BASIC ALGEBRAIC AND TOPOLOGICAL K-THEORY

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Abstract. These notes, prepared for a minicourse given in Swisk, the Sedano Winter School on K-theory held in Sedano, Spain, during the week January 22–27 of 2007, are themselves based on a course given in the University of Buenos Aires during the second semester of 2006. They intend to be an introduction to K-theory, with emphasis in the comparison between its algebraic and topological variants. We have tried to keep as elementary as possible. The first lecture introduces K_0, K_1 and $K_{\leq}0$, and discusses excision. Lecture II introduces K^{top} for Banach algebras, and sketches Cuntz' proof of Bott periodicity for C^*-algebras, via the C^*-Toeplitz extension. The parallelism between Bott periodicity and the fundamental theorem for nonpositive K-theory is emphasized by the use of the algebraic Toeplitz extension in the proof of the latter. Then an elementary definition of homotopy algebraic K-theory is given, and its basic properties are proved. In Lecture III we introduce Quillen's algebraic K-theory by means of the plus construction, and sketch a proof of the fundamental theorem using the algebraic Toeplitz extension. The excision theorems of Suslin and Wodzicki are discussed. After recalling the homotopy invariance theorem for C^*-algebras, we give a proof of Karoubi's conjecture in both the C^* and Banach cases. Next we review the spectrum definition of KH and of diffeotopy K-theory and the associated spectral sequences. The lecture ends with the homotopy invariance theorems for locally convex algebras and a proof of the isomorphism between KH and K^{top} in the stable case. In Lecture IV we introduce the various characters connecting K-theory to cyclic homology and prove an exact sequence which shows that for stable locally convex algebras, algebraic cyclic homology measures the obstruction for the comparison map $K \to K^{\mathrm{top}}$ to be an isomorphism. Then we discuss some open problems in connection with this exact sequence. Finally we review bivariant algebraic K-theory, its relation with KH, and the definition of the bivariant Chern character.

1. Lecture I

Notations 1.0.1. Throughout these notes, A, B, C will be rings and R, S, T will be rings with unit.

Let R be a ring with unit. Write M_nR for the matrix ring. Regard $M_nR \subset M_{n+1}R$ via

$$a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \tag{1}$$

Put

$$M_\infty R = \bigcup_{n=1}^{\infty} M_nR$$

Note $M_\infty R$ is a ring (without unit). We write $\mathrm{Idem}R$ for the set of idempotent elements of $M_\infty R$. Thus

$$M_\infty R \supset \mathrm{Idem}R := \{e : e^2 = e\} = \bigcup_{n=1}^{\infty} \mathrm{Idem}_nR.$$

We write $\mathrm{GL}_nR \subset M_nR$ for the group of invertible matrices. Regard $\mathrm{GL}_nR \subset \mathrm{GL}_{n+1}R$ via

$$g \mapsto \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}$$

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Put

$$\mathrm{GL}R := \bigcup_{n=1}^{\infty} \mathrm{GL}_n R$$

Note $\mathrm{GL}R$ acts by conjugation in $M_\infty R$, $\mathrm{Idem}R$ and, of course, $\mathrm{GL}R$.
For $a, b \in M_\infty R$ there is defined the *direct sum*

$$a \oplus b := \begin{bmatrix} a_{1,1} & 0 & a_{1,2} & 0 & a_{1,3} & 0 & \dots \\ 0 & b_{1,1} & 0 & b_{1,2} & 0 & b_{1,3} & \dots \\ a_{2,1} & 0 & a_{2,2} & 0 & a_{2,3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{2}$$

We remark that if $a \in M_p R$ and $b \in M_q R$ then $a \oplus b \in M_{p+q} R$ and is conjugate, by a permutation matrix, to the usual direct sum

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

One checks that $\oplus$ is associative and commutative up to conjugation. Thus the coinvariants under the conjugation action

$$I(R) := (\mathrm{Idem}(R)_{\mathrm{GL}R}, \oplus)$$

form an abelian monoid.

Lemma 1.0.2. *Let M be an abelian monoid. Then there exist an abelian group M^+ and a monoid homomorphism $M \to M^+$ such that if $M \to G$ is any other such homomorphism, then there exists a unique group homomorphism $M^+ \to G$ such that*

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & M^+ \arrow[d, dotted] \\ & G \end{tikzcd}$$

commutes.

Proof. Let F be the free abelian group on one generator e_m for each $m \in M$, and let $S \subset F$ be the subgroup generated by all elements of the form $e_{m_1} + e_{m_2} - e_{m_1+m_2}$. One checks that $M^+ = F/S$ satisfies the desired properties. $\square$

Definition 1.0.3.

$$K_0(R) := I(R)^+$$

$$K_1(R) := \mathrm{GL}R_{\mathrm{GL}R} = \frac{\mathrm{GL}R}{[\mathrm{GL}R, \mathrm{GL}R]} = \mathrm{GL}R_{ab}.$$

Facts 1.0.4. (see [29, Section 2.1])

- $[\mathrm{GL}R, \mathrm{GL}R] = ER := < 1 + ae_{i,j} : a \in R, i \neq j >$, the subgroup of $\mathrm{GL}R$ generated by elementary matrices.
- If $\alpha \in \mathrm{GL}_n R$ then

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \in E_{2n} R \qquad \text{(Whitehead's Lemma).}$$

As a consequence of the last fact, if $\beta \in \mathrm{GL}_n R$, then

$$\begin{aligned} \alpha\beta &= \begin{bmatrix} \alpha\beta & 0 \\ 0 & 1_{n\times n} \end{bmatrix} \\ &= \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{bmatrix} \\ &\equiv \alpha \oplus \beta \quad \mathrm{Mod} ER. \end{aligned} \tag{3}$$

1.1. **The maps $\mathbb{Z} \to K_0R$ and $R^*_{ab} \to K_1R$.** Let $r \geq 1$. Then

$$p_r = 1_{r\times r} \in \mathrm{Idem}R.$$

Also since $p_r \oplus p_s = p_{r+s}$, $r \mapsto p_r$ defines a monoid homomorphism $\mathbb{N} \to I(R)$. Applying the group completion functor we obtain a group homomorphism

$$\mathbb{Z} = \mathbb{N}^+ \to I(R)^+ = K_0R. \tag{4}$$

Similarly, the inclusion $R^* = \mathrm{GL}_1R \subset \mathrm{GL}R$ induces a homomorphism

$$R^*_{ab} \to K_1R \tag{5}$$

Example 1.1.1. If F is a field, and $e \in \mathrm{Idem}F$ is of rank r, then e is conjugate to p_r; moreover p_r and p_s are conjugate $\iff$ $r = s$. Thus (4) is an isomorphism in this case. Assume more generally that R is commutative. Then (4) is a split monomorphism. Indeed, there exists a surjective unital homomorphism $R \twoheadrightarrow F$ onto a field F; the induced map $K_0(R) \to K_0(F) = \mathbb{Z}$ is a left inverse of (4). Similarly, (5) is a split monomorphism, split by the map $det : K_1R \to R^*$ induced by the determinant.

Example 1.1.2. The following are examples of rings for which the maps (4) and (5) are isomorphisms: fields, division rings, principal ideal domains (PIDs) and local rings. Recall that a ring R is a *local ring* if $R\backslash R^*$ is an ideal of R. For instance if k is a field, then the k-algebra $k[\epsilon] := k \oplus k\epsilon$ with $\epsilon^2 = 0$ is a local ring. Indeed $k[\epsilon]^* = k^* + k\epsilon$ and $k[\epsilon]\backslash k[\epsilon]^* = k\epsilon \triangleleft k[\epsilon]$.

Example 1.1.3. Here is an example of a local ring involving operator theory. Let H be a separable Hilbert space over $\mathbb{C}$; put $\mathcal{B} = \mathcal{B}(H)$ for the algebra of bounded operators. Write $\mathcal{K} \subset \mathcal{B}$ for the ideal of compact operators, and $\mathcal{F}$ for that of finite rank operators. The Riesz-Schauder theorem from elementary operator theory implies that if $\lambda \in \mathbb{C}^*$ and $T \in \mathcal{K}$ then there exists an $f \in \mathcal{F}$ such that $\lambda + T + f$ is invertible in $\mathcal{B}$. In fact one checks that if $\mathcal{F} \subset I \subset \mathcal{K}$ is an ideal of $\mathcal{B}$ such that $T \in I$ then the inverse of $\lambda + T + f$ is again in $\mathbb{C} \oplus I$. Thus the ring

$$R_I := \mathbb{C} \oplus I/\mathcal{F}$$

is local.

1.2. **Infinite sum rings.** Recall from [35] that a *sum ring* is a unital ring R together with elements α_i, β_i, $i = 0, 1$ such that the following identities hold

$$\begin{aligned} \alpha_0\beta_0 = \alpha_1\beta_1 = 1 \\ \beta_0\alpha_0 + \beta_1\alpha_1 = 1 \end{aligned} \tag{6}$$

If R is a sum ring, then

$$\begin{aligned} \boxplus : R \times R \to R, \\ (a, b) \mapsto a \boxplus b = \beta_0 a\alpha_0 + \beta_1 b\alpha_1 \end{aligned} \tag{7}$$

is a unital homomorphism. In fact Wagoner has shown that if $a, b \in R$ then there is a matrix $Q \in \mathrm{GL}_3R$ which conjugates $a \boxplus b$ to $a \oplus b$ ([35, page 355]). But as K_n is a matrix stable functor of unital rings, inner automorphisms induce the identity on K_nR ([6, 5.1.2]), whence $\oplus$ and $\boxplus$ are the same operation on K_nR. Thus $\boxplus$ is the sum in K_nR. An *infinite sum ring* is a sum ring R together with a unit preserving ring homomorphism $\infty : R \to R$, $a \mapsto a^\infty$ such that

$$a \boxplus a^\infty = a^\infty \qquad (a \in R). \tag{8}$$

Because $\boxplus$ induces the group operation in K_nR, it follows that $K_nR = 0$ $(n = 0, 1)$.

Example 1.2.1. Let A be a ring. Write ΓA for the ring of all $\mathbb{N} \times \mathbb{N}$ matrices $a = (a_{i,j})_{i,j\geq 1}$ which satisfy the following two conditions:

(i) The set $\{a_{ij}, i, j \in \mathbb{N}\}$ is finite.
(ii) There exists a natural number $N \in \mathbb{N}$ such that each row and each column has at most N nonzero entries.

It is an exercise to show that ΓA is indeed a ring and that $M_\infty A \subset \Gamma A$ is an ideal. The ring ΓA is called (Karoubi's) *cone ring*; the quotient $\Sigma A := \Gamma A/M_\infty A$ is the *suspension* of A. Assume A has a unit; one checks that the the following elements of ΓA

$$\alpha_1 = \sum_{i=0}^{\infty} e_{i,2i}, \quad \beta_1 = \sum_{i=0}^{\infty} e_{2i,i}, \quad \alpha_2 = \sum_{i=0}^{\infty} e_{i,2i+1}, \quad \text{and} \quad \beta_2 = \sum_{i=0}^{\infty} e_{2i+1,i}.$$

satisfy the identities (6). Let $a \in \Gamma A$. Because the map $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(k,i) \mapsto 2^{k+1}i + 2^k - 1$, is injective, the following assignment defines gives a well-defined, $\mathbb{N} \times \mathbb{N}$-matrix

$$\phi^\infty(a) = \sum_{k=0}^{\infty} \beta_2^k \beta_1 a \alpha_1 \alpha_2^k = \sum_{k,i,j} e_{2^{k+1}i+2^k-1, 2^{k+1}j+2^k-1} \otimes a_{i,j}. \tag{9}$$

One checks that ϕ^∞ is a ring endomorphism of ΓA which satisfies (8) (see [6, 4.8.2]). In particular $K_n \Gamma A = 0$. A further useful fact about Γ and Σ is that the well-known isomorphism $M_\infty \mathbb{Z} \otimes A \cong M_\infty A$ extends to Γ, so that there are isomorphisms (see [6, 4.7.1])

$$\Gamma\mathbb{Z} \otimes A \xrightarrow{\cong} \Gamma A \text{ and } \Sigma\mathbb{Z} \otimes A \xrightarrow{\cong} \Sigma A.$$

Exercise 1.2.2. Let $\mathcal{B}$ and H be as in Example 1.1.3. Choose a Hilbert basis $\{e_i\}_{i\geq 1}$ of H, and regard $\mathcal{B}$ as a ring of $\mathbb{N} \times \mathbb{N}$ matrices. With these identifications, show that $\mathcal{B} \supset \Gamma\mathbb{C}$. Deduce from this that $\mathcal{B}$ is a sum ring. Further show that (9) extends to $\mathcal{B}$, so that the latter is in fact an infinite sum ring.

1.3. **Basic properties of K_n for $n = 0, 1$.**

- *Additivity*: If R_1 and R_2 are unital rings, then each of the projections $R := R_1 \times R_2 \to R_i$ induces a group homomorphism $K_n(R) \to K_n(R_i)$, which, when added up yield a homomorphism
$$K_n(R) \to K_n(R_1) \oplus K_n(R_2)$$
One checks that this map is in fact an isomorphism.
Application: extension to nonunital rings. If A is any (not necessarily unital) ring, then the abelian group $\tilde{A} = A \oplus \mathbb{Z}$ equipped with the multiplication
$$(a+n)(b+m) := ab + nm \qquad (a, b \in A, \quad n, m \in \mathbb{Z})$$
is a unital ring, with unit element $1 \in \mathbb{Z}$, and $\tilde{A} \to \mathbb{Z}$, $a+n \mapsto n$, is a unital homomorphism. Put
$$K_n(A) := \ker(K_n\tilde{A} \to K_n\mathbb{Z})$$
If A happens to have a unit, we have two definitions for K_nA. To check that they are the same, one observes that the map
$$\tilde{A} \to A \times \mathbb{Z}, \quad a + n \mapsto (a + n \cdot 1, n)$$
is a unital isomorphism. Under this isomorphism, $\tilde{A} \to \mathbb{Z}$ identifies with $A \times \mathbb{Z} \to \mathbb{Z}$, and $\ker(K_n(\tilde{A}) \to K_n\mathbb{Z})$ with $\ker(K_nA \oplus K_n\mathbb{Z} \to K_n\mathbb{Z}) = K_nA$.

- *Matrix stability*: We have a canonical isomorphism $K_n(M_pR) \cong K_n(R)$. The isomorphism is induced by the (nonunital) ring homomorphism (1).

- *Continuity*: K_n preserves colimits of filtered systems of (not necessarily unital) rings; that is, the canonical map
$$\operatorname*{colim}_i K_n(A_i) \to K_n(\operatorname*{colim}_i A_i)$$
is an isomorphism.
Note that matrix stability combined with continuity yields
$$K_n(M_\infty R) = K_nR.$$

- *Nilinvariance for* K_0: If $I \triangleleft R$ is a nilpotent ideal, then $K_0(R) \to K_0(R/I)$ is an isomorphism. This property is a consequence of the well-known fact that nilpotent extensions admit idempotent liftings, and that any two liftings of the same idempotent are conjugate. Note that K_1 does not have the same property, as the following example shows.

  Example 1.3.1. Let k be a field. Then by 1.1.2, $K_1(k[\epsilon]) = k^* + k\epsilon$ and $K_1(k) = k^*$. Thus $k[\epsilon] \to k[\epsilon]/\epsilon k[\epsilon] = k$ does not become an isomorphism under K_1.

Remark 1.3.2. The functor GL : $\mathrm{Rings}_1 \to \mathfrak{Grp}$ preserves products. Hence it extends to all rings by

$$\mathrm{GL}(A) := \ker(\mathrm{GL}(A \oplus \mathbb{Z}) \to \mathrm{GL}\mathbb{Z})$$

It is a straightforward exercise to show that, with this definition, GL becomes a left exact functor in Rings; thus if $A \triangleleft B$ is an ideal embedding, then $\mathrm{GL}(A) = \ker(\mathrm{GL}(B) \to \mathrm{GL}(B/A))$. It is straightforward from this that the group K_1A defined above can be described as

$$K_1A = \mathrm{GL}(A)/E(\tilde{A}) \cap \mathrm{GL}(A)$$

A little more work shows that $E(\tilde{A}) \cap \mathrm{GL}(A)$ is the smallest normal subgroup of $E(\tilde{A})$ generated by the elementary matrices $1 + ae_{i,j}$ with $a \in A$ [29, 2.5].

1.4. **Excision.** The reason one considers K_0 and K_1 as part of the same theory is that they are connected by a long exact sequence, as shown by the following theorem.

Theorem 1.4.1. *Let*

$$0 \to A \to B \to C \to 0 \tag{10}$$

be an exact sequence of rings. Then there is a long exact sequence

$$\begin{tikzcd} K_1A \arrow[r] & K_1B \arrow[r] & K_1C \arrow[d, "\partial"] \\ K_0C & K_0B \arrow[l] & K_0A \arrow[l] \end{tikzcd}$$

Remark 1.4.2. Let $g \in \mathrm{GL}_n(C) = \ker(\mathrm{GL}_n(\tilde{C}) \to \mathrm{GL}_n(\mathbb{Z}))$, and let $\hat{g}, \hat{g}^* \in M_n\tilde{B}$ be liftings of g and g^{-1}. Then p_n and its conjugate under

$$h := \begin{bmatrix} 2\hat{g} - \hat{g}\hat{g}^*\hat{g} & \hat{g}\hat{g}^* - 1 \\ 1 - \hat{g}^*\hat{g} & \hat{g}^* \end{bmatrix} \in GL_{2n}(\tilde{B})$$

both go to the class of p_n in $K_0(C)$. The connecting map ∂ of the long exact sequence above sends the class of g to the difference $[hp_nh^{-1}] - [p_n] \in \ker(K_0(\tilde{A}) \to K_0(\mathbb{Z})) = K_0(A)$.

Corollary 1.4.3. *Assume* (10) *is split by a ring homomorphism* $C \to B$. *Then* $K_0(A) \to K_0(B)$ *is injective, and induces an isomorphism*

$$K_0(A) = \ker(K_0(B) \to K_0(C))$$

Because of this we say that K_0 *is* split exact.

Remark 1.4.4. (Swan's example [33]) K_1 is not split exact. To see this, let k be a field, and consider the ring of upper triangular matrices

$$T := \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$$

The set I of strictly upper triangular matrices forms an ideal of T, isomorphic as a ring, to the ideal $k\epsilon \triangleleft k[\epsilon]$. By Example 1.1.2, $\ker(K_1(k[\epsilon]) \to K_1(k)) \cong k\epsilon$, the additive group underlying k. If K_1 were split exact, then also $\ker(K_1(T) \to K_1(k \times k))$ should be isomorphic to k. However we shall see presently that

$$\ker(K_1(T) \to K_1(k \times k)) = 0$$

It suffices to show that the inclusion $k \times k \subset T$

$$(\lambda, \mu) \mapsto \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

induces a surjection $K_1(k \times k) \to K_1(T)$. Note that if $g \in \mathrm{GL}_n T$, then its image in $M_n(k \times k)$ must be invertible. Thus each column and each row of g must have a coefficient (which is an element of T and thus a matrix) such that both its diagonal coefficients are nonzero. Note any such an element of T is invertible. Using row and column operations (which do not change the class in K_1) we can replace g by a diagonal matrix in the same conjugacy class. Using (3), we can further replace the diagonal matrix by a 1×1 matrix. Thus $T^*_{ab} \to K_1(T)$ is onto. To finish, we must show that if $\lambda \in k$ then the element $1 + \lambda e_{12} \in T^*$ is in ET. We leave this as an exercise.

Exercise 1.4.5. Use the previous corollary to prove that the matrix invariance and nilinvariance properties of K_0 cited above remain valid for nonunital rings, and that $K_0(\Gamma A) = 0$ for all rings A.

Example 1.4.6. Let R be a unital ring. Applying the theorem above to the cone sequence

$$0 \to M_\infty R \to \Gamma R \to \Sigma R \to 0 \tag{11}$$

we obtain an isomorphism

$$K_1(\Sigma R) = K_0(R). \tag{12}$$

The example above motivates the following definition.

Definition 1.4.7. Let A be a ring and $n \geq 0$. Put

$$K_{-n} A := K_0(\Sigma^n A).$$

Properties of $K_{\leq 0}$ 1.4.8.

- Using the fact that Σ preserves exact sequences, the isomorphism (12) and Theorem 1.4.1, one shows, first for $B \to C$ unital and then in general, that the sequence of 1.4.1 extends to all negative K_n. In particular it follows that K_n is split exact.
- From the fact, mentioned above, that $\Sigma A = \Sigma \mathbb{Z} \otimes A$, and that similarly $M_p A = M_p \mathbb{Z} \otimes A$, it follows that $\Sigma M_p A = M_p \Sigma A$. Taking this into consideration it is immediate that the isomorphism $K_0(M_p A) = K_0(A)$ is valid for all K_n with $n \leq 0$.
- Since tensor products preserve filtering colimits, the functor Σ does. Since K_0 is continous, it follows that the same is true for K_n $(n \leq 0)$.
- If $I \triangleleft A$ is a nilpotent ideal, then $K_n A \to K_n(A/I)$ is an isomorphism for all $n \leq 0$.
- Recall a noetherian unital ring R is called *regular* if every finitely generated R-module has a projective resolution of finite length. It is a theorem of Bass that if R is noetherian regular, then $K_n R = 0$ for $n < 0$ (see Schlichting's lectures).

Example 1.4.9. It is well-known that the map $\mathcal{M}_\infty \mathbb{C} \to \mathcal{F}$ induces an isomorphism in K_0; in particular $K_0(\mathcal{F}) = \mathbb{Z}$. Next we use this fact to show that $K_0(I) = \mathbb{Z}$ for any ideal $0 \neq I \subsetneq \mathcal{B}$. It is classical that $\mathcal{F} \subset I \subset \mathcal{K}$ for any such ideal. We have a commutative diagram with exact rows and split exact columns

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & I \arrow[r] \arrow[d] & I/\mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathbb{C} \oplus I \arrow[r] \arrow[d] & R_I \arrow[r] \arrow[d] & 0 \\
& & \mathbb{C} \arrow[r, equal] & \mathbb{C} &
\end{tikzcd}$$

It follows from the discussion of Example 1.1.3 that the map $K_1(\mathbb{C}\oplus I) \to K_1(R_I)$ is onto. Similarly, $K_0(R_I) \to K_0(\mathbb{C})$ is an isomorphism because R_I is local. Thus $K_0(I/\mathcal{F}) = 0$ by split exactness. It follows that $\mathbb{Z} = K_0(\mathcal{F}) \to K_0(I)$ is an isomorphism.

Exercise 1.4.10. A theorem of Karoubi asserts that $K_{-1}(\mathcal{K}) = 0$ [21]. Use this and excision to show that $K_{-1}(I) = 0$ for any operator ideal I.

2. Lecture II

We saw in the last lecture that K_1 is not split exact. It follows from this that there is no way of defining higher K-groups such that the long exact sequence of Theorem 1.4.1 can be extended to higher K-theory. This motivates the question of whether this problem could be fixed if we replaced K_1 by some other functor. This is succesfully done in topological K-theory of Banach algebras.

2.1. **Topological K-theory. Banach algebras.** A *Banach* ($\mathbb{C}$-) algebra is a $\mathbb{C}$-algebra together with a norm $||\ \ ||$ which makes it into a Banach space and is such that there exists a constant C such that $||xy|| \le C||x|| \cdot ||y||$ for all $x, y \in A$. If A is a Banach algebra then $A \oplus \mathbb{C}$ is a unital Banach algebra with norm $||a + \lambda|| := ||a|| + |\lambda|$. An algebra homomorphism is a morphism of Banach algebras if it is continuous. If X is a compact Hausdorff space and A is a Banach algebra, then the algebra $C(X, A)$ of continuous functions $X \to A$ is a Banach algebra with norm $||f||_\infty := \sup_x ||f(x)||$. If X is locally compact and X^+ its one point compactification then the algebra $C_0(X, A)$ of continuous functions on X^+ vanishing at infinity is again a Banach algebra, isomorphic to the kernel of the homomorphism $C(X^+, A) \to \mathbb{C}$, $f \mapsto f(\infty)$. We write $A[0, 1]$ for the algebra of continous functions $[0, 1] \to A$, and $A(0, 1]$ and $A(0, 1)$ for the ideals of those functions which vanish respectively at 0 and at both endpoints. Two homomorphisms $f_0, f_1 : A \to B$ of Banach algebras are called *homotopic* if there exists a homomorphism $H : A \to B[0, 1]$ such that the following diagram commutes.

$$\begin{tikzcd} & B[0,1] \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ A \arrow[ur, "H"] \arrow[r, "(f_0,f_1)"'] & B \times B \end{tikzcd}$$

A functor G from Banach algebras to abelian groups is called *homotopy invariant* if it maps homotopic maps to equal maps.

Exercise 2.1.1. Prove that G is homotopy invariant if and only if for every Banach algebra A the map $G(A) \to G(A[0, 1])$ induced by the natural inclusion $A \subset A[0, 1]$ is an isomorphism.

Theorem 2.1.2. *(*[29, 1.6.11]*) K_0 is a homotopy invariant functor on Banach algebras.*

Example 2.1.3. *K_1 is not homotopy invariant.* The algebra $A := \mathbb{C}[\epsilon]$ is a Banach algebra with norm $||a + b\epsilon|| = |a| + |b|$. Both the inclusion $\iota : \mathbb{C} \to A$ and the projection $\pi : A \to \mathbb{C}$ are homomorphisms of Banach algebras; they satisfy $\pi\iota = 1$. Moreover the map $H : A \to A[0, 1]$, $H(a + b\epsilon)(t) = a + tb\epsilon$ is also a Banach homomorphism, and satisfies $\mathrm{ev}_0 H = \iota\pi$, $\mathrm{ev}_1 H = 1$. Thus any homotopy invariant functor G sends ι and π to inverse homomorphisms; since K_1 does not do so, it is not homotopy invariant.

The previous theorem and the example above suggest that we may try to modify K_1 to obtain a homotopy invariant version.

Definition 2.1.4. Let R be a unital Banach algebra. Put

$$\mathrm{GL}(R)_0 := \{g \in \mathrm{GL}(R) : \exists h \in \mathrm{GL}(R[0, 1]) : h(0) = 1, h(1) = g\}.$$

The topological K_1 of R is

$$K_1^{\mathrm{top}} R = \mathrm{GL}(R)/\mathrm{GL}(R)_0$$

Exercise 2.1.5. Show that if we regard $\mathrm{GL}(R) = \mathrm{colim}_n \mathrm{GL}_n(R)$ with the topology inherited from that of R, then $K_1^{\mathrm{top}} R = \pi_0(\mathrm{GL}(R))$. Then show that K_1^{top} is homotopy invariant.

Note that if R is a unital Banach algebra, $a \in R$ and $i \neq j$, then $1 + tae_{i,j} \in E(R[0,1])$ is a path connecting 1 to the elementary matrix $1 + ae_{i,j}$. Thus $ER \subset \mathrm{GL}R$, whence we have a surjection

$$K_1 R \twoheadrightarrow K_1^{\mathrm{top}} R.$$

Example 2.1.6. Because $\mathbb{C}$ is a field, $K_1\mathbb{C} = \mathbb{C}^*$. Since on the other hand $\mathbb{C}^*$ is path connected, we have $K_1^{\mathrm{top}}\mathbb{C} = 0$.

Note that K_1^{top} is additive. Thus we can extend K_1^{top} to nonunital Banach algebras in the usual way, i.e.

$$K_1^{\mathrm{top}} A := \ker(K_1^{\mathrm{top}}(A \oplus \mathbb{C}) \to K_1^{\mathrm{top}}\mathbb{C}) = K_1^{\mathrm{top}}(A \oplus \mathbb{C})$$

Exercise 2.1.7. Show that if A is a (not necessarily unital) Banach algebra, then

$$K_1^{\mathrm{top}} A = \mathrm{GL}(A)/\mathrm{GL}(A)_0$$

(cf. 1.3.2).

Fact 2.1.8. If $R \twoheadrightarrow S$ then $\mathrm{GL}(R)_0 \twoheadrightarrow \mathrm{GL}(S)_0$. (See [2, 3.4.4]).

Let

$$0 \to A \to B \to C \to 0 \tag{13}$$

be an exact sequence of Banach algebras. Then

$$0 \to A \to \mathbb{C} \oplus B \to \mathbb{C} \oplus C \to 0$$

is again exact. By the fact above, the connecting map $\partial : K_1(\mathbb{C} \oplus C) \to K_0 A$ induces a homomorphism

$$\partial : K_1^{\mathrm{top}} C \to K_0 A.$$

Theorem 2.1.9. *The sequence*

$$\begin{tikzcd} K_1^{\mathrm{top}} A \arrow[r] & K_1^{\mathrm{top}} B \arrow[r] & K_1^{\mathrm{top}} C \arrow[d, "\partial"] \\ K_0 C & K_0 B \arrow[l] & K_0 A \arrow[l] \end{tikzcd}$$

is exact.

Proof. Straightforward from 1.4.1. □

Since the sequences

$$0 \to A(0,1] \to A[0,1] \to A \to 0$$
$$0 \to A(0,1) \to A(0,1] \to A \to 0$$

are exact, and since K_0 is homotopy invariant, we get an isomorphism

$$K_1^{\mathrm{top}} A = K_0(A(0,1)) \tag{14}$$

Since also K_1^{top} is homotopy invariant, we put

$$K_2^{\mathrm{top}}(A) = K_1^{\mathrm{top}}(A(0,1)). \tag{15}$$

Fact 2.1.10. If (13) is exact, then

$$0 \to A(0,1) \to B(0,1) \to C(0,1) \to 0$$

is exact too. (See [28, 10.1.2] for a proof in the C^*-algebra case; a similar argument works for Banach algebras.)

Taking into account the fact above, as well as (14) and (15), and applying 2.1.9, we obtain the following.

Corollary 2.1.11. *There is an exact sequence*

$$\begin{tikzcd} K_2^{\mathrm{top}}A \arrow[r] & K_2^{\mathrm{top}}B \arrow[r] & K_2^{\mathrm{top}}C \arrow[d, "\partial"] \\ K_1^{\mathrm{top}}C & K_1^{\mathrm{top}}B \arrow[l] & K_1^{\mathrm{top}}A \arrow[l] \end{tikzcd}$$

The sequence can be extended further by defining inductively

$$K_{n+1}^{\mathrm{top}}(A) := K_n^{\mathrm{top}}(A(0,1)).$$

2.2. **Bott periodicity.** Let R be a unital Banach algebra. Consider the map $\beta : \mathrm{Idem}_n R \to \mathrm{GL}_n C_0(S^1, R)$,

$$\beta(e)(z) = ze + 1 - e \tag{16}$$

This map induces a group homomorphism $K_0 R \to K_1^{\mathrm{top}}(C_0(S^1, R))$ (see [2, 9.1]). If A is any Banach algebra, we write β for the composite

$$K_0(A) \to K_0(\mathbb{C} \oplus A) \xrightarrow{\beta} K_1^{\mathrm{top}}(C_0(S^1, \mathbb{C} \oplus A)) = K_1^{\mathrm{top}}\mathbb{C}(0,1) \oplus K_1^{\mathrm{top}}A(0,1) \twoheadrightarrow K_1^{\mathrm{top}}A(0,1) \tag{17}$$

One checks that for unital A this defintion agrees with that given above.

Theorem 2.2.1. *(Bott periodicity) (*[2, 9.2.1]*) The map* (17) *is an isomorphism.*

Let

$$0 \to A \to B \to C \to 0 \tag{18}$$

be an exact sequence of Banach algebras. By 2.1.11 we have a map $\partial : K_1^{\mathrm{top}}(C(0,1)) \to K_1^{\mathrm{top}}(A)$. Composing with the Bott map, we obtain a homomorphism

$$\partial\beta : K_0(C) \to K_1^{\mathrm{top}}A$$

Theorem 2.2.2. *The sequence*

$$\begin{tikzcd} K_1^{\mathrm{top}}A \arrow[r] & K_1^{\mathrm{top}}B \arrow[r] & K_1^{\mathrm{top}}C \arrow[d, "\partial"] \\ K_0C \arrow[u, "\partial\beta"] & K_0B \arrow[l] & K_0A \arrow[l] \end{tikzcd}$$

is exact.

2.3. **Sketch of Cuntz' proof of Bott periodicity for C^*-algebras.** ([8, Sec. 2]) A C^*-algebra is a Banach algebra with an involution $*$ such that $||aa^*|| = ||a||^2$. The *Toeplitz* algebra is the free unital algebra $\mathcal{T}^{\mathrm{top}}$ on a generator α subject to $\alpha\alpha^* = 1$. Since the shift $s : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $s(e_1) = 0$, $s(e_{i+1}) = e_i$ satisfies $ss^* = 1$, there is a homomorphism $\mathcal{T}^{\mathrm{top}} \to \mathcal{B} = \mathcal{B}(\ell^2(\mathbb{N}))$. It turns out that this is a monomorphism, that its image contains the ideal $\mathcal{K}$, and that the latter is the kernel of the homomorphism $\mathcal{T}^{\mathrm{top}} \to \mathbb{C}(S^1)$ which sends α to the identity function $S^1 \to S^1$. We have a commutative diagram with exact rows and split exact columns:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d, equal] & \mathcal{T}_0^{\mathrm{top}} \arrow[r] \arrow[d] & \mathbb{C}(0,1) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{T}^{\mathrm{top}} \arrow[r] \arrow[d] & \mathbb{C}(S^1) \arrow[r] \arrow[d, "\mathrm{ev}_1"'] & 0 \\ & & \mathbb{C} \arrow[r, equal] & \mathbb{C} & \end{tikzcd}$$

Here we have used the identification $C_0(S^1,\mathbb{C}) = \mathbb{C}(0,1)$. If now A is any C^*-algebra, and we apply the functor $A\tilde{\otimes} := A\otimes_{\min}$ we obtain a commutative diagram whose columns are split exact and whose rows are still exact [2, Ex. 9.4.2]

$$\begin{tikzcd}
0 \arrow[r] & A\tilde{\otimes}\mathcal{K} \arrow[r] \arrow[d, equal] & A\tilde{\otimes}\mathcal{T}_0^{\mathrm{top}} \arrow[r] \arrow[d] & A(0,1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A\tilde{\otimes}\mathcal{K} \arrow[r] & A\tilde{\otimes}\mathcal{T}^{\mathrm{top}} \arrow[r] \arrow[d] & A(S^1) \arrow[r] \arrow[d] & 0 \\
 & & A \arrow[r, equal] & A &
\end{tikzcd}$$

Consider the inclusion $\mathbb{C}\subset M_\infty\mathbb{C}\subset\mathcal{K}=\mathcal{K}(\ell^2(\mathbb{N}))$, $\lambda\mapsto\lambda e_{1,1}$. A functor G from C^*-algebras to abelian groups is *$\mathcal{K}$-stable* if for every C^*-algebra A the map $G(A)\to G(A\tilde{\otimes}\mathcal{K})$ is an isomorphism. We say that G is *half exact* if for every exact sequence (18), the sequence

$$GA\to GB\to GC$$

is exact.

Remark 2.3.1. In general, there is no precedence between the notions of split exact and half exact. However a functor of C^*-algebras which is homotopy invariant, additive and half exact is automatically split exact (see [2, §21.4]).

The following theorem of Cuntz' is stated in the literature for half exact rather than split exact functors. However the proof uses only split exactness.

Theorem 2.3.2. *([8, 4.4]) Let G be a functor from C^*-algebras to abelian groups. Assume that*

- *G is homotopy invariant.*
- *G is $\mathcal{K}$-stable.*
- *G is split exact.*

Then for every C^*-algebra A,

$$G(A\tilde{\otimes}\mathcal{T}_0^{\mathrm{top}})=0$$

Fact 2.3.3. K_0 is $\mathcal{K}$-stable [28, 6.4.1].

It follows from the fact above, Cuntz' theorem and excision, that the connecting map $\partial: K_1^{\mathrm{top}}(A(0,1))\to K_0(A\tilde{\otimes}\mathcal{K})$ is an isomorphism. Further, from the explicit formulas for β and ∂ ((16), 1.4.2), one checks that the following diagram commutes

$$\begin{tikzcd}
K_1^{\mathrm{top}}(A(0,1)) \arrow[r, "\partial"] & K_0(A\tilde{\otimes}\mathcal{K}) \\
 & K_0(A) \arrow[ul, "\beta"'] \arrow[u, "\wr"]
\end{tikzcd}$$

This proves that β is an isomorphism.

2.4. **Back to the algebraic case.** Next we analyze to what extent the properties of topological K-theory of Banach algebras have analogues for algebraic K-theory of general rings.

2.5. **Homotopy invariance.** It does not make sense to consider continuous homotopies for general rings, among other reasons, because in general they do not carry any interesting topologies. What does make sense is to consider polynomial homotopies. We shall let the reader figure out the appropriate definitions for polynomial homotopy. Let us just say that a functor G from rings to abelian groups is *homotopy invariant* if for every ring A the map $GA \to G(A[t])$ induced by the inclusion $A \subset A[t]$ is an isomorphism. If G is any functor from rings to abelian groups, we call a ring A G-*regular* if $GA \to G(A[t_1, \dots, t_n])$ is an isomorphism for all n. We write $NG(A) = \operatorname{coker}(GA \to G(A[t]))$. Thus A is G-regular if $N^pG(A) = 0$ for all $p \geq 1$.

Example 2.5.1. Noetherian regular rings are K_0-regular ([29], as are all Banach algebras 2.1.2. If k is any field, then the ring $R = k[x,y]/ < y^2 - x^3 >$ is not K_0-regular. By the discussion of 2.1.3, the ring $k[\epsilon]$ is not K_1-regular.

The groups $GL(\ )_0$ and K_1^{top} have the following algebraic analogues. Let R be a unital ring. Put

$$\mathrm{GL}(R)'_0 = \{g \in \mathrm{GL}R : \exists h \in \mathrm{GL}(R[t]) : h(0) = 1, h(1) = g\}.$$

Set

$$KV_1(R) := \mathrm{GL}R/\mathrm{GL}(R)'_0$$

the group KV_1 is the K_1 of *Karoubi-Villamayor* [24]. The functor KV_1 is additive, split exact and matrix stable; furthermore it is homotopy invariant and nilinvariant [24]. However, unlike what happens with its topological analogue, the functor $GL(\ )'_0$ does not preserve surjections. As a consequence, the KV-analogue of 1.4.1 does not hold for general short exact sequences of rings. Higher KV-groups are defined as follows. We have exact sequences

$$0 \to PA \to A[t] \overset{\mathrm{ev}_0}{\to} A \to 0 \tag{19}$$

$$0 \to \Omega A \to PA \overset{\mathrm{ev}_1}{\to} A \to 0 \tag{20}$$

Here PA is defined as the kernel of ev_0, and ΩA as that of $\mathrm{ev}_{1|PA}$. One defines inductively

$$KV_{n+1}(A) = KV_1(\Omega^n A)$$

As said above, the KV-analogue of 1.4.1 does not hold in general. However it does hold for the sequences (19) and (20) ([24]). In particular we have a natural injective map

$$KV_1(A) \overset{\partial}{\to} K_0(\Omega A). \tag{21}$$

Exercise 2.5.2. Prove the analogue of 2.1.7 for KV_1.

2.6. **Toeplitz ring.** Write $\mathcal{T}$ for the free unital ring on two generators α, α^* subject to $\alpha\alpha^* = 1$. Mapping α to $\sum_i e_{i,i+1}$ and α^* to $\sum_i e_{i+1,i}$ yields a monomorphism $\mathcal{T} \to \Gamma := \Gamma\mathbb{Z}$ whose image contains the ideal $M_\infty := M_\infty\mathbb{Z}$ [6, 4.10]. There is a commutative diagram with exact rows and split exact columns:

$$\begin{tikzcd}
0 \arrow[r] & M_\infty \arrow[r] \arrow[d, equal] & \mathcal{T}_0 \arrow[r] \arrow[d] & \sigma \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_\infty \arrow[r] & \mathcal{T} \arrow[r] \arrow[d] & \mathbb{Z}[t,t^{-1}] \arrow[r] \arrow[d, "\mathrm{ev}_1"] & 0 \\
 & & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} &
\end{tikzcd}$$

Moreover the rows are split as sequences of abelian groups. Thus tensoring with any ring A yields an exact diagram

$$\begin{tikzcd} 0 \arrow[r] & M_\infty A \arrow[r] \arrow[d, equal] & \mathcal{T}_0 A \arrow[r] \arrow[d] & \sigma A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_\infty A \arrow[r] & \mathcal{T} A \arrow[r] \arrow[d] & A[t,t^{-1}] \arrow[r] \arrow[d] & 0 \\ & & A \arrow[r, equal] & A & \end{tikzcd} \tag{22}$$

We have the following algebraic analogue of Cuntz' theorem.

Theorem 2.6.1. *([6, 7.3.2]) Let G be a functor from rings to abelian groups. Assume that:*

- *G is homotopy invariant.*
- *G is split exact.*
- *G is M_∞-stable.*

  Then for any ring A, $G(\mathcal{T}_0 A) = 0$.

Unfortunately, we cannot apply the theorem above to K-theory, as the latter is not homotopy invariant. However, we can always consider the long exact sequence of K-groups associated to the sequences (22).

Lemma 2.6.2. *If A is a ring and $n \le 0$, then the map $K_n(M_\infty A) \to K_n(\mathcal{T} A)$ is zero.*

Proof. In view of 1.4.7 and of the fact that both M_∞ and Σ commute with $\mathcal{T}$, it suffices to prove the lemma for $n = 0$. Using excision, we reduce further to the unital case. Let R be a unital ring; we have to show that if $p \ge 1$ and $e \in M_p R$ is idempotent, the class of its image in $M_p(\mathcal{T}R) = \mathcal{T}(M_p R)$ is zero in $K_0(\mathcal{T}R)$. By matrix stability, it suffices to prove this for $p = 1$. Let $f \in \mathrm{Idem}_1 R$; then $f^\infty := \sum_i e_{i,i} \otimes f \in \mathrm{Idem}_1(\mathcal{T} \otimes R) = \mathrm{Idem}_1(\mathcal{T}R)$. Identify $f = e_{1,1} \otimes f \in \mathcal{T}R$. For the direct sum operation $\boxplus$ of ΓR, we have $f \boxplus f^\infty = f^\infty$. Moreover, one checks that the matrix

$$Q = \begin{bmatrix} 1 - \alpha^*\alpha & \alpha^* \\ \alpha & 0 \end{bmatrix} \in \mathrm{GL}_2 \mathcal{T}$$

satisfies $Q^2 = 1$ and

$$Q \begin{bmatrix} f & 0 \\ 0 & f^\infty \end{bmatrix} Q = \begin{bmatrix} f^\infty & 0 \\ 0 & 0 \end{bmatrix}$$

Thus

$$[e] + [e^\infty] = [e^\infty] \text{ in } K_0(\mathcal{T}R).$$

□

Corollary 2.6.3.

i) *If $n \le -1$, there is a short exact sequence*

$$0 \to K_{n+1}\mathcal{T}_0 A \to K_{n+1} A \to K_n A \to 0$$

ii) *There is a surjection $\partial : K_1(A[t,t^{-1}]) \to K_0(A[t,t^{-1}])$.*

Let R be a unital ring. Define a map $\beta : \mathrm{Idem}_n R \to \mathrm{GL}_n(R[t,t^{-1}])$,

$$\beta(e) = te + 1 - e$$

One checks that β induces a group homomorphism $K_0(R) \to K_1(R[t,t^{-1}])$ whose image lies in $\ker(K_1(R[t,t^{-1}]) \overset{\mathrm{ev}_1}{\to} K_1 R)$.

Lemma 2.6.4. *The composite $K_0 R \overset{\beta}{\to} K_1(R[t,t^{-1}]) \overset{\partial}{\to} K_0 R$ is the identity map.*

Proof. Follows from direct computation, using the explicit formulas for β and ∂ ((16), 1.4.2). □

Proposition 2.6.5. *For any ring A and any $n \le -1$, the sequence of Lemma 2.6.2 i), is split.*

Proof. (Sketch) For unital A this is immediate from Lemma 2.6.4, using 1.4.7. The general case follows from the unital case using split exactness of K_m $(m \le 0)$. □

Proposition 2.6.6. *Let A be a ring and $n \le 0$.*

$$K_n(\mathcal{T}_0 A) = NK_nA \oplus NK_nA.$$

Proof. Let $s \in \Gamma$ be the shift. Because $s^*s - 1 \in M_\infty$, there is a ring homomorphism $\mathbb{Z}[t, t^{-1}] \to \Sigma$ mapping t to the class of s. It follows from results of Bass [1] and Loday [26] and split exactness, that the kernel of the induced map $K_n(A[t, t^{-1}]) \to K_{n-1}A$ is $K_nA \oplus NK_nA \oplus NK_nA$. The proposition follows from split exactness and the commutativity of the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_\infty & \longrightarrow & \mathcal{T} & \longrightarrow & \mathbb{Z}[t,t^{-1}] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_\infty & \longrightarrow & \Gamma & \longrightarrow & \Sigma & \longrightarrow & 0 \end{array}$$

□

Corollary 2.6.7. $K_n(\sigma A) = K_{n-1}A \oplus NK_nA \oplus NK_nA$

Remark 2.6.8. In case A happens to be K_n-regular, the previous corollary says that $K_n(\sigma A) = K_{n-1}A$. We regard this as an algebraic analogue of Bott periodicity (at least for nonpositive K-theory). What is missing in the algebraic case is an analogue of the exponential map; where as for any Banach algebra A, $A(0, 1) \cong C_0(S^1, A)$, for a general ring A there is no isomorphism $\Omega A \to \sigma A$.

2.7. **Homotopy K-theory.** Let A be a ring. Consider the natural map

$$\partial : K_0(A) \to K_{-1}(\Omega A) \tag{23}$$

associated with the exact sequence (20). As $K_{-1} = K_0\Sigma$ we may iterate the construction and form the colimit

$$KH_0(A) := \operatorname*{colim}_n K_{-n}(\Omega^n A).$$

Put

$$KH_n(A) := \begin{cases} KH_0(\Omega^n A) & (n \ge 0) \\ KH_0(\Sigma^n A) & (n \le 0) \end{cases}$$

The groups KH_*A are Weibel's *homotopy K-theory* groups of A ([37]). Although this is not Weibel's original definition, it is equivalent to it ([6, 8.1.1]).

Theorem 2.7.1. *([37]) Homotopy K-theory has the following properties.*

i) *It is homotopy invariant, nilinvariant and M_∞-stable.*
ii) *It satisfies excision: to the sequence (10) there corresponds a long exact sequence ($n \in \mathbb{Z}$)*

$$KH_{n+1}C \to KH_nA \to KH_nB \to KH_nC \to KH_{n-1}A$$

iii) $KH_n(\sigma A) = KH_{n-1}A$ *($n \in \mathbb{Z}$).*

Proof. (Sketch) We know that $K_1(S) = 0$ for every infinite sum ring S; hence $KV_1(S) = 0$. In particular $KV_1(\Gamma R) = 0$ for unital R. Using split exactness of KV_1, it follows that $KV_1\Gamma A = 0$ for every ring A. Since $K_0(\Gamma A) = KV_1(\Gamma A) = 0$, the surjection $K_1(\Sigma A) \to KV_1(\Sigma A)$ factors through $K_0(A)$, obtaining an epimorphism

$$K_0(A) \twoheadrightarrow KV_1(\Sigma A). \tag{24}$$

One checks that the map (23) is the composite of (24) with (21). Thus

$$KH_n(A) = \operatorname*{colim}_r KV_1(\Omega^{n+r}\Sigma^r A).$$

It follows from this that KH is homotopy invariant. Nilinvariance, M_∞-stability and excision follow from the fact that these hold for nonpositive K-theory. Thus parts i) and ii) are proved. Part iii) follows from Theorem 2.6.1 and excision. □

3. Lecture III

3.1. Quillen's Higher K-theory. The *classifying space* of a group G is a connected CW-complex BG such that

$$\pi_n BG = \begin{cases} G & n = 1 \\ 0 & n \neq 1 \end{cases}$$

This property characterizes BG and makes it functorial up to homotopy. Further there are various strictly functorial models of BG ([29], [16]). The homology of BG is the same as the group homology of G; if M is $\pi_1 BG = G$-module, then

$$H_n(BG, M) = H_n(G, M) := \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M)$$

Let R be a unital ring. Quillen's *plus construction* applied to $B\mathrm{GL}R$ yields a cellular map of CW-complexes $\iota : B\mathrm{GL}R \to (B\mathrm{GL}R)^+$ such that ([26])

- At the level of π_1, ι induces the projection $\mathrm{GL}R \to K_1R$.
- At the level of homology, ι is an isomorphism $H_*(\mathrm{GL}R, M) \to H_*((B\mathrm{GL}R)^+, M)$ for each K_1R-module M.
- If $B\mathrm{GL}R \to X$ is any continuous function which at the level of π_1 maps $ER \to 0$, then the dotted arrow in the following diagram exists and is unique up to homotopy

$$\begin{tikzcd} B\mathrm{GL}R \arrow[r, "\iota"] \arrow[d] & (B\mathrm{GL}R)^+ \arrow[dl, dotted] \\ X & \end{tikzcd}$$

The higher K-groups of R are the homotopy groups of $(B\mathrm{GL}R)^+$:

$$K_nR := \pi_n(B\mathrm{GL}R)^+ \qquad (n \geq 1)$$

All the basic properties 1.3 which hold for K_1 hold also for higher K_n ([26]). In particular $K_nR = K_nM_pR = K_nM_\infty R$ for unital R, and K_n is additive on unital rings. Hence the direct sum of matrices induces the group operation in K_nR ([26]). Thus $K_*R = 0$ for every ring R with infinite sums [35], for all n, by the same argument as for $n = 0, 1$ (cf. [35]). In particular

$$K_*\Gamma R = 0 \tag{25}$$

for any unital ring R. The groups K_n have a lot of additional structure. For example, J.L. Loday has defined an associative product [26]

$$\cup : K_pR \otimes K_qS \to K_{p+q}(R \otimes S)$$

3.1.1. *Relative K-groups.* Let R be a unital ring, $I \triangleleft R$ an ideal, and $S = R/I$. Put

$$\overline{\mathrm{GL}}S := \mathrm{Im}(\mathrm{GL}R \to \mathrm{GL}S)$$

The plus construction applied to $B\overline{\mathrm{GL}}S$ yields a space whose fundamental group is the image of K_1R in K_1S, and whose higher homotopy groups are the K-groups of S. Consider the homotopy fiber

$$F(R : I) := \mathrm{hofiber}((B\mathrm{GL}R)^+ \to (B\overline{\mathrm{GL}}R)^+)$$

The *relative K-groups* of I with respect to the ideal embedding $I \triangleleft R$ are the homotopy groups

$$K_n(R : I) := \pi_n F(R : I) \qquad (n \geq 1).$$

We have an exact sequence $(n \geq 2)$

$$K_{n+1}S \to K_n(R : I) \to K_nR \to K_n(S) \to K_{n-1}(R : I) \tag{26}$$

which can be spliced together with the excision sequence 1.4.1 as follows

$$K_2S \to K_1(R:I) \to K_1R \to K_1S \to K_0I \to K_0R \to K_0S$$

Thus if we unify notation and set $K_n(R:I) = K_nI$ for $n \leq 0$, we get the sequence (26) for all $n \in \mathbb{Z}$.
For $n \geq 1$, we put

$$K_n(I) := K_n(\mathbb{Z} \oplus I : I)$$

One checks that for $n = 1$ this defintion agrees with that given above. The canonical map $\mathbb{Z}\oplus I \to R$ induces a map

$$K_n(I) \to K_n(R:I) \tag{27}$$

This map is an isomorphism for $n \leq 0$, but not in general (see Remark 1.4.4). The rings I so that this map is an isomorphism for all n and R are called K-excisive. Suslin and Wodzicki have completely characterized K-excisive rings ([39],[32],[31]). We have

$$\text{(27) is an isomorphism for all } n \text{ and } R \iff \operatorname{Tor}_n^{\mathbb{Z}\oplus I}(\mathbb{Z}, I) = 0 \ \ \forall n$$

Remark 3.1.1.1.

$$\operatorname{Tor}_0^{\mathbb{Z}\oplus I}(\mathbb{Z}, I) = I/I^2$$
$$\operatorname{Tor}_n^{\mathbb{Z}\oplus I}(\mathbb{Z}, I) = \operatorname{Tor}_{n+1}^{\mathbb{Z}\oplus I}(\mathbb{Z}, \mathbb{Z})$$

Example 3.1.1.2. Let G be a group, $IG \triangleleft \mathbb{Z}G$ the augmentation ideal. Then $\mathbb{Z}G = \mathbb{Z} \oplus IG$ is the unitalization of IG. Hence

$$\operatorname{Tor}_n^{\mathbb{Z}\oplus IG}(\mathbb{Z}, IG) = \operatorname{Tor}_{n+1}^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) = H_{n+1}(G, \mathbb{Z})$$

In particular

$$\operatorname{Tor}_0^{\mathbb{Z}\oplus I}(\mathbb{Z}, I) = G_{ab}$$

When this group is zero, we say that G is *perfect*. Note no nonzero abelian group can be perfect. In particular, the ring σ is not excisive, as it coincides with the augmentation ideal of $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$.

Example 3.1.1.3. Any unital ring R is excisive. If I is excisive then $M_\infty I$ is excisive too.

3.2. **Cone and Toeplitz rings and the fundamental theorem.** If R is a unital ring, then $M_\infty R$ is excisive, whence by (25), the connecting map in K-theory for the cone sequence (11) is an isomorphism

$$\partial : K_n(\Sigma R) \xrightarrow{\cong} K_{n-1}R \qquad (n \in \mathbb{Z}). \tag{28}$$

Remark 3.2.1. At the level of spaces, we obtain a weak equivalence between $(B\mathrm{GL}R)^+$ and the connected component of the loopspace of $(B\mathrm{GL}\Sigma R)^+$

$$(B\mathrm{GL}R)^+ \xrightarrow{\cong} (\Omega(B\mathrm{GL}\Sigma R)^+)_0$$

Taking loopspaces and iterating, we obtain a sequence of weak equivalences

$${}_n\mathbb{K}R := \Omega K(\Sigma^{n+1}R) \xrightarrow{\cong} \Omega({}_{n+1}\mathbb{K}R)$$

Thus $\mathbb{K}R := \{{}_n\mathbb{K}R\}$ is a spectrum, and its stable homotopy groups are the K-groups of R:

$$\pi_n\mathbb{K}R = \operatorname*{colim}_p \pi_{n+p}({}_n\mathbb{K}R) = K_nR$$

If A is any, not necessarily unital ring, one puts $\mathbb{K}A = \operatorname{hofiber}(\mathbb{K}(\mathbb{Z}\oplus A) \to \mathbb{K}\mathbb{Z})$.

Next we recall a description of the inverse of the map (28) in terms of products, given by J. L. Loday in [26]. We have a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M_\infty \arrow[r] \arrow[d, equal] & \mathcal{T} \arrow[r] \arrow[d] & \mathbb{Z}[t,t^{-1}] \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & M_\infty \arrow[r] & \Gamma \arrow[r] & \Sigma \arrow[r] & 0
\end{tikzcd}$$

The Bott map $\beta : K_0\mathbb{Z} \to K_1\mathbb{Z}[t, t^{-1}]$ sends $\beta(p_1) = t$. Loday shows that the composite

$$\begin{tikzcd} K_{n-1}R \arrow[r, dotted] \arrow[d, equal] & K_n\Sigma R \\ K_{n-1}R \otimes K_0\mathbb{Z} \arrow[r, "1\otimes j(t)"'] & K_{n-1}R \otimes K_1(\Sigma) \arrow[u, "\cup"'] \end{tikzcd}$$

is the inverse of (28). Similarly, the Toeplitz extension (22) yields an exact sequence. From the argument of the proof of 2.6.2 and the fact that K_* is matrix invariant, one shows that 2.6.2 holds for higher K_n, so one obtains an exact sequence

$$0 \to K_n(\mathcal{T}R) \to K_n(R[t, t^{-1}]) \xrightarrow{\partial} K_{n-1}R \to 0 \tag{29}$$

It follows from 2.6.4 and Loday's results that the composite

$$\begin{tikzcd} K_n(R) \arrow[r, "\beta"] \arrow[d, "\wr"'] & K_{n+1}R[t, t^{-1}] \\ K_nR \otimes K_0\mathbb{Z} \arrow[r, "1\cup\beta(p_1)"'] & K_nR \otimes K_1\mathbb{Z}[t, t^{-1}] \arrow[u, "\cup"'] \end{tikzcd}$$

is left inverse to the connecting map ∂ of (29). In particular

$$K_n(R[t, t^{-1}]) = K_{n-1}R \oplus K_n(\mathcal{T}R) = K_{n-1}R \oplus K_nR \oplus K_n(\mathcal{T}R : \mathcal{T}_0R)$$

Further, it follows from diagram (22) that

$$K_n(R[t, t^{-1}] : \sigma R) = K_n(\mathcal{T}R : \mathcal{T}_0R) \oplus K_{n-1}R$$

It follows from the fundamental theorem in higher algebraic K-theory [29] that

$$K_n(\mathcal{T}R : \mathcal{T}_0R) = NK_nR \oplus NK_nR.$$

If R happens to be K_n-regular, this gives (cf. 2.6.8).

$$K_n(R[t, t^{-1}] : \sigma R) = K_{n-1}R. \tag{30}$$

3.3. **Fréchet algebras with approximate units.** A *locally convex* algebra is a complete topological $\mathbb{C}$-algebra L with a locally convex topology. Such a topology is defined by a family of seminorms $\{\rho_\alpha\}$; continuity of the product means that for every α there exists a β such that

$$\rho_\alpha(xy) \le \rho_\beta(x)\rho_\beta(y) \qquad (x, y \in L)$$

Whenever a defining family of seminorms can be chosen so that the condition above is satisfied with $\alpha = \beta$ (i.e. the seminorms are *submultiplicative*) we say that L is an m-algebra. A *Fréchet* algebra is a locally convex algebra with a defining sequence of seminorms. A locally convex algebra whose topology is Fréchet will be called *Fréchet m-algebra.* A *uniformly bounded approximate left unit* (ubalu) in a locally convex algebra L is a net $\{e_\lambda\}$ of elements of L $e_\lambda a \mapsto a$ for all a and $\sup_\alpha \rho_\alpha(a) < \infty$. Right ubau's are defined analogously.
The following theorem is a direct consequence of results of Suslin and Wodzicki.

Theorem 3.3.1. *Every Fréchet m-algebra with left or right ubau is K-excisive.*

Proof. It is proved in [39, 8.1] that such algebras satisfy the hypothesis of [32, Thm. C]. □

Example 3.3.2. Every C^*-algebra has a two-sided ubau. If G is a locally compact group, then the group algebra $L^1(G)$ is a Banach algebra with ubau. If L_1 and L_2 are locally convex algebras with ublaus $\{e_\lambda\}$ and $\{f_\mu\}$, then $\{e_\lambda \otimes f_\mu\}$ is a ublau for the projective tensor product $L_1\hat{\otimes}L_2$, which is a (Fréchet) m-algebra if both L_1 and L_2 are.

3.4. Comparison between algebraic and topological K-theory I.

3.4.1. *Stable C^*-algebras.*

Theorem 3.4.1.1. [19, 3.2.2] *Let G be a functor from C^*-algebras to abelian groups. Assume that G is split exact and $\mathcal{K}$-stable. Then G is homotopy invariant.*

Proposition 3.4.1.2. *Let G be a functor from C^*-algebras to abelian groups. Assume that G is M_2-stable. Then the functor $A \mapsto G(A \tilde{\otimes} \mathcal{K})$ is stable.*

Proof. The argument of the proof of [11, 5.1.2] shows this. □

The following result, due to Suslin and Wodzicki, is (one of the variants of) what is known as Karoubi's conjecture [21].

Theorem 3.4.1.3. [32, 10.9] *Let A be a C^*-algebra. Then there is a natural isomorphism $K_n(A \tilde{\otimes} \mathcal{K}) = K_n^{\mathrm{top}}(A \tilde{\otimes} \mathcal{K})$.*

Proof. By definition $K_0 = K_0^{\mathrm{top}}$ on all C^*-algebras. By 3.4.1.1 and 3.3.1, $K_n(A(0,1] \tilde{\otimes} \mathcal{K}) = 0$. Hence

$$K_{n+1}(A \tilde{\otimes} \mathcal{K}) = K_n(A(0,1) \tilde{\otimes} \mathcal{K})$$

by excision (3.3.1). In particular, for $n \geq 0$,

$$K_n(A \tilde{\otimes} \mathcal{K}) = K_0(A \tilde{\otimes} \tilde{\otimes}_{i=1}^{n} \mathbb{C}(0,1) \tilde{\otimes} \mathcal{K}) = K_n^{\mathrm{top}}(A \tilde{\otimes} \mathcal{K}).$$

On the other hand, $K_{n+1}(A(0,1) \tilde{\otimes} \mathcal{K}) = K_n(A \tilde{\otimes} \mathcal{K} \tilde{\otimes} \mathcal{K}) = K_n(A \tilde{\otimes} \mathcal{K})$ by 2.3.2. It follows that for $i = 0, 1$ and $q \geq 0$,

$$K_{-2q-i}(A \tilde{\otimes} \mathcal{K}) = K_i(A \tilde{\otimes} \mathcal{K}) = K_i^{\mathrm{top}}(A \tilde{\otimes} \mathcal{K})$$

□

3.4.2. *Stable Banach algebras.* The following result is a particular case of a theorem of Wodzicki.

Theorem 3.4.2.1. [40] *Let L be Banach algebra with right or left ubau. Then there is an isomorphism $K_*(L \hat{\otimes} \mathcal{K}) = K_*^{\mathrm{top}}(L \hat{\otimes} \mathcal{K})$.*

Proof. Consider the functor $G_L : C^* \to \mathfrak{Ab}$, $A \mapsto K_*(L \hat{\otimes}(A \tilde{\otimes} \mathcal{K}))$. By 3.3.1 and 3.4.1.1, K_* is homotopy invariant. Hence $\mathbb{C} \to \mathbb{C}[0,1]$ induces an isomorphism

$$\begin{aligned} G_L(\mathbb{C}) =& K_*(L \hat{\otimes} \mathcal{K}) \xrightarrow{\cong} G_L(\mathbb{C}[0,1]) = K_*(L \hat{\otimes}(\mathbb{C}[0,1] \tilde{\otimes} \mathcal{K})) \\ =& K_*(L \hat{\otimes} \mathcal{K}[0,1]) = K_*((L \hat{\otimes} \mathcal{K})[0,1]). \end{aligned}$$

Hence $K_{n+1}(L \hat{\otimes} \mathcal{K}) = K_n(L \hat{\otimes} \mathcal{K}(0,1))$, by 3.3.1 and 3.3.2. Thus $K_n(L \hat{\otimes} \mathcal{K}) = K_n^{\mathrm{top}}(L \hat{\otimes} \mathcal{K})$ for $n \geq 0$. Consider the Calkin algebra $\mathcal{Q} := \mathcal{B}/\mathcal{K}$. Because $\mathcal{B}$ is an infinite sum ring, we have $K_n(L \hat{\otimes} \mathcal{B}) = 0$ for $n \leq 0$. Thus

$$K_{-n}(L \hat{\otimes} \mathcal{K}) = K_0(L \hat{\otimes} \hat{\otimes}_{i=1}^{n} \mathcal{Q}) = K_{-n}^{\mathrm{top}}(L \hat{\otimes} \mathcal{K})$$

□

Remark 3.4.2.2. The theorem above holds more generally for m-Fréchet algebras ([40], [7, 8.3.4]), with the appropriate definition of topological K-theory (see below).

3.4.3. *Various spectral sequences.*
If A is any ring, we write $\Delta^{\mathrm{alg}}A$ for the simplicial ring

$$[n] \mapsto \Delta_n^{\mathrm{alg}}A := A \otimes \mathbb{Z}[t_0, \dots, t_n]/ < \sum_i t_i - 1 >$$

According to Weibel's original definition [37], the groups KH_* are the homotopy groups of the spectrum

$$\mathbb{KH}(A) := |\mathbb{K}\Delta^{\mathrm{alg}}A|$$

It is proved in [6, 8.1.1] that this definition agrees with that given above. Applying $|\mathbb{K}(\ \ )|$ to the canonical map $A \to \Delta^{\mathrm{alg}}A$ and taking homotopy groups, one obtains a comparison homomorphism

$$K_*A \to KH_*A \tag{31}$$

The spectral sequence for a simplicial spectrum yields

$$E^2_{p,q} = \pi_p([n] \mapsto K_q\Delta_n^{\mathrm{alg}}A) \Rightarrow KH_{p+q}A \tag{32}$$

Theorem 3.4.3.1. [34] *If A is K_q-regular, then it is K_{q-1}-regular ($q \in \mathbb{Z}$).*

Remark 3.4.3.2. In [34], the theorem above is proved for $q > 0$. The case $q \le 0$ follows from this, using (28).

Corollary 3.4.3.3. *If A is K_n-regular then $K_mA \to KH_mA$ is an isomorphism for all $m \le n$ (see also* [13]*).*

Example 3.4.3.4. One can show, using 3.4.1.1, that if A is a C^*-algebra, then $A\tilde{\otimes}\mathcal{K}$ is K-regular (see [30, 3.4] for a proof of this result of Higson's). Thus $KH_*(A\tilde{\otimes}\mathcal{K}) = K_*(A\tilde{\otimes}\mathcal{K}) = K_*^{\mathrm{top}}(A\tilde{\otimes}\mathcal{K})$. Rosenberg has shown that any commutative C^*-algebra is K-regular [30, 3.5].

There are several definitions of topological K-theory for locally convex algebras. One variant is what we shall call *diffeotopy* K-theory [7, 4.1]. It is invariant under C^∞-homotopies (*diffeotopies*). If L is a locally convex algebra, we write $\Delta^{\mathrm{dif}}L$ for the simplicial algebra

$$[n] \mapsto \Delta_n^{\mathrm{dif}}L := L\hat{\otimes}C^\infty(\Delta^n)$$

The *diffeotopy K-theory* groups $KD_*(L)$, are the stable homotopy groups of the spectrum

$$\mathbb{KD}(L) = |\mathbb{K}\Delta^{\mathrm{dif}}L|$$

By the same reason as before, we get a comparison map $K_*L \to KD_*L$; further, from the natural map $\Delta^{\mathrm{alg}} \subset \Delta^{\mathrm{dif}}$ we also obtain $K_*L \to KH_*L$, so that there is a commutative diagram ([7, 4.3.1])

$$\begin{tikzcd} K_*L \arrow[rr] \arrow[dr] & & KD_*L \\ & KH_*L \arrow[ur] & \end{tikzcd}$$

The spectral sequence for a simplicial spectrum yields

$$E^2_{p,q} = \pi_p([n] \mapsto K_q\Delta_n^{\mathrm{dif}}L) \Rightarrow KD_{p+q}L \tag{33}$$

There is also a spectral sequence [7, 4.1]

$$E'^2_{p,q} = \pi_p([n] \mapsto KH_q\Delta_n^{\mathrm{dif}}L) \Rightarrow KD_{p+q}L \tag{34}$$

Hence we get:

Lemma 3.4.3.5. *If the degeneracy $L \to \Delta_p^{\mathrm{dif}}L$ induces an isomorphism $K_qL \to K_q\Delta_p^{\mathrm{dif}}L$ (resp. $KH_qL \to KH_q(\Delta_p^{\mathrm{dif}}L)$) for all $p \ge 0$ and all $q \le n$, then the map $K_nL \to KD_nL$ (resp. $KH_nL \to KD_nL$) is an isomorphism.*

3.4.4. *KH of Stable locally convex algebras.* Let H be a separable hilbert space; write $H \otimes_2 H$ for the completed tensor product of Hilbert spaces. Note any two Hilbert separable Hilbert spaces are isomorphic; hence we may regard any operator ideal $\mathcal{J} \triangleleft \mathcal{B}(H)$ as a functor on Hilbert spaces (see [20, 3.3]). Let $\mathcal{J} \triangleleft \mathcal{B}$ be an ideal.

- $\mathcal{J}$ is *multiplicative* if $\mathcal{B}\hat{\otimes}\mathcal{B} \to \mathcal{B}(H \otimes_2 H)$ maps $\mathcal{J}\hat{\otimes}\mathcal{J}$ to $\mathcal{J}$.
- $\mathcal{J}$ is *Fréchet* it is a Fréchet algebra and the inclusion $\mathcal{J} \to \mathcal{B}$ is continuous. A Fréchet ideal is a *Banach* ideal if it is a Banach algebra.

  Write $\omega = (1/n)_n$ for the harmonic sequence.
- $\mathcal{J}$ is *harmonic* if it is a multiplicative Banach ideal such that $\mathcal{J}(\ell^2(\mathbb{N}))$ contains $diag(\omega)$.

Example 3.4.4.1. Let $p \in \mathbb{R}_{>0}$; write $\mathcal{L}_p$ for the ideal of those compact operators whose sequence of singular values is p-summable; $\mathcal{L}_p$ is called the *p-Schatten ideal.* It is Banach $\iff p \geq 1$, and is harmonic $\iff p > 1$. There is no interesting locally convex topology on $\mathcal{L}_p$ for $p < 1$.

The following theorem, due to J. Cuntz and A. Thom, is a variant of 3.4.1.1, valid for locally convex algebras. The formulation we use here is a consequence of [11, 5.1.2] and [11, 4.2.1].

Theorem 3.4.4.2. *Let $\mathcal{J}$ be a harmonic operator ideal, and G a functor from locally convex algebras to abelian groups. Assume that*

i) *G is M_2-stable.*
ii) *G is split exact.*

Then $L \mapsto G(L\hat{\otimes}\mathcal{J})$ is diffeotopy invariant.

We shall need a variant of 3.4.4.2 which is valid for all Fréchet ideals $\mathcal{J}$. First we need some notation. Let $\alpha : A \to B$ be a homomorphism of locally convex algebras. We say that α is an *isomorphism up to square zero* if there exists a continous linear map $\beta : B\hat{\otimes}B \to A$ such that the compositions $\beta \circ (\alpha\hat{\otimes}\alpha)$ and $\alpha \circ \beta$ are the multiplication maps of A and B. Note that if α is an isomorphism up to square zero, then its image is a ideal of B, and both its kernel and its cokernel are square-zero algebras.

Definition 3.4.4.3. Let G be a functor from locally convex algebras to abelian groups. We call G *continously nilinvariant* if it sends isomorphisms up to square zero into isomorphisms.

Example 3.4.4.4. For any $n \in \mathbb{Z}$, KH_n is a continously nilinvariant functor of locally convex algebras. If $n \leq 0$, the same is true of K_n. In general, if H_* is the restriction to locally convex algebras of any excisive, nilinvariant homology theory of rings, then H_* is continously nilinvariant.

Theorem 3.4.4.5. [7, 6.1.6] *Let $\mathcal{J}$ be a Fréchet operator ideal, and G a functor from locally convex algebras to abelian groups. Assume that*

i) *G is M_2-stable.*
ii) *G is split exact.*
iii) *G is continuously nilinvariant.*

Then $L \mapsto G(L\hat{\otimes}\mathcal{J})$ is diffeotopy invariant.

Theorem 3.4.4.6. [7, 6.2.1 (iii)] *Let L be a locally convex algebra and $\mathcal{J}$ a Fréchet ideal. Then $KH_n(L\hat{\otimes}\mathcal{J}) = KD_n(L\hat{\otimes}\mathcal{J})$ for all $n \in \mathbb{Z}$ and $K_n(L\hat{\otimes}\mathcal{J}) = KH_n(L\hat{\otimes}\mathcal{J})$ for $n \leq 0$.*

Proof. Immediate from 3.4.4.5, 3.4.4.4, using the spectral sequences (33) and (32). □

Fact 3.4.4.7. Theorem [7, 6.2.1] states further that all the variants of topological K-theory coincide for algebras of the form $L\hat{\otimes}\mathcal{J}$ with $\mathcal{J}$ Fréchet. In particular, it agrees with the covariant version of Cuntz' bivariant K-theory of locally convex algebras [9]:

$$KH_n(L\hat{\otimes}\mathcal{J}) = kk^{\mathrm{top}}(\mathbb{C}, L\hat{\otimes}\mathcal{J}) =: K^{\mathrm{top}}(L\hat{\otimes}\mathcal{J}) \tag{35}$$

When restricted to Banach algebras L, this definition of K^{top} coincides with that given above.

Corollary 3.4.4.8. *Let $\mathcal{J}$ be a Fréchet operator ideal. Then*

$$KH_n(\mathcal{J}) = \begin{cases} \mathbb{Z} & n \text{ even.} \\ 0 & n \text{ odd.} \end{cases}$$

Proof. The cases $n = 0, 1$ are 1.4.9 and 1.4.10. The general case follows from this, from (35) and the fact that kk^{top} is 2-periodic [9]. □

Remark 3.4.4.9. The corollary above is valid more generally for "subharmonic" ideals (see [7, 6.5.1] for the definition of this term, and [7, 7.2.1] for the statment). This applies to the nonlocally convex Schatten ideals $\mathcal{L}_\epsilon$, $0 < \epsilon < 1$.

4. Lecture IV

4.1. **The Chern character and homotopization.** All rings considered from now on are implicitly (and some times also explicitly) assumed to be $\mathbb{Q}$-algebras. There are various maps connecting K-theory and its variants with cyclic homology and its variants. The variants of cyclic homology we shall be concerned with are cyclic, negative cyclic and periodic cyclic homology, denoted respectively HC, HN and HP. These homology groups are $\mathbb{Q}$-vectorspaces connected by a long exact sequence ($n \in \mathbb{Z}$)

$$HP_{n+1}A \xrightarrow{S} HC_{n-1}A \xrightarrow{B} HN_nA \xrightarrow{i} HP_nA \xrightarrow{S} HC_{n-2}A$$

This is Connes' SBI-sequence. Cyclic homology is defined as the homology of Connes' complex $C^\lambda A$. This is a nonnegatively graded chain complex, given in dimension n by the coinvariants

$$C_n^\lambda A := (A^{\otimes n+1})_{\mathbb{Z}/(n+1)\mathbb{Z}} \tag{36}$$

of the tensor power under the action of $\mathbb{Z}/(n+1)\mathbb{Z}$ given on a generator t by

$$t(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

The boundary map $b : C_n^\lambda A \to C_{n-1}^\lambda A$ is induced by

$$b : A^{\otimes n+1} \to A^{\otimes n}, \quad b(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$
$$+ (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}$$

Example 4.1.1. The map $C_1^\lambda(A) \to C_0^\lambda(A)$ sends the class of $a \otimes b$ to $[a, b] := ab - ba$. Hence

$$HC_0A = A/[A, A].$$

For a definition of HN and HP, see [25]. We'll just mention here some of their properties. The groups HN_* and HP_*, as well as HC_*, are $\mathbb{Q}$-vectorspaces. Periodic cyclic homology, like K^{top}, is periodic of period 2; like KH, it has some other desirable properties, including excision [12], nil- and homotopy invariance [17], and matrix invariance. As a consequence of this, and of the SBI-sequence, the homology groups measuring obstructions to desirable properties in HC and HN coincide up to a degree-shift. For example, if $I \triangleleft A$ is nilpotent, then $HP_*(A : I) = 0$ by nilinvariance, whence

$$HC_{*-1}(A : I) \cong HN_*(A : I)$$

A similar situation occurs in K-theory; the obstruction groups to the desirable properties of KH are the same for K-theory as for *nil* K-theory, which is defined by the spectrum

$$\mathbb{K}^{\mathrm{nil}}A := \mathrm{hofiber}(\mathbb{K}A \to \mathbb{KH}A)$$

The analogy between the maps $K \to KH$ and $HN \to HP$ is in fact more profound. Geller and Weibel noted in [15] that the spectrum $HP(A)$ is weakly equivalent to the total spectrum $HNH(A) := |HN(\Delta^{\mathrm{alg}}A)|$. Jones and Goodwillie defined a map $\mathbb{K}A \to HNA$, the *primary Chern* *character* [18]. Using the fact that $X \mapsto |X(\Delta^{\mathrm{alg}}\ \ )|$ preserves fibration sequences, we obtain a commutative diagram of spectra

$$\begin{tikzcd} \mathbb{K}^{\mathrm{inf,nil}}A \arrow[r] \arrow[d] & \mathbb{K}^{\mathrm{inf}}A \arrow[r] \arrow[d] & \mathbb{K}^{\mathrm{inf}}(\Delta^{\mathrm{alg}}A) \arrow[d] \\ \mathbb{K}^{\mathrm{nil}}A \arrow[r] \arrow[d, "\nu"'] & \mathbb{K}A \arrow[r] \arrow[d, "c^{\tau}"'] & \mathbb{KH}A \arrow[d, "ch^{\tau}"'] \\ \Omega^{-1}HC(A) \arrow[r] & HN(A) \arrow[r] & HP(\Delta^{\mathrm{alg}}A). \end{tikzcd} \tag{37}$$

Each row and each column of this diagram is a homotopy fibration; the right column is the result of applying the middle column to $\Delta^{\mathrm{alg}}A$ and then $|\ \ |$. The column of the left is the homotopy fiber of the map between the middle and right columns.

Remark 4.1.2. The approach to algebraic Chern characters presented above, taken from [7, 2.2], is essentially due to Weibel [36] (see also [22] for a different approach). For locally convex algebras, it makes sense to "diffeotopize" the Chern character; one can also replace the algebraic cyclic theories for their topological variants (using completed projective tensor products over $\mathbb{C}$ instead of algebraic tensor products over $\mathbb{Z}$, as in (36)). This approach was studied by Connes and Karoubi ([3],[4],[22],[23]); see [7, §4] for a comparison of the two approaches.

4.2. **Properties of K^{inf}.**

Theorem 4.2.1. [18] *K^{inf} is nilinvariant.*

Theorem 4.2.2. [5] *K^{inf} is excisive.*

We call a $\mathbb{Q}$-algebra A *K^{inf}-regular* if it is K^{inf}_n-regular for all n.

Lemma 4.2.3. *Let A be a $\mathbb{Q}$-algebra. If A is K^{inf}-regular, then $\nu_* : K^{\mathrm{nil}}_*A \to HC_{*-1}A$ is an isomorphism.*

Proof. It is immediate from the spectral sequence associated to the simplicial spectrum $\mathbb{K}^{\mathrm{inf}}(\Delta^{\mathrm{alg}}A)$, that if A is K^{inf}-regular, then $\mathbb{K}^{\mathrm{nil}}A \overset{\cong}{\to} 0$. □

4.3. **Comparison between algebraic and topological K-theory II.**

Theorem 4.3.1. [7, 6.2.1 (ii)] *Let L be a locally convex algebra, and $\mathcal{J}$ a Fréchet operator ideal. Then $L\hat{\otimes}\mathcal{J}$ is K^{inf}-regular.*

Proof. Let $n \in \mathbb{Z}$; we have to prove that $L \mapsto K^{\mathrm{inf}}_n(L\hat{\otimes}\mathcal{J})$ is polynomially homotopy invariant. But by 4.2.1 and 4.2.2, K^{inf}_n satisfies the hypothesis of 3.4.4.5, whence it is diffeotopy invariant, and in particular polynomially homotopy invariant. □

Theorem 4.3.2. [7, 6.3.1] *Let L be a locally convex algebra, and $\mathcal{J}$ a Fréchet operator ideal. For each $n \in \mathbb{Z}$, there is a natural 6-term exact sequence of abelian groups as follows:*

$$\begin{tikzcd} K^{\mathrm{top}}_1(L\hat{\otimes}\mathcal{J}) \arrow[r] & HC_{2n-1}(L\hat{\otimes}\mathcal{J}) \arrow[r] & K_{2n}(L\hat{\otimes}\mathcal{J}) \arrow[d] \\ K_{2n-1}(L\hat{\otimes}\mathcal{J}) \arrow[u] & HC_{2n-2}(L\hat{\otimes}\mathcal{J}) \arrow[l] & K^{\mathrm{top}}_0(L\hat{\otimes}\mathcal{J}). \arrow[l] \end{tikzcd}$$

Proof. Follows from 4.2.3, (37), and Bott periodicity of K^{top}. □

Remark 4.3.3. Applying the theorem above when L is an m-Fréchet algebra with right or left ubau and $\mathcal{J} = \mathcal{K}$, and using 3.4.2.1, we obtain

$$HC_*(L\hat{\otimes}\mathcal{K}) = 0.$$

One can also prove this statement independently, and use 4.3.2 to prove 3.4.2.1; see [7, 8.3.3].

Example 4.3.4. Setting $L = \mathbb{C}$ in 4.3.2, and using 3.4.4.8, we obtain an exact sequence

$$0 \to HC_{2n-1}(\mathcal{J}) \to K_{2n}(\mathcal{J}) \to \mathbb{Z} \overset{\alpha_n}{\to} HC_{2n-2}(\mathcal{J}) \to K_{2n-1}(\mathcal{J}) \to 0.$$

Note that, as the cyclic homology groups are $\mathbb{Q}$-vectorspaces, there are only two possibilities for map α_n; it is either injective or zero. It is shown in [7, 7.2.1 (iii)] that α_n is injective whenever $\mathcal{J} \subset \mathcal{L}_p$ for some $p \geq 1$ and $n \geq (p+1)/2$. Thus for example we have an exact sequence

$$0 \to \mathbb{Z} \to \mathcal{L}_1/[\mathcal{L}_1, \mathcal{L}_1] \to K_1(\mathcal{L}_1) \to 0$$

4.4. **Comments, questions and problems.**

The first question that comes to mind when presented with Theorem 4.3.2 is: can one actually compute the cyclic homology groups appearing there in some special cases, such as $L = \mathbb{C}$? In view of the formula 4.1.1, the first step in this direction is to understand the commutator structure of operator ideals; this has been done in great detail by Dykema, Figiel, Weiss and Wodzicki in [14]. One can extend vanishing results for lower cyclic homology to higher degrees, using multiplicativity properties (see [40] and [7, 8.2.3]). Other vanishing results can be derived from ring theoretic properties of Von Neumann algebras [41]. One can also consider the variants of the cyclic homology theories which come from replacing the algebraic tensor product over $\mathbb{Q}$ by the completed projective tensor product over $\mathbb{C}$ in the definition of the complex C^λ and those for HN and HP. The resulting periodic theory HP^{top} has been computed in several examples, including all Schatten ideals $\mathcal{L}^p$ ($p \geq 1$)(see [10]); one has $HP_*^{\mathrm{top}}(\mathcal{L}_p) = HP_*^{\mathrm{top}}(\mathcal{L}_1) = K_*^{\mathrm{top}}(\mathcal{L}_1) \otimes \mathbb{C}$. On the other hand, the groups $HP_*(\mathcal{L}_1)$ have not been computed, nor is much known about the map $HP_0(\mathcal{L}_1) \to HP_0^{\mathrm{top}}(\mathcal{L}_1)$, except that it is not zero. This is because the Connes-Karoubi character $\mathbb{Z} = KH_0(\mathcal{L}_1) = K_*^{\mathrm{top}}(\mathcal{L}^1) \to HP_*^{\mathrm{top}}(\mathcal{L}^1) = \mathbb{C}$ is an injection [4], which factors through the algebraic character $KH_0 \to HP_0$ mentioned above ([7, §4]).

4.5. **Algebraic kk.** Homology theories of rings are usually defined as the homology of some functorial complex or the homotopy of some functorial spectrum. The fact that the (stable) homotopy categories of chain complexes and spectra are triangulated, suggests that we deal in general with functors from rings to triangulated categories.

Let $\mathcal{T}$ be a triangulated category with loop (i.e. inverse suspension) functor Ω. Let $\mathcal{E}$ be the class of all short exact sequences of rings (10). An *excisive homology theory* for rings with values in $\mathcal{T}$ consists of a functor X : Rings $\to \mathcal{T}$, together with a collection $\{\partial_E : E \in \mathcal{E}\}$ of maps $\partial_E^X = \partial_E \in \hom_{\mathcal{T}}(\Omega X(C), X(A))$. The maps ∂_E are to satisfy the following requirements.

i) For all $E \in \mathcal{E}$ as in (10),

$$\Omega X(C) \xrightarrow{\partial_E} X(A) \xrightarrow{X(f)} X(B) \xrightarrow{X(g)} X(C)$$

is a distinguished triangle in $\mathcal{T}$.

ii) If

$$\begin{tikzcd} (E): & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"'] \\ (E'): & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \end{tikzcd}$$

is a map of extensions, then the following diagram commutes

$$\begin{tikzcd} \Omega X(C) \arrow[r, "\partial_E"] \arrow[d, "\Omega X(\gamma)"'] & X(A) \arrow[d, "X(\alpha)"] \\ \Omega X(C') \arrow[r, "\partial_{E'}"'] & X(A) \end{tikzcd}$$

We say that the functor $X : \mathrm{Rings} \to \mathcal{T}$ is *homotopy invariant* if it maps homotopic homomomorphisms to equal maps, or equivalently, if for every $A \in \mathrm{Rings}$, X maps the inclusion $A \subset A[t]$ to an isomorphism. Call X *M_∞-stable* if for every $A \in \mathrm{Rings}$, it maps the inclusion $\iota_\infty : A \to M_\infty A$ to an isomorphism. Note that if X is M_∞-stable, and $n \geq 1$, then X maps the inclusion $\iota_n : A \to M_n A$ to an isomorphism.

The homotopy invariant, M_∞-stable, excisive homology theories form a category, where a homomorphism between the theories $X : \mathrm{Rings} \to \mathcal{T}$ and $Y : \mathrm{Rings} \to \mathcal{U}$ is a triangulated functor $G : \mathcal{T} \to \mathcal{U}$ such that

$$\begin{tikzcd} \mathrm{Rings} \arrow[r, "X"] \arrow[dr, "Y"'] & \mathcal{T} \arrow[d, "G"] \\ & \mathcal{U} \end{tikzcd} \tag{38}$$

commutes, and such that for every extension (10), the natural isomorphism $\phi : G(\Omega X(C)) \to \Omega Y(C)$ makes the following into a commutative diagram

$$\begin{tikzcd} G(\Omega X(C)) \arrow[r, "G(\partial_E^X)"] \arrow[d, "\phi"'] & Y(A) \\ \Omega Y(C). \arrow[ur, "\partial_E^Y"'] & \end{tikzcd} \tag{39}$$

Theorem 4.5.1. [6, 6.6.2] *The category of all excisive, homotopy invariant and M_∞-stable homology theories has an initial object $j : \mathrm{Rings} \to kk$. We have*

$$j(\Omega A) = \Omega j A, \qquad \Omega^{-1} j A = j(\Sigma A) = j(\sigma A)$$

Example 4.5.2. As a consequence of the theorem above, we obtain that any excisive, homotopy invariant and M_∞-stable homology theory X satisfies the fundamental theorem

$$\Omega^{-1} X(A) \xrightarrow{\cong} X(\sigma A)$$

Definition 4.5.3. Let $A, B \in \mathrm{Rings}$. Put $kk(A, B) = \hom_{kk}(jA, jB)$, and

$$kk_n(A, B) := \hom_{kk}(jA, \Omega^n jB) = \begin{cases} kk(A, \Omega^n B) & n \geq 0 \\ kk(A, \Sigma^n B) & n \leq 0 \end{cases}$$

It follows from the definition and elementary properties of triangulated categories that any short exact sequence (10) induces a long exact sequence ($n \in \mathbb{Z}$)

$$kk_{n+1}(D, C) \longrightarrow kk_n(D, A) \longrightarrow kk_n(D, B) \longrightarrow kk_n(D, C) \longrightarrow kk_{n-1}(D, A)$$

and similarly in the other variable.

Theorem 4.5.4. [6, 6.6.6] *Let $\mathfrak{A}$ be an abelian category, and $G : \mathrm{Rings} \to \mathfrak{A}$ a half exact, homotopy invariant, M_∞-stable, additive functor. Then there exists a unique homological functor $\bar{G} : kk \to \mathfrak{A}$ such that the following diagram commutes.*

$$\begin{tikzcd} \mathrm{Rings} \arrow[r, "j"] \arrow[dr, "G"'] & kk \arrow[d, dotted, "\bar{G}"] \\ & \mathfrak{A} \end{tikzcd}$$

Theorem 4.5.5. [6, 8.2.1] *Let A be a ring, and $n \in \mathbb{Z}$. Then*

$$kk_n(\mathbb{Z}, A) = KH_n A.$$

4.5.1. *Bivariant Chern character.*

One can also consider homology theories of algebras over a (say commutative unital) ring H. Theorems 4.5.1 and 4.5.4 still hold, and give rise to a bivariant K-theory kk^{H} for H algebras, which is in principle different from the one defined for all rings. However, Theorem 4.5.5 still holds for H-algebras A with H substituted for $\mathbb{Z}$; we have

$$kk_n^{\mathrm{H}}(\mathrm{H}, A) = KH_n A.$$

In particular, if X is a homotopy invariant, M_∞-stable, excisive homology theory with values in some triangulated category $\mathcal{U}$, then by the universal property, we get a map

$$kk_n^{\mathrm{H}}(A, B) \to \hom_{\mathcal{U}}(X(A), \Omega^n X(B))$$

Setting $A = \mathrm{H}$, we obtain

$$KH_n(B) \to \hom_{\mathcal{U}}(X(\mathrm{H}), \Omega^n X(B))$$

The bivariant Chern character for $\mathbb{Q}$-algebras, of which the map $KH_* \to HP_*$ is the covariant version, is a particular case of this construction. Here $\mathcal{U}$ is the homotopy category of pro-supercomplexes and X is the Cuntz-Quillen pro-supercomplex.

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