d_\mu^{-\theta} \ge C_1 d_\mu^{-\theta} \ge (-\Delta)^s_{K_\mu} u \text{ in } (\Omega^\mu)_\delta,$$

if we choose $R > C_1 C_0^{-1}$. Moreover, $u = v = 0$ in $\mathbb{R}^N \setminus \Omega^\mu$ and $v \ge R\delta^{2s-\theta} \ge u$ in $\Omega^\mu \setminus (\Omega^\mu)_\delta$ if R is chosen so that $R\delta^{2s-\theta} \ge \|u\|_{L^\infty(\Omega^\mu)}$. Thus by comparison $u \le v$ in $(\Omega^\mu)_\delta$, which gives the desired result, with, for instance $C_2 = \delta^{\theta-2s} + C_0^{-1}$. This concludes the proof. $\square$

We close this section with a statement of the strong comparison principle for the operator $(-\Delta)^s_K$, which will be frequently used throughout the rest of the paper. We include a proof for completeness (cf. Lemma 12 in [40] for a similar proof).

Lemma 7. *Let K be a measurable function verifying* (1.1) *and assume $u \in C(\mathbb{R}^N)$, $u \geq 0$ in $\mathbb{R}^N$ verifies $(-\Delta)^s_K u \geq 0$ in the viscosity sense in Ω. Then $u > 0$ or $u \equiv 0$ in Ω.*

Proof. Assume $u(x_0) = 0$ for some $x_0 \in \Omega$ but $u \not\equiv 0$ in Ω. Choose a nonnegative test function $\phi \in C^2(\mathbb{R}^N)$ such that $u \geq \phi$ in a neighborhood U of x_0 with $\phi(x_0) = 0$ and let

$$\psi = \begin{cases} \phi & \text{in } U \\ u & \text{in } \mathbb{R}^N \setminus U. \end{cases}$$

Observe that ψ can be taken to be nontrivial since u is not identically zero, by diminishing U if necessary. Since $(-\Delta)^s_K u \geq 0$ in Ω in the viscosity sense, it follows that $(-\Delta)^s_K \psi(x_0) \geq 0$. Taking into account that for a nonconstant ψ we should have $(-\Delta)^s_K \psi < 0$ at a global minimum, we deduce that ψ is a constant function. Moreover, since $\psi(x_0) = \phi(x_0) = 0$ then $\psi \equiv 0$ in $\mathbb{R}^N$, which is a contradiction. Therefore if $u(x_0) = 0$ for some $x_0 \in \Omega$ we must have $u \equiv 0$ in Ω, as was to be shown. □

3. A priori bounds

In this section we will be concerned with our most important step: the obtention of a priori bounds for positive solutions for both problems (1.2) and (1.5). We begin with problem (1.2), with the essential assumption of subcriticality of p, that is equation (1.3) and assuming that g verifies the growth restriction

$$|g(x,z)| \leq C(1 + |z|^r), \quad x \in \Omega,\ z \in \mathbb{R}, \tag{3.1}$$

where $C > 0$ and $0 < r < p$.

Theorem 8. *Assume Ω is a C^2 bounded domain and K a measurable function verifying* (1.1) *and* (1.4)*. Suppose p is such that* (1.3) *holds and g verifies* (3.1)*. Then there exists a constant $C > 0$ such that for every positive viscosity solution u of* (1.2) *we have*

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

Proof. Assume on the contrary that there exists a sequence of positive solutions $\{u_k\}$ of (1.2) such that $M_k = \|u_k\|_{L^\infty(\Omega)} \to +\infty$. Let $x_k \in \Omega$ be points with $u_k(x_k) = M_k$ and introduce the functions

$$v_k(y) = \frac{u_k(x_k + \mu_k y)}{M_k}, \quad y \in \Omega^k,$$

where $\mu_k = M_k^{-\frac{p-1}{2s}} \to 0$ and

$$\Omega^k := \{y \in \mathbb{R}^N : \ x_k + \mu_k y \in \Omega\}.$$

Then v_k is a function verifying $0 < v_k \le 1$, $v_k(0) = 1$ and

$$(-\Delta)^s_{K_k} v_k = v_k^p + h_k \quad \text{in } \Omega^k \tag{3.2}$$

where $K_k(y) = K(\mu_k y)$ and $h_k \in C(\Omega^k)$ verifies $|h_k| \le C M_k^{r-p}$.

By passing to subsequences, two situations may arise: either $d(x_k)\mu_k^{-1} \to +\infty$ or $d(x_k)\mu_k^{-1} \to d \ge 0$.

Assume the first case holds, so that $\Omega^k \to \mathbb{R}^N$ as $k \to +\infty$. Since the right hand side in (3.2) is uniformly bounded and $v_k \le 1$, we may use estimates (2.2) with an application of Ascoli-Arzelá's theorem and a diagonal argument to obtain that $v_k \to v$ locally uniformly in $\mathbb{R}^N$. Passing to the limit in (3.2) and using that K is continuous at zero with $K(0) = 1$, we see that v solves $(-\Delta)^s v = v^p$ in $\mathbb{R}^N$ in the viscosity sense (use for instance Lemma 5 in [19]).

By standard regularity (cf. for instance Proposition 2.8 in [52]) we obtain $v \in C^{2s+\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. Moreover, since $v(0) = 1$, the strong maximum principle implies $v > 0$. Then by bootstrapping using again Proposition 2.8 in [52] we would actually have $v \in C^\infty(\mathbb{R}^N)$. In particular we deduce that v is a strong solution of $(-\Delta)^s v = v^p$ in $\mathbb{R}^N$ in the sense of [60]. However, since $p < \frac{N+2s}{N-2s}$, this contradicts for instance Theorem 4 in [60] (see also [25]).

If the second case holds then we may assume $x_k \to x_0 \in \partial\Omega$. With no loss of generality assume also $\nu(x_0) = -e_N$. In this case, rather than working with the functions v_k, it is more convenient to deal with

$$w_k(y) = \frac{u_k(\xi_k + \mu_k y)}{M_k}, \quad y \in D^k,$$

where $\xi_k \in \partial\Omega$ is the projection of x_k on $\partial\Omega$ and

$$D^k := \{y \in \mathbb{R}^N : \ \xi_k + \mu_k y \in \Omega\}. \tag{3.3}$$

Observe that

$$0 \in \partial D^k, \tag{3.4}$$

and

$$D^k \to \mathbb{R}^N_+ = \{y \in \mathbb{R}^N : \ y_N > 0\} \text{ as } k \to +\infty.$$

It also follows that w_k verifies (3.2) in D^k with a slightly different function h_k, but with the same bounds.

Moreover, setting

$$y_k := \frac{x_k - \xi_k}{\mu_k},$$

so that $|y_k| = d(x_k)\mu_k^{-1}$, we see that $w_k(y_k) = 1$. We claim that $d = \lim_{k\to+\infty} d(x_k)\mu_k^{-1} > 0$. This in particular guarantees that by passing to a further subsequence $y_k \to y_0$, where $|y_0| = d > 0$, thus y_0 is in the interior of the half-space $\mathbb{R}^N_+$.

Let us show the claim. Observe that by (3.2), and since $r < p$, we have

$$(-\Delta)^s_{K_k} w_k \le C \le C_1 d_k^{-\theta} \quad \text{in } D^k$$

for every $\theta \in (s, 2s)$, where $d_k(y) = \operatorname{dist}(y, \partial D^k)$. By Lemma 6, fixing any such θ, there exist constants $C_0 > 0$ and $\delta > 0$ such that $w_k(y) \le C_0 d_k(y)^{2s-\theta}$ if $d_k(y) < \delta$. In particular, since by (3.4) $|y_k| \ge d_k(y_k)$, if

$d_k(y_k) < \delta$, then $1 \leq C_0 d_k(y_k)^{2s-\theta} \leq C_0|y_k|^{2s-\theta}$, which implies $|y_k|$ is bounded from below so that $d > 0$.

Now we can employ (2.2) as above to obtain that $w_k \to w$ uniformly on compact sets of $\mathbb{R}^N_+$, where w verifies $0 \leq w \leq 1$ in $\mathbb{R}^N_+$, $w(y_0) = 1$ and $w(y) \leq C y_N^{2s-\theta}$ for $y_N < \delta$. Therefore $w \in C(\mathbb{R}^N)$ is a nonnegative, bounded solution of

$$\begin{cases} (-\Delta)^s w = w^p & \text{in } \mathbb{R}^N_+, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+. \end{cases}$$

Again by bootstrapping and the strong maximum principle we have $w \in C^\infty(\mathbb{R}^N_+)$, $w > 0$. Since $p < \frac{N+2s}{N-2s} < \frac{N-1+2s}{N-1-2s}$, this is a contradiction with Theorem 1.1 in [43] (cf. also Theorem 1.2 in [32]). This contradiction proves the theorem. $\square$

We now turn to analyze the a priori bounds for solutions of problem (1.5). We have already remarked that due to the expected singularity of the gradient of the solutions near the boundary we need to work in spaces with weights which take care of the singularity. Thus we fix $\sigma \in (0,1)$ verifying

$$0 < \sigma < 1 - \frac{s}{t} < 1 \tag{3.5}$$

and let

$$E_\sigma = \{u \in C^1(\Omega) : \ \|u\|_1^{(-\sigma)} < +\infty\}, \tag{3.6}$$

where $\|\cdot\|_1^{(-\sigma)}$ is given by (2.4) with $\theta = -\sigma$. As for the function h, we assume that it has a prescribed growth at infinity: there exists $C^0 > 0$ such that for every $x \in \Omega$, $z \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$,

$$|h(x,z,\xi)| \leq C^0(1 + |z|^r + |\xi|^t), \tag{3.7}$$

where $0 < r < p$ and $1 < t < \frac{2sp}{p+2s-1} < 2s$ (observe that there is no loss of generality in assuming $t > 1$). We recall that in the present situation we require the stronger restriction (1.6) on the exponent p.

Then we can prove:

Theorem 9. *Assume Ω is a C^2 bounded domain and K a measurable function verifying* (1.1) *and* (1.4). *Suppose that $s > \frac{1}{2}$, p verifies* (1.6) *and h is nonnegative and such that* (3.7) *holds. Then there exists a constant $C > 0$ such that for every positive solution u of* (1.5) *in E_σ with σ satisfying* (3.5) *we have*

$$\|u\|_1^{(-\sigma)} \leq C.$$

We prove the a priori bounds in two steps. In the first one we obtain rough bounds for all solutions of the equation which are universal, in the spirit of [41]. It is here where the restriction (1.6) comes in.

Lemma 10. *Assume Ω is a C^2 (not necessarily bounded) domain and K a measurable function verifying* (1.1) *and* (1.4). *Suppose that $s > \frac{1}{2}$ and p verifies* (1.6). *Then there exists a positive constant $C = C(N,s,p,r,t,C^0,\Omega)$ (where r, t and C^0 are given in* (3.7)*) such that for every positive function*

$u \in C^1(\Omega) \cap L^\infty(\mathbb{R}^N)$ verifying $(-\Delta)^s_K u = u^p + h(x,u,\nabla u)$ in the viscosity sense in Ω, we have

$$u(x) \le C(1 + \mathrm{dist}(x,\partial\Omega)^{-\frac{2s}{p-1}}), \quad |\nabla u(x)| \le C(1 + \mathrm{dist}(x,\partial\Omega)^{-\frac{2s}{p-1}-1})$$

for $x \in \Omega$.

Proof. Assume on the contrary that there exist sequences of positive functions $u_k \in C^1(\Omega) \cap L^\infty(\mathbb{R}^N)$ verifying $(-\Delta)^s_K u_k = u_k^p + h(x,u_k,\nabla u_k)$ in Ω and points $y_k \in \Omega$ such that

$$u_k(y_k)^{\frac{p-1}{2s}} + |\nabla u_k(y_k)|^{\frac{p-1}{p+2s-1}} > 2k\,(1 + \mathrm{dist}(y_k,\partial\Omega)^{-1}). \tag{3.8}$$

Denote $N_k(x) = u_k(x)^{\frac{p-1}{2s}} + |\nabla u_k(x)|^{\frac{p-1}{p+2s-1}}$, $x \in \Omega$. By Lemma 5.1 in [41] (cf. also Remark 5.2 (b) there) there exists a sequence of points $x_k \in \Omega$ with the property that $N_k(x_k) \ge N_k(y_k)$, $N_k(x_k) > 2k\,\mathrm{dist}(x_k,\partial\Omega)^{-1}$ and

$$N_k(z) \le 2N_k(x_k) \text{ in } B(x_k, kN_k(x_k)^{-1}). \tag{3.9}$$

Observe that, in particular, (3.8) implies that $N_k(x_k) \to +\infty$. Let $\nu_k := N_k(x_k)^{-1} \to 0$ and define

$$v_k(y) := \nu_k^{\frac{2s}{p-1}} u_k(x_k + \nu_k y), \quad y \in B_k := \{y \in \mathbb{R}^N : \ |y| < k\}. \tag{3.10}$$

Then the functions v_k verify $(-\Delta)^s_{K_k} v_k = v_k^p + h_k$ in B_k, where $K_k(y) = K(\mu_k y)$ and

$$h_k(y) = \nu_k^{\frac{2sp}{p-1}} h(\xi_k + \nu_k y, \nu_k^{-\frac{2s}{p-1}} v_k(y), \nu_k(x_k)^{-\frac{2s+p-1}{p-1}} \nabla v_k(y)).$$

Since h verifies (3.7), we have $|h_k| \le C_0 \nu_k^\gamma (1 + v_k^r + |\nabla v_k|^t)$ in B_k, where

$$\gamma = \max\left\{ \frac{2s(p-r)}{p-1}, \frac{2ps - (2s+p-1)t}{p-1} \right\} > 0.$$

Moreover by (3.9) it follows that

$$v_k(y)^{\frac{p-1}{2s}} + |\nabla v_k(y)|^{\frac{p-1}{p+2s-1}} \le 2, \quad y \in B_k. \tag{3.11}$$

Also it is clear that

$$v_k(0)^{\frac{p-1}{2s}} + |\nabla v_k(0)|^{\frac{p-1}{p+2s-1}} = 1. \tag{3.12}$$

Since $\nu_k \to 0$ and v_k and $|\nabla v_k|$ are uniformly bounded in B_k, we see that h_k is also uniformly bounded in B_k. We may then use estimate (2.3) to obtain, again with the use of Ascoli-Arzelá's theorem and a diagonal argument, that there exists a subsequence, still labeled v_k such that $v_k \to v$ in $C^1_{\rm loc}(\mathbb{R}^N)$ as $k \to +\infty$. Since $v(0)^{\frac{p-1}{2s}} + |\nabla v(0)|^{\frac{p-1}{p+2s-1}} = 1$, we see that v is nontrivial.

Now let w_k be the functions obtained by extending v_k to be zero outside B_k. Then it is easily seen that $(-\Delta)^s_{K_k} w_k \ge w_k^p$ in B_k. Passing to the limit using again Lemma 5 of [19], we arrive at $(-\Delta)^s v \ge v^p$ in $\mathbb{R}^N$, which contradicts Theorem 1.3 in [33] since $p < \frac{N}{N-2s}$. This concludes the proof. □

Remarks 1.
(a) With a minor modification in the above proof, it can be seen that the constants given by Lemma 10 can be taken independent of the domain Ω (cf. the proof of Theorem 2.3 in [41]).

(b) We expect Lemma 10 to hold in the full range given by (1.3). Unfortunately, this method of proof seems purely local and needs to be properly adapted to deal with nonlocal equations. Observe that there is no information available for the functions v_k defined in (3.10) in $\Omega \setminus B_k$, which makes it difficult to pass to the limit appropriately in the equation satisfied by v_k.

We now come to the proof of the a priori bounds for positive solutions of (1.5).

Proof of Theorem 9. Assume that the conclusion of the theorem is not true. Then there exists a sequence of positive solutions $u_k \in E_\sigma$ of (1.5) such that $\|u_k\|_1^{(-\sigma)} \to +\infty$, where σ satisfies (3.5). Define

$$M_k(x) = d(x)^{-\sigma} u_k(x) + d(x)^{1-\sigma} |\nabla u_k(x)|.$$

Now choose points $x_k \in \Omega$ such that $M_k(x_k) \geq \sup_\Omega M_k - \frac{1}{k}$ (this supremum may not be achieved). Observe that our assumption implies $M_k(x_k) \to +\infty$.

Let ξ_k be a projection of x_k on $\partial\Omega$ and introduce the functions:

$$v_k(y) = \frac{u_k(\xi_k + \mu_k y)}{\mu_k^\sigma M_k(x_k)}, \quad y \in D^k,$$

where $\mu_k = M_k(x_k)^{-\frac{p-1}{2s+\sigma(p-1)}} \to 0$ and D^k is the set defined in (3.3). It is not hard to see that

$$\left\{ \begin{array}{ll} (-\Delta)^s_{K_k} v_k = v_k^p + h_k & \text{in } D^k, \\ v_k = 0 & \text{in } \mathbb{R}^N \setminus D^k, \end{array} \right. \tag{3.13}$$

where $K_k(y) = K(\mu_k y)$ and

$$h_k(y) = M_k(x_k)^{-\frac{2sp}{2s+\sigma(p-1)}} h(\xi_k + \mu_k y, M_k(x_k)^{\frac{2s}{2s+\sigma(p-1)}} v_k, M_k(x_k)^{\frac{2s+p-1}{2s+\sigma(p-1)}} \nabla v_k).$$

By assumption (3.7) on h, it is readily seen that h_k verifies the inequality $|h_k| \leq C M_k(x_k)^{-\bar{\gamma}} (1 + v_k^r + |\nabla v_k|^t)$ for some positive constant C independent of k, where

$$\bar{\gamma} = \frac{2sp}{2s+\sigma(p-1)} - \frac{\max\{2sr, (2s+p-1)t\}}{2s+\sigma(p-1)} > 0.$$

Moreover, the functions v_k verify

$$\mu_k^\sigma d(\xi_k + \mu_k y)^{-\sigma} v_k(y) + \mu_k^{\sigma-1} d(\xi_k + \mu_k y)^{1-\sigma} |\nabla v_k(y)| = \frac{M_k(\xi_k + \mu_k y)}{M_k(x_k)}.$$

Then, using that $\mu_k^{-1} d(\xi_k + \mu_k y) = \operatorname{dist}(y, \partial D^k) =: d_k(y)$ and the choice of the points x_k, we obtain for large k

$$d_k(y)^{-\sigma} v_k(y) + d_k(y)^{1-\sigma} |\nabla v_k(y)| \leq 2 \quad \text{in } D^k \tag{3.14}$$

and

$$d_k(y_k)^{-\sigma} v_k(y_k) + d_k(y_k)^{1-\sigma} |\nabla v_k(y_k)| = 1, \tag{3.15}$$

where, as in the proof of Theorem 8, $y_k := \mu_k^{-1}(x_k - \xi_k)$.

Next, since u_k solves (1.5), we may use Lemma 10 to obtain that $M_k(x_k) \leq Cd(x_k)^{-\sigma}(1+d(x_k)^{-\frac{2s}{p-1}})$ for some positive constant independent of k, which implies $d(x_k)\mu_k^{-1} \leq C$. This bound immediately entails that (passing to subsequences) $x_k \to x_0 \in \partial\Omega$ and $|y_k| = d(x_k)\mu_k^{-1} \to d \geq 0$ (in particular the points ξ_k are uniquely determined at least for large k). Assuming that the outward unit normal to $\partial\Omega$ at x_0 is $-e_N$, we also obtain then that $D^k \to \mathbb{R}^N_+$ as $k \to +\infty$.

We claim that $d > 0$. To show this, notice that from (3.13) and (3.14) we have $(-\Delta)^s_{K_k} v_k \leq Cd_k^{(\sigma-1)t}$ in D^k, for some constant C not depending on k. By our choice of σ and t, we get that

$$\sigma > \frac{t-2s}{t}. \tag{3.16}$$

That is, we have

$$s < (1-\sigma)t < 2s, \tag{3.17}$$

so that Lemma 6 can be applied to give $\delta > 0$ and a positive constant C such that

$$v_k(y) \leq Cd_k(y)^{2s+(\sigma-1)t}, \quad \text{when } d_k(y) < \delta. \tag{3.18}$$

Moreover, since $1 < t < 2s$, (3.16) in particular implies that

$$\sigma > \frac{t-2s}{t-1}, \tag{3.19}$$

and, therefore, $-\sigma + 2s + (\sigma-1)t = \sigma(t-1) + 2s - t > 0$. Thus, by (3.14) we have

$$v_k(y) \leq 2d_k(y)^{\sigma} \leq 2\delta^{\sigma-2s-(\sigma-1)t} d_k(y)^{2s+(\sigma-1)t} \text{ when } d_k(y) \geq \delta.$$

Hence $\|v_k\|_0^{(-2s-(\sigma-1)t)}$ is bounded. We can then use Lemma 5, with $\theta = (1-\sigma)t$, to obtain that

$$|\nabla v_k(y)| \leq Cd_k(y)^{2s+(\sigma-1)t-1} \quad \text{in } D^k, \tag{3.20}$$

where C is also independent of k. Taking inequalities (3.18) and (3.20) in (3.15), we deduce

$$1 \leq Cd_k(y_k)^{-\sigma+2s+(\sigma-1)t},$$

thus, by (3.19) we see that $d_k(y_k)$ is bounded away from zero. Hence, by (3.4), $|y_k|$ also is, so that $d > 0$, as claimed.

Finally, we can use (2.3) together with Ascoli-Arzelá's theorem and a diagonal argument to obtain that $v_k \to v$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N_+)$, where by (3.15), the function v verifies $d^{-\sigma}v(y_0) + d^{1-\sigma}|\nabla v(y_0)| = 1$ for some $y_0 \in \mathbb{R}^N_+$, hence it is nontrivial and $v(y) \leq Cy_N^{2s+(\sigma-1)t}$ if $0 < y_N < \delta$. Thus $v \in C(\mathbb{R}^N)$ and $v = 0$ outside $\mathbb{R}^N_+$. Passing to the limit in (3.13) with the aid of Lemma 5 in [19] and using that K is continuous at zero with $K(0) = 1$, we obtain

$$\begin{cases} (-\Delta)^s v = v^p & \text{in } \mathbb{R}^N_+, \\ v = 0 & \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+. \end{cases}$$

Using again bootstrapping and the strong maximum principle we have $v > 0$ and $v \in C^\infty(\mathbb{R}^N_+)$, therefore it is a classical solution. Moreover, by Lemma

10, we also see that $v(y) \le C y_N^{-\frac{2s}{p-1}}$ in $\mathbb{R}^N_+$, so that v is bounded. This is a contradiction with Theorem 1.2 in [32] (see also [43]), because we are assuming $p < \frac{N}{N-2s} < \frac{N-1+2s}{N-1-2s}$. The proof is therefore concluded. □

4. Existence of solutions

This final section is devoted to the proof of our existence results, Theorems 1 and 2. Both proofs are very similar, only that that of Theorem 2 is slightly more involved. Therefore we only show this one.

Thus we assume $s > \frac{1}{2}$. Fix σ verifying (3.5) and consider the Banach space E_σ, defined in (3.6), which is an ordered Banach space with the cone of nonnegative functions $P = \{u \in E_\sigma : \ u \ge 0 \text{ in } \Omega\}$. For the sake of brevity, we will drop the subindex σ throughout the rest of the section and will denote E and $\|\cdot\|$ for the space and its norm.

We will assume that h is nonnegative and verifies the growth condition in the statement of Theorem 2:

$$h(x,z,\xi) \le C(|z|^r + |\xi|^t), \quad x \in \Omega,\ z \in \mathbb{R},\ \xi \in \mathbb{R}^N, \tag{4.1}$$

where $1 < r < p$ and $1 < t < \frac{2sp}{2s+p-1}$. Observe that for every $v \in P$ we have

$$h(x, v(x), \nabla v(x)) \le C(\|v\|) d(x)^{(\sigma-1)t}. \tag{4.2}$$

Moreover, by (3.17) we may apply Lemma 3 to deduce that the problem

$$\begin{cases} (-\Delta)^s_K u = v^p + h(x,v,\nabla v) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

admits a unique nonnegative solution u, with $\|u\|_0^{(-\sigma)} < +\infty$. By Lemma 5 we also deduce $\|\nabla u\|_0^{(1-\sigma)} < +\infty$. Hence $u \in E$. In this way, we can define an operator $T : P \to P$ by means of $u = T(v)$. It is clear that nonnegative solutions of (1.2) in E coincide with the fixed points of this operator.

We begin by showing a fundamental property of T.

Lemma 11. *The operator $T : P \to P$ is compact.*

Proof. We show continuity first: let $\{u_n\} \subset P$ be such that $u_n \to u$ in E. In particular, $u_n \to u$ and $\nabla u_n \to \nabla u$ uniformly on compact sets of Ω, so that the continuity of h implies

$$h(\cdot, u_n, \nabla u_n) \to h(\cdot, u, \nabla u) \text{ uniformly on compact sets of } \Omega \tag{4.3}$$

Moreover, since u_n is bounded in E, similarly as in (4.2) we also have that $h(\cdot, u_n, \nabla u_n) \le C d^{(\sigma-1)t}$ in Ω, for a constant that does not depend on n (and the same is true for u after passing to the limit). This implies

$$\sup_\Omega d^\theta |h(\cdot, u_n, \nabla u_n) - h(\cdot, u, \nabla u)| \to 0, \tag{4.4}$$

for every $\theta > (1-\sigma)t > s$. Indeed, if we take $\varepsilon > 0$ then

$$d^\theta |h(\cdot, u_n, \nabla u_n) - h(\cdot, u, \nabla u)| \le C d^{\theta-(1-\sigma)t} \le C \delta^{\theta-(1-\sigma)t} \le \varepsilon,$$

if $d \le \delta$, by choosing a small δ. When $d \ge \delta$,

$$d^\theta |h(\cdot, u_n, \nabla u_n) - h(\cdot, u, \nabla u)| \le (\sup_\Omega d)^\theta |h(\cdot, u_n, \nabla u_n) - h(\cdot, u, \nabla u)| \le \varepsilon,$$

just by choosing $n \geq n_0$, by (4.3). This shows (4.4).

From Lemmas 3 and 5 for every $(1-\sigma)t < \theta < 2s$, we obtain

$$\sup_{\Omega} d^{\theta-2s}|T(u_n) - T(u)| + d^{\theta-2s+1}|\nabla(T(u_n) - T(u))| \to 0.$$

The desired conclusion follows by choosing θ such that

$$(1-\sigma)t < \theta \leq 2s - \sigma.$$

This shows continuity.

To prove compactness, let $\{u_n\} \subset P$ be bounded. As we did before, $h(\cdot, u_n, \nabla u_n) \leq C d^{(\sigma-1)t}$ in Ω. By (2.3) we obtain that for every $\Omega' \subset\subset \Omega$ the $C^{1,\beta}$ norm of $T(u_n)$ in Ω' is bounded. Therefore, we may assume by passing to a subsequence that $T(u_n) \to v$ in $C^1_{\rm loc}(\Omega)$.

From Lemmas 3 and 5 we deduce that $T(u_n) \leq C d^{(\sigma-1)t+2s}$, $|\nabla T(u_n)| \leq C d^{(\sigma-1)t+2s-1}$ in Ω, and the same estimates hold for v and ∇v by passing to the limit. Hence

$$\sup_{\Omega} d^{-\sigma}|T(u_n) - v| + d^{1-\sigma}|\nabla(T(u_n) - v)| \to 0,$$

which shows compactness. The proof is concluded. □

The proof of Theorem 2 relies in the use of topological degree, with the aid of the bounds provided by Theorem 9. The essential tool is the following well-known result (see for instance Theorem 3.6.3 in [21]).

Theorem 12. *Suppose that E is an ordered Banach space with positive cone P, and $U \subset P$ is an open bounded set containing 0. Let $\rho > 0$ be such that $B_\rho(0) \cap P \subset U$. Assume $T : U \to P$ is compact and satisfies*

(a) *for every $\mu \in [0,1)$, we have $u \neq \mu T(u)$ for every $u \in P$ with $\|u\| = \rho$;*

(b) *there exists $\psi \in P \setminus \{0\}$ such that $u - T(u) \neq t\psi$, for every $u \in \partial U$, for every $t \geq 0$.*

Then T has a fixed point in $U \setminus B_\rho(0)$.

The final ingredient in our proof is some knowledge on the principal eigenvalue for the operator $(-\Delta)^s_K$. The natural definition of such eigenvalue in our context resembles that of [8] for linear second order elliptic operators, that is:

$$\lambda_1 := \sup \left\{ \lambda \in \mathbb{R} : \begin{array}{c} \text{there exists } u \in C(\mathbb{R}^N),\ u > 0 \text{ in } \Omega, \text{ with} \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \text{ and } (-\Delta)^s_K u \geq \lambda u \text{ in } \Omega \end{array} \right\}. \tag{4.5}$$

At the best of our knowledge, there are no results available for the eigenvalues of $(-\Delta)^s_K$, although it seems likely that the first one will enjoy the usual properties (see [42]).

For our purposes here, we only need to show the finiteness of λ_1:

Lemma 13. $\lambda_1 < +\infty$.

Proof. We begin by constructing a suitable subsolution. The construction relies in a sort of "implicit" Hopf's principle (it is to be noted that Hopf's principle is not well understood for general kernels K verifying (1.1); see for instance Lemma 7.3 in [44] and the comments after it). However, a relaxed version is enough for our purposes.

Let $B' \subset\subset B \subset\subset \Omega$ and consider the unique solution ϕ of

$$\begin{cases} (-\Delta)^s_K \phi = 0 & \text{in } B \setminus B', \\ \phi = 1 & \text{in } B', \\ \phi = 0 & \text{in } \mathbb{R}^N \setminus B. \end{cases}$$

given for instance by Theorem 3.1 in [34], and the unique viscosity solution of

$$\begin{cases} (-\Delta)^s_K v = \phi & \text{in } B, \\ v = 0 & \text{in } \mathbb{R}^N \setminus B. \end{cases}$$

given by the same theorem. By Lemma 7 we have both $\phi > 0$ and $v > 0$ in B, so that there exists $C_0 > 0$ such that $C_0 v \geq \phi$ in B'. Hence by comparison $C_0 v \geq \phi$ in $\mathbb{R}^N$. In particular,

$$(-\Delta)^s_K v \leq C_0 v \text{ in } B. \tag{4.6}$$

We claim that $\lambda_1 \leq C_0$. Indeed, if we assume $\lambda_1 > C_0$, then there exist $\lambda > C_0$ and a positive function $u \in C(\mathbb{R}^N)$ vanishing outside Ω such that

$$(-\Delta)^s_K u \geq \lambda u \text{ in } \Omega. \tag{4.7}$$

Since $u > 0$ in $\overline{B}$, the number

$$\omega = \sup_B \frac{v}{u}$$

is finite. Moreover, $\omega u \geq v$ in $\mathbb{R}^N$. Observe that, since we are assuming $\lambda > C_0$, by (4.6) and (4.7) it follows that

$$\begin{cases} (-\Delta)^s_K (\omega u - v) \geq 0 & \text{in } B, \\ \omega u - v > 0 & \text{in } \mathbb{R}^N \setminus B. \end{cases}$$

Hence the strong maximum principle (Lemma 7) implies $\omega u - v > 0$ in $\overline{B}$. However this would imply $(\omega - \varepsilon) u > v$ in $\overline{B}$ for small ε, contradicting the definition of ω. Then $\lambda_1 \leq C_0$ and the lemma follows. □

Now we are in a position to prove Theorem 2.

Proof of Theorem 2. As already remarked, we will show that Theorem 12 is applicable to the operator T in $P \subset E$.

Let us check first hypothesis (a) in Theorem 12. Assume we have $u = \mu T(u)$ for some $\mu \in [0, 1)$ and $u \in P$. This is equivalent to

$$\begin{cases} (-\Delta)^s_K u = \mu(u^p + h(x, u, \nabla u)) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

By our hypotheses on h we get that the right hand side of the previous equation can be bounded by

$$\begin{aligned}\mu(u^p + h(x,u,\nabla u)) &\le d^{\sigma p}\|u\|^p + C_0(d^{\sigma r}\|u\|^r + d^{(\sigma-1)t}\|u\|^t)\\ &\le Cd^{(\sigma-1)t}(\|u\|^p + \|u\|^r + \|u\|^t).\end{aligned}$$

Therefore, by Lemmas 3 and 5 and (3.17), we have $\|u\| \le C(\|u\|^p + \|u\|^r + \|u\|^t)$. Since $p, r, t > 1$, this implies that $\|u\| > \rho$ for some small positive ρ. Thus there are no solutions of $u = \mu T(u)$ if $\|u\| = \rho$ and $\mu \in [0,1)$, and (a) follows.

To check (b), we take $\psi \in P$ to be the unique solution of the problem:

$$\begin{cases} (-\Delta)^s_K \psi = 1 & \text{in } \Omega,\\ \psi = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

given by Theorem 3.1 in [34]. We claim that there are no solutions in P of the equation $u - T(u) = t\psi$ if t is large enough. For that purpose we note that this equation is equivalent to

$$\begin{cases} (-\Delta)^s_K u = u^p + h(x,u,\nabla u) + t & \text{in } \Omega,\\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.8}$$

Fix $\mu > \lambda_1$, where λ_1 is given by (4.5). Using the nonnegativity of h, and since $p > 1$, there exists a positive constant C such that $u^p + h(x,u,\nabla u) + t \ge \mu u - C + t$. If $t \ge C$, then $(-\Delta)^s_K u \ge \mu u$ in Ω, which is against the choice of μ and the definition of λ_1. Therefore $t < C$, and (4.8) does not admit positive solutions in E if t is large enough.

Finally, since $h + t$ also verifies condition (3.7) for $t \le C$, we can apply Theorem 9 to obtain that the solutions of (4.8) are a priori bounded, that is, there exists $M > \rho$ such that $\|u\| < M$ for every positive solution of (4.8) with $t \ge 0$. Thus Theorem 12 is applicable with $U = B_M(0) \cap P$ and the existence of a solution in P follows. This solution is positive by Lemma 7. The proof is concluded. □

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