HIGHER ALGEBRAIC K-THEORY

(AFTER QUILLEN, THOMASON AND OTHERS)

MARCO SCHLICHTING

ABSTRACT. We give a short introduction (with a few proofs) to higher algebraic K-theory (mainly of schemes) based on the work of Quillen, Waldhausen, Thomason and others.

CONTENTS

INTRODUCTION

These are the notes for a course taught at the Sedano Winter school on K-theory, January 23 – 26, 2007, in Sedano, Spain.

Section 1 is an introduction to Quillen's fundamental article [Qui73]. Here the algebraic K-theory of exact categories is introduced via Quillen's Q-construction. We state some fundamental theorems, and state/derive results about the G-theory of noetherian schemes and the K-theory of smooth schemes.

Sections 2 and 3 are an introduction to Thomason's fundamental paper [TT90]. In section 2 we introduce the abstract concepts. and define the K-theory of complicial exact categories with weak equivalences. Then we state (the connective and non-connective versions 2.21 and 2.29 of) Thomason's localization theorem in this context. In section 3, we use the localization theorem to derive basic properties of the K-theory of quasi-compact and separated schemes most of which are beyond the reach of Quillen's methods.

In section 4, we state results that go beyond the methods explained in sections 1 – 3.

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1. Quillen's Q-construction

1.1. **Exact categories.** An *exact category* is an additive category $\mathcal{E}$ equipped with a family of sequences of maps in $\mathcal{E}$, called *conflation* (or admissible exact sequences),

$$X \stackrel{i}{\to} Y \stackrel{p}{\to} Z \tag{1}$$

satisfying the properties (a) – (f) below. In a conflation (1), the map i is called *inflation* (or admissible monomorphism) and may be depicted as $\rightarrowtail$, and the map p is called *deflation* (or admissible epimorphism) and may be depicted as $\twoheadrightarrow$.

(a) In a conflation (1), the map i is a kernel of p, and p is a cokernel of i.
(b) Conflations are closed under isomorphisms.
(c) Inflations are closed under compositions, and deflations are closed under compositions.
(d) Any diagram $Z \leftarrow X \stackrel{i}{\rightarrowtail} Y$ with i an inflation can be completed to a cocartesian square

$$\begin{array}{ccc} X & \stackrel{i}{\rightarrowtail} & Y \\ \downarrow & & \downarrow \\ Z & \stackrel{j}{\rightarrowtail} & W \end{array}$$

with j an inflation.
(e) Dually, any diagram $X \longrightarrow Z \stackrel{p}{\twoheadleftarrow} Y$ with p a deflation can be completed to a cartesian square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \longrightarrow & Z \end{array}$$

with q a deflation.
(f) The following sequence is a conflation

$$X \stackrel{\binom{1}{0}}{\to} X \oplus Y \stackrel{(0\ 1)}{\to} Y. \tag{2}$$

An additive functor between exact categories is called *exact* if it sends conflations to conflations.

Let $\mathcal{A}$, $\mathcal{B}$ be exact categories such that $\mathcal{B} \subset \mathcal{A}$ is a full subcategory. We say that $\mathcal{B}$ is a *fully exact subcategory* of $\mathcal{A}$ if $\mathcal{B}$ is closed under extensions in $\mathcal{A}$ (that is, if in a conflation (1) in $\mathcal{A}$, X and Z are isomorphic to objects in $\mathcal{B}$ then Y is isomorphic to an object in $\mathcal{B}$), and if the inclusion $\mathcal{B} \subset \mathcal{A}$ preserves and detects conflations.

1.2. **Examples.**

(a) Abelian categories are exact categories when equipped with the family of conflations (1) where $0 \to X \to Y \to Z \to 0$ is a short exact sequence. Examples of abelian (thus exact) categories are: the category R-Mod of all (left) R-modules, R a ring; the category R-mod of all finitely generated (left) R-modules, R a noetherian ring; the category O_X-Mod (Qcoh(X)) of (quasi-coherent) O_X-modules, X a scheme; the category Coh(X) of coherent O_X-modules, X a noetherian scheme.

(b) Let $\mathcal{A}$ be an exact category, and let $\mathcal{B} \subset \mathcal{A}$ be a full additive subcategory closed under extensions in $\mathcal{A}$. Call a sequence (1) in $\mathcal{B}$ a conflation if it is a conflation in $\mathcal{A}$. One checks that $\mathcal{B}$ equipped with this family of conflations is an exact category making $\mathcal{B}$ into a fully exact subcategory of $\mathcal{A}$. In particular, any extension closed subcategory of an abelian category is canonically an exact category.

(c) The category $R-$proj of finitely generated projective left R-modules is extension closed in the category of all R-modules. Similarly, the category $\mathrm{Vect}(X)$ of vector bundles (that is, locally free sheaves of finite rank) on a scheme X is extension closed in the category of all O_X-modules. In this way, we consider $R-$proj and $\mathrm{Vect}(X)$ as exact categories where a sequence is a conflation if it is in its ambient abelian category.

(d) An additive category can be made into an exact category by declaring a sequence (1) to be a conflation if it is isomorphic to a sequence of the form (2). Such exact categories are referred to as *split exact categories*.

(e) Let $\mathcal{E}$ be an exact category. We let $\mathrm{Ch}\,\mathcal{E}$ be the category of chain complexes in $\mathcal{E}$. Objects are sequences (A, d) :

$$\cdots \to A^{i-1} \stackrel{d^{i-1}}{\to} A^i \stackrel{d}{\to} A^{i+1} \to \cdots$$

of maps in $\mathcal{E}$ such that $d \circ d = 0$. A map $f : (A, d_A) \to (B, d_B)$ is a collection of maps $f^i : A^i \to B^i$, $i \in \mathbb{Z}$, such that $f \circ d_A = d_B \circ f$. A sequence $(A, d_A) \to (B, d_B) \to (C, d_C)$ is a conflation if $A^i \to B^i \to C^i$ is a conflation for all $i \in \mathbb{Z}$. This makes $\mathrm{Ch}\,\mathcal{E}$ into an exact category.

The full subcategory $\mathrm{Ch}^b\,\mathcal{E} \subset \mathrm{Ch}\,\mathcal{E}$ of bounded chain complexes ((A, d_A) is bounded if $A^i = 0$ for $i >> 0$ and $i << 0$) is a fully exact subcategory.

It turns out that the examples in 1.2 (c) are typical as the following lemma shows. The proof of the lemma can be found in [TT90, Appendix A] and [Kel90, Appendix A].

1.3. **Lemma.** *Every small exact category can be embedded into an abelian category as a fully exact subcategory.*

1.4. **Exercise.** Use the axioms 1.1 (a) – (f) of an exact category or lemma 1.3 above to show the following (and their duals).

(a) A cartesian square as in 1.1 (e) with p a deflation is also cocartesian. If, moreover, $X \to Z$ is is an inflation, then $W \to Y$ is also an inflation.

(b) If the composition ab of two maps in an exact category is an inflation, and if b has a cokernel, then b is also an inflation.

1.5. **Definition of K_0.** Let $\mathcal{E}$ be a small exact category. The *Grothendieck group* $K_0(\mathcal{E})$ of $\mathcal{E}$ is the abelian group freely generated by symbols $[X]$ for every object X of $\mathcal{E}$ modulo the relation $[Y] = [X] + [Z]$ for every conflation $X \to Y \to Z$. An exact functor $F : \mathcal{A} \to \mathcal{B}$ between exact categories induces a homomorphism $F : K_0(\mathcal{A}) \to K_0(\mathcal{B}) : [X] \mapsto [FX]$.

1.6. **Remark.** The conflation $0 \to 0 \to 0$ shows that $0 = [0]$ in $K_0(\mathcal{E})$. Let $X \to Y$ be an isomorphism, then we have a conflation $0 \to X \to Y$, and thus $[X] = [Y]$ in $K_0(\mathcal{E})$, so $K_0(\mathcal{E})$ is in fact generated by isomorphism classes of objects in $\mathcal{E}$. The split conflation 1.1 (2) shows that $[X \oplus Y] = [X] + [Y]$.

1.7. **Exercise.** Let $F : \mathcal{A} \to \mathcal{B}$ be an equivalence of small exact categories, that is, an exact functor which is an equivalence of categories and which detects conflations. Then F induces an isomorphism of K_0-groups. In particular, the group $K_0(\mathcal{E})$ makes sense for any essentially small (that is, equivalent to a small) exact category $\mathcal{E}$.

1.8. **Definition.** The groups $K_0(R)$, $K_0(X)$, and $G_0(X)$ are the Grothendieck groups of the (essentially small) exact categories $R-$proj (for R any ring), Vect(X) (for X a quasi-projective or separated regular noetherian scheme), and Coh(X) (for X a noetherian scheme) defined in 1.2.

In order to define higher K-groups, one constructs a topological space $K(\mathcal{E})$ and defines $K_i(\mathcal{E})$ as the homotopy groups $\pi_i K(\mathcal{E})$. We start with describing the space $K(\mathcal{E})$.

1.9. **Quillen's Q-construction.** Let $\mathcal{E}$ be a small exact category. We define a new category $Q\mathcal{E}$ as follows. The objects of $Q\mathcal{E}$ are the objects of $\mathcal{E}$. A map $X \to Y$ in $Q\mathcal{E}$ is an equivalence class of data $X \overset{p}{\twoheadleftarrow} W \overset{i}{\rightarrowtail} Y$ where p is a deflation and i an inflation. The datum (W, p, i) is equivalent to the datum (W', p', i') if there is an isomorphism $g : W \to W'$ such that $p = p'g$ and $i = i'g$. The composition of $(W, p, i) : X \to Y$ and $(V, q, j) : Y \to Z$ in $Q\mathcal{E}$ is the map $X \to Z$ represented by the datum $(U, p\bar{q}, j\bar{i})$ where U is the pull-back of q along i as in the diagram

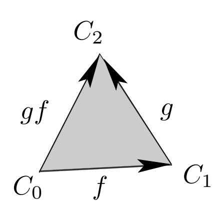
$$\begin{array}{ccccccc} X & \overset{p}{\twoheadleftarrow} & W & \overset{\bar{q}}{\twoheadleftarrow} & U & & \\ & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} & & \\ & & Y & \underset{q}{\twoheadleftarrow} & V & \underset{j}{\rightarrowtail} & Z \end{array}$$

which exists, by 1.1 (e). The map $\bar{q}$ (and hence $p\bar{q}$, by 1.1 (c)) is a deflation, by 1.1 (e), and the map $\bar{i}$ (and hence $j\bar{i}$) is an inflation, by 1.4 (a). The universal property of cartesian squares implies that composition is well-defined and associative. The identity map id_X of an object X of $Q\mathcal{E}$ is represented by the datum $(X, 1, 1)$.

1.10. **The classifying space of a category.** To any small category $\mathcal{C}$, one associates a topological space $B\mathcal{C}$, its classifying space. This is a CW-complex.

- Its 0-cells are the objects of $\mathcal{C}$.
- Its 1-cells are the non-identity morphisms attached to their source and target.
- Its 2-cells are the 2-simplices (see the figure below) corresponding to pairs (f, g) of composable morphisms such that neither f nor g is an identity morphism.

(f, g) :



The edges f, g and gf, which make up the boundary of the 2-simplex (f, g), are attached to the 1-cells corresponding to f, g, and gf. In case $gf = id_{C_0}$, the whole edge gf is identified with the 0-cell corresponding to C_0.

- Its 3-cells are the 3-simplices corresponding to triples $C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3$ of composable arrows such that none of the maps f_0, f_1, f_2 is an identity morphism. They are attached in a similar way as in the case of 2-cells, etc.

For a precise definition, see appendix A.

Since we have a category $Q\mathcal{E}$, we have a topological space $BQ\mathcal{E}$. We make the classifying space $BQ\mathcal{E}$ of $Q\mathcal{E}$ into a pointed topological space by choosing a 0-object of $\mathcal{E}$ as base-point.

To an object X of $\mathcal{E}$, we associate a loop $l_X = (0,0,0)^{-1}(X,0,1)$

$$l_X : \qquad 0 \underset{(X,0,1)}{\overset{(0,0,0)}{\rightrightarrows}} X$$

in $BQ\mathcal{E}$, and thus an element $[l_X]$ in $\pi_1 BQ\mathcal{E}$.

1.11. **Proposition.** *The assignment which sends an object X to the loop l_X induces a well-defined homomorphism of abelian groups $K_0(\mathcal{E}) \to \pi_1 BQ\mathcal{E}$ which is an isomorphism.*

Proof. In order to see that the assignment $[X] \mapsto [l_X]$ yields a well defined group homomorphism $K_0(\mathcal{E}) \to \pi_1 BQ\mathcal{E}$, we observe that we could have defined $K_0(\mathcal{E})$ as the free group generated by symbols $[X]$, $X \in \mathcal{E}$, modulo the relation $[Y] = [X][Z]$ for any conflation $X \xrightarrow{i} Y \xrightarrow{p} Z$ (commutativity is forced by axiom 1.1 (f)). So we have to check the relation $[l_Y] = [l_X][l_Z]$ in $\pi_1 BQ\mathcal{E}$ (alternatively, direct sum operation makes $BQ\mathcal{E}$ into a connected H-space, hence an H-group, so that $\pi_1 BQ\mathcal{E}$ is an abelian group [Whi78, III.4.17]). The loops l_X and l_Z are homotopic to the loops

$$0 \underset{(X,0,1)}{\overset{(0,0,0)}{\rightrightarrows}} X \xrightarrow{(X,1,i)} Y \qquad \text{and} \qquad 0 \underset{(Z,0,1)}{\overset{(0,0,0)}{\rightrightarrows}} Z \xrightarrow{(Y,p,1)} Y \qquad \text{which are}$$

$$0 \underset{(X,0,i)}{\overset{(0,0,0)}{\rightrightarrows}} Y \qquad \text{and} \qquad 0 \underset{(Y,0,1)}{\overset{(X,0,i)}{\rightrightarrows}} Y. \qquad \text{Therefore,}$$

$$[l_X][l_Z] = [(0,0,0)^{-1}(X,0,i)][(X,0,i)^{-1}(Y,0,1)] = [(0,0,0)^{-1}(Y,0,1)] = [l_Y],$$

and the map $K_0(\mathcal{E}) \to \pi_1 BQ\mathcal{E}$ is well-defined.

It is easy to see that $K_0(\mathcal{E}) \to \pi_1 BQ\mathcal{E}$ is surjective. To show injectivity, we construct a map $\pi_1 BQ\mathcal{E} \to K_0(\mathcal{E})$ such that the composition $K_0(\mathcal{E}) \to K_0(\mathcal{E})$ is the identity. To this end, we introduce a little notation. For a group G, we let $\underline{G}$ be the category with one object $*$, and $\mathrm{Hom}(*,*) = G$. Recall from appendix A.5 that $\pi_i B\underline{G} = 0$ for $i \neq 1$ and $\pi_1 B\underline{G} = G$ where the isomorphism $G \to \pi_1 B\underline{G}$ sends an element $g \in G$ to the loop l_g represented by the morphism $g : * \to *$. In order to obtain a map $\pi_1 BQ\mathcal{E} \to K_0(\mathcal{E})$, we construct a functor $F : Q\mathcal{E} \to \underline{K_0(\mathcal{E})}$. The functor sends an object X of $Q\mathcal{E}$ to the object $*$ of $\underline{K_0(\mathcal{E})}$. A map $(W,p,i) : X \to Y$ in $Q\mathcal{E}$ is sent to the map represented by the element $[\ker(p)] \in K_0(\mathcal{E})$. For a composition in the notation of 1.9, we have $F[(V,q,j)\circ(W,p,i)] = F(U, p\bar{q}, j\bar{i}) = [\ker(p\bar{q})] = [\ker(\bar{q})] + [\ker(p)] = [\ker(q)] + [\ker(p)] = F(V,q,j) \circ F(W,p,i)$, because there is a conflation $\ker(\bar{q}) \to \ker(p\bar{q}) \to \ker(p)$ (5-lemma), and

$\ker(\bar{q}) = \ker(q)$ (universal property of pull-backs). So F is a functor, and induces a map on fundamental groups of classifying spaces $\pi_1 BQ\mathcal{E} \to \pi_1 \underline{K_0(\mathcal{E})} = K_0(\mathcal{E})$. It is easy to check that the composition $K_0(\mathcal{E}) \to K_0(\mathcal{E})$ is the identity. □

1.12. **Definition of $K(\mathcal{E})$.** Let $\mathcal{E}$ be a small exact category. Its K-theory space is the topological space

$$K(\mathcal{E}) = \Omega BQ\mathcal{E}.$$

Its K-groups are the homotopy groups $K_i(\mathcal{E}) = \pi_i K(\mathcal{E}) = \pi_{i+1} BQ\mathcal{E}$ of its K-theory space. An exact functor $\mathcal{E} \to \mathcal{E}'$ induces a functor $Q\mathcal{E} \to Q\mathcal{E}'$ on Q-constructions, and thus continuous maps $BQ\mathcal{E} \to BQ\mathcal{E}'$ and $K(\mathcal{E}) \to K(\mathcal{E}'$, compatible with composition of exact functors. The K-theory space and the K-groups are thus functorial with respect to exact functors between small exact categories. By Proposition 1.11, the group $K_0(\mathcal{E})$ defined in this way coincides with the group defined in 1.5.

For a ring R and a quasi-projective (or separated regular noetherian) scheme X, the K-theory spaces $K(R)$ and $K(X)$ are the K-theory spaces associated with (a small model of) the exact categories $R-$proj of finitely generated projective R-modules and $\mathrm{Vect}(X)$ of vector bundles on X. For a noetherian scheme X, its G-theory space $G(X)$ is the K-theory space associated with (a small model of) the abelian category $\mathrm{Coh}(X)$ of coherent O_X-modules.

1.13. **Remark.** An equivalence of small exact categories induces an equivalence of associated Q-constructions, and thus a homotopy equivalence of associated K-theory spaces (see A.6). Therefore, changing the small models of $R-$proj, $\mathrm{Vect}(X)$ and $\mathrm{Coh}(X)$ in the definition of $K(R)$, $K(X)$ and $G(X)$ results in homotopy equivalent K-theory spaces and isomorphic K-groups.

In order to reconcile the definition of $K(R)$ given above with the plus-construction of Cortiñas' lecture, we cite the following theorem of Quillen, a proof of which can be found in [Gra76].

1.14. **Theorem ($Q = +$).** *There is a natural homotopy equivalence*

$$BGL(R)^+ \simeq \Omega_0 BQ(R-\mathrm{proj}),$$

where Ω_0 stands for the connected component of the constant loop. In particular, there are natural isomorphisms for $i \geq 1$

$$\pi_i BGL(R)^+ \cong \pi_{i+1} BQ(R-\mathrm{proj}).$$

1.15. **Warning.** Some authors define $K(R)$ to be $K_0(R) \times BGL(R)^+$ as functors in R. Strictly speaking, this is wrong; there is no zig-zag of homotopy equivalences between $K_0(R) \times BGL(R)^+$ and $\Omega BQ(R-\mathrm{proj})$ which is functorial in R.

1.16. **Exact sequences of abelian categories.** Let $\mathcal{A}$ be an abelian category. A *Serre subcategory* of $\mathcal{A}$ is a fully exact subcategory $\mathcal{B} \subset \mathcal{A}$ of $\mathcal{A}$ which is closed under taking subobjects and quotient objects in $\mathcal{A}$. In particular, $\mathcal{B}$ is itself an abelian category. In this situation, one can (up to set theoretical issues which don't exist when $\mathcal{A}$ is small) construct a quotient abelian category $\mathcal{A}/\mathcal{B}$ which has the universal property of a quotient object in the category of exact categories. The quotient abelian category $\mathcal{A}/\mathcal{B}$ is equivalent to the localization $\mathcal{A}[S^{-1}]$ of $\mathcal{A}$ w.r.t. the set S of morphisms f in $\mathcal{A}$ for which $\ker(f)$ and $\mathrm{coker}(f)$ are isomorphic to objects in $\mathcal{B}$. The set S satisfies a calculus of fractions so that $\mathcal{A}[S^{-1}]$ has a very explicit description, see C.3. We may call $\mathcal{B} \to \mathcal{A} \to \mathcal{A}/\mathcal{B}$ an "exact sequence of abelian categories". More details can be found in [Gab62], [Pop73].

The following two theorems are proved in [Qui73, §5 Theorem 5] and [Qui73, §5 Theorem 4].

1.17. **Theorem** (Quillen's Localization Theorem). *Let $\mathcal{A}$ be a small abelian category, and let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Then the sequence*

$$BQ(\mathcal{B}) \to BQ(\mathcal{A}) \to BQ(\mathcal{A}/\mathcal{B})$$

is a homotopy fibration (see appendix B for a definition). In particular, there is a long exact sequence

$$\cdots \to K_{n+1}(\mathcal{A}) \to K_{n+1}(\mathcal{A}/\mathcal{B}) \to K_n(\mathcal{B}) \to K_n(\mathcal{A}) \to K_n(\mathcal{A}/\mathcal{B}) \to \cdots$$
$$\cdots \to K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B}) \to 0.$$

1.18. **Theorem** (Dévissage). *Let $\mathcal{A}$ be a small abelian category, and $\mathcal{B} \subset \mathcal{A}$ full abelian subcategory such that the inclusion $\mathcal{B} \subset \mathcal{A}$ is exact. Assume that every object A of $\mathcal{A}$ has a finite filtration*

$$0 = A_0 \subset A_1 \subset ... \subset A_n = A$$

such that the quotients A_i/A_{i-1} are in $\mathcal{B}$. Then the inclusion $\mathcal{B} \subset \mathcal{A}$ induces a homotopy equivalence

$$K(\mathcal{B}) \xrightarrow{\sim} K(A).$$

In particular, it induces an isomorphism of K-groups $K_i(\mathcal{B}) \cong K_i(\mathcal{A})$.

The following are two applications of Quillen's localization and devissage theorems.

1.19. **Nilpotent extensions.** Let X be a noetherian scheme, and $i : Z \hookrightarrow X$ a closed subscheme corresponding to a nilpotent sheaf of ideals $I \subset O_X$. Assume $I^n = 0$. Then $i_* : \mathrm{Coh}(Z) \to \mathrm{Coh}(X)$ satisfies the hypothesis of the *dévissage* theorem, since $\mathrm{Coh}(Z)$ can be identified with the subcategory of those coherent sheaves F on X for which $IF = 0$, and every sheaf $F \in \mathrm{Coh}(X)$ has a filtration $0 = I^nF \subset I^{n-1}F \subset ... \subset IF \subset F$ with quotients in $\mathrm{Coh}(Z)$. We conclude that i_* induces a homotopy equivalence $G(Z) \simeq G(X)$. In particular,

$$G(X) \simeq G(X_{red}).$$

1.20. **G-theory localization.** Let X be a noetherian scheme, and $j : U \subset X$ an open subscheme with $i : Z \subset X$ its closed complement $X - U$. Let $\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X)$ be the (fully exact) subcategory of those coherent sheaves on X which have support in Z, that is, for which $F_{|U} = 0$. Then the sequence

$$\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X) \xrightarrow{j^*} \mathrm{Coh}(U) \tag{3}$$

is an exact sequence of abelian categories (see below). By theorem 1.17, we obtain a homotopy fibration $K\,\mathrm{Coh}_Z(X) \to K\,\mathrm{Coh}(X) \to K\,\mathrm{Coh}(Z)$. Moreover, the inclusion $i_* : \mathrm{Coh}(Z) \subset \mathrm{Coh}_Z(X)$ satisfies *dévissage*, so that we have a homotopy equivalence $K\,\mathrm{Coh}(Z) \simeq K\,\mathrm{Coh}_Z(X)$. Put together, we obtain a homotopy fibration

$$G(Z) \xrightarrow{i_*} G(X) \xrightarrow{j^*} G(U)$$

and an associated long exact sequence of G-theory groups.

Proof (that (3) is an exact sequence of abelian categories). As the "kernel" of the exact functor $\mathrm{Coh}(X) \to \mathrm{Coh}(U)$, the category $\mathrm{Coh}_Z(X)$ is obviously a Serre subcategory of $\mathrm{Coh}(X)$. Moreover, the composition $\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X) \to \mathrm{Coh}(U)$ is trivial, so that we obtain an induced functor $\mathrm{Coh}(X)/\mathrm{Coh}_Z(X) \to \mathrm{Coh}(U)$ which we have to show is an equivalence.

The functor is essentially surjective on objects, since for any $F \in \mathrm{Coh}(U)$, the O_X-module j_*F is quasi-coherent (because X is noetherian), and thus it is a filtered colimit $\mathrm{colim}\, G_i$ of its coherent sub-O_X-modules G_i. Every ascending chain of subobjects of a coherent sheaf eventually stops, so that we must have $j^*G_i \cong j^*j_*F = F$ for some i.

The functor is full, because for $F, G \in \mathrm{Coh}(X)$, any map $f : j^*F \to j^*G$ in $\mathrm{Coh}(U)$ is the image of the fraction $F \overset{t}{\leftarrow} H \to G$ which which we can take to be the pull-back of $F \to j_*j^*F \xrightarrow{j_*f} j_*j^*G \leftarrow G$. The unit of adjunction $G \to j_*j^*G$ has kernel and cokernel in $\mathrm{Coh}_Z(X)$, hence the same is true for its pull back $t : F \to H$. Moreover, H is coherent as it is a quasi-coherent subsheaf of the coherent sheaf $F \oplus G$.

By construction, the "kernel category" of $\mathrm{Coh}(X)/\mathrm{Coh}_Z(X) \to \mathrm{Coh}(U)$ is trivial. This implies that the functor is conservative (that is, detects isomorphisms). Let $f : F \to G$ be a map in $\mathrm{Coh}(X)/\mathrm{Coh}_Z(X)$ such that $j^*(f) = 0$. Then $\ker(f) \to F$ and $G \to \mathrm{coker}(f)$ are isomorphisms in $\mathrm{Coh}(U)$, hence are isomorphisms themselves, so that $f = 0$. It follows that the functor is fully faithful. Since it is also full and essentially surjective, it has to be an equivalence. □

1.21. **Theorem** (Homotopy invariance of G-theory [Qui73, Proposition 4.1]). *Let X and P be a noetherian schemes and $f : P \to X$ a flat map whose fibres are affine spaces (for instance, a geometric vector bundle). Then*

$$f^* : G(X) \xrightarrow{\sim} G(P)$$

is a homotopy equivalence. In particular, $G_i(X \times \mathbb{A}^1) \cong G_i(X)$.

Proof (sketch). Using noetherian induction, 1.19 and 1.20, one reduces to X an integral noetherian scheme. By a limit argument, one reduces the claim to $X = \mathrm{Spec}\, k$, and $P = \mathrm{Spec}(k[T_1, ..., T_n])$, k a field. The homotopy equivalence $G(k) \to G(k(T_1, ..., T_n))$ is treated separately [Qui73, §6 theorem 8]. □

1.22. **Remark.** Besides the theorems mentioned above, Quillen proves further fundamental theorems among which "Additivity" [Qui73, §3 Theorem 2 and Corollary 1] and "Resolution" [Qui73, §4 Theorem 3]. Both can – a posteriori – be deduced from Thomason's Localization Theorem 2.21 (see [Kel99]), so we will refrain from stating them here.

1.23. **K-theory of regular schemes.** Let X be a regular noetherian and separated scheme, then the inclusion $\mathrm{Vect}(X) \subset \mathrm{Coh}(X)$ induces a homotopy equivalence $K\,\mathrm{Vect}(X) \simeq K\,\mathrm{Coh}(X)$, that is, $K(X) \simeq G(X)$ (see Poincaré duality 3.1, it also follows more classically from Quillen's resolution theorem). Thus, 1.20 and 1.21 translate into theorems about $K(X)$ when X is regular, noetherian and separated.

Besides the groups K_i, $i \geq 0$, one can also define groups K_i, $i < 0$, which extend certain K_0 exact sequences to the right (see Cortiñas lecture). For rings they were introduced by Bass [Bas68] and Karoubi [Kar68]. The treatment for exact categories below follows [Sch04].

1.24. **Idempotent completion.** Let $\mathcal{A}$ be an exact category, and $\mathcal{B} \subset \mathcal{A}$ a fully exact subcategory. We call the inclusion $\mathcal{B} \subset \mathcal{A}$ *cofinal* if every object of $\mathcal{A}$ is a direct factor of an object of $\mathcal{B}$. For instance, the category of (finitely generated) free R-modules is cofinal in the category of (finitely generated) projective R-modules.

Given an additive category $\mathcal{A}$, there is a "largest" category $\tilde{\mathcal{A}}$ of $\mathcal{A}$ such that the inclusion $\mathcal{A} \subset \tilde{\mathcal{A}}$ is cofinal. We call $\tilde{\mathcal{A}}$ the idempotent completion of $\mathcal{A}$ (an additive category is called *idempotent complete* if every idempotent map $p = p^2 : A \to A$ has an image, that is, is isomorphic to $\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right) : X \oplus Y \to X \oplus Y$). The objects of $\tilde{\mathcal{A}}$ are pairs (A, p) with A an object of $\mathcal{A}$ and $p = p^2 : A \to A$ an idempotent endomorphism. Maps $(A,p) \to (B,q)$ are the maps $f : A \to B$ such that $fp = f = qf$. Composition is composition in $\mathcal{A}$, and $id_{(A,p)} = p$. We have a fully faithful embedding $\mathcal{A} \subset \tilde{\mathcal{A}} : A \mapsto (A, 1)$. Every idempotent $q = q^2 : (A,p) \to (A,p)$ has an image in $\tilde{\mathcal{A}}$, namely (A,q), so that $\tilde{\mathcal{A}}$ is indeed idempotent complete.

If $\mathcal{E}$ is an exact category, its idempotent completion $\tilde{\mathcal{E}}$ becomes an exact category when we declare a sequence in $\tilde{\mathcal{E}}$ to be a conflation if it is a direct factor of a conflation of $\mathcal{E}$. For more details, see [TT90, Appendix A].

1.25. **Proposition** (Cofinality [Gra79, Theorem 1.1]). *Let $\mathcal{A}$ be an exact category and $\mathcal{B} \subset \mathcal{A}$ be a cofinal fully exact subcategory. Then the maps $K_i(\mathcal{B}) \to K_i(\mathcal{A})$ are isomorphisms for $i > 0$ and a monomorphism for $i = 0$. This holds in particular for $K_i(\mathcal{E}) \to K_i(\tilde{\mathcal{E}})$.*

1.26. **Negative K-theory and the spectrum $\mathbb{K}(\mathcal{E})$.** To any exact category $\mathcal{E}$, one can associate a new exact category $S\mathcal{E}$ (see 1.28, called suspension of $\mathcal{E}$, such that there is a natural homotopy equivalence [Sch04]

$$K(\tilde{\mathcal{E}}) \xrightarrow{\sim} \Omega K(S\mathcal{E}). \tag{4}$$

If $\mathcal{E} = R-\mathrm{proj}$ one can take $S\mathcal{E} = (SR)-\mathrm{proj}$ where SR is the suspension ring of R (see Cortiñas' lecture).

One uses the suspension construction to slightly modify the definition of algebraic K-theory in order to incorporate negative K-groups as follows. One sets $\mathbb{K}_i(\mathcal{E}) = K_i(\mathcal{E})$ for $i \geq 1$, $\mathbb{K}_0(\mathcal{E}) = K_0(\tilde{\mathcal{E}})$, and $\mathbb{K}_i(\mathcal{E}) = K_0(\widetilde{S^{-i}\mathcal{E}})$, $i < 0$. Since $\mathrm{Vect}(X)$, $\mathrm{Coh}(X)$, and $R-\mathrm{proj}$ are all idempotent complete, we have $\mathbb{K}_0\,\mathrm{Vect}(X) = K_0\,\mathrm{Vect}(X) = K_0(X)$, $\mathbb{K}_0\,\mathrm{Coh}(X) = K_0\,\mathrm{Coh}(X) = G_0(X)$, and $\mathbb{K}_0(R-\mathrm{proj}) = K_0(R-\mathrm{proj}) = K_0(R)$, so that in these cases, we have not changed the definition of K-theory, and we have merely introduced "negative K-groups" $\mathbb{K}_i$, $i < 0$. That's why, we may write $K_i(X)$, and $K_i(R)$ instead of $\mathbb{K}_i(X)$ and $\mathbb{K}_i(R)$, $i \in \mathbb{Z}$.

In fancy language, one constructs a spectrum $I\!K(\mathcal{E})$ whose homotopy groups are the groups $I\!K_i(\mathcal{E})$, $i \in \mathbb{Z}$. Its n-th space is $K(S^n\mathcal{E})$ with structure maps given by (4).

It turns out (see Bass' fundamental theorem 3.10 below) that there is a split exact sequence

$$0 \to K_i(R) \to K_i(R[T]) \oplus K_i(R[T^{-1}]) \to K_i(R[T, T^{-1}]) \to K_{i-1}(R) \to 0, \quad i \in \mathbb{Z}.$$

One can use the exact sequence to give a recursive definition of the groups $K_i(R)$, $i < 0$, starting with the functor K_0. This was Bass' original way of defining $K_i(R)$, $i < 0$.

1.27. **Remark.** Not much is known about $I\!K_i(\mathcal{E})$, $i < 0$, though their calculation should be easier than that of $K_i(\mathcal{E})$, $i \geq 0$. However, we do know the following. $K_i(R) = 0$, $i < 0$, for R a regular noetherian ring [Bas68]. $I\!K_{-1}(\mathcal{A}) = 0$ for any abelian category $\mathcal{A}$ [Sch06, Theorem 6], and $I\!K_{-1}(\mathcal{A}) = 0$, $i < 0$, for $\mathcal{A}$ a noetherian abelian category [Sch06, Theorem 7]. In particular, $K_{-1}(R) = 0$ for a regular coherent ring R, and $K_{-i}(X) = 0$, $i < 0$ for any regular noetherian and separated scheme X. In [CHSW05] it is shown that $K_i(X) = 0$, $i < -d$, for X a d-dimensional scheme essentially of finite type over a field of characteristic 0, and $K_{-d}(X) = H^d_{cdh}(X, \mathbb{Z})$ which may be non-zero [Rei87]. Similar statements are conjectured to be true in arbitrary and mixed characteristic [Wei80]. It is also conjectured that $K_i(\mathbb{Z}G) = 0$, $i < -1$, and G a finitely presented group [Hsi84]. For results in this direction, see [LR05].

1.28. **Construction of the suspension $S\mathcal{E}$.** Let $\mathcal{E}$ be an exact category. The countable envelope $\mathcal{F}\mathcal{E}$ of $\mathcal{E}$ is an exact category whose objects are sequences $A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow ...$ of inflations in $\mathcal{E}$. The morphism set from a sequence A_* to B_* is $\lim_i \operatorname{colim}_j \operatorname{Hom}_{\mathcal{E}}(A_i, B_j)$. A sequence in $\mathcal{F}\mathcal{E}$ is a conflation iff it is isomorphic (in $\mathcal{F}\mathcal{E}$) to the maps of sequences $A_* \to B_* \to C_*$ with $A_i \to B_i \to C_i$ a conflation in $\mathcal{E}$. Colimits of sequences of inflations exists in $\mathcal{F}\mathcal{E}$ and are exact. In particular, $\mathcal{F}\mathcal{E}$ has exact countable direct sums. There is a fully faithful exact functor $\mathcal{E} \to \mathcal{F}\mathcal{E}$ which sends an object $X \in \mathcal{E}$ to the constant sequence $X \xrightarrow{1} X \xrightarrow{1} X \xrightarrow{1} \cdots$. For details, see [Kel90, Appendix B] (where $\mathcal{F}\mathcal{E}$ was denoted $\mathcal{E}^{\sim}$).

The suspension $S\mathcal{E}$ of $\mathcal{E}$ is the quotient $\mathcal{F}\mathcal{E}/\mathcal{E}$ (in the category of small exact categories) of the countable envelope $\mathcal{F}\mathcal{E}$ of $\mathcal{E}$ by the subcategory $\mathcal{E}$. The proof of the existence of the quotient $\mathcal{F}\mathcal{E}/\mathcal{E}$ and an explicit description is given in [Sch04]. One shows that there is a homotopy fibration $K(\tilde{\mathcal{E}}) \to K(\mathcal{F}\mathcal{E}) \to K(S\mathcal{E})$ of K-theory spaces [Sch04]. Since $\mathcal{F}\mathcal{E}$ has exact countable direct sums, $K(\mathcal{F}\mathcal{E}) \simeq 0$, which implies the homotopy equivalence 1.26 (4).

2. Algebraic K-theory of complicial exact categories

Most calculations in the early days of K-theory were based on Quillen's localization theorem 1.17 for abelian categories (together with dévissage 1.18). Unfortunately, not all K-groups are (not even equivalent to) the K-groups associated to some abelian category, notably $K(X)$ where X is some singular variety. Also, there is no satisfactory generalization of Quillen's localization theorem to exact categories which would apply to all situations K-theorists had in mind.

Here is where triangulated categories come in. They provide a flexible framework within which to state theorems which allowed to prove many results that cannot be proved with Quillen's methods.

We start this section with the definition of the Grothendieck group of a triangulated category, and recall two basic properties in exercise 2.3 and 2.7 which motivate their generalizations to higher K-theory in theorems 2.16 and 2.21.

Unfortunately, there is no good definition of higher algebraic K-theory of triangulated categories [Sch02] (see however the work of Neeman [Nee05], Breuning [Bre]). That's why we have to work with categories carrying more structure. These are the "complicial exact categories with weak equivalences". They are close enough to triangulated categories, and they allow a good definition of higher algebraic K-theory for which the analog of Quillen's localization theorem holds. Their definition is in 2.13 and the analog is in 2.21.

At this point, the reader is advised to be acquainted with some background on triangulated categories which is summarized in appendix C.1 – 2.2.

2.1. **Definition of $K_0(\mathcal{T})$.** Let $\mathcal{T}$ be a small triangulated category. The group $K_0(\mathcal{T})$ is the abelian group freely generated by symbols $[X]$, where X is an object of $\mathcal{T}$, modulo the relation $[X] + [Z] = [Y]$ for every distinguished triangle $X \to Y \to Z \to TX$ in $\mathcal{T}$.

As in remark 1.6, one shows that $[X] = [Y]$ if there is an isomorphism $X \cong Y$. We also have $[X \oplus Y] = [X] + [Y]$. Moreover, the distinguished triangle $X \to 0 \to TX \to TX$ shows that $[TX] = -[X]$. In particular, every element in $K_0(\mathcal{T})$ can be represented as $[X]$ for some object X in $\mathcal{T}$.

2.2. **The bounded derived category $D^b\mathcal{E}$.** A standard example of a triangulated category is the bounded derived category $\mathcal{D}^b(\mathcal{E})$ of an exact category $\mathcal{E}$, see for example [Kel96]. Let $\mathrm{Ch}^b\,\mathcal{E}$ be the exact category of bounded chain complexes in $\mathcal{E}$, see example 1.2 (e). Call a bounded chain complex (A, d) in $\mathcal{E}$ *strictly acyclic* if every differential $d^i : A^i \to A^{i+1}$ can be factored into $A^i \to Z^{i+1} \to A^{i+1}$ such that $Z^i \to A^i \to Z^{i+1}$ is a conflation in $\mathcal{E}$, $i \in \mathbb{Z}$. A bounded chain complex is called *acyclic* if it is homotopy equivalent to a strictly acyclic chain complex. A map $f : (A, d) \to (B, d)$ is called *quasi-isomorphism* if its cone $C(f)$ (C.2) is acyclic.

As a category, the bounded derived category $\mathcal{D}^b(\mathcal{E})$ is $\mathrm{quis}^{-1}\,\mathrm{Ch}^b\,\mathcal{E}$ – the category obtained from the category of bounded chain complexes $\mathrm{Ch}^b\,\mathcal{E}$ by formally inverting quasi-isomorphisms.

A more explicit description of $\mathcal{D}^b(\mathcal{E})$ is obtained as follows. Let $\mathcal{K}^b(\mathcal{E})$ be the homotopy category of bounded chain complexes in $\mathcal{E}$. Its objects are bounded chain complexes in $\mathcal{E}$, and maps are chain maps up to chain homotopy. With the same definitions as in C.2, the homotopy category $\mathcal{K}^b(\mathcal{E})$ is a triangulated category.

Let $\mathcal{K}^b_{ac}(\mathcal{E}) \subset \mathcal{K}^b(\mathcal{E})$ be the full subcategory of acyclic chain complexes. The category $\mathcal{K}^b_{ac}(\mathcal{E})$ is closed under taking cones and shifts T, T^{-1} in $\mathcal{K}^b(\mathcal{E})$, and is therefore a full triangulated subcategory of $\mathcal{K}^b(\mathcal{E})$. The bounded derived category of the exact category $\mathcal{E}$ is the Verdier quotient $\mathcal{K}^b(\mathcal{E})/\mathcal{K}^b_{ac}(\mathcal{E})$.

An inflation $X \stackrel{i}{\to} Y \stackrel{p}{\to} Z$ of chain complexes in $\mathrm{Ch}^b\,\mathcal{E}$ yields a canonical triangle

$$X \xrightarrow{i} Y \xrightarrow{p} Z \xrightarrow{q \circ s^{-1}} TX$$

in $D^b(\mathcal{E})$, where s is the quasi-isomorphism $C(i) \to C(i)/C(id_X) \cong Z$ and q is the canonical map $C(f) \to TX$ as in C.2.

2.3. **Exercise.** Let $\mathcal{E}$ be an exact category, and $\mathcal{D}^b\mathcal{E}$ its bounded derived category, see appendix 2.2. Consider objects of $\mathcal{E}$ as chain complexes concentrated in degree zero. Show that the map $K_0(\mathcal{E}) \to K_0(D^b\mathcal{E}) : [X] \mapsto [X]$ is an isomorphism. *Hint*: The inverse $K_0(D^b\mathcal{E}) \to K_0(\mathcal{E})$ is given by $[A, d] \mapsto \Sigma_i(-1)^i[A^i]$. The point is to show that this map is well-defined.

2.4. **Definition.** A sequence of triangulated categories $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is called *exact* if the composition sends $\mathcal{A}$ to 0, $\mathcal{A} \to \mathcal{B}$ is fully faithful and coincides (up to equivalence) with the subcategory of those objects in $\mathcal{B}$ which are zero in $\mathcal{C}$, and if the induced map $\mathcal{B}/\mathcal{A} \to \mathcal{C}$ from the Verdier quotient $\mathcal{B}/\mathcal{A}$ to $\mathcal{C}$ is an equivalence.

2.5. **Example.** Let R be a ring with unit, and let $S \subset R$ a multiplicative set of central non zero divisors in R. Let $\mathcal{H}_S(R)$ be the exact category of finitely presented S-torsion left R-modules of projective dimension at most 1. It is an extension closed full subcategory of the category of all left R-modules, and we therefore consider it as an exact category. Let $\mathcal{P}^1(R) \subset R$-Mod be the full subcategory of those left R-modules M which fit into an exact sequence $0 \to P \to M \to H \to 0$ of R-modules with P finitely generated projective, and $H \in \mathcal{H}_S(R)$. The inclusion $\mathcal{P}^1(R) \subset R$-Mod is closed under extensions which makes $\mathcal{P}^1(R)$ into a fully exact subcategory of R-Mod. Let $\mathcal{P}'(S^{-1}R) \subset S^{-1}R-$proj be the full subcategory of those finitely generated projective $S^{-1}R$-modules which are localizations of finitely generated projective R-modules. Then the sequence $\mathcal{H}_S(R) \to \mathcal{P}^1(R) \to \mathcal{P}'(S^{-1}R)$ induces an exact sequence of associated bounded derived categories. Moreover, the inclusion $R-$proj $\subset \mathcal{P}^1(R)$ induces an equivalence of bounded derived categories. In summary, we have an exact sequence of triangulated categories

$$\mathcal{D}^b(\mathcal{H}_S(R)) \to \mathcal{D}^b(R\text{--proj}) \to \mathcal{D}^b(\mathcal{P}'(S^{-1}R)). \tag{5}$$

For instance, let R be a Dedekind ring, and $S \subset R$ the set of non-zero elements. Then $S^{-1}R = K$, the field of fractions of R, $\mathcal{P}'(S^{-1}R)$ the category of finitely generated K-vector spaces, and $\mathcal{H}_S(R) \subset R$-Mod is the category of finitely generated torsion R-modules.

2.6. **Two useful criteria.** Let $\mathcal{A} \to \mathcal{B}$ be an exact functor between exact categories.

(a) If $\mathcal{B}$ is obtained from $\mathcal{A}$ by a calculus of left (or right) fractions, then $\mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{B})$ is a localization. This shows for instance, that in the diagram 2.5 (5), the category $\mathcal{D}^b(\mathcal{P}'(S^{-1}R))$ is a quotient triangulated category of $\mathcal{D}^b(R-$proj$)$.

(b) Suppose that $\mathcal{A}$ is a fully exact subcategory of $\mathcal{B}$. If for any inflation $A \to B$ in $\mathcal{B}$ with $A \in \mathcal{A}$, there is a map $B \to A'$ with $A' \in \mathcal{A}$ such that the composition $A \to A'$ is an inflation in $\mathcal{A}$, then the functor $\mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{B})$ is fully faithful [Kel96, 12.1]. This shows that in the diagram 2.5 (5), the left hand map is fully faithful.

2.7. **Exercise.** Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be an exact sequence of triangulated categories. Then the following sequence of abelian groups is exact

$$K_0(\mathcal{A}) \to K_0(\mathcal{B}) \to K_0(\mathcal{C}) \to 0. \tag{6}$$

Hint: Show that the map $K_0(\mathcal{C}) \to \mathrm{coker}(K_0(\mathcal{A}) \to K_0(\mathcal{B})) : [C] \mapsto [B]$ is well-defined, where $B \in \mathcal{B}$ is such that its image in $\mathcal{C}$ is isomorphic to C.

We would like to extend the exact sequence 2.7 (6) to the left. To this end, we need to introduce more structure.

2.8. **Definition.** An *exact category with weak equivalences* is an exact category $\mathcal{E}$ together with a set $w \subset \operatorname{Mor}\mathcal{E}$ of morphisms in $\mathcal{E}$. Morphisms in w are called *weak equivalences*. The set of weak equivalences is required to contain all identity morphisms; to be closed under isomorphisms, retracts, push-outs along inflations, pull-backs along deflations, composition; and to satisfy the "two out of three" property for composition (if two of the three maps among a, b, ab are weak equivalences, then so is the third).

2.9. **Example.** Let $\mathcal{E}$ be an exact category. The exact category $\mathrm{Ch}^b\,\mathcal{E}$ of bounded chain complexes in $\mathcal{E}$ of example 1.2 (e) together with the set quis of quasi-isomorphisms (as defined in 2.2) is an exact category with weak equivalences, $\mathbf{Ch}^b\,\mathcal{E} = (\mathrm{Ch}^b\,\mathcal{E}, \mathrm{quis})$.

2.10. **Notation.** We write $\mathrm{Ch}^b(\mathbb{Z})$ for the exact category with weak equivalences of bounded chain complexes of finitely generated free $\mathbb{Z}$-modules, see example 2.9. A quasi-isomorphism here is just an ordinary chain homotopy equivalence. A sequence here is a conflation if it splits in each degree (that is, is isomorphic to the sequence 1.1 (2).

There is a symmetric monoidal tensor product $\otimes : \mathrm{Ch}^b(\mathbb{Z}) \times \mathrm{Ch}^b(\mathbb{Z}) \to \mathrm{Ch}^b(\mathbb{Z})$ which extends the usual tensor product of free $\mathbb{Z}$-modules [Wei94, 2.7.1]. The unit of the tensor product is the chain complex $\mathbb{1}$ which is $\mathbb{Z}$ in degree 0 and 0 elsewhere.

Besides $\mathbb{1}$, we have two other distinguished objects in $\mathrm{Ch}^b(\mathbb{Z})$. The complex C is $\mathbb{Z}$ in degrees 0 and -1, and is 0 otherwise. The only non-trivial differential is $d^{-1} = id_{\mathbb{Z}}$. The complex T is $\mathbb{Z}$ in degree -1 and 0 elsewhere. Note that there is a short exact sequences of chain complexes $0 \to \mathbb{1} \to C \to T \to 0$.

2.11. **Definition.** An exact category $\mathcal{E}$ is called *complicial* if it is equipped with a bi-exact tensor product

$$\otimes : \mathrm{Ch}^b(\mathbb{Z}) \times \mathcal{E} \to \mathcal{E} \tag{7}$$

which is associative and unital in the sense that there are natural isomorphisms $A \otimes (B \otimes X) \cong (A \otimes B) \otimes X$ and $\mathbb{1} \otimes X \cong X$ such that a pentagonal and a triangular diagram [ML98, VII.1.] commute. In other words, a complicial exact category is an exact category $\mathcal{E}$ equipped with a bi-exact action of the symmetric monoidal category $\mathrm{Ch}^b(\mathbb{Z})$ on $\mathcal{E}$, see also [Gra76, p. 218] for actions of monoidal categories.

For an object X of $\mathcal{E}$, we write CX and TX instead of $C \otimes X$ and $T \otimes X$. Note that there is a functorial conflation $X \to CX \to TX$ which is the tensor product of $\mathbb{1} \to C \to T$ with X. For a map $f : X \to Y$ in $\mathcal{E}$, we write $C(f)$ for the push-out of f along the inflation $X \to CX$, and call it the cone of f. As a push-out of an inflation, $Y \to C(f)$ is also an inflation with cokernel TX. In particular, the following is a conflation in $\mathcal{E}$

$$Y \to C(f) \to TX.$$

2.12. **Example.** Write $F(\mathbb{Z})$ for the category of finitely generated free $\mathbb{Z}$-modules. So we have $\mathrm{Ch}^b(\mathbb{Z}) = \mathrm{Ch}^b F(\mathbb{Z})$.

Let $\mathcal{E}$ be an exact category. We define an associative and unital tensor product $F(\mathbb{Z}) \times \mathcal{E} \to \mathcal{E}$ by $\mathbb{Z}^n \otimes X = Xe_1 \oplus ... \oplus Xe_n$ where Xe_i stands for a copy of X corresponding the the base element e_i of the based free module $\mathbb{Z}^n = \mathbb{Z}e_1 \oplus ... \oplus \mathbb{Z}e_n$. On maps, the tensor product is defined by $(a_{ij}) \otimes f = (a_{ij} f)$.

Using the usual formulas for tensor products of chain complexes, this tensor product extends to an associative, unital and bi-exact pairing

$$\otimes : \mathrm{Ch}^b(\mathbb{Z}) \times \mathrm{Ch}^b \mathcal{E} \to \mathrm{Ch}^b \mathcal{E}$$

making the category of bounded chain complexes $\mathrm{Ch}^b \mathcal{E}$ into a complicial exact category.

2.13. **Definition.** An exact category with weak equivalences $(\mathcal{E}, w)$ is *complicial*, if $\mathcal{E}$ is complicial, and if the tensor product 2.11 (7) preserves weak equivalences (in both variables). For instance, the exact category with weak equivalences $(\mathrm{Ch}^b \mathcal{E}, \mathrm{quis})$ of examples 2.9 and 2.12 is complicial.

2.14. **Definition of $K(\mathcal{E}, w)$, $K(\mathbf{E})$.** Let $\mathbf{E} = (\mathcal{E}, w)$ be a complicial exact category with weak equivalences. We write $\mathcal{E}^w \subset \mathcal{E}$ for the fully exact subcategory of w-acyclic objects in $\mathcal{E}$, that is of those objects X in $\mathcal{E}$ for which the map $0 \to X$ is a weak equivalence.

The algebraic K-theory space $K(\mathbf{E}) = K(\mathcal{E}, w)$ of $(\mathcal{E}, w)$ is the homotopy fibre of the map of pointed topological spaces $BQ\mathcal{E}^w \to BQ\mathcal{E}$, that is,

$$K(\mathbf{E}) = K(\mathcal{E}, w) = F(g) \quad \text{where} \quad g : BQ\mathcal{E}^w \to BQ\mathcal{E}.$$

The higher algebraic K-groups $K_i(\mathbf{E})$ of $\mathbf{E}$ are the homotopy groups $\pi_i K(\mathbf{E})$ of the K-theory space of $\mathbf{E}$, $i \geq 0$.

Exact functors preserving weak equivalences induce maps between algebraic K-theory spaces of complicial exact categories with weak equivalences.

2.15. **Remark.** Definition 2.14, though different from the one in [Wal85, p. 330, Definition], is equivalent to it, by [Wal85, Theorem 1.6.4]. Here we use that $\mathbf{E} = (\mathcal{E}, w)$ is complicial. The reader might have noticed that definition 2.14 makes sense for any exact category with weak equivalences – the complicial structure is not needed. The reason, we define it only for complicial exact categories with weak equivalences is that otherwise it would not necessarily be equivalent to [Wal85, p. 330, Definition].

2.16. **Theorem** [TT90, Theorem 1.11.7]. *Let $\mathcal{E}$ be an exact category. The embedding of $\mathcal{E}$ into $\mathrm{Ch}^b \mathcal{E}$ as degree-zero complexes induces a homotopy equivalence*

$$K(\mathcal{E}) \simeq K(\mathrm{Ch}^b \mathcal{E}, \mathrm{quis}).$$

2.17. **The triangulated category $\mathcal{T}(\mathbf{E})$ associated with $\mathbf{E} = (\mathcal{E}, w)$.** Let $\mathbf{E} = (\mathcal{E}, w)$ be a complicial exact category with weak equivalences. As in 2.2, one can associate a triangulated category $\mathcal{T}(\mathbf{E})$ with $\mathbf{E}$. The construction is analogous. As a category, $\mathcal{T}(\mathbf{E})$ is just $w^{-1}\mathcal{E}$ – the category obtained from $\mathcal{E}$ by formally inverting the weak equivalences. In order to obtain a more explicit description, we will first construct the homotopy category $\underline{\mathcal{E}}$ of $\mathcal{E}$. This is a triangulated category which only depends on the complicial structure of $\mathcal{E}$, but not on the set of weak equivalences. Two maps $f, g : X \to Y$ in $\mathcal{E}$ are homotopic if their difference factors through $X \to CX$. One easily checks that this defines an equivalence relation compatible with composition of maps. The homotopy category $\underline{\mathcal{E}}$ of $\mathcal{E}$ has the same objects as $\mathcal{E}$, and morphisms homotopy classes of maps in $\mathcal{E}$. A sequence in $\underline{\mathcal{E}}$ is a distinguished triangle if it is isomorphic (in $\underline{\mathcal{E}}$) to a sequence of the form

$$X \xrightarrow{f} Y \to C(f) \to TX \tag{8}$$

defined in 2.11. The category $\underline{\mathcal{E}}$ equipped with this set of distinguished triangles is a triangulated category (see remark 2.18 below).

Since $\mathcal{E}^w$ is also complicial, its homotopy category $\underline{\mathcal{E}}^w$ is a triangulated category as well. Moreover, the functor $\underline{\mathcal{E}}^w \to \underline{\mathcal{E}}$ is a fully faithful triangle functor. As a category $w^{-1}\mathcal{E}$ can be identified with $w^{-1}\underline{\mathcal{E}}$ which is the Verdier quotient $\underline{\mathcal{E}}/\underline{\mathcal{E}}^w$ (see C.4), and thus carries a structure of a triangulated category which we define to be $\mathcal{T}(\mathbf{E})$.

2.18. **Remark.** Let $\mathcal{E}$ be a complicial exact category. Call conflation $X \to Y \to Z$ in $\mathcal{E}$ a *Frobenius conflation* if every map $X \to CU$ extends to a map $Y \to CU$, and if every map $CU \to Z$ lifts to a map $CU \to Y$. One can show that $\mathcal{E}$ together with the Frobenius conflations is a Frobenius exact category, that is, an exact category which has enough injectives, enough projectives, and injectives and projectives coincide. The injective-projective objects are the direct factors of objects of the form CU, $U \in \mathcal{E}$. The homotopy category $\underline{\mathcal{E}}$ of $\mathcal{E}$ is now the stable category of the Frobenius category $\mathcal{E}$, which is always a triangulated category [Kel96], [Hap87, Section 9].

2.19. **Exercise.** Let C be a dg-category. Show that the pre-triangulated category $\mathcal{C}^{pretr}$ (see Toen's lecture) of $\mathcal{C}$ can be given the structure of a complicial exact category such that its associated triangulated category is isomorphic to the triangulated category $H^0(\mathcal{C}^{pretr})$.

2.20. **Proposition.** *Let $\mathbf{E} = (\mathcal{E}, w)$ complicial exact category with weak equivalences. Then the map $K_0(\mathbf{E}) \to K_0(\mathcal{T}(\mathbf{E})) : [X] \mapsto [X]$ is well defined and an isomorphism of abelian groups.*

Proof. By definition 2.14 and proposition 1.11, the group $K_0(\mathbf{E})$ is the cokernel of $K_0(\mathcal{E}^w) \to K_0(\mathcal{E})$. Since inflations in $\mathcal{E}$ yield distinguished triangles in $\mathcal{T}(\mathbf{E})$, the map $K_0(\mathcal{E}) \to K_0(\mathcal{T}(\mathbf{E})) : [X] \mapsto [X]$ is well defined. It clearly sends $K_0(\mathcal{E}^w)$ to zero, so that we obtain a well-defined map $K_0(\mathbf{E}) \to K_0(\mathcal{T}(\mathbf{E}))$.

In order to see that the inverse $K_0(\mathcal{T}(\mathbf{E})) \to K_0(\mathbf{E}) : [X] \mapsto [X]$ is also well-defined, we first observe that the existence of a quasi-isomorphism $f : X \to Y$ implies that $[X] = [Y]$ in $K_0(\mathbf{E})$. This is because, by the definition of the mapping cone $C(f)$, there is a conflation $X \to CX \oplus Y \to C(f)$ in $\mathcal{E}$. Since CX and $C(f)$ are in $\mathcal{E}^w$, we have $[X] = [Y]$. The conflation $X \to CX \to TX$ in $\mathcal{E}$ with $CX \in \mathcal{E}^w$ shows that $[X] = -[TX]$ in $K_0(\mathbf{E})$. Since every distinguished triangle in $\mathcal{T}(\mathbf{E})$ is isomorphic (in $\mathcal{T}(\mathbf{E})$) to one of the form 2.17 (8) where $Y \to C(f) \to TX$ is a conflation in $\mathcal{E}$, the inverse $K_0(\mathcal{T}(\mathbf{E})) \to K_0(\mathbf{E})$ is well-defined. □

2.21. **Theorem** (Thomason's Localization Theorem, connective version [TT90, 1.9.8., 1.8.2]). *Let* $\mathbf{A} \to \mathbf{B} \to \mathbf{C}$ *be a sequence of complicial exact categories with weak equivalences such that the associated sequence of triangulated categories* $\mathcal{T}\mathbf{A} \to \mathcal{T}\mathbf{B} \to \mathcal{T}\mathbf{C}$ *is exact. Then the sequence of* K*-theory spaces*

$$K(\mathbf{A}) \to K(\mathbf{B}) \to K(\mathbf{C})$$

is a homotopy fibration. In particular, there is a long exact sequence of K*-groups*

$$\cdots \to K_{i+1}(\mathbf{C}) \to K_i(\mathbf{A}) \to K_i(\mathbf{B}) \to K_i(\mathbf{C}) \to K_{i-1}(\mathbf{A}) \to \cdots$$

ending in $K_0(\mathbf{B}) \to K_0(\mathbf{C}) \to 0$.

2.22. **Remark.** As a special case of theorem 2.21, a functor $\mathbf{A} \to \mathbf{B}$ of complicial exact categories with weak equivalences such that $\mathcal{T}\mathbf{A} \to \mathcal{T}\mathbf{B}$ is an equivalence, induces a homotopy equivalence of K-theory spaces $K(\mathbf{A}) \to K(\mathbf{B})$ and isomorphisms $K_i(\mathbf{A}) \cong K_i(\mathbf{B})$ on K-groups.

2.23. **Example.** Theorem 2.21 applied to example 2.5 yields a homotopy fibration

$$K(\mathcal{H}_S(R)) \to K(R) \to K(\mathcal{P}'(S^{-1}R)).$$

This example illustrates a slight inconvenience. In the homotopy fibration, one would like to have $K(S^{-1}R)$ instead of $K(\mathcal{P}'(S^{-1}R))$. However, the map $K_0(R) \to K_0(S^{-1}R)$ in not surjective, in general. The problem is that $\mathcal{D}^b(S^{-1}R\text{-proj})$ is not a quotient triangulated category of $\mathcal{D}^b(R\text{-proj})$, in general. However, it is a quotient of $\mathcal{D}^b(R\text{-proj})$ – *up to factors*. This leads to the following definition.

2.24. **Definition.** A sequence of triangulated categories $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is *exact up to factors*, if the composition is zero, the functor $\mathcal{A} \to \mathcal{B}$ is fully faithful, and the induced map $\mathcal{B}/\mathcal{A} \to \mathcal{C}$ is an *equivalence up to factors* (or *cofinal*), that is, $\mathcal{B}/\mathcal{A} \to \mathcal{C}$ is fully faithful, and every object of $\mathcal{C}$ is a direct factor of an object of $\mathcal{B}/\mathcal{A}$.

2.25. **Example.** Keep the hypothesis and notation of example 2.5. The following sequence of triangulated categories is exact up to factors

$$\mathcal{D}^b(\mathcal{H}_S(R)) \to \mathcal{D}^b(R\text{-proj}) \to \mathcal{D}^b(S^{-1}R\text{-proj})$$

but not exact, in general.

2.26. **Proposition** (Cofinality, [TT90, 1.10.1, 1.9.8]). *Let* $\mathbf{A} \to \mathbf{B}$ *be a functor of complicial exact categories with weak equivalences such that* $\mathcal{T}\mathbf{A} \to \mathcal{T}\mathbf{B}$ *is cofinal (that is, an equivalence up to factors). Then* $K_i(\mathbf{A}) \to K_i(\mathbf{B})$ *is an isomorphism for* $i \geq 1$ *and a monomorphism for* $i = 0$.

2.27. **Idempotent completion of triangulated categories.** A triangulated category $\mathcal{A}$ is in particular an additive category. So we can speak of its idempotent completion $\tilde{\mathcal{A}}$, see 1.24. It turns out that $\tilde{\mathcal{A}}$ can be equipped with the structure of a triangulated category such that the inclusion $\mathcal{A} \subset \tilde{\mathcal{A}}$ is a triangle functor [BS01]. A sequence in $\tilde{\mathcal{A}}$ is a distinguished triangle if it is a direct factor of a distinguished triangle in $\mathcal{A}$.

If $\mathcal{E}$ is an idempotent complete exact category, then $\mathcal{D}^b\mathcal{E}$ is also idempotent complete [BS01]. In particular, $\mathcal{D}^b \operatorname{Vect}(X)$, $\mathcal{D}^b \operatorname{Coh}(X)$ and $\mathcal{D}^b(R\text{-proj})$ are all idempotent complete.

2.28. **Negative K-theory and the functor $I\!K$.** To any complicial exact category with weak equivalences $\mathbf{E}$, one can associate a new complicial exact category with weak equivalences $S\mathbf{E}$, called suspension of $\mathbf{E}$, such that there is a natural map

$$K(\mathbf{E}) \to \Omega K(S\mathbf{E}) \tag{9}$$

which is an isomorphism on π_i, $i \geq 1$ and a monomorphism on π_0 [Sch06], see the construction in 2.31 below. In fact, $K_1(S\mathbf{E}) = K_0((\mathcal{T}\mathbf{E})^{\sim})$, where $(\mathcal{T}\mathbf{E})^{\sim}$ denotes the idempotent completion of $\mathcal{T}\mathbf{E}$. Moreover, the suspension functor sends sequences of complicial exact categories with weak equivalences such that the associated sequence of triangulated categories exact up to factors to sequences with the same property.

One uses the suspension construction to slightly modify the definition of algebraic K-theory in order to incorporate negative K-groups as follows. One sets $I\!K_i(\mathbf{E}) = K_i(\mathbf{E})$ for $i \geq 1$, $I\!K_0(\mathbf{E}) = K_0((\mathcal{T}\mathbf{E})^{\sim})$, and $I\!K_i(\mathbf{E}) = K_0((\mathcal{T}S^{-i}\mathbf{E})^{\sim})$, $i < 0$.

In fancy language, one constructs a spectrum $I\!K(\mathbf{E})$ whose homotopy groups are the groups $I\!K_i(\mathbf{E})$, $i \in \mathbb{Z}$. Its n-th space is $K(S^n\mathbf{E})$ with structure maps given by (9). Then one has the following theorem.

2.29. **Theorem** (Thomason's Localization Theorem, non-connective version). *Let $\mathbf{A} \to \mathbf{B} \to \mathbf{C}$ be a sequence of complicial exact categories with weak equivalences such that the associated sequence of triangulated categories $\mathcal{T}\mathbf{A} \to \mathcal{T}\mathbf{B} \to \mathcal{T}\mathbf{C}$ is exact up to factors. Then the sequence of K-theory spectra*

$$I\!K(\mathbf{A}) \to I\!K(\mathbf{B}) \to I\!K(\mathbf{C})$$

is a homotopy fibration. In particular, there is a long exact sequence of K-groups for $i \in \mathbb{Z}$

$$\cdots \to I\!K_{i+1}(\mathbf{C}) \to I\!K_i(\mathbf{A}) \to I\!K_i(\mathbf{B}) \to I\!K_i(\mathbf{C}) \to I\!K_{i-1}(\mathbf{A}) \to \cdots$$

2.30. **Remark.** Theorem 2.29 is proved in [Sch06, Theorem 9] for "Frobenius pairs". The proof for complicial exact categories with weak equivalences is (almost *mutatis mutandis*) the same. Alternatively, one can use the fact that $(\mathcal{E}, \mathcal{E}^w)$ is a Frobenius pair for (E, w) a complicial exact category with weak equivalences, see remark 2.18.

2.31. **Construction of the suspension $S\mathbf{E}$.** Let $\mathbf{E} = (\mathcal{E}, w)$ be a complicial exact category with weak equivalences. In particular, $\mathcal{E}$ is an exact category so that we can construct its countable envelope $\mathcal{F}\mathcal{E}$, see 1.28. The complicial structure on $\mathcal{E}$ extends to a complicial structure on $\mathcal{F}\mathcal{E}$ setting $A \otimes (E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots) = (A \otimes E_0 \hookrightarrow A \otimes E_1 \hookrightarrow A \otimes E_2 \hookrightarrow \cdots)$ for $A \in \mathrm{Ch}^b(\mathbb{Z})$ and $E_* \in \mathcal{F}\mathcal{E}$. Call a map in $\mathcal{F}\mathcal{E}$ a weak equivalence if its cone is a direct factor of an object of $\mathcal{F}(\mathcal{E}^w)$, and write w for the set of weak equivalences in $\mathcal{F}\mathcal{E}$. The pair $\mathcal{F}\mathbf{E} = (\mathcal{F}\mathcal{E}, w)$ defines a complicial exact category with weak equivalences. The functor $\mathcal{E} \to \mathcal{F}\mathcal{E}$ of exact categories defines a functor $\mathbf{E} \to \mathcal{F}\mathbf{E}$ of complicial exact categories with weak equivalences such that the functor $\mathcal{T}(\mathbf{E}) \to \mathcal{T}(\mathcal{F}\mathbf{E})$ of associated triangulated categories is fully faithful.

The suspension $S\mathbf{E}$ is the complicial exact category with underlying exact category $\mathcal{F}\mathcal{E}$ and set of weak equivalences those maps in $\mathcal{F}\mathcal{E}$ which are isomorphisms in the Verdier quotient $\mathcal{T}(\mathcal{F}\mathbf{E})/\mathcal{T}(\mathbf{E})$. For more details, see [Sch06].

2.32. **Remark.** If $\mathcal{C}$ is a dg-category, one can more easily define its suspension as $S \otimes \mathcal{C}$, where $S = S\mathbb{Z}$ is the suspension ring of $\mathbb{Z}$ (see Cortiñas' lecture).

2.33. **K-theory-like functors.** Versions of theorems 2.21 and 2.29 also hold for functors other than K-theory. For instance, a version of theorem 2.21 holds for Witt groups [Bal00]. A version of theorem 2.29 holds for cyclic homology and its variants [Kel99] and also for hermitian K-theory [Scha]. All K-theory calculations based on theorems 2.21 and 2.29 – some of which are explained in the next section – have therefore (potential) analogues in all these theories.

3. Applications of the Localization Theorem

All results in this section are due to Thomason [TT90], [Tho93] with the exception of theorem 3.1 – which is due to Quillen [Qui73]. Their proofs are based on Thomason's localization theorem 2.21, or rather on its non-connective version 2.29. This has the advantage that they also apply *mutatis mutandis* to cyclic homology, and – in suitably adapted form – to Witt-groups and hermitian K-theory, see remark 2.33.

3.1. **Theorem** (Poincaré duality, [Qui73]). *Let X be a regular noetherian separated scheme. Then the fully exact inclusion* $\operatorname{Vect}(X) \subset \operatorname{Coh}(X)$ *induces an equivalence of triangulated categories*

$$\mathcal{D}^b \operatorname{Vect}(X) \cong \mathcal{D}^b \operatorname{Coh}(X).$$

In particular, it induces a homotopy equivalence $K(X) \simeq G(X)$.

Proof. One first shows that every coherent sheaf F on X admits a surjective map $V \to F$ of sheaves with V a vector bundle. This implies that the dual of criterion 2.6 (b) is satisfied, so that $\mathcal{D}^b \operatorname{Vect}(X) \to \mathcal{D}^b \operatorname{Coh}(X)$ is fully faithful. It also implies that any coherent sheaf F admits a resolution

$$\cdots \to V_i \to V_{i-1} \to \cdots \to V_0 \to F \to 0$$

by vector bundles V_i. By Serre's theorem [Wei94, Theorem 4.4.16], [Mat89, Theorem 19.2], the image of the map $V_i \to V_{i-1}$ is a vector bundle for $i \geq \dim X$, so that we may assume $V_i = 0$ for $i > n$ in the above resolution. This shows that $\mathcal{D}^b \operatorname{Vect}(X) \to \mathcal{D}^b \operatorname{Coh}(X)$ is also essentially surjective, hence an equivalence. In particular, $K(X) \simeq G(X)$, by 2.22 and 2.16.

To see the existence of a surjection $V \to F$, we can assume X connected, hence integral. The local rings $O_{X,x}$ are regular noetherian, hence UFD's. This implies that for any closed $Z \subset X$ of pure codimension 1, there is a line bundle $\mathcal{L}$ and a section $s : O_X \to \mathcal{L}$ such that $Z = X - X_s$, where $X_s = \{x \in X |\ s_x : O_{X,x} \cong \mathcal{L}_x\} \subset X$ [Har77, Propositions II 6.11, 6.13]. Since any proper closed subset of X is in such a Z, the open subsets X_s, $(\mathcal{L}, s)$, form a basis for the topology of X where $\mathcal{L}$ runs through the line bundles of X, and $s \in \Gamma(X, \mathcal{L})$. For $a \in F(X_s)$, there is an integer $n \geq 0$ such that $a \otimes s^n \in \Gamma(X_s, F \otimes \mathcal{L}^n)$ extends to a global section of $F \otimes \mathcal{L}^n$ [Har77, Lemma II.5.14], that is, $a \in F(X_s)$ is in the image of $\mathcal{L}^{-n}(X_s) \to F(X_s)$. It follows that there is a surjection $\bigoplus \mathcal{L}_i \to F$ from a sum of line bundles $\mathcal{L}_i$ to F. Since F is coherent, and X quasi-compact, finitely many of the $\mathcal{L}_i$'s are sufficient to yield a surjection onto F. □

3.2. **Remark.** The derived equivalence of theorem 3.1 also yields an equivalence $\mathbb{K}\,\mathrm{Vect}(X) \cong \mathbb{K}\,\mathrm{Coh}(X)$, by theorem 2.29. Since negative K-theory of noetherian abelian categories are trivial [Sch06], we have $\mathbb{K}_i(X) = \mathbb{K}_i\,\mathrm{Coh}(X) = 0$ for $i < 0$ and X a regular noetherian separated scheme, see also [TT90, Proposition 6.8].

3.3. **K-theory of schemes.** Any reasonable cohomology theory for schemes should satisfy a Mayer-Vietoris long exact sequence for open covers. For K-theory this means that for scheme $X = U \cup V$ covered by two open subschemes U, V, we should have a long exact sequence

$$\cdots \to \mathbb{K}_{i+1}(U\cap V) \to \mathbb{K}_i(X) \to \mathbb{K}_i(U)\oplus\mathbb{K}_i(V) \to \mathbb{K}_i(U\cap V) \to \mathbb{K}_{i-1}(X) \to \cdots$$

If we defined $\mathbb{K}(X)$ naively as $\mathbb{K}\,\mathrm{Vect}(X)$ we would not have such a long exact sequence, in general. Therefore, one has to give a slightly different definition.

3.4. **Definition.** Let X be a quasi-compact and separated scheme. A complex (A, d) of quasi-coherent O_X-modules is called *perfect*, if there is an open covering $X = \bigcup_{i\in I} U_i$ of X such that $(A, d)_{|U_i}$ is quasi-isomorphic to a bounded complex of vector bundles on U_i, $i \in I$.

Let $Z \subset X$ be a closed subset of X. We write $\mathrm{Perf}_Z(X) \subset \mathrm{Ch}\,\mathrm{Qcoh}(X)$ for the full subcategory of perfect complexes on X which are acyclic over $X - Z$. The inclusion is extension closed, so that we can consider $\mathrm{Perf}_Z X$ as a fully exact subcategory of the abelian category $\mathrm{Ch}\,\mathrm{Qcoh}(X)$ of chain complexes of quasi-coherent O_X-modules. Ordinary tensor product of chain complexes makes $(\mathrm{Perf}_Z(X), \mathrm{quis})$ into a complicial exact category with weak equivalences. It is customary to write $\mathcal{D}\,\mathrm{Perf}_Z(X)$ for $\mathcal{T}(\mathrm{Perf}_Z(X), \mathrm{quis})$. We define

$$\mathbb{K}_Z(X) = \mathbb{K}\,\mathrm{Perf}_Z(X)$$

where $\mathrm{Perf}_Z(X)$ denotes (up to derived equivalence, a small model of) the complicial exact category with weak equivalences $(\mathrm{Perf}(X), \mathrm{quis})$. In case $Z = X$, we may write $\mathbb{K}(X)$ instead of $\mathbb{K}_Z(X)$.

In many interesting cases, $\mathbb{K}(X)$ is equivalent to vector bundle K-theory. To be more precise, we say that X has an *ample family of line bundles* if the sets $X_s = \{x \in X \mid s_x : O_{X,x} \cong \mathcal{L}_x\} \subset X$ form a basis for the topology of X where $\mathcal{L}$ runs through the line bundles of X, and $s \in \Gamma(X, \mathcal{L})$. For instance, a regular noetherian separated scheme has an ample family of line bundles (see proof of theorem 3.1), and every quasi-projective scheme has an ample family of line bundles.

3.5. **Proposition** [TT90, Corollary 3.9]**.** *Let X be a quasi-compact and separated scheme that has an ample family of line bundles. Then the inclusion* $\mathrm{Ch}^b\,\mathrm{Vect}(X) \subset \mathrm{Perf}(X)$ *induces an equivalence of triangulated categories* $\mathcal{D}^b\,\mathrm{Vect}(X) \cong \mathcal{D}\,\mathrm{Perf}(X)$. *In particular,*

$$\mathbb{K}\,\mathrm{Vect}(X) \simeq \mathbb{K}\,\mathrm{Perf}(X) \quad (= \mathbb{K}(X)).$$

Proof. The derived category $\mathcal{D}(\mathrm{Qcoh}(X))$ of quasi-coherent O_X-modules is a compactly generated triangulated category with $\mathcal{D}\,\mathrm{Perf}(X) \subset \mathcal{D}(\mathrm{Qcoh}(X))$ the subcategory of compact objects, see appendix C.11. Since X has an ample family of line bundles, every quasi-coherent sheaf F on X admits a surjective map $\bigoplus \mathcal{L}_i \to F$ from a direct sum of line bundles to F (see proof of theorem 3.1). This implies firstly that the dual of criterion 2.6 (b) is satisfied, and we have fully faithful functors

$\mathcal{D}^b \operatorname{Vect}(X) \subset \mathcal{D}^b \operatorname{Qcoh}(X) \subset \mathcal{D} \operatorname{Qcoh}(X)$, and secondly that the compact objects $\operatorname{Vect}(X)$ generates $\mathcal{D} \operatorname{Qcoh}(X)$ as a triangulated category with infinite sums. Since $\mathcal{D}^b \operatorname{Vect}(X)$ is idempotent complete [BS01], the functor $\mathcal{D}^b \operatorname{Vect}(X) \to \mathcal{D} \operatorname{Perf}(X)$ is an equivalence, see theorem C.10 □

3.6. **Theorem** [TT90, Theorem 5.1, 7.4]**.** *Let X be a quasi-compact and separated scheme, and $U \subset X$ a quasi-compact open subscheme. Let $Z = X - U$. Then U is separated, and the sequence $\mathcal{D} \operatorname{Perf}_Z(X) \to \mathcal{D} \operatorname{Perf}(X) \to \mathcal{D} \operatorname{Perf}(U)$ of triangulated categories is exact up to factors. In particular, we have a homotopy fibration*

$$\mathbb{K}_Z(X) \to \mathbb{K}(X) \to \mathbb{K}(U)$$

and its associated long exact sequence of K-groups.

Proof. Write $j : U \subset X$ for the open immersion. Restriction $Lj^* = j^* : \mathcal{D} \operatorname{Qcoh}(X) \to \mathcal{D} \operatorname{Qcoh}(U)$ has a right adjoint $Rj_* : \mathcal{D} \operatorname{Qcoh}(U) \to \mathcal{D} \operatorname{Qcoh}(X)$ which is computed as $Rj_*(A) = j_*(I)$ where $A \to I$ is a $\mathcal{K}$-injective resolution (for terminology, see C.7). The counit of adjunction $Lj^* Rj_*(A) \to A$ is $j^* j_*(I) = I \leftarrow A$ which is a (quasi-) isomorphism. Therefore, Rj_* is fully faithful and Lj^* makes $\mathcal{D} \operatorname{Qcoh}(U)$ into a Verdier quotient of $\mathcal{D} \operatorname{Qcoh}(X)$. Let $\mathcal{D}_Z \operatorname{Qcoh}(X) \subset \mathcal{D} \operatorname{Qcoh}(X)$ be the subcategory of those complexes which are acyclic outside of Z, then we have an exact sequence of triangulated categories

$$\mathcal{D}_Z \operatorname{Qcoh}(X) \to \mathcal{D} \operatorname{Qcoh}(X) \to \mathcal{D} \operatorname{Qcoh}(U).$$

In all three categories infinite direct sums exist and the functors preserve them. By theorem C.11, all three categories are compactly generated by their subcategory of perfect complexes. Therefore, the sequence

$$\mathcal{D} \operatorname{Perf}_Z(X) \to \mathcal{D} \operatorname{Perf}(X) \to \mathcal{D} \operatorname{Perf}(U)$$

of subcategories of compact objects is exact up to factors, by theorem C.10. The $\mathbb{K}$-theory statement follows from 2.29. □

3.7. **Theorem** (Excision). *Let $X \to Y$ be a map of quasi-compact and separated schemes, and let $Z \hookrightarrow X$ be a closed immersion such that the composition $Z \to Y$ is also a closed immersion. Assume that $X - Z$ and $Y - Z$ are quasi-compact, and that f is flat in a neighborhood of $Z \subset X$. Then $Lf^* : \operatorname{Perf}_Z(Y) \to \operatorname{Perf}_Z(X)$ induces an equivalence*

$$\mathbb{K}_Z(Y) \xrightarrow{\sim} \mathbb{K}_Z(X)$$

Proof. This is because $Lf^* : \operatorname{Perf}_Z(Y) \to \operatorname{Perf}_Z(X)$ induces an equivalence $Lf^* : D \operatorname{Perf}_Z(Y) \to D \operatorname{Perf}_Z(X)$ of triangulated categories [TT90, Proposition 3.19]. □

3.8. **Zariski descent.** Let $X = U \cup V$ be a quasi-compact and separated scheme covered by two quasi-compact open subschemes U and V. Then U, V, $U \cap V$ are quasi-compact and separated, and the square

$$\begin{array}{ccc} \mathbb{K}(X) & \longrightarrow & \mathbb{K}(U) \\ \downarrow & & \downarrow \\ \mathbb{K}(V) & \longrightarrow & \mathbb{K}(U \cap V) \end{array}$$

is homotopy cartesian. In particular, there is a long exact sequence of K-groups

$$\cdots \to I\!\!K_{i+1}(U\cap V) \to I\!\!K_i(X) \to I\!\!K_i(U)\oplus I\!\!K_i(V) \to I\!\!K_i(U\cap V) \to I\!\!K_{i-1}(X) \to \cdots$$

This is because the horizontal homotopy fibres are $I\!\!K_Z(X)$ and $I\!\!K_Z(V)$ with $Z = X - U = V - U\cap V$, and $I\!\!K_Z(X) \to I\!\!K_Z(V)$ is an equivalence, by excision 3.7.

3.9. **Theorem** (Projective bundle theorem [TT90, Theorem 4.1, 7.3]). *Let X be a quasi-compact and separated scheme, and let $\mathcal{E} \to X$ be a geometric vector bundle over X of rank $n+1$. Let $p : \mathbb{P}\mathcal{E} \to X$ be the associated projective bundle with twisting sheaf $O_{\mathcal{E}}(1)$. Then we have an equivalence*

$$\prod_{l=0}^{n} O_{\mathcal{E}}(-l)\otimes Lp^* : \quad \prod_{l=0}^{n} I\!\!K(X) \xrightarrow{\sim} I\!\!K(\mathbb{P}\mathcal{E}).$$

Proof. By Zariski descent 3.8, the question is local in X, so that we may assume $X = \operatorname{Spec} A$ affine, and $p : \mathbb{P}\mathcal{E} \to X$ the canonical projection $\operatorname{Proj}(A[T_0,...,T_n]) = \mathbb{P}^n_A \xrightarrow{p} \operatorname{Spec} A$. Moreover, $\mathcal{D}\operatorname{Perf}(\operatorname{Spec} A) = \mathcal{D}^b(A\text{−proj})$ and $\mathcal{D}\operatorname{Perf}(\mathbb{P}^n) = \mathcal{D}^b\operatorname{Vect}(\mathbb{P}^n)$, by 3.5. In this case, the twisting sheaf $O(1)$ is ample so that for every quasi-coherent sheaf F there is a surjection $\bigoplus_{i\in I} O(-l_i) \to F$ with $l_i \geq 0$. This implies that $\mathcal{D}\operatorname{Vect}(\mathbb{P}^n)$ is generated as an idempotent complete triangulated category by the family $\{O_{\mathbb{P}^n}(-l)|\ l\geq 0\}$, see theorem C.10. The sequence $T_0,...,T_n$ is a regular sequence in $S = A[T_1,...,T_n]$ so that the Koszul complex $\bigotimes_{i=0}^n(\ S(-1) \xrightarrow{T_i} S\ )$ induces an exact sequence of graded S-modules

$$0\to S(-n-1)\to \bigoplus_1^{n+1} S(-n) \to \bigoplus_1^{\binom{n+1}{2}} S(-n+1) \to \cdots \to \bigoplus_1^{n+1} S(-1)\to S\to A\to 0.$$

Taking associated sheaves, we obtain an exact sequence of quasi-coherent sheaves on $\mathbb{P}^n_A$

$$0\to O(-n-1)\to \bigoplus_1^{n+1} O(-n) \to \bigoplus_1^{\binom{n+1}{2}} O(-n+1) \to \cdots \to \bigoplus_1^{n+1} O(-1)\to O_{\mathbb{P}^n}\to 0.$$

This shows that $\mathcal{D}^b\operatorname{Vect}(\mathbb{P}^n_A)$ is generated as an idempotent complete triangulated category by $O(-n),...,O(-1),O_{\mathbb{P}^n}$. For $i\leq j$, let $\mathcal{D}^b_{[i,j]}\subset \mathcal{D}^b\operatorname{Vect}(\mathbb{P}^n)$ be the full idempotent complete triangulated subcategory generated by $O(l)$, $i\leq l\leq j$. We have a filtration

$$0\subset \mathcal{D}^b_{[0,0]}\subset \mathcal{D}^b_{[-1,0]}\subset ... \subset \mathcal{D}^b_{[-n,0]} = \mathcal{D}^b\operatorname{Vect}(\mathbb{P}^n).$$

Since $H^*(\mathbb{P}^n_A, O(-l)) = 0$ for $l = 1,...,n$ [Gro61, Proposition III 2.1.12], we have $\operatorname{Hom}(O(-j)[r], O(-l)[s]) = 0$ in $\mathcal{D}^b\operatorname{Vect}(\mathbb{P}^n_A)$ for $0\leq i < l\leq n$. This implies that the composition

$$\mathcal{D}^b_{[-l,-l]}\subset \mathcal{D}^b_{[-l,0]} \to \mathcal{D}^b_{[-l,0]}/\mathcal{D}^b_{[-l+1,0]}$$

is an equivalence, by exercise C.6.

The unit of adjunction $F\to Rp_*Lp^*F$ is an (quasi-) isomorphism for $F = A$ because $A\to H^0(Rp_*Lp^*A) = H^0(Rp_*O_{\mathbb{P}^n}) = H^0(\mathbb{P}^n, O_{\mathbb{P}^n})$ is an isomorphism, and $H^i(\mathbb{P}^n, O_{\mathbb{P}^n}) = 0$ for $i\neq 0$ [Gro61, Proposition III 2.1.12]. Since $\mathcal{D}^b(A\text{−proj})$ is generated as an idempotent complete triangulated category by A, we see that the unit of adjunction $F\to Rp_*Lp^*F$ is an (quasi-) isomorphism for all $F\in$

$\mathcal{D}^b(A-\mathrm{proj})$. This implies that $Lp^* = p^* : \mathcal{D}^b(A-\mathrm{proj}) \to \mathcal{D}^b \operatorname{Vect}(\mathbb{P}^n)$ is fully faithful. Since tensoring with $O(l)$ is an equivalence, we see that $O(-l) \otimes Lp^* : \mathcal{D}^b(A-\mathrm{proj}) \to \mathcal{D}^b_{[-l,-l]}$ is an equivalence (the functor is fully faithful, and both categories have the same set of generators).

Now we are ready to prove the theorem. For $i \leq j$, let $\mathrm{Ch}^b_{[i,j]} \subset \mathrm{Ch}^b \operatorname{Vect}(\mathbb{P}^n)$ be the full subcategory of those chain complexes which lie in $\mathcal{D}^b_{[i,j]}$. Let w be the set of maps in $\mathrm{Ch}^b_{[-l,0]}$ which are isomorphisms in the quotient category $\mathcal{D}^b_{[-l,0]}/\mathcal{D}^b_{[-l+1,0]}$. By construction,

$$(\mathrm{Ch}^b_{[-l+1,0]}, \mathrm{quis}) \to (\mathrm{Ch}^b_{[-l,0]}, \mathrm{quis}) \to (\mathrm{Ch}^b_{[-l,0]}, w)$$

induces an exact sequence of associated triangulated categories and thus a homotopy fibration in $I\!K$-theory (theorem 2.29) for $l = 1, ..., n$. Moreover, the argument above shows that the composition

$$O(-l) \otimes p^* : (\mathrm{Ch}^b(A-\mathrm{proj}), \mathrm{quis}) \to (\mathrm{Ch}^b_{[-l,0]}, \mathrm{quis}) \to (\mathrm{Ch}^b_{[-l,0]}, w)$$

induces an equivalence of associated triangulated categories, and thus an equivalence in $I\!K$-theory. It follows that the $I\!K$-theory fibration splits, and we have a homotopy equivalence

$$(O(-l) \otimes p^*, 1) : I\!K(A) \times I\!K(\mathrm{Ch}^b_{[-l+1,0]}, \mathrm{quis}) \xrightarrow{\sim} I\!K(\mathrm{Ch}^b_{[-l,0]}, \mathrm{quis})$$

for $l = 1, ..., n$. Since $\mathrm{Ch}^b_{[-n,0]} = \mathrm{Ch}^b \operatorname{Vect}(\mathbb{P}^n_A)$, this implies the theorem. □

3.10. **Theorem** (Bass fundamental theorem). *Let X be a quasi-compact and separated scheme. Then there is a split exact sequence for all $n \in \mathbb{Z}$*

$$0 \to I\!K_n(X) \to I\!K_n(X[T]) \oplus I\!K_n(X[T^{-1}]) \to I\!K_n(X[T, T^{-1}]) \to I\!K_{n-1}(X) \to 0$$

Proof. The projective line $\mathbb{P}^1_X$ over X has standard open covering by $X[T]$ and $X[T^{-1}]$ with intersection $X[T, T^{-1}]$. Zariski descent 3.8 yields a long exact sequence

$$\cdots \to I\!K_n(\mathbb{P}^1_X) \to I\!K_n(X[T]) \oplus I\!K_n(X[T^{-1}]) \to I\!K_n(X[T, T^{-1}]) \to I\!K_{n-1}(\mathbb{P}^1_X) \to \cdots$$

By the projective bundle theorem 3.9, the group $I\!K_n(\mathbb{P}^1_X)$ is $I\!K_n(X) \oplus I\!K_n(X)$ with basis $[O_{\mathbb{P}^1}]$ and $[O_{\mathbb{P}^1}(-1)]$. Making a base-change, we can write $I\!K_n(\mathbb{P}^1_X)$ as $I\!K_n(X) \oplus I\!K_n(X)$ with basis $[O_{\mathbb{P}^1}]$ and $[O_{\mathbb{P}^1}] - [O_{\mathbb{P}^1}(-1)]$. Since on $X[T]$ and on $X[T^{-1}]$ the two line-bundles $O_{\mathbb{P}^1}$ and $O_{\mathbb{P}^1}(-1)$ are isomorphic, the left map in the long Mayer-Vietoris exact sequence above is trivial on the direct summand $K(X)$ corresponding to the base $[O_{\mathbb{P}^1}] - [O_{\mathbb{P}^1}(-1)]$. Since the map is clearly (split) injective on the other summand, the long Mayer-Vietoris exact sequence breaks up into shorter exact sequences which give Bass fundamental exact sequences. The splitting of the map $I\!K_n(X[T, T^{-1}]) \to I\!K_{n-1}(X)$ is given by the cup product with the element $[T] \in K_1(\mathbb{Z}[T, T^{-1}])$. □

For a proof of the following theorem, see [Tho93], and [CHSW05].

3.11. **Theorem** (Blow-ups along regularly embedded centers). *Let $i : Y \subset X$ be a regular embedding of pure codimension d with X quasi-compact and separated. Let $p : X' \to X$ be the blow-up of X along Y and $j : Y' \subset X'$ the exceptional divisor. Write $q : Y' \to Y$ for the induced map. Then the square*

$$\begin{array}{ccc} I\!\!K(X) & \xrightarrow{Li^*} & I\!\!K(Y) \\ {\scriptstyle Lp^*}\downarrow & & \downarrow{\scriptstyle Lq^*} \\ I\!\!K(X') & \xrightarrow[Lj^*]{} & I\!\!K(Y') \end{array}$$

is homotopy cartesian. Moreover, there is a homotopy equivalence

$$I\!\!K(X') \simeq I\!\!K(X) \times \prod_{1}^{d-1} I\!\!K(Y).$$

3.12. **More derived equivalences.** There exist semi-orthogonal decompositions of the derived categories of grassmannians, smooth quadrics [Kap88], for Severi Brauer varieties [Ber05], toric varieties [CMR04], [Kaw06] and many more. Also, Neeman and Ranicki describe the exact conditions under which a Cohn localization $R \to S^{-1}R$ of rings induces a localization $\mathcal{D}^b(R\text{−proj}) \to \mathcal{D}^b(S^{-1}R\text{−proj})$ of triangulated categories [NR04]. All these theorems yield results in K-theory, by an application of theorems 2.21 and 2.29. Historically however, many of them were obtained first as applications of Quillen's methods, see for instance [Swa85], [Pan94], [Qui73, §8 4].

4. Beyond the Localization theorem

In this section we simply state some important theorems in the K-theory of schemes the proofs of which go beyond the methods explained in the previous sections. For more overviews on a variety of topics in K-theory, we refer the reader to the K-theory handbooks [FG05].

4.1. **Brown-Gersten-Quillen spectral sequence** [Qui73]. Let X be a noetherian scheme, and write $X^p \subset X$ for the set of points of codimension p in X. There is a filtration $0 \subset ... \subset \mathrm{Coh}^2(X) \subset \mathrm{Coh}^1(X) \subset \mathrm{Coh}^0(X) = \mathrm{Coh}(X)$ of $\mathrm{Coh}(X)$ by the Serre abelian subcategories $\mathrm{Coh}^i(X) \subset \mathrm{Coh}(X)$ of those coherent sheaves F whose support has codimension $\geq i$. This filtration, together with Quillen's localization and devissage theorems, leads to the Brown-Gersten-Quillen (BGQ) spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X^p} K_{-p-q}(k(x)) \Rightarrow G_{-p-q}(X)$$

If X is of finite type over a field, inspection of the differential d_1 yields an isomorphism

$$E_2^{p,-p} \cong \mathrm{CH}^p(X),$$

where $\mathrm{CH}^p(X)$ is the Chow-group of codimension p cycles modulo rational equivalence as defined in [Ful98].

4.2. **Gersten's conjecture and Bloch's formula.** The Brown-Gersten-Quillen spectral sequence yields a complex

$$0 \to G_n(X) \to \bigoplus_{x\in X^0} K_n(k(x)) \stackrel{d_1}{\to} \bigoplus_{x\in X^1} K_{n-1}(k(x)) \stackrel{d_1}{\to} \cdots$$

Gersten conjectured that this complex is exact for $X = \operatorname{Spec} R$, where R is a regular noetherian ring. The conjecture is proved in case R (is regular noetherian and) contains a field [Pan03] building on the geometric case proved in [Qui73].

As a corollary [Qui73], one obtains an isomorphism for the E_2-term of the BGQ-spectral sequence $E_2^{p,q} \cong H^p_{Zar}(X, \mathcal{K}_{-q,X})$, and Bloch's formula

$$\mathrm{CH}^p(X) \cong H^p_{Zar}(X, \mathcal{K}_{p,X}),$$

where $\mathcal{K}_{p,X}$ denotes the Zariski sheaf associated to the presheaf $U \mapsto K_p(U)$, and X is a regular scheme of finite type over a field.

As an application, one can use products $K_p(X)\otimes K_q(X) \to K_{p+q}(X)$ in K-theory and Bloch's formula to define the intersection product on Chow-groups (without using any moving lemma nor deformation to the normal cone).

4.3. **The motivic spectral sequence.** Let X be a smooth scheme over a perfect field. Then there is a spectral sequence [FS02], [Lev05]

$$E_2^{p,q} = H^{p-q}_{mot}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X).$$

Here $H^p_{mot}(X, \mathbb{Z}(q))$ is the motivic cohomology of X as defined in [VSF00], [MVW06]. It is proved in *loc.cit* that this group is isomorphic to Bloch's higher Chow group $\mathrm{CH}^q(X, 2q-p)$ as defined in [Blo86]. Rationally, the spectral sequence collapses, and yields an isomorphism [Blo86], [Lev94]

$$K_n(X)_{\mathbb{Q}} \cong \bigoplus_i \mathrm{CH}^i(X, n)_{\mathbb{Q}}.$$

4.4. **Milnor K-theory and the Bloch-Kato conjecture.** Let F be a (commutative) field. The Milnor K-theory $K^M_*(F)$ of F is the graded ring generated in degree 1 by symbols $\{a\}$ for $a \in F^\times$ a unit in F, modulo the relations $\{ab\} = \{a\}+\{b\}$ and $\{a\}\cdot\{1-a\} = 0$ for $a \neq 1$. One easily computes $K^M_0(F) = \mathbb{Z}$ and $K^M_1(F) = F^\times$. Since $K_1(F) = F^\times$, and since Quillen's K-groups define a graded ring $K_*(F)$ (which is commutative – in the graded sense), we obtain a morphism $K^M_*(F) \to K_*(F)$ extending the isomorphisms on K_0 and K_1. Matsumotos' theorem says that this map is also an isomorphism for $* = 2$, that is, $K^M_2(F) \to K_2(F)$, see [Mil71].

Let $m = p^\nu$ be a prime power, with p different from the characteristic of F, and let F_s be a separable closure of F. Then we have an exact sequence of Galois modules $1 \to \mu_m \to F_s^\times \stackrel{m}{\to} F_s^\times \to 1$ where μ_m are the m-th roots of unity. The sequence induces a map on etale cohomology $F^\times \to H^1_{et}(F, \mu_m)$. Using the multiplicative structure of etale cohomology $H^*_{et}(F, \mu_m^{\otimes *})$, this map extends to a map of rings $K^M_*(F) \to H^*_{et}(F, \mu_m^{\otimes *})$ which induces the "norm residue homomorphism"

$$K^M_n(F)/m \to H^n_{et}(F, \mu_m^{\otimes n}).$$

The Bloch-Kato conjecture for the prime p says that this map is an isomorphism for all n.

The conjecture for $m = 2^\nu$ was proved by Voevodsky [Voe03], and proofs for $m = p^\nu$ odd have been announced by Rost and Voevodsky.

4.5. **Computation of $K(\mathbb{F}_q)$.** Quillen computed the K-groups of finite fields in [Qui72]. They are given by the formulas $K_0(\mathbb{F}_q) \cong \mathbb{Z}$, $K_{2n}(\mathbb{F}_q) = 0$, $n > 0$, and $K_{2n-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^n - 1)\mathbb{Z}$, $n > 0$.

4.6. **Computation of $K(\mathbb{Z})$.** Modulo the Bloch-Kato conjecture 4.4 for odd primes (which is believed to be proved...) and the Vandiver conjecture (which seems to be wide open), the K-groups of $\mathbb{Z}$ for $n \geq 2$ are given as follows [Wei05], [Kur92], [Mit97]

n *mod* 8	1	2	3	4	5	6	7	8
$K_n(\mathbb{Z})$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2c_k$	$\mathbb{Z}/2w_{2k}$	0	$\mathbb{Z}$	$\mathbb{Z}/c_k$	$\mathbb{Z}/w_{2k}$	0

where k is the integer part of $1 + \frac{n}{4}$, c_k and w_{2k} are the numerator and denominator of $\frac{B_k}{4k}$ with B_k the k-th Bernoulli number. The B_k's are the coefficients of the power series

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{t^{2k}}{(2k)!}$$

4.7. **Cdh descent** [CHSW05]. The following is due to Häsemeyer [Hae04]. Let k be a field of characteristic 0, and write Sch_k for the category of separated schemes of finite type over k. Let F be a contravariant functor from Sch_k to the category of spectra (or chain complexes of abelian groups). Let $Y \to X \leftarrow X'$ be maps of schemes in Sch_k and $Y' = Y \times_X X'$ the fibre product. Consider the following square of spectra (or chain complexes)

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ F(X') & \longrightarrow & F(Y') \end{array} \tag{10}$$

obtained by functoriality of F. Suppose that F satisfies the following.

(a) *Nisnevich Descent.* Let $f : X' \to X$ be an etale map, and $Y \to X$ an open immersion. Assume that f induces an isomorphism $f : (X' - Y')_{red} \cong (X - Y)_{red}$. Then the square (10) is homotopy cartesian.
(b) *Invariance under nilpotent extensions.* The map $X_{red} \to X$ induces an equivalence $F(X) \simeq F(X_{red})$.
(c) *Excision for ideals.* Let $f : R \to S$ be a map of commutative rings, $I \subset R$ an ideal such that $f : I \to f(I)$ is an isomorphism, and an ideal in S. Set $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} R/I$, $X' = \operatorname{Spec} S$, $Y' = \operatorname{Spec} S/f(I)$ with the induced maps between them. Then (10) is homotopy cartesian.
(d) *Excision for blow-ups along regularly embedded centers.* Let $Y \subset X$ be a regular embedding of pure codimension (a closed immersion is regular of pure codimension d if, locally, its ideal sheaf is generated by a regular sequence of length d), X' the blow-up of X along Y, and $Y' \subset X'$ the exceptional divisor. Then (10) is homotopy cartesian.

Then for any closed embedding $Y \subset X$ in Sch_k, the square (10) is homotopy cartesian, where X' is the blow-up of X along Y, and $Y' \subset X'$ the exceptional divisor. In this case we say that F satisfies *cdh descent*.

4.8. **Example** (Infinitesimal K-theory [CHSW05]). By theorems 3.7, 2.29, and 3.11, $I\!K$-theory satisfies (a) and (d). But neither (b) nor (c) holds for $I\!K$-theory (see Cortiñas' lecture). The same holds for cyclic homology and its variants, since (a) and (d) are formal consequences of the localization theorem 2.29. Therefore, the homotopy fibre K^{inf} of the Chern character $I\!K \to HN$ from $I\!K$-theory to negative cyclic homology satisfies (a) and (d). By a theorem of Goodwillie [Goo86], K^{inf} satisfies (b), and by a theorem of Cortiñas [Cor06], K^{inf} satisfies (c). Therefore, infinitesimal K-theory K^{inf} satisfies cdh-descent in characteristic 0.

This was used in [CHSW05] to prove that $K_i(X) = 0$, $i < -d$, for X a d-dimensional scheme essentially of finite type over a field of characteristic 0, and $K_{-d}(X) = H^d_{cdh}(X, \mathbb{Z})$.

4.9. **Examples.** Cdh-descent in characteristic 0 also holds for homotopy K-theory KH [Hae04], periodic cyclic homology HP [CHSW05], and stabilized Witt groups [Schb].

Appendix A. The classifying space of a category

We recall the definition of simplicial sets and that of a classifying space of a category. Details can be found for instance in [FP90], [GJ99], [May67], [Wei94].

A.1. **Simplicial sets.** Let Δ be the category whose objects are the sets $[n] = \{0, 1, 2, ..., n\}$, $n \geq 0$, and where a map $[n] \to [m]$ is an order preserving map of sets $[n] \to [m]$. Composition in Δ is composition of (order preserving) maps. The unique order preserving injective maps $d_i : [n-1] \to [n]$ which leave out i, are called *face maps*, $i = 0, ..., n$. The unique order preserving surjective maps $s_j : [n] \to [n-1]$ for which the pre-image of $j \in [n-1]$ contains two elements, are called *degeneracy maps*, $j = 0, ...n-1$. Every map in Δ is a composition of face and degeneracy maps. Thus Δ is generated by face and degeneracy maps modulo some relations (see references above).

A *simplicial set* is a functor $X : \Delta^{op} \to \text{Sets}$ where Sets stands for the category of sets. Thus, for every integer $n \geq 0$, we are given a set X_n, and for every order preserving map $\theta : [n] \to [m]$, we are given a map of sets $\theta^* : X_m \to X_n$ such that $(\theta \circ \sigma)^* = (\sigma)^* \circ (\theta)^*$. Since Δ is generated by face and degeneracy maps, it suffices to specify θ^* for face and degeneracy maps, and to check the relations alluded to above. A map of simplicial sets $X \to Y$ is a natural transformation of functors.

A *cosimplicial space* is a functor $\Delta \to \text{Top}$, where Top stands for the category of (compactly generated Hausdorff) topological spaces (a Hausdorff topological space is compactly generated if a subset is closed iff its intersection with every compact subset is closed in that compact subset; every compact Hausdorff space, and every CW-complex is compactly generated, [ML98, VIII.8], [Whi78, I.4]). The standard cosimplicial space is the functor $\Delta_* : \Delta \to \text{Top}$ where

$$\Delta_n = \{(t_0, ..., t_n) \in \mathbb{R}^n \mid t_i \geq 0, \ \ t_0 + \cdots t_n = 1\} \subset \mathbb{R}^n$$

is equipped with the subspace topology coming from $\mathbb{R}^n$.

An order preserving map $\theta : [n] \to [m]$ defines a continuous map

$$\theta_* : \Delta_n \to \Delta_m : (s_0, ..., s_n) \mapsto (t_0, ..., t_m) \ \text{ with } \ t_i = \sum_{\theta(j)=i} s_j$$

such that $(\theta \circ \sigma)_* = \theta_* \circ \sigma_*$. The space Δ_n is homeomorphic to the usual n-dimensional ball with boundary $\partial\Delta_n = \bigcup_{0\le i\le n}(d_i)_*\Delta_{n-1} \subset \Delta_n$ homeomorphic to the $n-1$-dimensional sphere.

The *topological realization* of a simplicial set X is the quotient topological space

$$|X| = \bigsqcup_{j\ge 0} X_j \times \Delta_j / \sim$$

where the equivalence relation $\sim$ is generated by $(\theta^* x, t) = (x, \theta_* t)$. A simplex $x \in X_n$ is called *non-degenerate* if $x \notin s_j^* X_{n-1}$, $j = 0, ...n-1$. Write $X_n^{nd} \subset X_n$ for the set of non-degenerate n-simplices. Let $|X|_n \subset |X|$ be the image of $\bigsqcup_{n\ge j\ge 0} X_j \times \Delta_j$ in $|X|$. Note that $|X|_0 = X_0$. One checks that the square

$$\begin{array}{ccc} X_n^{nd} \times \partial\Delta_n & \hookrightarrow & X_n^{nd} \times \Delta_n \\ \downarrow & & \downarrow \\ |X|_{n-1} & \hookrightarrow & |X|_n \end{array}$$

is cocartesian, so that $|X|_n$ is obtained from $|X|_{n-1}$ by attaching exactly one n-cell Δ_n along its boundary $\partial\Delta_n$ for each non-degenerate n-simplex in X. In particular, $|X|$ has a structure of a CW-complex.

If X and Y are simplicial sets, the product simplicial set $X \times Y$ has n-simplices $X_n \times Y_n$ with structure maps $\theta^*(x, y) = (\theta^* x, \theta^* y)$.

A.2. **Proposition.** *The projection maps $X \times Y \to X$ and $X \times Y \to Y$ induce a map of topological spaces $|X \times Y| \to |X| \times |Y|$ which is a homeomorphism, provided the cartesian product $|X| \times |Y|$ is taken in the category of compactly generated topological spaces.*

A.3. **The classifying space of a category.** Consider the sets $[n]$ as categories with objects the integers 0, 1, ..., n. There is a unique map $i \to j$ if $i \le j$. Then a functor $[n] \to [m]$ is nothing else than an order preserving map. Thus, we can consider Δ as the category with objects the categories $[n]$, $n \ge 0$, and maps the functors $[n] \to [m]$.

Let $\mathcal{C}$ be a small category. Its nerve is the simplicial set $N_*\mathcal{C}$ whose n-simplices $N_n\mathcal{C}$ are the functors $[n] \to \mathcal{C}$. A functor $\theta : [n] \to [m]$ defines a map $N_m\mathcal{C} \to N_n\mathcal{C}$: $F \mapsto F \circ \theta$ such that $(\theta \circ \sigma)^* = (\sigma)^* \circ (\theta)^*$. An n-simplex in $N_*\mathcal{C}$, that is, a functor $[n] \to \mathcal{C}$, is nothing else than a string of maps

$$C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} C_n \tag{11}$$

in $\mathcal{C}$. The face map d_i^* deletes the object C_i and, if $i \ne 0, n$, composes the maps f_{i-1} and f_i. The degeneracy map s_i doubles C_i and inserts the identity map 1_{C_i}. In particular, the n-simplex (11) is non-degenerate iff none of the maps f_i, $i = 0, ..., n$, is an identity map.

The *classifying space* $B\mathcal{C}$ of a small category $\mathcal{C}$ is the topological realization

$$B\mathcal{C} = |N_*\mathcal{C}|$$

of the nerve $N_*\mathcal{C}$ of $\mathcal{C}$. Any functor $\mathcal{C} \to \mathcal{C}'$ induces a map on associated nerves and classifying spaces.

The classifying space construction commutes with products. This is because a functor $[n] \to \mathcal{C} \times \mathcal{C}'$ is the same as a pair of functors $[n] \to \mathcal{C}$, $[n] \to \mathcal{C}'$, so that we have $N_*(\mathcal{C} \times \mathcal{C}') = N_*\mathcal{C} \times N_*\mathcal{C}'$, hence $B(\mathcal{C} \times \mathcal{C}') = B\mathcal{C} \times B\mathcal{C}'$, by proposition A.2.

A.4. **Example** $B[1]$**.** The nerve of the category $[1]$ has two non-degenerate 0-simplices, namely the objects 0 and 1, and exactly one non-degenerate 1-simplex, namely $0 \to 1$. All other simplices are degenerate. Thus, the classifying space $B[1]$ is obtained from the two point set $\{0,1\}$ by attaching a 1-cell Δ_1 along its boundary $\partial\Delta_1$. The attachment is such that the two points of $\partial\Delta_1$ are identified with the two points $\{0,1\}$. We see that $B[1]$ is homeomorphic to the usual interval $\Delta_1 \cong [0,1]$.

A.5. **Example** $B\underline{G}$**.** For a group G, we let $\underline{G}$ be the category with one object $*$, and $\mathrm{Hom}(*,*) = G$. Then $\pi_i B\underline{G} = 0$ for $i \neq 1$ and $\pi_1 B\underline{G} = G$ where the isomorphism $G \to \pi_1 B\underline{G}$ sends an element $g \in G$ to the loop l_g represented by the morphism $g : * \to *$. For details, see for instance [Wei94, Exercise 8.2.4, Example 8.3.3].

A.6. **Lemma.** *A natural transformation $\eta : F_0 \to F_1$ between functors $F_0, F_1 : \mathcal{C} \to \mathcal{C}'$ induces a homotopy $BF_0 \simeq BF_1$ between the associated maps $BF_0, BF_1 : B\mathcal{C} \to B\mathcal{C}'$ on classifying spaces. In particular, an equivalence of categories $\mathcal{C} \to \mathcal{C}'$ induces a homotopy equivalence $\mathcal{B}\mathcal{C} \to \mathcal{B}\mathcal{C}'$.*

Proof. A natural transformation $\eta : F_0 \to F_1$ defines a functor $H : [1] \times \mathcal{C} \to \mathcal{C}'$ which sends the object (i, X), $i = 0, 1$, $X \in \mathcal{C}$ to $F_i(X)$. There are two types of morphisms in $[1] \times \mathcal{C}$, namely (id_i, f) and $(0 \to 1, f)$ where $i = 0, 1$ and $f : X \to Y$ is a map in $\mathcal{C}$. They are sent to $F_i(f)$, $i = 0, 1$ and $\eta_Y F_0(f) = F_1(f)\eta_X$. It is easy to check that H is indeed a functor. Now H induces a map $[0,1] \times B\mathcal{C} = B[1] \times B\mathcal{C} = B([1] \times \mathcal{C}) \to B\mathcal{C}'$ whose restriction to $\{0\} \times B\mathcal{C}$ and $\{1\} \times B\mathcal{C}$ are BF_0 and BF_1. Thus BF_0 and BF_1 are homotopic maps.

If $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories, then there is a functor $G : \mathcal{C}' \to \mathcal{C}$ and natural isomorphisms $FG \cong 1$ and $1 \cong GF$. Thus $BG : B\mathcal{C}' \to B\mathcal{C}$ is a homotopy inverse of BF. □

Appendix B. Homotopy fibres and homotopy fibrations

We recall the definition of the homotopy fibre (or mapping fibre) of a map of spaces [Whi78, chapter I.7]. Let $g : Y \to Z$ be a map of pointed topological spaces. The homotopy fibre $F(g)$ of g is the pointed topological space

$$F(g) = \{(\gamma, y) |\ \gamma : [0,1] \to Z \ \text{ s.t. } \ \gamma(0) = *, \gamma(1) = g(y)\} \subset Z^{[0,1]} \times Y$$

with base point the pair $(*, *)$, where the first $*$ is the constant path $t \mapsto *$, $t \in [0,1]$. There is a continuous map of pointed spaces $F(g) \to Y : (\gamma, y) \mapsto y$ which fits into a natural long exact sequence of homotopy groups [Whi78, Corollary IV.8.9]

$$\cdots \to \pi_{i+1} Z \to \pi_i F(g) \to \pi_i Y \to \pi_i Z \to \pi_{i-1} F(g) \to \cdots \qquad (12)$$

ending in $\pi_0 Y \to \pi_0 Z$.

A sequence of pointed spaces $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that the composition is the constant map to the base point of Z is called *homotopy fibration* if the natural map $X \to F(g) : x \mapsto (*, x)$ is a homotopy equivalence. In this case, there is a long exact sequence of homotopy groups as in (12) with X in place of $F(g)$.

Appendix C. Background on triangulated categories

C.1. **Definition.** References here are [Ver96] and [Kel96]. A *triangulated category* is an additive category $\mathcal{A}$ together with an auto-equivalence $T : \mathcal{A} \to \mathcal{A}$ and a class of sequences

$$X \stackrel{u}{\to} Y \stackrel{v}{\to} Z \stackrel{w}{\to} TX \tag{13}$$

of maps in $\mathcal{A}$ called distinguished triangles. They are to satisfy the axioms TR1 – TR4 below.

- TR1 Every sequence of the form (13) which is isomorphic to a distinguished triangle is a distinguished triangle. For every object A of $\mathcal{A}$, the sequence $A \stackrel{1}{\to} A \to 0 \to TA$ is a distinguished triangle. Every map $u : X \to Y$ in $\mathcal{A}$ is part of a distinguished triangle (13).
- TR2 A sequence (13) is distinguished if and only if $Y \stackrel{v}{\to} Z \stackrel{w}{\to} TX \stackrel{-Tu}{\to} TY$ is a distinguished triangle.
- TR3 For any two distinguished triangles $X \stackrel{u}{\to} Y \stackrel{v}{\to} Z \stackrel{w}{\to} TX$ an $X' \stackrel{u'}{\to} Y' \stackrel{v'}{\to} Z' \stackrel{w'}{\to} TX'$ and any pair of maps $f : X \to X'$, $g : Y \to Y'$ such that $gu = u'f$ there is a map $h : Z \to Z'$ such that $hv = v'g$ and $(Tf)w = w'h$.
- TR4 Octahedron axiom, see [Ver96], [Kel96]

In a distinguished triangle (13) the object Z is determined by the map u up to (non-canonical) isomorphism. We call Z "the" cone of u.

C.2. **Example.** Let $\mathcal{A}$ be an additive category, and let $\mathcal{K}(\mathcal{A})$ be the homotopy category of chain complexes in $\mathcal{A}$. Its objects are chain complexes in $\mathcal{A}$, and maps are chain maps up to chain homotopy. The category $\mathcal{K}(\mathcal{A})$ is a triangulated category where a sequence is a distinguished triangle if it is isomorphic in $\mathcal{K}(\mathcal{A})$ to a cofibre sequence

$$X \stackrel{f}{\to} Y \stackrel{j}{\to} C(f) \stackrel{q}{\to} TX.$$

Here, $C(f)$ is the mapping cone of the chain map $f : X \to Y$, which has $C(f)^i = Y^i \oplus X^{i+1}$ and differential $d^i = \begin{pmatrix} d_Y & f \\ 0 & -d_X \end{pmatrix}$, and TX is the "shift" of X which has $(TX)^i = X^{i+1}$ and differential $d^i = -d_X^{i+1}$. The maps $j : Y \to C(f)$ and $q : C(f) \to TX$ are the canonical inclusions and projections in each degree.

C.3. **Calculus of fractions.** Let $\mathcal{C}$ be a category and $w \subset \operatorname{Mor}\mathcal{C}$ a class of morphisms in $\mathcal{C}$. The localization of $\mathcal{C}$ with respect to w is the category obtained from $\mathcal{C}$ by formally inverting the morphisms in w, that is, it is the category $\mathcal{C}[w^{-1}]$ together with a functor $\mathcal{C} \to \mathcal{C}[w^{-1}]$ which satisfies the following universal property. For any functor $\mathcal{C} \to \mathcal{D}$ which sends maps in w to isomorphisms, there is a unique functor $\mathcal{C}[w^{-1}] \to \mathcal{D}$ such that the composition $\mathcal{C} \to \mathcal{C}[w^{-1}] \to \mathcal{D}$ is the given functor $\mathcal{C} \to \mathcal{D}$. In general, the category $\mathcal{C}[w^{-1}]$ may, or may not exist. It always exists if $\mathcal{C}$ is a small category.

If the class w satisfies a "calculus of right (or left) fractions", there is an explicit description of $\mathcal{C}[w^{-1}]$ as we shall explain now. A class w of morphisms in a category $\mathcal{C}$ is said to satisfy a *calculus of right fractions* if

(a) it is closed under composition, and $1_X \in w$ for every object X of $\mathcal{C}$.
(b) For all pairs of maps $u : X \to Y$, $s : Z \to Y$ with $s \in w$, there are maps $v : W \to Z$, $t : W \to X$ with $t \in w$ and $sv = ut$.

(c) For any three maps $f, g : X \to Y$, $s : Y \to Z$ with $s \in w$ and $sf = sg$ there is a map $t : W \to X$ with $t \in w$ and $ft = gt$.

If w satisfies the dual of (a) – (c) then w is said to satisfy a *calculus of left fractions*. If w satisfies both, a calculus of left and right fractions, then w is said to satisfy a *calculus of fractions*.

If w satisfies a calculus of right fractions, then the localized category $\mathcal{C}[w^{-1}]$ has the following description. Objects are the same as in $\mathcal{C}$. A map $X \to Y$ in $\mathcal{C}[w^{-1}]$ is an equivalence class of data $X \stackrel{s}{\leftarrow} M \stackrel{f}{\to} Y$ written as a "right fraction" fs^{-1}, where f, s are maps in $\mathcal{C}$ and $s \in w$. The datum fs^{-1} is equivalent to the datum $X \stackrel{t}{\leftarrow} N \stackrel{g}{\to} Y$ iff there are map $\bar{s} : P \to N$ and $\bar{t} : P \to M$ with $\bar{s}$ (or $\bar{t}$) in w and such that $s\bar{t} = \bar{s}t$ and $f\bar{t} = g\bar{s}$. Composition $(fs^{-1})(gt^{-1})$ is defined as follows. By (b) above, there are maps h, r in $\mathcal{C}$ with $r \in w$ and $sh = gr$. Then $(fs^{-1})(gt^{-1}) = (fh)(tr)^{-1}$.

In this description, it is not clear whether or not $\mathrm{Hom}_{\mathcal{C}[w^{-1}]}(X, Y)$ is actually a set. It is a set if $\mathcal{C}$ is a small category, but in general, this issue has to be dealt with separately.

C.4. **Verdier quotient.** Let $\mathcal{A}$ be a triangulated category and $\mathcal{B} \subset \mathcal{A}$ a full triangulated subcategory. The class w of maps whose cones are (isomorphic to objects) in $\mathcal{B}$ satisfies a calculus of fractions. The Verdier quotient $\mathcal{A}/\mathcal{B}$ is, by definition, the localized category $\mathcal{A}[w^{-1}]$. It is a triangulated category where a sequence is a distinguished triangle if it is isomorphic to the image of a distinguished triangle of $\mathcal{A}$.

C.5. **Example.** Let $\mathcal{A}$ be an abelian category. Its (unbounded) derived category $\mathcal{D}(\mathcal{A})$ is obtained from the category $\mathrm{Ch}\,\mathcal{A}$ of chain complexes in $\mathcal{A}$ by formally inverting the quasi-isomorphisms (a chain map is a quasi-isomorphism if it induces an isomorphism in homology). Since homotopy equivalences are quasi-isomorphisms, $\mathcal{D}(\mathcal{A})$ is also obtained from $\mathcal{K}(\mathcal{A})$ by formally inverting quasi-isomorphisms. Let $\mathcal{K}_{ac}(\mathcal{A}) \subset \mathcal{K}(\mathcal{A})$ be the full subcategory of acyclic chain complexes (that is, those chain complexes which have trivial homology). It is closed under cones, and a chain complex A is acyclic iff TA is. Therefore, $\mathcal{K}_{ac}(\mathcal{A})$ is a full triangulated subcategory of $\mathcal{K}(\mathcal{A})$. Since a map is a quasi-isomorphism iff its cone is acyclic, we see that $\mathcal{D}(\mathcal{A})$ is the Verdier quotient $\mathcal{K}(\mathcal{A})/\mathcal{K}_{ac}(\mathcal{A})$. In particular, $D(\mathcal{A})$ is a triangulated category (provided it exists, that is, has small hom sets).

C.6. **Exercise.** Let $\mathcal{A}$ be a (idempotent complete) triangulated category, and $\mathcal{A}_0, \mathcal{A}_1 \subset \mathcal{A}$ full (idempotent complete) triangulated subcategories. Assume that $\mathrm{Hom}(A_0, A_1) = 0$ for all objects $A_0 \in \mathcal{A}_0$ and $A_1 \in \mathcal{A}_1$. If $\mathcal{A}$ is generated as a (idempotent complete) triangulated category by the union of $\mathcal{A}_0$ and $\mathcal{A}_1$, then the the composition $\mathcal{A}_1 \subset \mathcal{A} \to \mathcal{A}/\mathcal{A}_0$ is an equivalence. Moreover, an inverse induces a left adjoint $\mathcal{A} \to \mathcal{A}/A_0 \cong \mathcal{A}_1$ to the inclusion $\mathcal{A}_1 \subset \mathcal{A}$.

C.7. **Grothendieck abelian categories.** References here are [Spa88] and [Fra01]. Let $\mathcal{A}$ be a Grothendieck abelian category (that is, an abelian category which has all set-indexed colimits, filtered colimits are exact, and has a generator (U is a generator if for every object X there is a surjection $\bigoplus_I U \to X$ with I some index set; a set of objects is called "set of generators" if their direct sum is a generator). Then $D\mathcal{A}$ has small hom sets [Fra01]. It has the following explicit description.

A $\mathcal{K}$-injective complex is a complex $I \in \mathrm{Ch}\,\mathcal{A}$ such that for every map $f : X \to I$ and every quasi-isomorphism $s : X \to Y$ there is a unique map (up to homotopy) $g : Y \to I$ such that $gs = f$ in $\mathcal{K}(\mathcal{A})$. This is equivalent to the requirement that $\mathrm{Hom}_{\mathcal{K}\mathcal{A}}(A, I) = 0$ for all acyclic chain complexes A. For instance, a bounded below chain complex of injective objects in $\mathcal{A}$ is $\mathcal{K}$-injective. But $\mathcal{K}$-injective chain complexes don't need to consist of injective objects (for instance every contractible chain complex is $\mathcal{K}$-injective), nor does an unbounded chain complex of injective objects need to be $\mathcal{K}$-injective.

In a Grothendieck abelian category, every chain complex A has a $\mathcal{K}$-injective resolution, that is, admits a quasi-isomorphism $A \to I$ with I a $\mathcal{K}$-injective complex [Fra01].

Let $\mathcal{K}_{inj}(\mathcal{A}) \subset \mathcal{K}(\mathcal{A})$ be the full subcategory of $\mathcal{K}$-injective chain complexes. It is a triangulated subcategory. Using the fact that every chain complex in $\mathcal{A}$ has a $\mathcal{K}$-injective resolution, it is easy to see that the composition $\mathcal{K}_{inj}(\mathcal{A}) \subset \mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is an equivalence.

C.8. **The derived category of quasi-coherent sheaves.** Let X be a quasi-compact and separated scheme. In the category $\mathrm{Qcoh}(X)$ of quasi-coherent O_X-modules, all small colimits exists, and filtered colimits are exact (as they can be calculated locally). Every quasi-coherent O_X-module is a filtered colimit of its quasi-coherent submodules of finite type [Gro60, 9.4.9]. Therefore, the set of quasi-coherent O_X-module of finite type forms a set of generators for $\mathrm{Qcoh}(X)$. Hence, the category $\mathrm{Qcoh}(X)$ is a Grothendieck abelian category. In particular, its derived category $\mathcal{D}\,\mathrm{Qcoh}(X)$ exists, and has an explicit description as in C.7.

C.9. **Compactly generated triangulated categories.** References here are [Nee96] and [Nee92]. Let $\mathcal{A}$ be a triangulated category in which all (set indexed) direct sums exist. An object A of $\mathcal{A}$ is called *compact* if the canonical map

$$\bigoplus_{i\in I} \mathrm{Hom}(A, E_i) \to \mathrm{Hom}(A, \bigoplus_{i\in I} E_i)$$

is an isomorphism, for any set of objects E_i, $i \in I$. Let $\mathcal{A}^c \subset \mathcal{A}$ be the full subcategory of compact objects. It is easy to see that $\mathcal{A}^c$ is an idempotent complete triangulated subcategory of $\mathcal{A}$.

A set $S = \{A_i, i \in I\}$, of compact objects is said to generate $\mathcal{A}$ (or $\mathcal{A}$ is compactly generated (by S)) if for every object $E \in \mathcal{A}$ we have

$$\mathrm{Hom}(A_i, E) = 0 \ \ \forall i \in I \implies E = 0$$

C.10. **Theorem** [Nee92].

(a) *Let $\mathcal{A}$ be a compactly generated triangulated category with generating set S of compact objects. Then $\mathcal{A}^c$ is the smallest idempotent complete triangulated subcategory of $\mathcal{A}$ containing S.*

(b) *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be an exact sequence of compactly generated triangulated categories. Assume that the functors in the sequence preserve direct sums and compact objects. Then the sequence*

$$\mathcal{A}^c \to \mathcal{B}^c \to \mathcal{C}^c$$

of triangulated categories is exact up to factors.

Let $j : U \subset X$ be an open immersion of quasi-compact and separated schemes, and let $Z = X - U$ be the closed complement. Write $\mathcal{D}_Z \operatorname{Qcoh}(X) \subset \mathcal{D} \operatorname{Qcoh}(X)$ for the full subcategory of those complexes A which have support in Z, that is, which are acyclic over U: $Lj^*(A) = 0$. It is obviously a triangulated subcategory of $\mathcal{D} \operatorname{Qcoh}(X)$. For a proof of the following theorem, see [TT90, proof of Theorem 5.1] or [Nee96].

C.11. **Theorem** (Thomason). *Let X be a quasi-compact and separated scheme, and let $U \subset X$ be a quasi-compact open subscheme with closed complement $Z = X - U$. Then $\mathcal{D}_Z \operatorname{Qcoh}(X)$ is compactly generated by the (essentially small) subcategory of perfect complexes on X which are acyclic over U. In particular, $\mathcal{D} \operatorname{Perf}_Z(X) \cong \mathcal{D}^c_Z \operatorname{Qcoh}(X)$.*

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Marco Schlichting, Department of Mathematics, Louisiana State University, Baton Rouge, LA

E-mail address: mschlich@math.lsu.edu