Universal Coefficient Theorems and assembly maps in KK-theory

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Abstract. We introduce equivariant Kasparov theory using its universal property and construct the Baum–Connes assembly map by localising the Kasparov category at a suitable subcategory. Then we explain a general machinery to construct derived functors and spectral sequences in triangulated categories. This produces various generalisations of the Rosenberg–Schochet Universal Coefficient Theorem.

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Contents

1. Introduction

We may view Kasparov theory and its equivariant generalisations as categories. These categories are non-commutative analogue of (equivariant generalisations of) the stable homotopy category of spectra. These equivariant Kasparov categories can be described in two ways:

Abstractly, as the universal split-exact C^*-stable functor on the appropriate category of C^*-algebras — this approach is due to Cuntz and Higson [9, 10, 14, 15].

It is useful for *general* constructions like the descent functor or the adjointness between induction and restriction functors (see §2.6 or [25]).

Concretely, using Fredholm operators on equivariant Hilbert bimodules — this is the original definition of Kasparov [16, 17].

It is useful for *specific* constructions that use, say, geometric properties of a group to construct elements in Kasparov groups.

We mainly treat Kasparov theory as a black box. We define G-equivariant Kasparov theory via its universal property and equip it with a triangulated category structure. This formalises some basic properties of the stable homotopy category that are needed for algebraic topology. We later apply this structure to construct spectral sequences in KASPAROV theory.

We use the universal property to construct the descent functor and induction and restriction functors for closed subgroups, and to verify that the latter are adjoint for open subgroups.

Then we turn to the BAUM–CONNES assembly map for a locally compact group G, which we treat as in [25]. GREEN's Imprimitivity Theorem suggests that we understand crossed products for compactly induced actions much better than general crossed products. We want to construct more general actions out of compactly induced actions by an analogue of the construction of CW-complexes. The notion of *localising subcategory* makes this idea precise.

The orthogonal complement of the compactly induced actions consists of actions that are KK^H-equivalent to 0 for all compact subgroups H of G. We call such actions *weakly contractible*.

The compactly induced and weakly contractible objects together generate the whole KASPAROV category. This allows us to compute the localisation of a functor at the weakly contractible objects. The general machinery of localisation yields the BAUM–CONNES assembly map

$$\mu_*\colon \mathrm{K}_*^{\mathrm{top}}(G, A) \to \mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$$

when we apply it to the functor $A \mapsto \mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$. Roughly speaking, this means that $A \mapsto \mathrm{K}_*^{\mathrm{top}}(G, A)$ is the best possible approximation to $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$ that vanishes for weakly contractible objects. The above statements involve functors and the BAUM–CONNES assembly map with coefficients. The above approach only works if we study this generalisation right away.

The groups $\mathrm{K}_*^{\mathrm{top}}(G, A)$ are supposed to be computable by topological methods. We present one approach to make this precise that works completely within equivariant KASPAROV theory and is a special case of a very general machinery for constructing spectral sequences. We carry over notions from homological algebra like exact chain complexes and projective objects to our category and use them to define derived functors (see [26]). The derived functors of $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$ and $\mathrm{K}_*^{\mathrm{top}}(G, A)$ agree and form the E^2-term of a spectral sequence that converges towards $\mathrm{K}_*^{\mathrm{top}}(G, A)$. Many other spectral sequences like the ADAMS spectral sequence in topology can be constructed with the same machinery. In simple special cases, the spectral sequence degenerates to an exact sequence. The Universal Coefficient Theorem by ROSENBERG–SCHOCHET in [33] and the PIMSNER–VOICULESCU exact sequence are special cases of this machinery.

Part I
Kasparov theory and Baum–Connes conjecture

2. Kasparov theory via its universal property

This section is mostly taken from [24], where more details can be found.

Let G be a locally compact group.

Definition 1. A *G-C^*-algebra* is a C^*-algebra with a strongly continuous representation of G by *-automorphisms.

Let $G\text{-}\mathfrak{C}^*\mathfrak{alg}$ be the category of G-C^*-algebras; its objects are G-C^*-algebras and its morphisms $A \to B$ are the G-equivariant *-homomorphisms $A \to B$; we sometimes denote this morphism set by $\mathrm{Hom}_G(A, B)$.

A C^*-algebra is *separable* if it has a countable dense subset. We often restrict attention to the full subcategory $G\text{-}\mathfrak{C}^*\mathfrak{sep} \subseteq G\text{-}\mathfrak{C}^*\mathfrak{alg}$ of separable G-C^*-algebras.

Homology theories for C^*-algebras are usually required to be homotopy invariant, stable, and exact in a suitable sense. We can characterise G-equivariant Kasparov theory as the *universal functor* on $G\text{-}\mathfrak{C}^*\mathfrak{sep}$ with these properties, in the following sense.

Definition 2. Let P be a property for functors defined on $G\text{-}\mathfrak{C}^*\mathfrak{sep}$. A *universal functor* with P is a functor $u\colon G\text{-}\mathfrak{C}^*\mathfrak{sep} \to \mathfrak{U}_P(G\text{-}\mathfrak{C}^*\mathfrak{sep})$ such that

- $\bar{F} \circ u$ has P for each functor $\bar{F}\colon \mathfrak{U}_P(G\text{-}\mathfrak{C}^*\mathfrak{sep}) \to \mathfrak{C}$;
- any functor $F\colon G\text{-}\mathfrak{C}^*\mathfrak{sep} \to \mathfrak{C}$ with P factors uniquely as $F = \bar{F} \circ u$ for some functor $\bar{F}\colon \mathfrak{U}_P(G\text{-}\mathfrak{C}^*\mathfrak{sep}) \to \mathfrak{C}$.

Of course, such a functor need not exist. If it does, then it restricts to a bijection between objects of $G\text{-}\mathfrak{C}^*\mathfrak{sep}$ and $\mathfrak{U}_P(G\text{-}\mathfrak{C}^*\mathfrak{sep})$. Hence we can completely describe it by the sets of morphisms $\mathfrak{U}_P(A, B)$ from A to B in $\mathfrak{U}_P(G\text{-}\mathfrak{C}^*\mathfrak{sep})$ and the maps $G\text{-}\mathfrak{C}^*\mathfrak{sep}(A, B) \to \mathfrak{U}_P(A, B)$ for $A, B \in\in G\text{-}\mathfrak{C}^*\mathfrak{sep}$. The universal property means that for any functor $F\colon G\text{-}\mathfrak{C}^*\mathfrak{sep} \to \mathfrak{C}$ with P there is a unique functorial way to extend the maps $\mathrm{Hom}_G(A, B) \to \mathfrak{C}\big(F(A), F(B)\big)$ to $\mathfrak{U}_P(A, B)$.

2.1. Some basic homotopy theory. We define cylinders, cones, and suspensions of objects and mapping cones and mapping cylinders of morphisms in $G\text{-}\mathfrak{C}^*\mathfrak{alg}$. Then we define homotopy invariance for functors. Mapping cones will be used later to introduce the triangulated category structure on Kasparov theory.

Notation 3. Let A be a G-C^*-algebra. We define the *cylinder*, *cone*, and *suspension* over A by

$$\mathrm{Cyl}(A) := \mathcal{C}([0,1], A), \qquad \mathrm{Sus}(A) := \mathcal{C}_0\bigl([0,1] \setminus \{0,1\}, A\bigr) \cong \mathcal{C}_0(\mathbb{S}^1, A),$$
$$\mathrm{Cone}(A) := \mathcal{C}_0\bigl([0,1] \setminus \{0\}, A\bigr),$$

If $A = \mathcal{C}_0(X)$ for a pointed compact space, then the cylinder, cone, and suspension of A are $\mathcal{C}_0(Y)$ with Y equal to the usual cylinder $[0,1]_+ \wedge X$, cone $[0,1] \wedge X$, or suspension $\mathbb{S}^1 \wedge X$, respectively; here $[0,1]$ has the base point 0.

Definition 4. Let $f\colon A \to B$ be a morphism in G-$\mathfrak{C}^*\mathfrak{alg}$. The *mapping cylinder* $\mathrm{Cyl}(f)$ and the *mapping cone* $\mathrm{Cone}(f)$ of f are the limits of the diagrams

$$A \xrightarrow{f} B \xleftarrow{\mathrm{ev}_1} \mathrm{Cyl}(B), \qquad A \xrightarrow{f} B \xleftarrow{\mathrm{ev}_1} \mathrm{Cone}(B)$$

in G-$\mathfrak{C}^*\mathfrak{alg}$. More concretely,

$$\mathrm{Cone}(f) = \bigl\{(a,b) \in A \times \mathcal{C}_0\bigl((0,1], B\bigr) \bigm| f(a) = b(1)\bigr\},$$
$$\mathrm{Cyl}(f) = \bigl\{(a,b) \in A \times \mathcal{C}\bigl([0,1], B\bigr) \bigm| f(a) = b(1)\bigr\}.$$

If $f\colon X \to Y$ is a morphism of pointed compact spaces, then the mapping cone and mapping cylinder of the induced *-homomorphism $\mathcal{C}_0(f)\colon \mathcal{C}_0(Y) \to \mathcal{C}_0(X)$ agree with $\mathcal{C}_0\bigl(\mathrm{Cyl}(f)\bigr)$ and $\mathcal{C}_0\bigl(\mathrm{Cone}(f)\bigr)$, respectively.

The familiar maps relating mapping cones and cylinders to cones and suspensions continue to exist in our case. For any morphism $f\colon A \to B$ in G-$\mathfrak{C}^*\mathfrak{alg}$, we get a morphism of extensions

$$\begin{tikzcd}
\mathrm{Sus}(B) \arrow[r, tail] \arrow[d] & \mathrm{Cone}(f) \arrow[r, two heads] \arrow[d] & A \arrow[d, equal] \\
\mathrm{Cone}(B) \arrow[r, tail] & \mathrm{Cyl}(f) \arrow[r, two heads] & A
\end{tikzcd}$$

The bottom extension splits and the maps $A \leftrightarrow \mathrm{Cyl}(f)$ are inverse to each other up to homotopy. The composite map $\mathrm{Cone}(f) \to A \to B$ factors through $\mathrm{Cone}(\mathrm{id}_B) \cong \mathrm{Cone}(B)$ and hence is homotopic to the zero map.

Definition 5. Let $f_0, f_1\colon A \rightrightarrows B$ be two parallel morphisms in G-$\mathfrak{C}^*\mathfrak{alg}$. We write $f_0 \sim f_1$ and call f_0 and f_1 *homotopic* if there is a morphism $f\colon A \to \mathrm{Cyl}(B)$ with $\mathrm{ev}_t \circ f = f_t$ for $t = 0, 1$.

A functor $F\colon G\text{-}\mathfrak{C}^*\mathfrak{alg} \to \mathfrak{C}$ is called *homotopy invariant* if $f_0 \sim f_1$ implies $F(f_0) = F(f_1)$.

It is easy to check that homotopy is an equivalence relation on $\mathrm{Hom}_G(A, B)$. We let $[A, B]$ be the set of equivalence classes. The composition of morphisms in G-$\mathfrak{C}^*\mathfrak{alg}$ descends to maps

$$[B, C] \times [A, B] \to [A, C], \qquad \bigl([f], [g]\bigr) \mapsto [f \circ g],$$

that is, $f_1 \sim f_2$ and $g_1 \sim g_2$ implies $f_1 \circ f_2 \sim g_1 \circ g_2$. Thus the sets $[A, B]$ form the morphism sets of a category, called the *homotopy category of G-C^*-algebras*. A functor is homotopy invariant if and only if it descends to the homotopy category. Other characterisations of homotopy invariance are listed in [24, §3.1].

Of course, our notion of homotopy restricts to the usual one for pointed compact spaces or to *proper* homotopy for locally compact spaces.

2.2. Morita–Rieffel equivalence and stable isomorphism. One of the basic ideas of non-commutative geometry is that $G \ltimes \mathcal{C}_0(X)$ (or $G \ltimes_{\mathrm{r}} \mathcal{C}_0(X)$) should be a substitute for the quotient space $G\backslash X$, which may have bad singularities. In the special case of a free and proper G-space X, we expect that $G \ltimes \mathcal{C}_0(X)$ and $\mathcal{C}_0(G\backslash X)$ are "equivalent" in a suitable sense. Already the simplest possible case $X = G$ shows that we cannot expect an isomorphism here because

$$G \ltimes \mathcal{C}_0(G) \cong G \ltimes_{\mathrm{r}} \mathcal{C}_0(G) \cong \mathbb{K}(L^2 G).$$

The right notion of equivalence is a C^*-version of Morita equivalence introduced by Marc A. Rieffel ([30–32]); therefore, we call it Morita–Rieffel equivalence.

The definition of Morita–Rieffel equivalence involves Hilbert modules over C^*-algebras and the C^*-algebras of compact operators on them; these notions are crucial for Kasparov theory as well. We refer to [19] for the definition and a discussion of their basic properties.

Definition 6. Two G-C^*-algebras A and B are called Morita–Rieffel equivalent if there are a full G-equivariant Hilbert B-module $\mathcal{E}$ and a G-equivariant *-isomorphism $\mathbb{K}(\mathcal{E}) \cong A$.

It is possible (and desirable) to express this definition more symmetrically: $\mathcal{E}$ is an A, B-bimodule with two inner products taking values in A and B, satisfying various conditions (see also [30]). Two Morita–Rieffel equivalent G-C^*-algebras hve equivalent categories of G-equivariant Hilbert modules via $\mathcal{E} \otimes_B \text{\textvisiblespace}$. The converse is not so clear.

Example 7. The following is a more intricate example of a Morita–Rieffel equivalence. Let Γ and P be two subgroups of a locally compact group G. Then Γ acts on G/P by left translation and P acts on $\Gamma\backslash G$ by right translation. The corresponding orbit space is the double coset space $\Gamma\backslash G/P$. Both $\Gamma \ltimes \mathcal{C}_0(G/P)$ and $P \ltimes \mathcal{C}_0(\Gamma\backslash G)$ are non-commutative models for this double coset space. They are indeed Morita–Rieffel equivalent; the bimodule that implements the equivalence is a suitable completion of $\mathcal{C}_c(G)$.

These examples suggest that Morita–Rieffel equivalent C^*-algebras are different ways to describe the *same* non-commutative space. Therefore, we expect that reasonable functors on $\mathfrak{C}^*\mathfrak{alg}$ should not distinguish between Morita–Rieffel equivalent C^*-algebras.

Definition 8. Two G-C^*-algebras A and B are called *stably isomorphic* if there is a G-equivariant *-isomorphism $A \otimes \mathbb{K}(\mathcal{H}_G) \cong B \otimes \mathbb{K}(\mathcal{H}_G)$, where $\mathcal{H}_G := L^2(G \times \mathbb{N})$

is the direct sum of countably many copies of the regular representation of G; we let G act on $\mathbb{K}(\mathcal{H}_G)$ by conjugation, of course.

The following technical condition is often needed in connection with Morita–Rieffel equivalence.

Definition 9. A C^*-algebra is called *σ-unital* if it has a countable approximate identity or, equivalently, contains a strictly positive element.

All separable C^*-algebras and all unital C^*-algebras are σ-unital; the algebra $\mathbb{K}(\mathcal{H})$ is σ-unital if and only if $\mathcal{H}$ is separable.

Theorem 10 ([7]). *σ-unital G-C^*-algebras are G-equivariantly* Morita–Rieffel *equivalent if and only if they are stably isomorphic.*

In the non-equivariant case, this theorem is due to Brown–Green–Rieffel ([7]). A simpler proof that carries over to the equivariant case appeared in [27].

2.3. C^*-stable functors.

The definition of C^*-stability is more intuitive in the non-equivariant case:

Definition 11. Fix a rank-one projection $p \in \mathbb{K}(\ell^2\mathbb{N})$. The resulting embedding $A \to A \otimes \mathbb{K}(\ell^2\mathbb{N})$, $a \mapsto a \otimes p$, is called a *corner embedding* of A.

A functor $F\colon \mathfrak{C}^*\mathfrak{alg} \to \mathfrak{C}$ is called *C^*-stable* if any corner embedding induces an isomorphism $F(A) \cong F\bigl(A \otimes \mathbb{K}(\ell^2\mathbb{N})\bigr)$.

The correct equivariant generalisation is the following:

Definition 12. A functor $F\colon G\text{-}\mathfrak{C}^*\mathfrak{alg} \to \mathfrak{C}$ is called *C^*-stable* if the canonical embeddings $\mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2 \leftarrow \mathcal{H}_2$ induce isomorphisms

$$F\bigl(A \otimes \mathbb{K}(\mathcal{H}_1)\bigr) \xrightarrow{\cong} F\bigl(A \otimes \mathbb{K}(\mathcal{H}_1 \oplus \mathcal{H}_2)\bigr) \xleftarrow{\cong} F\bigl(A \otimes \mathbb{K}(\mathcal{H}_2)\bigr)$$

for all non-zero G-Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

Of course, it suffices to require $F\bigl(A \otimes \mathbb{K}(\mathcal{H}_1)\bigr) \xrightarrow{\cong} F\bigl(A \otimes \mathbb{K}(\mathcal{H}_1 \oplus \mathcal{H}_2)\bigr)$. It is not hard to check that Definitions 11 and 12 are equivalent for trivial G.

Our next goal is to describe the *universal* C^*-stable functor. We abbreviate $A_{\mathbb{K}} := \mathbb{K}(L^2G) \otimes A$.

Definition 13. A *correspondence* from A to B (or $A \dashrightarrow B$) is a G-equivariant Hilbert $B_{\mathbb{K}}$-module $\mathcal{E}$ together with a G-equivariant essential (or non-degenerate) *-homomorphism $f\colon A_{\mathbb{K}} \to \mathbb{K}(\mathcal{E})$.

Given correspondences $\mathcal{E}$ from A to B and $\mathcal{F}$ from B to C, their *composition* is the correspondence from A to C with underlying Hilbert module $\mathcal{E} \mathbin{\bar\otimes}_{B_{\mathbb{K}}} \mathcal{F}$ and map $A_{\mathbb{K}} \to \mathbb{K}(\mathcal{E}) \to \mathbb{K}(\mathcal{E} \mathbin{\bar\otimes}_{B_{\mathbb{K}}} \mathcal{F})$, where the last map sends $T \mapsto T \otimes 1$; this yields compact operators because $B_{\mathbb{K}}$ maps to $\mathbb{K}(\mathcal{F})$. See [19] for the definition of the relevant completed tensor product of Hilbert modules.

The composition of correspondences is only defined up to isomorphism. It is associative and the identity maps $A \to A = \mathbb{K}(A)$ act as unit elements, so that we get a category $\mathfrak{Corr}_G$ whose morphisms are the *isomorphism classes* of correspondences. Any *-homomorphism $\varphi\colon A \to B$ yields a correspondence: let $\mathcal{E}$ be the right ideal $\varphi(A_{\mathbb{K}}) \cdot B_{\mathbb{K}}$ in $B_{\mathbb{K}}$, viewed as a HILBERT B-module, and let $\varphi(a) \cdot b = \varphi(a) \cdot b$; this restricts to a compact operator on $\mathcal{E}$. This defines a canonical functor $\natural\colon G\text{-}\mathfrak{C}^*\mathfrak{alg} \to \mathfrak{Corr}_G$.

Proposition 14. *The functor $\natural\colon G\text{-}\mathfrak{C}^*\mathfrak{alg} \to \mathfrak{Corr}_G$ is the universal C^*-stable functor on $G\text{-}\mathfrak{C}^*\mathfrak{alg}$; that is, it is C^*-stable, and any other such functor factors uniquely through $\natural$.*

Proof. First we sketch the proof in the non-equivariant case. First we must verify that $\natural$ is C^*-stable. The MORITA–RIEFFEL equivalence between $\mathbb{K}(\ell^2\mathbb{N}) \otimes A \cong \mathbb{K}\bigl(\ell^2(\mathbb{N}, A)\bigr)$ and A is implemented by the HILBERT module $\ell^2(\mathbb{N}, A)$, which yields a correspondence $\bigl(\mathrm{id}, \ell^2(\mathbb{N}, A)\bigr)$ from $\mathbb{K}(\ell^2\mathbb{N}) \otimes A$ to A; this is inverse to the correspondence induced by a corner embedding $A \to \mathbb{K}(\ell^2\mathbb{N}) \otimes A$.

A HILBERT B-module $\mathcal{E}$ with an essential *-homomorphism $A \to \mathbb{K}(\mathcal{E})$ is countably generated because A is assumed σ-unital. KASPAROV's Stabilisation Theorem yields an isometric embedding $\mathcal{E} \to \ell^2(\mathbb{N}, B)$. Hence we get *-homomorphisms

$$A \to \mathbb{K}(\ell^2\mathbb{N}) \otimes B \leftarrow B.$$

This diagram induces a map $F(A) \to F(\mathbb{K}(\ell^2\mathbb{N}) \otimes B) \cong F(B)$ for any stable functor F. Now we should check that this well-defines a functor $\bar{F}\colon \mathfrak{Corr}_G \to \mathfrak{C}$ with $\bar{F} \circ \natural = F$, and that this yields the only such functor. We omit these computations.

The generalisation to the equivariant case uses the crucial property of the left regular representation that $L^2(G) \otimes \mathcal{H} \cong L^2(G \times \mathbb{N})$ for any countably infinite-dimensional G-HILBERT space $\mathcal{H}$. Since we replace A and B by $A_{\mathbb{K}}$ and $B_{\mathbb{K}}$ in the definition of correspondence right away, we can use this to repair a possible lack of G-equivariance; similar ideas appear in [22]. ∎

Example 15. Let u be a G-invariant multiplier of B. Then the identity map and the *inner automorphism* $B \to B$, $b \mapsto ubu^*$, defined by u define isomorphic correspondences $B \dashrightarrow B$ (via u). Hence inner automorphims act trivially on C^*-stable functors. Actually, this is one of the computations that we have omitted in the proof above; the argument can be found in [11].

2.4. Exactness properties.

Definition 16. A diagram $I \to E \to Q$ in $G\text{-}\mathfrak{C}^*\mathfrak{alg}$ is an *extension* if it is isomorphic to the canonical diagram $I \to A \to A/I$ for some G-invariant ideal I in a G-C^*-algebra A. We write $I \rightarrowtail E \twoheadrightarrow Q$ to denote extensions. A *section* for an extension

$$I \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q \tag{17}$$

in $G\text{-}\mathfrak{C}^*\mathfrak{alg}$ is a map (of sets) $Q \to E$ with $p \circ s = \mathrm{id}_Q$. We call (17) *split* if there is a section that is a G-equivariant *-homomorphism. We call (17) *G-equivariantly cp-split* if there is a G-equivariant, completely positive, contractive, linear section.

Sections are also often called *lifts*, *liftings*, or *splittings*.

Definition 18. A functor F on $G\text{-}\mathfrak{C}^*\mathfrak{alg}$ is *split-exact* if, for any split extension $K \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q$ with section $s\colon Q \to E$, the map $\bigl(F(i), F(s)\bigr)\colon F(K)\oplus F(Q) \to F(E)$ is invertible.

Split-exactness is useful because of the following construction of Joachim Cuntz ([9]).

Let $B \triangleleft E$ be a G-invariant ideal and let $f_+, f_-\colon A \rightrightarrows E$ be G-equivariant *-homomorphisms with $f_+(a) - f_-(a) \in B$ for all $a \in A$. Equivalently, f_+ and f_- both lift the same morphism $\bar{f}\colon A \to E/B$. The data (A, f_+, f_-, E, B) is called a *quasi-homomorphism* from A to B.

Pulling back the extension $B \rightarrowtail E \twoheadrightarrow E/B$ along $\bar{f}$, we get an extension $B \rightarrowtail E' \twoheadrightarrow A$ with two sections $f'_+, f'_-\colon A \rightrightarrows E'$. The split-exactness of F shows that $F(B) \rightarrowtail F(E') \twoheadrightarrow F(A)$ is a split extension in $\mathfrak{C}$. Since both $F(f'_-)$ and $F(f'_+)$ are sections for it, we get a map $F(f'_+) - F(f'_-)\colon F(A) \to F(B)$. *Thus a quasi-homomorphism induces a map $F(A) \to F(B)$ if F is split-exact.* The formal properties of this construction are summarised in [11].

Given a C^*-algebra A, there is a *universal quasi-homomorphism* out of A. Let $Q(A) := A * A$ be the free product of two copies of A and let $\pi_A\colon Q(A) \to A$ be the *folding homomorphism* that restricts to id_A on both factors. Let $q(A)$ be its kernel. The two canonical embeddings $A \to A * A$ are sections for the folding homomorphism. Hence we get a quasi-homomorphism $A \rightrightarrows Q(A) \triangleright q(A)$. The universal property of the free product shows that any quasi-homomorphism yields a G-equivariant *-homomorphism $q(A) \to B$.

Theorem 19. *Functors that are C^*-stable and split-exact are automatically homotopy invariant.*

This is a deep result of Nigel Higson ([15]); a simple proof can be found in [11]. Besides basic properties of quasi-homomorphisms, it only uses that inner endomorphisms act identically on C^*-stable functors.

Definition 20. We call F *exact* if $F(K) \to F(E) \to F(Q)$ is exact (at $F(E)$) for any extension $K \rightarrowtail E \twoheadrightarrow Q$ in $\mathfrak{S}$. More generally, given a class $\mathcal{E}$ of extensions in $\mathfrak{S}$ like, say, the class of equivariantly cp-split extensions, we define exactness for extensions in $\mathcal{E}$.

Most functors we are interested in satisfy homotopy invariance and Bott periodicity, and these two properties prevent a functor from being exact in the stronger sense of being *left* or *right* exact. This explains why our notion of exactness is much weaker than usual in homological algebra.

It is reasonable to require that a functor be part of a *homology* theory, that is, a sequence of functors $(F_n)_{n\in\mathbb{Z}}$ together with natural long exact sequences for all extensions. We do not require this because this additional information tends to be hard to get *a priori* but often comes for free *a posteriori*:

Proposition 21. *Suppose that F is homotopy invariant and exact (or exact for equivariantly cp-split extensions). Then F has long exact sequences of the form*

$$\cdots \to F\bigl(\mathrm{Sus}(K)\bigr) \to F\bigl(\mathrm{Sus}(E)\bigr) \to F\bigl(\mathrm{Sus}(Q)\bigr) \to F(K) \to F(E) \to F(Q)$$

for any (equivariantly cp-split) extension $K \rightarrowtail E \twoheadrightarrow Q$. In particular, F is split-exact.

See §21.4 in [4] for the proof.

Together with Bott periodicity, this yields long exact sequences that extend towards $\pm\infty$ in *both* directions, showing that an exact homotopy invariant functor that satisfies Bott periodicity is part of a homology theory in a canonical way.

2.5. Definition of Kasparov theory. Kasparov theory associates to two $\mathbb{Z}/2$-graded C^*-algebras an Abelian group $\mathrm{KK}_0^G(A,B)$; this is a vast generalisation of K-theory and K-homology. The most remarkable feature of this theory is an associative product on KK called Kasparov *product*, which generalises various known product constructions in K-theory and K-homology. We do not discuss KK^G for $\mathbb{Z}/2$-graded G-C^*-algebras here because it does not fit so well with the universal property approach.

Fix a locally compact group G. The Kasparov groups $\mathrm{KK}_0^G(A,B)$ for $A,B \in G$-$\mathfrak{C}^*\mathfrak{sep}$ form morphisms sets $A \to B$ of a category, which we denote by $\mathfrak{KK}^G$; the composition in $\mathfrak{KK}^G$ is the Kasparov product. The categories G-$\mathfrak{C}^*\mathfrak{sep}$ and $\mathfrak{KK}^G$ have the same objects. We have a canonical functor

$$\mathrm{KK}\colon G\text{-}\mathfrak{C}^*\mathfrak{sep} \to \mathfrak{KK}^G$$

that acts identically on objects.

Theorem 22. *The functor $\mathrm{KK}^G\colon G\text{-}\mathfrak{C}^*\mathfrak{sep} \to \mathfrak{KK}^G$ is the universal split-exact C^*-stable functor; in particular, $\mathfrak{KK}^G$ is an additive category. In addition, KK^G also has the following properties and is, therefore, universal among functors with some of these extra properties: KK^G is*

- *homotopy invariant;*
- *exact for G-equivariantly cp-split extensions;*
- *satisfies* Bott *periodicity, that is, in $\mathfrak{KK}^G$ there are natural isomorphisms $\mathrm{Sus}^2(A) \cong A$ for all $A \Subset \mathfrak{KK}^G$.*

Definition 23. A G-equivariant *-homomorphism $f\colon A \to B$ is called a KK^G-*equivalence* if $\mathrm{KK}(f)$ is invertible in $\mathfrak{KK}^G$.

Corollary 24. *Let $F\colon G\text{-}\mathfrak{C}^*\mathfrak{sep} \to \mathfrak{C}$ be split-exact and C^*-stable. Then F factors uniquely through KK^G, is homotopy invariant, and satisfies* Bott *periodicity. A KK^G-equivalence $A \to B$ induces an isomorphism $F(A) \to F(B)$.*

We will take the universal property of Theorem 22 as a definition of $\mathfrak{KK}^G$ and thus of the groups $\mathrm{KK}_0^G(A, B)$. We also let

$$\mathrm{KK}_n^G(A, B) := \mathrm{KK}^G\bigl(A, \mathrm{Sus}^n(B)\bigr);$$

since the Bott periodicity isomorphism identifies $\mathrm{KK}_2^G \cong \mathrm{KK}_0^G$, this yields a $\mathbb{Z}/2$-graded theory.

By the universal property, K-theory descends to a functor on $\mathfrak{KK}$, that is, we get canonical maps

$$\mathrm{KK}_0(A, B) \to \mathrm{Hom}\bigl(\mathrm{K}_*(A), \mathrm{K}_*(B)\bigr)$$

for all separable C^*-algebras A, B, where the right hand side denotes grading-preserving group homomorphisms. For $A = \mathbb{C}$, this yields a map $\mathrm{KK}_0(\mathbb{C}, B) \to \mathrm{Hom}\bigl(\mathbb{Z}, \mathrm{K}_0(B)\bigr) \cong \mathrm{K}_0(B)$. Using suspensions, we also get a corresponding map $\mathrm{KK}_1(\mathbb{C}, B) \to \mathrm{K}_1(B)$.

Theorem 25. *The maps* $\mathrm{KK}_*(\mathbb{C}, B) \to \mathrm{K}_*(B)$ *constructed above are isomorphisms for all* $B \Subset \mathfrak{C}^*\mathfrak{sep}$.

Thus Kasparov theory is a bivariant generalisation of K-theory. Roughly speaking, $\mathrm{KK}_*(A, B)$ is the place where maps between K-theory groups live. Most constructions of such maps, say, in index theory can in fact be improved to yield elements of $\mathrm{KK}_*(A, B)$. One reason for this is the *Universal Coefficient Theorem* (UCT) by Rosenberg and Schochet [33], which computes $\mathrm{KK}_*(A, B)$ from $\mathrm{K}_*(A)$ and $\mathrm{K}_*(B)$ for many C^*-algebras A, B. If A satisfies the UCT, then any group homomorphism $\mathrm{K}_*(A) \to \mathrm{K}_*(B)$ lifts to an element of $\mathrm{KK}_*(A, B)$ of the same parity.

With our definition, it is not obvious how to construct elements in $\mathrm{KK}_0^G(A, B)$. The only source we know so far are G-equivariant *-homomorphisms. Another important source are *extensions*, more precisely, equivariantly cp-split extensions. Any such extension $I \rightarrowtail E \twoheadrightarrow Q$ yields a class in $\mathrm{KK}_1^G(Q, I) \cong \mathrm{KK}_0^G(\mathrm{Sus}(Q), I) \cong \mathrm{KK}_0^G\bigl(Q, \mathrm{Sus}(I)\bigr)$. Conversely, any element in $\mathrm{KK}_1^G(Q, I)$ comes from an extension in this fashion in a rather transparent way.

Thus it may seem that we can understand all of Kasparov theory from an abstract, category theoretic point of view. But this is not the case. To get a category, we must *compose* extensions; this leads to extensions of higher length. If we allow such higher-length extensions, we can easily construct a category that is isomorphic to Kasparov theory; this generalisation still works for more general algebras than C^*-algebras (see [11]) because it does not involves any difficult analysis any more. But such a setup offers no help to *compute* products. Here computing products means identifying them with other simple things like, say, the identity morphism. This is why the more concrete approach to Kasparov theory is still necessary for the interesting applications of the theory.

In connection with the Baum–Connes conjecture, our abstract approach allows us to formulate it and analyse its consequences. But to verify it, say, for amenable groups, we must show that a certain morphism in KK^G is invertible.

This involves constructing its inverse and checking that the two KASPAROV products in both order are 1. These computations require the concrete description of KASPAROV theory that we omit here. We merely refer to [4] for a detailed treatment.

2.6. Extending functors and identities to $\mathfrak{KK}^G$. We use the universal property to extend functors from G-$\mathfrak{C}^*\mathfrak{alg}$ to $\mathfrak{KK}^G$ and check identities in $\mathfrak{KK}^G$ without computing KASPAROV products. As our first example, consider the full and reduced crossed product functors

$$G \ltimes_{\mathrm{r}} \text{␣}, G \ltimes \text{␣}\colon G\text{-}\mathfrak{C}^*\mathfrak{alg} \to \mathfrak{C}^*\mathfrak{alg}.$$

Proposition 26. *These two functors extend to functors*

$$G \ltimes_{\mathrm{r}} \text{␣}, G \ltimes \text{␣}\colon \mathfrak{KK}^G \to \mathfrak{KK}$$

called descent functors*.*

KASPAROV constructs these functors directly using the concrete description of KASPAROV cycles. This requires a certain amount of work; in particular, checking functoriality involves knowing how to compute KASPAROV products.

Proof. We only write down the argument for *reduced* crossed products, the other case is similar. It is well-known that $G \ltimes_{\mathrm{r}} \bigl(A \otimes \mathbb{K}(\mathcal{H})\bigr) \cong (G \ltimes_{\mathrm{r}} A) \otimes \mathbb{K}(\mathcal{H})$ for any G-HILBERT space $\mathcal{H}$. Therefore, the composite functor

$$G\text{-}\mathfrak{C}^*\mathfrak{sep} \xrightarrow{G\ltimes_{\mathrm{r}}\text{␣}} \mathfrak{C}^*\mathfrak{sep} \xrightarrow{\mathrm{KK}} \mathfrak{KK}$$

is C^*-stable. This functor is split-exact as well (we omit the proof). Now the universal property provides an extension to a functor $\mathfrak{KK}^G \to \mathfrak{KK}$. ☐

Similarly, we get functors

$$A \otimes_{\min} \text{␣}, A \otimes_{\max} \text{␣}\colon \mathfrak{KK}^G \to \mathfrak{KK}^G$$

for any G-C^*-algebra A. Since these extensions are natural, we even get bifunctors

$$\otimes_{\min}, \otimes_{\max}\colon \mathfrak{KK}^G \times \mathfrak{KK}^G \to \mathfrak{KK}^G.$$

For the BAUM–CONNES assembly map, we need the *induction functors*

$$\mathrm{Ind}_H^G\colon \mathfrak{KK}^H \to \mathfrak{KK}^G$$

for closed subgroups $H \subseteq G$. For a finite group H, $\mathrm{Ind}_H^G(A)$ is the H-fixed point algebra of $\mathcal{C}_0(G, A)$, where H acts by $h \cdot f(g) = \alpha_h\bigl(f(gh)\bigr)$. For infinite H, we have

$$\begin{aligned}\mathrm{Ind}_H^G(A) = \{f \in \mathcal{C}_b(G, A) \mid \\ \alpha_h f(gh) = f(g) \text{ for all } g \in G,\ h \in H, \text{ and } gH \mapsto \|f(g)\| \text{ is } \mathcal{C}_0\};\end{aligned}$$

the group G acts by translations on the left. This construction is clearly functorial for equivariant *-homomorphisms. Furthermore, it commutes with C^*-stabilisations and maps split extensions again to split extensions. Therefore, the same argument as above allows us to extend it to a functor

$$\mathrm{Ind}_H^G \colon \mathfrak{KK}^H \to \mathfrak{KK}^G$$

The following examples are more trivial. Let $\tau \colon \mathfrak{C}^*\mathfrak{alg} \to G\text{-}\mathfrak{C}^*\mathfrak{alg}$ equip a C^*-algebra with the trivial G-action; it extends to a functor $\tau \colon \mathfrak{KK} \to \mathfrak{KK}^G$. The *restriction functors*

$$\mathrm{Res}_G^H \colon \mathfrak{KK}^G \to \mathfrak{KK}^H$$

for closed subgroups $H \subseteq G$ are defined by forgetting part of the equivariance.

The universal property also allows us to *prove identities* between functors. For instance, Green's Imprimitivity Theorem provides *-isomorphisms

$$G \ltimes \mathrm{Ind}_H^G(A) \sim_M H \ltimes A, \qquad G \ltimes_{\mathrm{r}} \mathrm{Ind}_H^G(A) \sim_M H \ltimes_{\mathrm{r}} A \tag{27}$$

for any H-C^*-algebra A. This is proved by completing $\mathcal{C}_0(G, A)$ to an imprimitivity bimodule for both C^*-algebras. This equivalence is clearly natural for H-equivariant *-homomorphisms. Since all functors involved are C^*-stable and split exact, the uniqueness part of the universal property of KK^H shows that the KK-equivalences $G \ltimes \mathrm{Ind}_H^G(A) \cong H \ltimes A$ and $G \ltimes_{\mathrm{r}} \mathrm{Ind}_H^G(A) \cong H \ltimes_{\mathrm{r}} A$ are natural for morphisms in KK^H. That is, the diagram

$$\begin{tikzcd}
G \ltimes_{\mathrm{r}} \mathrm{Ind}_H^G(A_1) \arrow[r, "\cong"] \arrow[d, "G \ltimes_{\mathrm{r}} \mathrm{Ind}_H^G(f)"'] & H \ltimes_{\mathrm{r}} A_1 \arrow[d, "H \ltimes_{\mathrm{r}} f"] \\
G \ltimes_{\mathrm{r}} \mathrm{Ind}_H^G(A_2) \arrow[r, "\cong"] & H \ltimes_{\mathrm{r}} A_2
\end{tikzcd}$$

in $\mathfrak{KK}$ commutes for any $f \in \mathrm{KK}_0^H(A_1, A_2)$. More examples of this kind are discussed in §4.1 of [24].

We can also prove *adjointness* relations in Kasparov theory in an abstract way by constructing the unit and counit of the adjunction. An important example is the adjointness between induction and restriction functors (see also §3.2 of [25]).

Proposition 28. *Let $H \subseteq G$ be a closed subgroup. If H is open, then we have natural isomorphisms*

$$\mathrm{KK}^G(\mathrm{Ind}_H^G A, B) \cong \mathrm{KK}^H(A, \mathrm{Res}_G^H B) \tag{29}$$

for all $A \in\in H\text{-}\mathfrak{C}^*\mathfrak{alg}$, $B \in\in G\text{-}\mathfrak{C}^*\mathfrak{alg}$. If $H \subseteq G$ is cocompact, then we have natural isomorphisms

$$\mathrm{KK}^G(A, \mathrm{Ind}_H^G B) \cong \mathrm{KK}^H(\mathrm{Res}_G^H A, B) \tag{30}$$

for all $A \in\in G\text{-}\mathfrak{C}^*\mathfrak{alg}$, $B \in\in H\text{-}\mathfrak{C}^*\mathfrak{alg}$.

Proof. We will not use (30) later and therefore only prove (29). We must construct natural elements

$$\alpha_A \in \mathrm{KK}_0^G(\mathrm{Ind}_H^G \mathrm{Res}_G^H A, A), \qquad \beta_B \in \mathrm{KK}_0^H(B, \mathrm{Res}_G^H \mathrm{Ind}_H^G B)$$

that satisfy the conditions for unit and counit of adjunction ([21]).

We have a natural G-equivariant *-isomorphism $\mathrm{Ind}_H^G \mathrm{Res}_H^G(A) \cong \mathcal{C}_0(G/H) \otimes A$ for any G-C^*-algebra A. Since H is open in G, the homogeneous space G/H is discrete. We represent $\mathcal{C}_0(G/H)$ on the HILBERT space $\ell^2(G/H)$ by pointwise multiplication operators. This is G-equivariant for the representation of G on $\ell^2(G/H)$ by left translations. Thus we get a correspondence from $\mathrm{Ind}_H^G \mathrm{Res}_G^H(A)$ to A, which yields $\alpha_A \in \mathrm{KK}_0^G(\mathrm{Ind}_H^G \mathrm{Res}_G^H(A), A)$ because KK^G is C^*-stable.

For any H-C^*-algebra B, we may embed B in $\mathrm{Res}_G^H \mathrm{Ind}_H^G(B)$ as the subalgebra of functions supported on the single coset H. This embedding is H-equivariant and provides $\beta_B \in \mathrm{KK}_0^H(B, \mathrm{Res}_G^H \mathrm{Ind}_H^G B)$.

Now we have to check that the following two composite maps are identity morphisms in $\mathfrak{KK}^G$ and $\mathfrak{KK}^H$, respectively:

$$\mathrm{Ind}_H^G(B) \xrightarrow{\mathrm{Ind}_H^G(\beta_B)} \mathrm{Ind}_H^G \mathrm{Res}_G^H \mathrm{Ind}_H^G(B) \xrightarrow{\alpha_{\mathrm{Ind}_H^G(B)}} \mathrm{Ind}_H^G(B)$$

$$\mathrm{Res}_G^H A \xrightarrow{\beta_{\mathrm{Res}_G^H A}} \mathrm{Res}_G^H \mathrm{Ind}_H^G \mathrm{Res}_G^H(A) \xrightarrow{\mathrm{Res}_G^H \alpha_A} \mathrm{Res}_G^H A$$

This yields the desired adjointness by a general argument from category theory (see [21]). In fact, both composites are already equal to the identity as *correspondences.* Hence we need no knowledge of KASPAROV theory except for its C^*-stability to prove (29). □

The following example is discussed in detail in §4.1 of [24]. If G is compact, then the trivial action functor $\tau\colon \mathfrak{KK} \to \mathfrak{KK}^G$ is left adjoint to $G \ltimes \text{␣} = G \ltimes_{\mathrm{r}} \text{␣}$, that is, we have natural isomorphisms

$$\mathrm{KK}_*^G(\tau(A), B) \cong \mathrm{KK}_*(A, G \ltimes B). \tag{31}$$

This is also known as the GREEN–JULG *Theorem.* For $A = \mathbb{C}$, it specialises to a natural isomorphism $\mathrm{K}_*^G(B) \cong \mathrm{K}_*(G \ltimes B)$.

2.7. Triangulated category structure. We can turn $\mathfrak{KK}^G$ into a triangulated category by extending standard constructions for topological spaces (see [25]). But some arrows change direction because the functor $\mathcal{C}_0$ from spaces to C^*-algebras is contravariant. We have already observed that $\mathfrak{KK}^G$ is additive. The *suspension automorphism* is $\Sigma^{-1}(A) := \mathrm{Sus}(A)$. Since $\mathrm{Sus}^2(A) \cong A$ in $\mathfrak{KK}^G$ by BOTT periodicity, we have $\Sigma = \Sigma^{-1}$. Thus we do not need formal desuspensions as for the stable homotopy category.

Definition 32. A triangle $A \to B \to C \to \Sigma A$ in $\mathfrak{KK}^G$ is called *exact* if it is isomorphic as a triangle to the *mapping cone triangle*

$$\mathrm{Sus}(B) \to \mathrm{Cone}(f) \to A \xrightarrow{f} B$$

for some G-equivariant *-homomorphism f.

Alternatively, we can use G-equivariantly cp-split extensions in G-$\mathfrak{C}^*\mathfrak{sep}$. Any such extension $I \rightarrowtail E \twoheadrightarrow Q$ determines a class in $\mathrm{KK}_1^G(Q, I) \cong \mathrm{KK}_0^G(\mathrm{Sus}(Q), I)$, so that we get a triangle $\mathrm{Sus}(Q) \to I \to E \to Q$ in $\mathfrak{KK}^G$. Such triangles are called *extension triangle*. A triangle in $\mathfrak{KK}^G$ is exact if and only if it is isomorphic to the extension triangle of a G-equivariantly cp-split extension.

Theorem 33. *With the suspension automorphism and exact triangles defined above, $\mathfrak{KK}^G$ is a triangulated category.*

Proof. This is proved in detail in [25]. □

Triangulated categories clarify the basic bookkeeping with long exact sequences. Mayer-Vietoris exact sequences and inductive limits are discussed from this point of view in [25]. More importantly, this framework sheds light on more advanced constructions like the Baum–Connes assembly map.

3. Subcategories in $\mathfrak{KK}^G$

Now we turn to the construction of the Baum–Connes assembly map by studying various subcategories of $\mathfrak{KK}^G$ that are related to it.

3.1. Compactly induced actions.

Definition 34. Let G be a locally compact group. A G-C^*-algebra is *compactly induced* if it is of the form $\mathrm{Ind}_H^G(A)$ for some compact subgroup H of G and some H-C^*-algebra A. We let $\mathcal{CI}$ be the class of all G-C^*-algebras that are KK^G-equivalent to a direct summand of $\bigoplus_{i\in\mathbb{N}} A_i$ with compactly induced G-C^*-algebras A_i for $i \in \mathbb{N}$.

Equivalently, $\mathcal{CI}$ is the smallest class of objects in $\mathfrak{KK}^G$ that is closed under direct sums, direct summands and isomorphism and contains all compactly induced G-C^*-algebras.

Green's Imprimitivity Theorem (27) tells us that the (reduced) crossed product for a compactly induced action $\mathrm{Ind}_H^G(A)$ is equivalent to the crossed product $H \ltimes_{\mathrm{r}} A$ for the compact group H. Hence we have

$$\mathrm{K}_*(G \ltimes \mathrm{Ind}_H^G A) \cong \mathrm{K}_*(G \ltimes_{\mathrm{r}} \mathrm{Ind}_H^G A) \cong \mathrm{K}_*(H \ltimes A) \cong \mathrm{K}_*^H(A)$$

by the Green–Julg Theorem, compare (31).

Since the computation of equivariant K-theory for compact groups is a problem of *classical* topology, operator algebraists can pretend that it is *Somebody Else's Problem.* We are more fascinated by the analytic difficulties created by crosssed products by infinite groups. For instance, it is quite hard to see which Laurent series $\sum_{n\in\mathbb{Z}} a_n z^n$ correspond to an element of $C^*_{\mathrm{red}}\mathbb{Z} = C^*\mathbb{Z}$ or, equivalently, which of them are the Fourier series of a continuous function on the unit circle. The Baum–Connes conjecture, when true, implies that such analytic difficulties do not influence the K-theory.

3.2. Two simple examples. It is best to explain our goals with two examples, namely, the groups $\mathbb{R}$ and $\mathbb{Z}$. The BAUM–CONNES conjectures for these groups hold and are equivalent to the CONNES–THOM *isomorphism* and a PIMSNER–VOICULESCU *exact sequence*. Although the BAUM–CONNES conjecture only concerns the K-theory of $C^*_{\mathrm{red}}G$ and, more generally, of crossed products $G \ltimes_{\mathrm{r}} A$, we get much stronger statements in this case.

Both $\mathbb{R}$ and $\mathbb{Z}$ are torsion-free, that is, they have no non-trivial compact subgroups. Hence the compactly induced actions are of the form $\mathcal{C}_0(G, A)$ with $G \in \{\mathbb{R}, \mathbb{Z}\}$ acting by translation. If A carries another action of G, then it makes no difference whether we let G act on $\mathcal{C}_0(G, A)$ by $t \cdot f(x) := f(t^{-1}x)$ or $t \cdot f(x) := \alpha_t\bigl(f(t^{-1}x)\bigr)$: both definitions yield isomorphic G-C^*-algebras.

Theorem 35. *Any $\mathbb{R}$-C^*-algebra is $\mathrm{KK}^{\mathbb{R}}$-equivalent to a compactly induced one. More briefly, $\mathcal{CI} = \mathrm{KK}^{\mathbb{R}}$.*

Proof. Let A be any $\mathbb{R}$-C^*-algebra. Let $\mathbb{R}$ act on $\mathbb{R}$ by translation and extend this to an action on $X = (-\infty, \infty]$ by $t \cdot \infty := \infty$ for all $t \in \mathbb{R}$. Then we get an extension of $\mathbb{R}$-C^*-algebras

$$\mathcal{C}_0(\mathbb{R}, A) \rightarrowtail \mathcal{C}_0(X, A) \twoheadrightarrow A,$$

where we let $\mathbb{R}$ act diagonally. It does not yet have an $\mathbb{R}$-*equivariant* completely positive section, but it becomes equivariantly cp-split if we tensor with $\mathbb{K}(L^2G)$. Therefore, it yields an extension triangle in $\mathrm{KK}^{\mathbb{R}}$.

The DIRAC operator on $\mathcal{C}_0(\mathbb{R}, A)$ for the standard RIEMANNian metric on $\mathbb{R}$ defines a class in $\mathrm{KK}^{\mathbb{R}}_1(\mathcal{C}_0(\mathbb{R}), \mathbb{C})$ which we may then map to $\mathrm{KK}^{\mathbb{R}}_1(\mathcal{C}_0(\mathbb{R}, A), A)$ by exterior product. This yields another cp-split extension

$$\mathbb{K}(L^2\mathbb{R}) \otimes A \rightarrowtail \mathcal{T} \otimes A \twoheadrightarrow \mathcal{C}_0(\mathbb{R}, A).$$

The resulting classes in $\mathrm{KK}^G_1\bigl(A, \mathcal{C}_0(\mathbb{R}, A)\bigr)$ and $\mathrm{KK}^G_1(\mathcal{C}_0(\mathbb{R}, A), A)$ are inverse to each other; this is checked by computing their KASPAROV products in both orders. Thus A is $\mathrm{KK}^{\mathbb{R}}$-equivalent to the induced $\mathbb{R}$-C^*-algebra $\mathcal{C}_0(\mathbb{R}, A)$. $\square$

Since the crossed product is functorial on KASPAROV categories, this implies

$$\mathbb{R} \ltimes A = \mathbb{R} \ltimes_{\mathrm{r}} A \sim \mathbb{R} \ltimes_{\mathrm{r}} \mathrm{Sus}\bigl(\mathcal{C}_0(\mathbb{R}, A)\bigr) \cong \mathrm{Sus}(\mathbb{K}(L^2\mathbb{R}) \otimes A) \sim \mathrm{Sus}(A),$$

where $\sim$ denotes KK-equivalence. Taking K-theory, we get the CONNES–THOM Isomorphism $\mathrm{K}_*(\mathbb{R} \ltimes A) \cong \mathrm{K}_{*+1}(A)$.

For most groups, we have $\mathcal{CI} \neq \mathrm{KK}^G$. We now study the simplest case where this happens, namely, $G = \mathbb{Z}$.

We have seen above that $\mathcal{C}_0(\mathbb{R})$ with the translation action of $\mathbb{R}$ is $\mathrm{KK}^{\mathbb{R}}$-equivalent to $\mathcal{C}_0(\mathbb{R})$ with trivial action. This equivalence persists if we restrict the action from $\mathbb{R}$ to the subgroup $\mathbb{Z} \subseteq \mathbb{R}$. Hence we get a $\mathrm{KK}^{\mathbb{Z}}$-equivalence

$$A \sim \mathrm{Sus}\bigl(\mathcal{C}_0(\mathbb{R}, A)\bigr),$$

where $n \in \mathbb{Z}$ acts on $\mathcal{C}_0(\mathbb{R}, A) \cong \mathcal{C}_0(\mathbb{R}) \otimes A$ by $(\alpha_n f)(x) := \alpha_n\bigl(f(x-n)\bigr)$. Although the $\mathbb{Z}$-action on $\mathbb{R}$ is free and proper, the action of $\mathbb{Z}$ on $\mathcal{C}_0(\mathbb{R}, A)$ need not be induced from the trivial subgroup.

Theorem 36. *For any $\mathbb{Z}$-C^*-algebra A, there is an exact triangle*

$$P_1 \to P_0 \to A \to \Sigma P_1$$

in $\mathfrak{KK}^{\mathbb{Z}}$ with compactly induced P_0 and P_1; more explicitly, $P_0 = P_1 = \mathcal{C}_0(\mathbb{Z}, A)$.

Proof. Restriction to $\mathbb{Z} \subseteq \mathbb{R}$ provides a surjection $\mathcal{C}_0(\mathbb{R}, A) \twoheadrightarrow \mathcal{C}_0(\mathbb{Z}, A)$, whose kernel may be identified with $\mathcal{C}_0((0,1)) \otimes \mathcal{C}_0(\mathbb{Z}, A)$. The resulting extension

$$\mathcal{C}_0((0,1)) \otimes \mathcal{C}_0(\mathbb{Z}, A) \rightarrowtail \mathcal{C}_0(\mathbb{R}, A) \twoheadrightarrow \mathcal{C}_0(\mathbb{Z}, A)$$

is $\mathbb{Z}$-equivariantly cp-split and hence provides an extension triangle in $\mathrm{KK}^{\mathbb{Z}}$. Since $\mathcal{C}_0(\mathbb{R}, A)$ is $\mathrm{KK}^{\mathbb{Z}}$-equivalent to the suspension of A, we get an exact triangle of the desired form. $\square$

When we apply a homological functor $\mathfrak{KK}^G \to \mathfrak{C}$ such as $\mathrm{K}_*(\mathbb{Z} \ltimes \text{␣})$ to the exact triangle in Theorem 36, then we get the Pimsner–Voiculescu *exact sequence*

$$\begin{tikzcd}
\mathrm{K}_1(A) \arrow[r] & \mathrm{K}_0(\mathbb{Z} \ltimes A) \arrow[r] & \mathrm{K}_0(A) \arrow[d, "\alpha_* - 1"] \\
\mathrm{K}_1(A) \arrow[u, "\alpha_* - 1"] & \mathrm{K}_1(\mathbb{Z} \ltimes A) \arrow[l] & \mathrm{K}_0(A). \arrow[l]
\end{tikzcd}$$

Here $\alpha_* \colon \mathrm{K}_*(A) \to \mathrm{K}_*(A)$ is the map induced by the automorphism $\alpha(1)$ of A. It is not hard to identify the boundary map for the above extension with this map. Our approach yields such exact sequences for any homological functor.

Now we formulate some structural results for $\mathbb{R}$ and $\mathbb{Z}$ that have a chance to generalise to other groups.

Theorem 37. *Let G be $\mathbb{R}$ or $\mathbb{Z}$. Let A_1 and A_2 be G-C^*-algebras and let $f \in \mathrm{KK}^G(A_1, A_2)$. If $\mathrm{Res}_G(f) \in \mathrm{KK}(A_1, A_2)$ is invertible, then so is f itself. In particular, if $\mathrm{Res}_G(A_1) \cong 0$ in KK, then already $A \cong 0$ in KK^G.*

Proof. We only write down the proof for $G = \mathbb{Z}$; the case $G = \mathbb{R}$ is similar but simpler. If f were an equivariant *-homomorphisms, then it would induce a morphism of extensions

$$\begin{tikzcd}
\mathcal{C}_0\bigl((0,1) \times \mathbb{Z}, A_1\bigr) \arrow[r] \arrow[d, "f_*"] & \mathcal{C}_0\bigl(\mathbb{R}, A_1\bigr) \arrow[r] \arrow[d, "f_*"] & \mathcal{C}_0\bigl(\mathbb{Z}, A_1\bigr) \arrow[d, "f_*"] \\
\mathcal{C}_0\bigl((0,1) \times \mathbb{Z}, A_2\bigr) \arrow[r] & \mathcal{C}_0\bigl(\mathbb{R}, A_2\bigr) \arrow[r] & \mathcal{C}_0\bigl(\mathbb{Z}, A_2\bigr)
\end{tikzcd}$$

and hence a morphism of triangles between the resulting extension triangles. The latter morphism still exists even if f is merely a morphism in $\mathrm{KK}^{\mathbb{Z}}$. This can be checked directly or deduced in a routine fashion from the uniqueness part of the universal property of $\mathrm{KK}^{\mathbb{Z}}$. If $\mathrm{Res}_G(f)$ is invertible, then so are the induced maps $\mathcal{C}_0\bigl((0,1) \times \mathbb{Z}, A_1\bigr) \to \mathcal{C}_0\bigl((0,1) \times \mathbb{Z}, A_2\bigr)$ and $\mathcal{C}_0(\mathbb{Z}, A_1) \to \mathcal{C}_0(\mathbb{Z}, A_2)$ because $\mathcal{C}_0(\mathbb{Z}, A) \cong \mathrm{Ind}^{\mathbb{Z}} \mathrm{Res}_{\mathbb{Z}}(A)$. Hence the Five Lemma in triangulated categories shows that f itself is invertible. To get the second statement, apply the first one to the zero maps $0 \to A_1 \to 0$. $\square$

Definition 38. A path of G-actions $(\alpha_t)_{t\in[0,1]}$ is *continuous* if its pointwise application defines a strongly continuous action of G on $\mathrm{Cyl}(A) := \mathcal{C}([0,1], A)$.

Corollary 39. *Let $G = \mathbb{R}$ or $\mathbb{Z}$. If $(\alpha_t)_{t\in[0,1]}$ is a continuous path of G-actions on A, then there is a canonical KK^G-equivalence $(A, \alpha_0) \sim (A, \alpha_1)$. As a consequence, the crossed products for both actions are* KK*-equivalent.*

Proof. Equip $\mathrm{Cyl}(A)$ with the automorphism α. Evaluation at 0 and 1 provides elements in $\mathrm{KK}^{\mathbb{Z}}\big(\mathrm{Cyl}(A), (A, \alpha_t)\big)$ that are non-equivariantly invertible because KK is homotopy invariant. Hence they are invertible in KK^G by Theorem 37. Their composition yields the desired KK^G-equivalence $(A, \alpha_0) \sim (A, \alpha_1)$. □

It is not hard to extend Theorem 37 and hence Corollary 39 to the groups $\mathbb{R}^n$ and $\mathbb{Z}^n$ for any $n \in \mathbb{Z}$. With a bit more work, we could also treat solvable Lie groups. But Theorem 37 as stated above fails for finite groups: there exists a space X and two homotopic actions α_0, α_1 of $\mathbb{Z}/2$ on X for which $\mathrm{K}^*_{\mathbb{Z}/2}(X, \alpha_t)$ are different for $t = 0, 1$. Reversing the argument in the proof of Corollary 39, this provides the desired counterexample. Less complicated counterexamples can be constructed where A is a UHF C^*-algebra. Such counterexamples force us to amend our question:

> Suppose $\mathrm{Res}^H_G(A) \cong 0$ for all compact subgroups $H \subseteq G$. Does it follow that $A \cong 0$ in KK^G? Or at least that $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A) \cong 0$?

It is shown in [25] that the second question has a positive answer if and only if the Baum–Connes *conjecture* holds for *G with arbitrary coefficients.* For many groups for which we know the Baum–Connes conjecture with coefficients, we also know that the first question has a positive answer. But the first question can only have a positive answer if the group is K-amenable, that is, if reduced and full crossed products have the same K-theory. The Lie group $\mathrm{Sp}(n, 1)$ and its cocompact subgroups are examples where we know the Baum–Connes conjecture with coefficients although the group is not K-amenable.

Definition 40. A G-C^*-algebra A is called *weakly contractible* if $\mathrm{Res}^H_G(A) \cong 0$ for all compact subgroups $H \subseteq G$. Let $\mathcal{CC}$ be the class of weakly contractible objects.

A morphism $f \in \mathrm{KK}^G(A_1, A_2)$ is called a *weak equivalence* if $\mathrm{Res}^H_G(f)$ is invertible for all compact subgroups $H \subseteq G$.

Recall that any $f \in \mathrm{KK}^G(A_1, A_2)$ is part of an exact triangle $A_1 \to A_2 \to C \to \Sigma A_1$ in KK^G. We have $C \in \mathcal{CC}$ if and only if f is a weak equivalence. Hence our two questions above are equivalent to:

> Are all weak equivalences invertible in KK^G? Do they at least act invertibly on $\mathrm{K}_*(G \ltimes_{\mathrm{r}} \text{␣})$?

The second question is equivalent to the Baum–Connes conjecture.

Suppose now that G is discrete. Then any subgroup is open, so that the adjointness isomorphism (29) always applies. It asserts that the subcategories $\mathcal{CI}$ and $\mathcal{CC}$ are orthogonal, that is, $\mathrm{KK}^G(A, B) = 0$ if $A \Subset \mathcal{CI}$, $B \Subset \mathcal{CC}$. Even more, if $\mathrm{KK}^G(A,B)=0$ for all $A \in\in \mathcal{CI}$, then it follows that $B \in\in \mathcal{CC}$. A more involved argument in [25] extends these observations to all locally compact groups G.

Definition 41. Let $\langle\mathcal{CI}\rangle$ be the smallest full subcategory of KK^G that contains $\mathcal{CI}$ and is closed under suspensions, (countable) direct sums, and exact triangles.

We may think of objects of $\langle\mathcal{CI}\rangle$ as generalised CW-complexes that are built out of the cells in $\mathcal{CI}$.

Theorem 42. *The pair of subcategories* $(\langle\mathcal{CI}\rangle, \mathcal{CC})$ *is* complementary *in the following sense (see* [25]*):*

- $\mathrm{KK}^G(P,N)=0$ *if* $P \in\in \langle\mathcal{CI}\rangle$, $N \in\in \mathcal{CC}$;
- *for any* $A \in\in \mathrm{KK}^G$, *there is an exact triangle* $P \to A \to N \to \Sigma P$ *with* $P \in\in \langle\mathcal{CI}\rangle$, $N \in\in \mathcal{CC}$.

Moreover, the exact triangle $P \to A \to N \to \Sigma P$ *above is unique up to a canonical isomorphism and depends functorially on* A, *and the ensuing functors* $A \mapsto P(A)$, $A \mapsto N(A)$ *are exact functors on* $\mathfrak{KK}^G$.

Proof. The orthogonality of $\langle\mathcal{CI}\rangle$ and $\mathcal{CC}$ follows easily from the orthogonality of $\mathcal{CI}$ and $\mathcal{CC}$. The existence of an exact triangle decomposition is more difficult. The proof in [25] reduces this to the special case $A=\mathbb{C}$. A more elementary construction of this exact triangle is explained in [11]. $\square$

Theorem 42 asserts that $\mathcal{CI}$ and $\mathcal{CC}$ together generate all of $\mathfrak{KK}^G$. This is why the vanishing of $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$ for $A \in\in \mathcal{CC}$ is so useful: it allows us to replace an arbitrary object by one in $\langle\mathcal{CI}\rangle$. The latter is built out of objects in $\mathcal{CI}$. We have already agreed that the computation of $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$ for $A \in\in \mathcal{CI}$ is Somebody Else's Problems. Once we understand a mechanism for decomposing objects of $\langle\mathcal{CI}\rangle$ into objects of $\mathcal{CI}$, the computation of $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$ for $A \in\in \langle\mathcal{CI}\rangle$ becomes a purely topological affair and hence Somebody Else's Problem as well.

For the groups $\mathbb{Z}^n$ and $\mathbb{R}^n$, the subcategory $\mathcal{CC}$ is trivial, so that Theorem 42 simply asserts that $\mathfrak{KK}^G = \langle\mathcal{CI}\rangle$ is *generated* by the compactly induced actions. More generally, this is the case for all amenable groups; the proof of the Baum–Connes conjecture by Higson and Kasparov for such groups also yields this stronger assertion (see [25]).

Definition 43. Let $F\colon \mathfrak{KK}^G \to \mathfrak{C}$ be a functor. Its *localisation* at $\mathcal{CC}$ (or at the weak equivalences) is the functor

$$\mathbb{L}F := F \circ P \colon \mathfrak{KK}^G \to \langle\mathcal{CI}\rangle \subseteq \mathfrak{KK}^G \to \mathfrak{C},$$

where we use the functors $P\colon \mathfrak{KK}^G \to \langle\mathcal{CI}\rangle$ and $N\colon \mathfrak{KK}^G \to \mathcal{CC}$ that are part of a natural exact triangle $P(A) \to A \to N(A) \to \Sigma P(A)$.

The natural transformation $P(A) \to A$ furnishes a natural transformation $\mathbb{L}F(A) \to F(A)$. If F is homological or exact, then $F \circ N(A)$ is the *obstruction* to invertibility of this map.

The localisation $\mathbb{L}F$ can be characterised by a universal property. First of all, it vanishes on $\mathcal{CC}$ because $P(A) \cong 0$ whenever $A \in\in \mathcal{CC}$. If $\tilde{F}$ is another functor with this property, then any natural transformation $\tilde{F} \to F$ factors uniquely through $\mathbb{L}F \to F$. This universal property characterises $\mathbb{L}F$ uniquely up to natural isomorphism of functors.

Theorem 44. *The natural transformation $\mathbb{L}F(A) \to F(A)$ for $F(A) := \mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$ is equivalent to the* Baum–Connes *assembly map. That is, there is a natural isomorphism $\mathbb{L}F(A) \cong \mathrm{K}_*^{\mathrm{top}}(G, A)$ compatible with the maps to $F(A)$.*

Proof. It is known (but not obvious) that $\mathrm{K}_*^{\mathrm{top}}(G, A)$ vanishes for $\mathcal{CC}$ and that the Baum–Connes assembly map is an isomorphism for coefficients in $\mathcal{CI}$. These two facts together imply the result. □

The Baum–Connes *conjecture* asserts that the assembly map $\mathbb{L}F(A) \to F(A)$ is invertible for all A if $F(A) := \mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$. This follows if $\mathcal{CI} = \mathfrak{KK}^G$, of course. In particular, the Baum–Connes *conjecture* is trivial if G itself is compact.

Part II
Homological algebra

It is well-known that many basic constructions from homotopy theory extend to categories of C^*-algebras. As we argued in [25], the framework of *triangulated categories* is ideal for this purpose. The notion of triangulated category was introduced by Jean-Louis Verdier to formalise the properties of the derived category of an Abelian category. Stable homotopy theory provides further classical examples of triangulated categories. The triangulated category structure encodes basic information about manipulations with long exact sequences and (total) derived functors. The main point of [25] is that the domain of the Baum–Connes assembly map is the total left derived functor of the functor that maps a G-C^*-algebra A to $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$.

Projective resolutions are among the most fundamental concepts in homological algebra; several others like derived functors are based on it. Projective resolutions seem to live in the underlying Abelian category and not in its derived category. This is why *total* derived functor make more sense in triangulated categories than the derived functors themselves. Nevertheless, we can define derived functors in triangulated categories and far more general categories. This goes back to S. Eilenberg and J. C. Moore ([12]). We learned about this theory in articles by Apostolos Beligiannis ([3]) and J. Daniel Christensen ([8]).

Homological algebra in non-Abelian categories is always *relative*, that is, we need additional structure to get started. This is useful because we may fit the additional data to our needs. In a triangulated category $\mathfrak{T}$, there are several kinds of additional data that yield equivalent theories; following [8], we use an *ideal* in $\mathfrak{T}$. We only consider ideals $\mathfrak{I} \subseteq \mathfrak{T}$ of the form

$$\mathfrak{I}(A, B) := \{x \in \mathfrak{T}(A, B) \mid F(x) = 0\}$$

for a stable homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$ into a stable Abelian category $\mathfrak{C}$. Here *stable* means that $\mathfrak{C}$ carries a suspension automorphism and that F intertwines the suspension automorphisms on $\mathfrak{T}$ and $\mathfrak{C}$, and homological means that exact triangles yield exact sequences. Ideals of this form are called *homological ideals*.

A basic example is the ideal in the Kasparov category KK defined by

$$\mathfrak{I}_{\mathrm{K}}(A,B) := \{f \in \mathrm{KK}(A,B) \mid 0 = \mathrm{K}_*(f)\colon \mathrm{K}_*(A) \to \mathrm{K}_*(B)\}. \tag{45}$$

For a locally compact group G and a (suitable) family of subgroups $\mathcal{F}$, we define the homological ideal

$$\mathcal{VC}_{\mathcal{F}}(A,B) := \{f \in \mathrm{KK}^G(A,B) \mid \\ \mathrm{Res}_G^H(f) = 0 \text{ in } \mathrm{KK}^H(A,B) \text{ for all } H \in \mathcal{F}\}. \tag{46}$$

If $\mathcal{F}$ is the family of compact subgroups, then $\mathcal{VC}_{\mathcal{F}}$ is related to the Baum–Connes assembly map ([25]). Of course, there are analogous ideals in more classical categories of (spectra of) G-CW-complexes.

All these examples can be analysed using the machinery we explain; but we only carry this out in some cases.

We use an ideal $\mathfrak{I}$ to carry over various notions from homological algebra to our triangulated category $\mathfrak{T}$. In order to see what they mean in examples, we characterise them using a stable homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$ with $\ker F = \mathfrak{I}$. This is often easy. For instance, a chain complex with entries in $\mathfrak{T}$ is $\mathfrak{I}$-*exact* if and only if F maps it to an exact chain complex in the Abelian category $\mathfrak{C}$, and a morphism in $\mathfrak{T}$ is an $\mathfrak{I}$-*epimorphism* if and only if F maps it to an epimorphism. Here we may take any functor F with $\ker F = \mathfrak{I}$.

But the most crucial notions like projective objects and resolutions require a more careful choice of the functor F. Here we need the *universal* $\mathfrak{I}$-*exact functor*, which is a stable homological functor F with $\ker F = \mathfrak{I}$ such that any other such functor factors uniquely through F (up to natural equivalence). The universal $\mathfrak{I}$-exact functor and its applications are due to Apostolos Beligiannis ([3]).

If $F\colon \mathfrak{T} \to \mathfrak{C}$ is universal, then F detects $\mathfrak{I}$-projective objects, and it identifies $\mathfrak{I}$-derived functors with derived functors in the Abelian category $\mathfrak{C}$. Thus all our homological notions reduce to their counterparts in the Abelian category $\mathfrak{C}$.

In order to apply this, we need to know when a functor F with $\ker F = \mathfrak{I}$ is the universal one. We develop a new, useful criterion for this purpose here, which uses partially defined adjoint functors.

Our criterion shows that the universal $\mathfrak{I}_{\mathrm{K}}$-exact functor for the ideal $\mathfrak{I}_{\mathrm{K}} \subseteq \mathrm{KK}$ in (45) is the K-theory functor K_*, considered as a functor from KK to the category $\mathfrak{Ab}_{\mathrm{c}}^{\mathbb{Z}/2}$ of countable $\mathbb{Z}/2$-graded Abelian groups. Hence the derived functors for $\mathfrak{I}_{\mathrm{K}}$ only involve Ext and Tor for Abelian groups.

The derived functors that we have discussed above appear in a spectral sequence which—in favourable cases—computes morphism spaces in $\mathfrak{T}$ (like $\mathrm{KK}^G(A,B)$) and other homological functors. This spectral sequence is a generalisation of the *Adams spectral sequence* in stable homotopy theory and is the main motivation for [8]. Much earlier, such spectral sequences were studied by Hans-Berndt Brinkmann in [6]. In a sequel to this article, we plan to apply this spectral sequence to our bivariant K-theory examples. Here we only consider the much easier case where this spectral sequence degenerates to an exact sequence. This generalises the familiar Universal Coefficient Theorem for $\mathrm{KK}_*(A, B)$.

4. Homological ideals in triangulated categories

After fixing some basic notation, we introduce several interesting ideals in bivariant Kasparov categories; we are going to discuss these ideals throughout this article. Then we briefly recall what a triangulated category is and introduce homological ideals. Before we begin, we should point out that the choice of ideal is important because all our homological notions depend on it. It seems to be a matter of experimentation and experience to find the right ideal for a given purpose.

4.1. Generalities about ideals in additive categories. All categories we consider will be *additive*, that is, they have a zero object and finite direct products and coproducts which agree, and the morphism spaces carry Abelian group structures such that the composition is additive in each variable ([21]).

Notation 47. Let $\mathfrak{C}$ be an additive category. We write $\mathfrak{C}(A, B)$ for the group of morphisms $A \to B$ in $\mathfrak{C}$, and $A \Subset \mathfrak{C}$ to denote that A is an object of the category $\mathfrak{C}$.

Definition 48. An *ideal* $\mathfrak{I}$ in $\mathfrak{C}$ is a family of subgroups $\mathfrak{I}(A, B) \subseteq \mathfrak{C}(A, B)$ for all $A, B \Subset \mathfrak{C}$ such that

$$\mathfrak{C}(C, D) \circ \mathfrak{I}(B, C) \circ \mathfrak{C}(A, B) \subseteq \mathfrak{I}(A, D) \qquad \text{for all } A, B, C, D \Subset \mathfrak{C}.$$

We write $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$ if $\mathfrak{I}_1(A, B) \subseteq \mathfrak{I}_2(A, B)$ for all $A, B \Subset \mathfrak{C}$. Clearly, the ideals in $\mathfrak{T}$ form a complete lattice. The largest ideal $\mathfrak{C}$ consists of all morphisms in $\mathfrak{C}$; the smallest ideal 0 contains only zero morphisms.

Definition 49. Let $\mathfrak{C}$ and $\mathfrak{C}'$ be additive categories and let $F\colon \mathfrak{C} \to \mathfrak{C}'$ be an additive functor. Its *kernel* $\ker F$ is the ideal in $\mathfrak{C}$ defined by

$$\ker F(A, B) := \{f \in \mathfrak{C}(A, B) \mid F(f) = 0\}.$$

This should be distinguished from the *kernel on objects*, consisting of all objects with $F(A) \cong 0$, which is used much more frequently. This agrees with the class of $\ker F$-contractible objects that we introduce below.

Definition 50. Let $\mathfrak{I} \subseteq \mathfrak{T}$ be an ideal. Its *quotient category* $\mathfrak{C}/\mathfrak{I}$ has the same objects as $\mathfrak{C}$ and morphism groups $\mathfrak{C}(A, B)/\mathfrak{I}(A, B)$.

The quotient category is again additive, and the obvious functor $F\colon \mathfrak{C} \to \mathfrak{C}/\mathfrak{I}$ is additive and satisfies $\ker F = \mathfrak{I}$. Thus any ideal $\mathfrak{I} \subseteq \mathfrak{C}$ is of the form $\ker F$ for a canonical additive functor F.

The additivity of $\mathfrak{C}/\mathfrak{I}$ and F depends on the fact that any ideal $\mathfrak{I}$ is compatible with *finite* products in the following sense: the natural isomorphisms

$$\mathfrak{C}(A, B_1 \times B_2) \xrightarrow{\cong} \mathfrak{C}(A, B_1) \times \mathfrak{C}(A, B_2), \quad \mathfrak{C}(A_1 \times A_2, B) \xrightarrow{\cong} \mathfrak{C}(A_1, B) \times \mathfrak{C}(A_2, B)$$

restrict to isomorphisms

$$\mathfrak{I}(A, B_1 \times B_2) \xrightarrow{\cong} \mathfrak{I}(A, B_1) \times \mathfrak{I}(A, B_2), \quad \mathfrak{I}(A_1 \times A_2, B) \xrightarrow{\cong} \mathfrak{I}(A_1, B) \times \mathfrak{I}(A_2, B).$$

4.2. Examples of ideals.

Example 51. Let KK be the *Kasparov category*, whose objects are the separable C^*-algebras and whose morphism spaces are the Kasparov groups $\mathrm{KK}_0(A, B)$, with the Kasparov product as composition. Let $\mathfrak{Ab}^{\mathbb{Z}/2}$ be the category of $\mathbb{Z}/2$-graded Abelian groups. Both categories are evidently additive.

K-theory is an additive functor $\mathrm{K}_*\colon \mathrm{KK} \to \mathfrak{Ab}^{\mathbb{Z}/2}$. We let $\mathfrak{I}_{\mathrm{K}} := \ker \mathrm{K}_*$ (as in (45)). Thus $\mathfrak{I}_{\mathrm{K}}(A, B) \subseteq \mathrm{KK}(A, B)$ is the kernel of the natural map

$$\gamma\colon \mathrm{KK}(A, B) \to \mathrm{Hom}\big(\mathrm{K}_*(A), \mathrm{K}_*(B)\big) := \prod_{n \in \mathbb{Z}/2} \mathrm{Hom}\big(\mathrm{K}_n(A), \mathrm{K}_n(B)\big).$$

There is another interesting ideal in KK, namely, the kernel of a natural map

$$\kappa\colon \mathfrak{I}_{\mathrm{K}}(A, B) \to \mathrm{Ext}\big(\mathrm{K}_*(A), \mathrm{K}_{*+1}(B)\big) := \prod_{n \in \mathbb{Z}/2} \mathrm{Ext}\big(\mathrm{K}_n(A), \mathrm{K}_{n+1}(B)\big)$$

due to Lawrence Brown (see [33]), whose definition we now recall. We represent $f \in \mathrm{KK}(A, B) \cong \mathrm{Ext}\big(A, \mathcal{C}_0(\mathbb{R}, B)\big)$ by a C^*-algebra extension $\mathcal{C}_0(\mathbb{R}, B) \otimes \mathbb{K} \rightarrowtail E \twoheadrightarrow A$. This yields an exact sequence

$$\begin{tikzcd} \mathrm{K}_1(B) \arrow[r] & \mathrm{K}_0(E) \arrow[r] & \mathrm{K}_0(A) \arrow[d, "f_*"] \\ \mathrm{K}_1(A) \arrow[u, "f_*"] & \mathrm{K}_1(E) \arrow[l] & \mathrm{K}_0(B). \arrow[l] \end{tikzcd} \tag{52}$$

The vertical maps in (52) are the two components of $\gamma(f)$. If $f \in \mathfrak{I}_{\mathrm{K}}(A, B)$, then (52) splits into two extensions of Abelian groups, which yield an element $\kappa(f)$ in $\mathrm{Ext}\big(\mathrm{K}_*(A), \mathrm{K}_{*+1}(B)\big)$.

Example 53. Let G be a second countable, locally compact group. Let KK^G be the associated *equivariant Kasparov category*; its objects are the separable G-C^*-algebras and its morphism spaces are the groups $\mathrm{KK}^G(A, B)$, with the Kasparov product as composition. If $H \subseteq G$ is a closed subgroup, then there is a *restriction functor* $\mathrm{Res}_G^H\colon \mathrm{KK}^G \to \mathrm{KK}^H$, which simply forgets part of the equivariance.

If $\mathcal{F}$ is a set of closed subgroups of G, we define an ideal $\mathcal{VC}_{\mathcal{F}} \subseteq \mathrm{KK}^G$ by

$$\mathcal{VC}_{\mathcal{F}}(A, B) := \{f \in \mathrm{KK}^G(A, B) \mid \mathrm{Res}_G^H(f) = 0 \text{ for all } H \in \mathcal{F}\}$$

as in (46). Of course, the condition $\mathrm{Res}_G^H(f) = 0$ is supposed to hold in $\mathrm{KK}^H(A, B)$. We are mainly interested in the case where $\mathcal{F}$ is the family of all compact subgroups of G and simply denote the ideal by $\mathcal{VC}$ in this case.

This ideal arises if we try to compute G-equivariant homology theories in terms of H-equivariant homology theories for $H \in \mathcal{F}$. The ideal $\mathcal{VC}$ is closely related to the approach to the Baum–Connes assembly map in [25].

Since I feel more at home with Kasparov theory than with spectra. Many readers will prefer to work in categories of spectra of, say, G-CW-complexes. We do not introduce these categories here; but it shoud be clear enough that they support similar restriction functors, which provide analogues of the ideals $\mathcal{VC}_{\mathcal{F}}$.

Finally, we consider a classic example from homological algebra.

Example 54. Let $\mathfrak{C}$ be an Abelian category. Let $\mathrm{Ho}(A)$ be the homotopy category of unbounded chain complexes

$$\cdots \to C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} C_{n-2} \xrightarrow{\delta_{n-2}} C_{n-3} \to \cdots$$

over $\mathfrak{C}$. The space of morphisms $A \to B$ in $\mathrm{Ho}(\mathfrak{C})$ is the space $[A, B]$ of *homotopy classes* of chain maps $A \to B$.

Taking homology defines functors $H_n\colon \mathrm{Ho}(\mathfrak{C}) \to \mathfrak{C}$ for $n \in \mathbb{Z}$, which we combine to a single functor $\mathrm{H}_*\colon \mathrm{Ho}(\mathfrak{C}) \to \mathfrak{C}^{\mathbb{Z}}$. We let $\mathfrak{I}_{\mathrm{H}} \subseteq \mathrm{Ho}(\mathfrak{C})$ be its kernel:

$$\mathfrak{I}_{\mathrm{H}}(A, B) := \{f \in [A, B] \mid \mathrm{H}_*(f) = 0\}. \tag{55}$$

We also consider the category $\mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p)$ of *p-periodic* chain complexes over $\mathfrak{C}$ for $p \in \mathbb{N}_{\geq 1}$; its objects satisfy $C_n = C_{n+p}$ and $\delta_n = \delta_{n+p}$ for all $n \in \mathbb{Z}$, and chain maps and homotopies are required p-periodic as well. The category $\mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/2)$ plays a role in connection with cyclic cohomology, especially with local cyclic cohomology ([23, 29]). The category $\mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/1)$ is isomorphic to the category of chain complexes without grading. By convention, we let $\mathbb{Z}/0 = \mathbb{Z}$, so that $\mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/0) = \mathrm{Ho}(\mathfrak{C})$.

The homology of a periodic chain complex is, of course, periodic, so that we get a homological functor $\mathrm{H}_*\colon \mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p) \to \mathfrak{C}^{\mathbb{Z}/p}$; here $\mathfrak{C}^{\mathbb{Z}/p}$ denotes the category of $\mathbb{Z}/p$-graded objects of $\mathfrak{C}$. We let $\mathfrak{I}_{\mathrm{H}} \subseteq \mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p)$ be the kernel of H_* as in (55).

4.3. What is a triangulated category?.

A *triangulated category* is a category $\mathfrak{T}$ with a *suspension automorphism* $\Sigma\colon \mathfrak{T} \to \mathfrak{T}$ and a class of *exact triangles*, subject to various axioms (see [25, 28, 35]). An exact triangle is a diagram in $\mathfrak{T}$ of the form

$$A \to B \to C \to \Sigma A \qquad \text{or} \qquad \begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C, \arrow[ul, "{[1]}"] & \end{tikzcd}$$

where the [1] in the arrow $C \to A$ warns us that this map has *degree* 1. A *morphism* of triangles is a triple of maps α, β, γ making the obvious diagram commute.

A typical example is the homotopy category $\mathrm{Ho}(\mathfrak{C};\mathbb{Z}/p))$ of $\mathbb{Z}/p$-graded chain complexes. Here the suspension functor is the (signed) *translation* functor

$$\begin{aligned} \Sigma\big((C_n, d_n)\big) &:= (C_{n-1}, -d_{n-1}) && \text{on objects,} \\ \Sigma\big((f_n)\big) &:= (f_{n-1}) && \text{on morphisms;} \end{aligned}$$

a triangle is exact if it is isomorphic to a *mapping cone triangle*

$$A \xrightarrow{f} B \to \mathrm{Cone}(f) \to \Sigma A$$

for some chain map f; the maps $B \to \mathrm{Cone}(f) \to \Sigma A$ are the canonical ones. It is well-known that this defines a triangulated category for $p = 0$; the arguments for $p \geq 1$ are essentially the same.

Another classical example is the stable homotopy category, say, of compactly generated pointed topological spaces (it is not particularly relevant which category of spaces or spectra we use). The suspension is $\Sigma(A) := \mathbb{S}^1 \wedge A$; a triangle is exact if it is isomorphic to a *mapping cone triangle*

$$A \xrightarrow{f} B \to \mathrm{Cone}(f) \to \Sigma A$$

for some map f; the maps $B \to \mathrm{Cone}(f) \to \Sigma A$ are the canonical ones.

We are mainly interested in the categories KK and KK^G introduced in §4.2. Their triangulated category structure is discussed in detail in [25]. We are facing a notational problem because the functor $X \mapsto \mathcal{C}_0(X)$ from pointed compact spaces to C^*-algebras is *contravariant*, so that *mapping cone triangles* now have the form

$$A \xleftarrow{f} B \leftarrow \mathrm{Cone}(f) \leftarrow \mathcal{C}_0(\mathbb{R}, A)$$

for a $*$-homomorphism $f\colon B \to A$; here

$$\mathrm{Cone}(f) = \big\{(a, b) \in \mathcal{C}_0\big((0, \infty], A\big) \times B \bigm| a(\infty) = f(b)\big\}$$

and the maps $\mathcal{C}_0(\mathbb{R}, A) \to \mathrm{Cone}(f) \to B$ are the obvious ones, $a \mapsto (a, 0)$ and $(a, b) \mapsto b$.

It is reasonable to view a $*$-homomorphism from A to B as a morphism from B to A. Nevertheless, we prefer the convention that an algebra homorphism $A \to B$ is a morphism $A \to B$. But then the most natural triangulated category structure lives on the opposite category $\mathrm{KK}^{\mathrm{op}}$. This creates only notational difficulties because the opposite category of a triangulated category inherits a canonical triangulated category structure, which has “the same” exact triangles. However, the passage to opposite categories exchanges suspensions and desuspensions and modifies some sign conventions. Thus the functor $A \mapsto \mathcal{C}_0(\mathbb{R}, A)$, which is the suspension functor in $\mathrm{KK}^{\mathrm{op}}$, becomes the *desuspension* functor in KK. Fortunately, Bott periodicity implies that $\Sigma^2 \cong \mathrm{id}$, so that Σ and Σ^{-1} agree.

Depending on your definition of a triangulated category, you may want the suspension to be an equivalence or isomorphism of categories. In the latter case, you must replace $\mathrm{KK}^{(G)}$ by an equivalent category (see [25]); since this is not important here, we do not bother about this issue.

A triangle in $\mathrm{KK}^{(G)}$ is called *exact* if it is isomorphic to a mapping cone triangle

$$\mathcal{C}_0(\mathbb{R}, B) \to \mathrm{Cone}(f) \to A \xrightarrow{f} B$$

for some (equivariant) $*$-homomorphism f.

An important source of exact triangles in KK^G are *extensions*. If $A \rightarrowtail B \twoheadrightarrow C$ is an extension of G-C^*-algebras with an equivariant completely positive contractive section, then it yields a class in $\mathrm{Ext}(C, A) \cong \mathrm{KK}(\Sigma^{-1}C, A)$; the resulting triangle

$$\Sigma^{-1}C \to A \to B \to C$$

in KK^G is exact and called an *extension triangle*. It is easy to see that any exact triangle is isomorphic to an extension triangle.

It is shown in [25] that KK and KK^G for a locally compact group G are triangulated categories with this extra structure.

The triangulated category axioms are discussed in greater detail in [25, 28, 35]. They encode some standard machinery for manipulating long exact sequences. Most of them amount to formal properties of mapping cones and mapping cylinders, which we can prove as in classical topology. The only axiom that requires more care is that any morphism $f\colon A \to B$ should be part of an exact triangle.

Unlike in [25], we prefer to construct this triangle as an extension triangle because this works in greater generality; we have learned this idea from Alexander Bonkat ([5]). Any element in $\mathrm{KK}^S_0(A, B) \cong \mathrm{KK}^S_1\bigl(A, \mathcal{C}_0(\mathbb{R}, B)\bigr)$ can be represented by an extension $\mathbb{K}(\mathcal{H}) \rightarrowtail E \twoheadrightarrow A$ with an equivariant completely positive contractive section, where $\mathcal{H}$ is a full S-equivariant Hilbert $\mathcal{C}_0(\mathbb{R}, B)$-module, so that $\mathbb{K}(\mathcal{H})$ is KK^S-equivalent to $\mathcal{C}_0(\mathbb{R}, B)$. Hence the resulting extension triangle in KK^S is isomorphic to one of the form

$$\mathcal{C}_0(\mathbb{R}, A) \to \mathcal{C}_0(\mathbb{R}, B) \to E \to A;$$

by construction, it contains the suspension of the given class in $\mathrm{KK}^S_0(A, B)$; it is easy to remove the suspension.

Definition 56. Let $\mathfrak{T}$ be a triangulated and $\mathfrak{C}$ an Abelian category. A covariant functor $F\colon \mathfrak{T} \to \mathfrak{C}$ is called *homological* if $F(A) \to F(B) \to F(C)$ is exact at $F(B)$ for all exact triangles $A \to B \to C \to \Sigma A$. A contravariant functor with the analogous exactness property is called *cohomological*.

Let $A \to B \to C \to \Sigma A$ be an exact triangle. Then a homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$ yields a natural long exact sequence

$$\cdots \to F_{n+1}(C) \to F_n(A) \to F_n(B) \to F_n(C) \to F_{n-1}(A) \to F_{n-1}(B) \to \cdots$$

with $F_n(A) := F(\Sigma^{-n}A)$ for $n \in \mathbb{Z}$, and a cohomological functor $F\colon \mathfrak{T}^{\mathrm{op}} \to \mathfrak{C}$ yields a natural long exact sequence

$$\cdots \leftarrow F^{n+1}(C) \leftarrow F^n(A) \leftarrow F^n(B) \leftarrow F^n(C) \leftarrow F^{n-1}(A) \leftarrow F^{n-1}(B) \leftarrow \cdots$$

with $F^n(A) := F(\Sigma^{-n}A)$.

Proposition 57. *Let $\mathfrak{T}$ be a triangulated category. The functors*

$$\mathfrak{T}(A, \text{␣})\colon \mathfrak{T} \to \mathfrak{Ab}, \qquad B \mapsto \mathfrak{T}(A, B)$$

are homological for all $A \in\in \mathfrak{T}$. Dually, the functors

$$\mathfrak{T}(\text{␣}, B)\colon \mathfrak{T}^{\mathrm{op}} \to \mathfrak{Ab}, \qquad A \mapsto \mathfrak{T}(A, B)$$

are cohomological for all $B \in\in \mathfrak{T}$.

Observe that

$$\mathfrak{T}^n(A, B) = \mathfrak{T}(\Sigma^{-n}A, B) \cong \mathfrak{T}(A, \Sigma^n B) \cong \mathfrak{T}_{-n}(A, B).$$

Definition 58. A *stable additive category* is an additive category equipped with an (additive) automorphism $\Sigma\colon \mathfrak{C} \to \mathfrak{C}$, called *suspension*.

A *stable homological functor* is a homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$ into a stable Abelian category $\mathfrak{C}$ together with natural isomorphisms $F\bigl(\Sigma_{\mathfrak{T}}(A)\bigr) \cong \Sigma_{\mathfrak{C}}\bigl(F(A)\bigr)$ for all $A \in\in \mathfrak{T}$.

Example 59. The category $\mathfrak{C}^{\mathbb{Z}/p}$ of $\mathbb{Z}/p$-graded objects of an Abelian category $\mathfrak{C}$ is stable for any $p \in \mathbb{N}$; the suspension automorphism merely shifts the grading. The functors $\mathrm{K}_*\colon \mathrm{KK} \to \mathfrak{Ab}^{\mathbb{Z}/2}$ and $\mathrm{H}_*\colon \mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p) \to \mathfrak{C}^{\mathbb{Z}/p}$ introduced in Examples 51 and 54 are stable homological functors.

If $F\colon \mathfrak{T} \to \mathfrak{C}$ is any homological functor, then

$$F_*\colon \mathfrak{T} \to \mathfrak{C}^{\mathbb{Z}}, \qquad A \mapsto \bigl(F_n(A)\bigr)_{n\in\mathbb{Z}}$$

is a stable homological functor. Many of our examples satisfy *Bott periodicity*, that is, there is a natural isomorphism $F_2(A) \cong F(A)$. Then we get a stable homological functor $F_*\colon \mathfrak{T} \to \mathfrak{C}^{\mathbb{Z}/2}$. A typical example for this is the functor K_*.

Definition 60. A functor $F\colon \mathfrak{T} \to \mathfrak{T}'$ between two triangulated categories is called *exact* if it intertwines the suspension automorphisms (up to specified natural isomorphisms) and maps exact triangles in $\mathfrak{T}$ again to exact triangles in $\mathfrak{T}'$.

Example 61. The restriction functor $\mathrm{Res}_G^H\colon \mathrm{KK}^G \to \mathrm{KK}^H$ for a closed quantum subgroup H of a locally compact quantum group G and the crossed product functors $G \ltimes \text{␣}, G \ltimes_{\mathrm{r}} \text{␣}\colon \mathrm{KK}^G \to \mathrm{KK}$ are exact because they preserve mapping cone triangles.

Let $F\colon \mathfrak{T}_1 \to \mathfrak{T}_2$ be an exact functor. If $G\colon \mathfrak{T}_2 \to ?$ is exact, homological, or cohomological, then so is $G \circ F$.

Using Examples 59 and 61, we see that the functors that define the ideals $\ker \gamma$ in Example 51, $\mathcal{VC}_{\mathcal{F}}$ in Example 53, and $\mathfrak{I}_{\mathrm{H}}$ in Example 54 are all stable and either homological or exact.

4.4. The universal homological functor. The following general construction of Peter Freyd ([13]) plays an important role in [3]. For an additive category $\mathfrak{C}$, let $\mathfrak{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Ab})$ be the category of contravariant additive functors $\mathfrak{C}^{\mathrm{op}} \to \mathfrak{Ab}$, with natural transformations as morphisms. Unless $\mathfrak{C}$ is essentially small, this is not quite a category because the morphisms may form classes instead of sets. We may ignore this set-theoretic problem because the bivariant Kasparov categories that we are interested in are essentially small, and the subcategory $\mathfrak{Coh}(\mathfrak{C}) \subseteq \mathfrak{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Ab})$ is an honest category for any $\mathfrak{C}$.

The category $\mathfrak{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Ab})$ is Abelian: if $f\colon F_1 \to F_2$ is a natural transformation, then its kernel, cokernel, image, and co-image are computed pointwise on the objects of $\mathfrak{C}$, so that they boil down to the corresponding constructions with Abelian groups.

The *Yoneda embedding* is an additive functor

$$\mathbb{Y}\colon \mathfrak{C} \to \mathfrak{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Ab}), \qquad B \mapsto \mathfrak{T}(\_, B).$$

This functor is fully faithful, and there are natural isomorphisms

$$\operatorname{Hom}(\mathbb{Y}(B), F) \cong F(B) \qquad \text{for all } F \in\in \mathfrak{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Ab}),\ B \in\in \mathfrak{T}$$

by the *Yoneda lemma.* A functor $F \in\in \mathfrak{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Ab})$ is called *representable* if it is isomorphic to $\mathbb{Y}(B)$ for some $B \in\in \mathfrak{C}$. Hence $\mathbb{Y}$ yields an equivalence of categories between $\mathfrak{C}$ and the subcategory of representable functors in $\mathfrak{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Ab})$.

A functor $F \in\in \mathfrak{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Ab})$ is called *finitely presented* if there is an exact sequence $\mathbb{Y}(B_1) \to \mathbb{Y}(B_2) \to F \to 0$ with $B_1, B_2 \in\in \mathfrak{T}$. Since $\mathbb{Y}$ is fully faithful, this means that F is the cokernel of $\mathbb{Y}(f)$ for a morphism f in $\mathfrak{C}$. We let $\mathfrak{Coh}(\mathfrak{C})$ be the full subcategory of finitely presented functors in $\mathfrak{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Ab})$. Since representable functors belong to $\mathfrak{Coh}(\mathfrak{C})$, we still have a Yoneda embedding $\mathbb{Y}\colon \mathfrak{C} \to \mathfrak{Coh}(\mathfrak{C})$. Although the category $\mathfrak{Coh}(\mathfrak{T})$ tends to be very big and therefore unwieldy, it plays an important theoretical role.

Theorem 62 (Freyd's Theorem). *Let $\mathfrak{T}$ be a triangulated category.*

Then $\mathfrak{Coh}(\mathfrak{T})$ is a stable Abelian category that has enough projective and enough injective objects, and the projective and injective objects coincide.

The functor $\mathbb{Y}\colon \mathfrak{T} \to \mathfrak{Coh}(\mathfrak{T})$ is fully faithful, stable, and homological. Its essential range $\mathbb{Y}(\mathfrak{T})$ consists of projective-injective objects. Conversely, an object of $\mathfrak{Coh}(\mathfrak{T})$ is projective-injective if and only if it is a retract of an object of $\mathbb{Y}(\mathfrak{T})$.

The functor $\mathbb{Y}$ is the universal (stable) homological functor in the following sense: any (stable) homological functor $F\colon \mathfrak{T} \to \mathfrak{C}'$ to a (stable) Abelian category $\mathfrak{C}'$ factors uniquely as $F = \bar{F} \circ \mathbb{Y}$ for a (stable) exact functor $F\colon \mathfrak{Coh}(\mathfrak{T}) \to \mathfrak{C}'$.

If idempotents in $\mathfrak{T}$ split — as in all our examples — then $\mathbb{Y}(\mathfrak{T})$ is closed under retracts, so that $\mathbb{Y}(\mathfrak{T})$ is equal to the class of projective-injective objects in $\mathfrak{Coh}(\mathfrak{T})$.

4.5. Homological ideals in triangulated categories. Let $\mathfrak{T}$ be a triangulated category, let $\mathfrak{C}$ be a stable additive category, and let $F\colon \mathfrak{T} \to \mathfrak{C}$ be a stable homological functor. Then $\ker F$ is a stable ideal in the following sense:

Definition 63. An ideal $\mathfrak{I} \subseteq \mathfrak{T}$ is called *stable* if the suspension isomorphisms $\Sigma\colon \mathfrak{T}(A,B) \xrightarrow{\cong} \mathfrak{T}(\Sigma A, \Sigma B)$ for $A, B \in\in \mathfrak{T}$ restrict to isomorphisms

$$\Sigma\colon \mathfrak{I}(A,B) \xrightarrow{\cong} \mathfrak{I}(\Sigma A, \Sigma B).$$

If $\mathfrak{I}$ is stable, then there is a unique suspension automorphism on $\mathfrak{T}/\mathfrak{I}$ for which the canonical functor $\mathfrak{T} \to \mathfrak{T}/\mathfrak{I}$ is stable. Thus the stable ideals are exactly the kernels of stable additive functors.

Definition 64. An ideal $\mathfrak{I} \subseteq \mathfrak{T}$ in a triangulated category is called *homological* if it is the kernel of a stable homological functor.

Remark 65. Freyd's Theorem shows that $\mathbb{Y}$ induces a bijection between (stable) exact functors $\mathfrak{Coh}(\mathfrak{T}) \to \mathfrak{C}'$ and (stable) homological functors $\mathfrak{T} \to \mathfrak{C}'$ because $\bar{F} \circ \mathbb{Y}$ is homological if $\bar{F}\colon \mathfrak{Coh}(\mathfrak{T}) \to \mathfrak{C}'$ is exact. Hence the notion of homological functor is independent of the triangulated category structure on $\mathfrak{T}$ because the Yoneda embedding $\mathbb{Y}\colon \mathfrak{T} \to \mathfrak{Coh}(\mathfrak{T})$ does not involve any additional structure. Hence the notion of homological ideal only uses the suspension automorphism, not the class of exact triangles.

All the ideals considered in §4.2 except for $\ker \kappa$ in Example 51 are kernels of stable homological functors or exact functors. Those of the first kind are homological by definition. If $F\colon \mathfrak{T} \to \mathfrak{T}'$ is an exact functor between two triangulated categories, then $\mathbb{Y} \circ F\colon \mathfrak{T} \to \mathfrak{Coh}(\mathfrak{T}')$ is a stable homological functor with $\ker \mathbb{Y} \circ F = \ker F$ by Freyd's Theorem 62. Hence kernels of exact functors are homological as well.

Is any homological ideal the kernel of an exact functor? This is *not* the case:

Proposition 66. *Let $\mathfrak{Der}(\mathfrak{Ab})$ be the derived category of the category $\mathfrak{Ab}$ of Abelian groups. Define the ideal $\mathfrak{I}_{\mathrm{H}} \subseteq \mathfrak{Der}(\mathfrak{Ab})$ as in Example 54. This ideal is not the kernel of an exact functor.*

We postpone the proof to the end of §5.1 because it uses the machinery of §5.1.

It takes some effort to characterise homological ideals because $\mathfrak{T}/\mathfrak{I}$ is almost never Abelian. The results in [3, §2–3] show that an ideal is homological if and only if it is *saturated* in the notation of [3]. We do not discuss this notion here because most ideals that we consider are obviously homological. The only example where we could profit from an abstract characterisation is the ideal $\ker \kappa$ in Example 51.

There is no obvious homological functor whose kernel is $\ker \kappa$ because κ is not a functor on KK. Nevertheless, $\ker \kappa$ is the kernel of an exact functor; the relevant functor is the functor $\mathrm{KK} \to \mathrm{UCT}$, where UCT is the variant of KK that satisfies the Universal Coefficient Theorem in complete generality. This functor can be constructed as a localisation of KK (see [25]). The Universal Coefficient Theorem implies that its kernel is exactly $\ker \kappa$.

5. From homological ideals to derived functors

Once we have a stable homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$, it is not surprising that we can do a certain amount of homological algebra in $\mathfrak{T}$. For instance, we may call a chain complex of objects of $\mathfrak{T}$ F-*exact* if F maps it to an exact chain complex in $\mathfrak{C}$; and we may call an object F-*projective* if F maps it to a projective object in $\mathfrak{C}$. But are these definitions reasonable?

We propose that a reasonable homological notion should depend only on the ideal $\ker F$. We will see that the notion of F-exact chain complex is reasonable and only depends on $\ker F$. In contrast, the notion of projectivity above depends on F and is only reasonable in special cases. There is another, more useful, notion of projective object that depends only on the ideal $\ker F$.

Various notions from homological algebra still make sense in the context of homological ideals in triangulated categories. Our discussion mostly follows [1, 3, 8, 12]. All our definitions involve only the ideal, not a stable homological functor that defines it. We reformulate them in terms of an exact or a stable homological functor defining the ideal in order to understand what they mean in concrete cases. Following [12], we construct projective objects using adjoint functors.

The most sophisticated concept in this section is the *universal* $\mathfrak{I}$-*exact functor*, which gives us complete control over projective resolutions and derived functors. We can usually describe such functors very concretely.

5.1. Basic notions.

We introduce some useful terminology related to an ideal:

Definition 67. Let $\mathfrak{I}$ be a homological ideal in a triangulated category $\mathfrak{T}$.

- Let $f\colon A \to B$ be a morphism in $\mathfrak{T}$; embed it in an exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$. We call f
  - $\mathfrak{I}$-*monic* if $h \in \mathfrak{I}$;
  - $\mathfrak{I}$-*epic* if $g \in \mathfrak{I}$;
  - an $\mathfrak{I}$-*equivalence* if it is both $\mathfrak{I}$-monic and $\mathfrak{I}$-epic, that is, $g, h \in \mathfrak{I}$;
  - an $\mathfrak{I}$-*phantom map* if $f \in \mathfrak{I}$.
- An object $A \in\in \mathfrak{T}$ is called $\mathfrak{I}$-*contractible* if $\mathrm{id}_A \in \mathfrak{I}(A, A)$.
- An exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ in $\mathfrak{T}$ is called $\mathfrak{I}$-*exact* if $h \in \mathfrak{I}$.

The notions of monomorphism (or monic morphism) and epimorphism (or epic morphism) — which can be found in any book on category theory such as [21] — are categorical ways to express injectivity or surjectivity of maps. A morphism in an Abelian category that is both monic and epic is invertible.

The classes of $\mathfrak{I}$-phantom maps, $\mathfrak{I}$-monics, $\mathfrak{I}$-epics, and of $\mathfrak{I}$-exact triangles determine each other uniquely because we can embed any morphism in an exact triangle in any position. It is a matter of taste which of these is considered most fundamental. Following Daniel Christensen ([8]), we favour the phantom maps. Other authors prefer exact triangles instead ([1, 3, 12]). Of course, the notion of an $\mathfrak{I}$-phantom map is redundant; it becomes more relevant if we consider, say, the class of $\mathfrak{I}$-exact triangles as our basic notion.

Notice that f is $\mathfrak{I}$-epic or $\mathfrak{I}$-monic if and only if $-f$ is. If f is $\mathfrak{I}$-epic or $\mathfrak{I}$-monic, then so are $\Sigma^n(f)$ for all $n \in \mathbb{Z}$ because $\mathfrak{I}$ is stable. Similarly, (signed) suspensions of $\mathfrak{I}$-exact triangles remain $\mathfrak{I}$-exact triangles.

Lemma 68. *Let $F\colon \mathfrak{T} \to \mathfrak{C}$ be a stable homological functor into a stable Abelian category $\mathfrak{C}$.*

- *A morphism f in $\mathfrak{T}$ is*
  - *a $\ker F$-phantom map if and only if $F(f) = 0$;*
  - *$\ker F$-monic if and only if $F(f)$ is monic;*
  - *$\ker F$-epic if and only if $F(f)$ is epic;*
  - *a $\ker F$-equivalence if and only if $F(f)$ is invertible.*
- *An object $A \in\in \mathfrak{T}$ is $\ker F$-contractible if and only if $F(A) = 0$.*
- *An exact triangle $A \to B \to C \to \Sigma A$ is $\ker F$-exact if and only if*

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

  is a short exact sequence in $\mathfrak{C}$.

Proof. Sequences in $\mathfrak{C}$ of the form $X \xrightarrow{0} Y \xrightarrow{f} Z$ or $X \xrightarrow{f} Y \xrightarrow{0} Z$ are exact at Y if and only if f is monic or epic, respectively. Moreover, a sequence of the form $X \xrightarrow{0} Y \to Z \to U \xrightarrow{0} W$ is exact if and only if $0 \to Y \to Z \to U \to 0$ is exact.

Combined with the long exact homology sequences for F and suitable exact triangles, these observations yield the assertions about monomorphisms, epimorphisms, and exact triangles. The description of equivalences and contractible objects follows, and phantom maps are trivial, anyway. $\square$

Now we specialise these notions to the ideal $\mathfrak{I}_{\mathrm{K}} \subseteq \mathrm{KK}$ of Example 51, replacing $\mathfrak{I}_{\mathrm{K}}$ by K in our notation to avoid clutter.

- Let $f \in \mathrm{KK}(A, B)$ and let $\mathrm{K}_*(f)\colon \mathrm{K}_*(A) \to \mathrm{K}_*(B)$ be the induced map. Then f is
  - a K-*phantom map* if and only if $\mathrm{K}_*(f) = 0$;
  - K-*epic* if and only if $\mathrm{K}_*(f)$ is surjective;
  - K-*monic* if and only if $\mathrm{K}_*(f)$ is injective;
  - a K-*equivalence* if and only if $\mathrm{K}_*(f)$ is invertible.
- A C^*-algebra $A \in\in \mathrm{KK}$ is K-*contractible* if and only if $\mathrm{K}_*(A) = 0$.
- An exact triangle $A \to B \to C \to \Sigma A$ in KK is K-*exact* if and only if

$$0 \to \mathrm{K}_*(A) \to \mathrm{K}_*(B) \to \mathrm{K}_*(C) \to 0$$

  is a short exact sequence (of $\mathbb{Z}/2$-graded Abelian groups).

Similar things happen for the other ideals in §4.2 that are *naturally* defined as kernels of stable homological functors.

Remark 69. It is crucial for the above theory that we consider functors that are both *stable* and *homological*. Everything fails if we drop either assumption and consider functors such as $\mathrm{K}_0(A)$ or $\mathrm{Hom}\big(\mathbb{Z}/4, \mathrm{K}_*(A)\big)$.

Lemma 70. *An object $A \Subset \mathfrak{T}$ is $\mathfrak{I}$-contractible if and only if $0\colon 0 \to A$ is an $\mathfrak{I}$-equivalence. A morphism f in $\mathfrak{T}$ is an $\mathfrak{I}$-equivalence if and only if its generalised mapping cone is $\mathfrak{I}$-contractible.*

Thus the classes of $\mathfrak{I}$-equivalences and of $\mathfrak{I}$-contractible objects determine each other. But they do not allow us to recover the ideal itself. For instance, the ideals $\mathfrak{I}_{\mathrm{K}}$ and $\ker \kappa$ in Example 51 have the same contractible objects and equivalences.

Proof. Recall that the generalised mapping cone of f is the object C that fits in an exact triangle $A \xrightarrow{f} B \to C \to \Sigma A$. The long exact sequence for this triangle yields that $F(f)$ is invertible if and only if $F(C) = 0$, where F is some stable homological functor F with $\ker F = \mathfrak{I}$. Now the second assertion follows from Lemma 68. Since the generalised mapping cone of $0 \to A$ is A, the first assertion is a special case of the second one. ☐

Many ideals are defined as $\ker F$ for an exact functor $F\colon \mathfrak{T} \to \mathfrak{T}'$ between triangulated categories. We can also use such a functor to describe the above notions:

Lemma 71. *Let $\mathfrak{T}$ and $\mathfrak{T}'$ be triangulated categories and let $F\colon \mathfrak{T} \to \mathfrak{T}'$ be an exact functor.*

- *A morphism $f \in \mathfrak{T}(A, B)$ is*
  - *a $\ker F$-phantom map if and only if $F(f) = 0$;*
  - *$\ker F$-monic if and only if $F(f)$ is (split) monic.*
  - *$\ker F$-epic if and only if $F(f)$ is (split) epic;*
  - *a $\ker F$-equivalence if and only if $F(f)$ is invertible.*
- *An object $A \Subset \mathfrak{T}$ is $\ker F$-contractible if and only if $F(A) = 0$.*
- *An exact triangle $A \to B \to C \to \Sigma A$ is $\ker F$-exact if and only if the exact triangle $F(A) \to F(B) \to F(C) \to F(\Sigma A)$ in $\mathfrak{T}'$ splits.*

We will explain the notation during the proof.

Proof. A morphism $f\colon X \to Y$ in $\mathfrak{T}'$ is called *split epic* (*split monic*) if there is $g\colon Y \to X$ with $f \circ g = \mathrm{id}_Y$ ($g \circ f = \mathrm{id}_X$). An exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is said to *split* if $h = 0$. This immediately yields the characterisation of $\ker F$-exact triangles. Any split triangle is isomorphic to a direct sum triangle, so that f is split monic and g is split epic ([28, Corollary 1.2.7]). Conversely, either of these conditions implies that the triangle is split.

Since the $\ker F$-exact triangles determine the $\ker F$-epimorphisms and $\ker F$-monomorphisms, the latter are detected by $F(f)$ being split epic or split monic, respectively. It is clear that split epimorphisms and split monomorphisms are epimorphisms and monomorphisms, respectively. The converse holds in a triangulated category because if we embed a monomorphism or epimorphism in an exact triangle, then one of the maps is forced to vanish, so that the exact triangle splits.

Finally, a morphism is invertible if and only if it is both split monic and split epic, and the zero map $F(A) \to F(A)$ is invertible if and only if $F(A) = 0$.

We may also prove Lemma 68 using the Yoneda embedding $\mathbb{Y}\colon \mathfrak{T}' \to \mathfrak{Coh}(\mathfrak{T}')$. The assertions about phantom maps, equivalences, and contractibility boil down to the observation that $\mathbb{Y}$ is fully faithful. The assertions about monomorphisms and epimorphisms follow because a map $f\colon A \to B$ in $\mathfrak{T}'$ becomes epic (monic) in $\mathfrak{Coh}(\mathfrak{T}')$ if and only if it is split epic (monic) in $\mathfrak{T}'$. $\square$

There is a similar description for $\bigcap \ker F_i$ for a set $\{F_i\}$ of exact functors. This applies to the ideal $\mathcal{VC}_\mathcal{F}$ for a family of (quantum) subgroups $\mathcal{F}$ in a locally compact (quantum) group G (Example 53). Replacing $\mathcal{VC}_\mathcal{F}$ by $\mathcal{F}$ in our notation to avoid clutter, we get:

- A morphism $f \in \mathrm{KK}^G(A, B)$ is
  - an *$\mathcal{F}$-phantom map* if and only if $\mathrm{Res}_G^H(f) = 0$ in KK^H for all $H \in \mathcal{F}$;
  - *$\mathcal{F}$-epic* if and only if $\mathrm{Res}_G^H(f)$ is (split) epic in KK^H for all $H \in \mathcal{F}$;
  - *$\mathcal{F}$-monic* if and only if $\mathrm{Res}_G^H(f)$ is (split) monic in KK^H for all $H \in \mathcal{F}$;
  - an *$\mathcal{F}$-equivalence* if and only if $\mathrm{Res}_G^H(f)$ is a KK^H-equivalence for all $H \in \mathcal{F}$.
- A G-C^*-algebra $A \in\in \mathrm{KK}^G$ is *$\mathcal{F}$-contractible* if and only if $\mathrm{Res}_G^H(A) \cong 0$ in KK^H for all $H \in \mathcal{F}$.
- An exact triangle $A \to B \to C \to \Sigma A$ in KK^G is *$\mathcal{F}$-exact* if and only if

$$\mathrm{Res}_G^H(A) \to \mathrm{Res}_G^H(B) \to \mathrm{Res}_G^H(C) \to \Sigma\,\mathrm{Res}_G^H(A)$$

  is a split exact triangle in KK^H for all $H \in \mathcal{F}$.

Lemma 71 allows us to prove that the ideal $\mathfrak{I}_\mathrm{H}$ in $\mathfrak{Der}(\mathfrak{Ab})$ cannot be the kernel of an exact functor:

Proof of 66. We embed $\mathfrak{Ab} \to \mathfrak{Der}(\mathfrak{Ab})$ as chain complexes concentrated in degree 0. The generator $\tau \in \mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}/2)$ corresponds to the extension of Abelian groups $\mathbb{Z}/2 \rightarrowtail \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$, where the first map is multiplication by 2 and the second map is the natural projection. We get an exact triangle

$$\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \xrightarrow{\tau} \mathbb{Z}/2[1]$$

in $\mathfrak{Der}(\mathfrak{Ab})$. This triangle is $\mathfrak{I}_\mathrm{H}$-exact because the map $\mathbb{Z}/2 \to \mathbb{Z}/4$ is injective as a group homomorphism and hence $\mathfrak{I}_\mathrm{H}$-monic in $\mathfrak{Der}(\mathfrak{Ab})$.

Assume there were an exact functor $F\colon \mathfrak{Der}(\mathfrak{Ab}) \to \mathfrak{T}'$ with $\ker F = \mathfrak{I}_{\mathrm{H}}$. Then $F(\tau) = 0$, so that F maps our triangle to a split triangle and $F(\mathbb{Z}/4) \cong F(\mathbb{Z}/2) \oplus F(\mathbb{Z}/2)$ by Lemma 71. It follows that $F(2 \cdot \mathrm{id}_{\mathbb{Z}/4}) = 2 \cdot \mathrm{id}_{F(\mathbb{Z}/4)} = 0$ because $2 \cdot \mathrm{id}_{F(\mathbb{Z}/2)} = F(2 \cdot \mathrm{id}_{\mathbb{Z}/2}) = 0$. Hence $2 \cdot \mathrm{id}_{\mathbb{Z}/4} \in \ker F = \mathfrak{I}_{\mathrm{H}}$, which is false. This contradiction shows that there is no exact functor F with $\ker F = \mathfrak{I}_{\mathrm{H}}$. □

One of the most interesting questions about an ideal is whether all $\mathfrak{I}$-contractible objects vanish or, equivalently, whether all $\mathfrak{I}$-equivalences are invertible. These two questions are equivalent by Lemma 70. The answer is negative for the ideal $\mathfrak{I}_{\mathrm{K}} \subseteq \mathrm{KK}$ because the Universal Coefficient Theorem does not hold for arbitrary separable C^*-algebras. If G is an amenable group, then $\mathcal{VC}$-equivalences in KK^G are invertible; this follows from the proof of the Baum-Connes conjecture for these groups by Nigel Higson and Gennadi Kasparov (see [25]). These examples show that this question is subtle and may involve difficult analysis.

5.2. Exact chain complexes. We are going to extend to chain complexes the notion of $\mathfrak{I}$-exactness, which we have only defined for exact triangles so far. Our definition differs from Beligiannis' one ([1, 3]), which we recall first.

Let $\mathfrak{T}$ be a triangulated category and let $\mathfrak{I}$ be a homological ideal in $\mathfrak{T}$.

Definition 72. A chain complex

$$C_\bullet := (\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \cdots)$$

in $\mathfrak{T}$ is called $\mathfrak{I}$-*decomposable* if there is a sequence of $\mathfrak{I}$-exact triangles

$$K_{n+1} \xrightarrow{g_n} C_n \xrightarrow{f_n} K_n \xrightarrow{h_n} \Sigma K_{n+1}$$

with $d_n = g_{n-1} \circ f_n\colon C_n \to C_{n-1}$.

Such complexes are called $\mathfrak{I}$-exact in [1, 3]. This definition is inspired by the following well-known fact: a chain complex over an Abelian category is exact if and only if it splits into short exact sequences of the form $K_n \rightarrowtail C_n \twoheadrightarrow K_{n-1}$ as in Definition 72.

We prefer another definition of exactness because we have not found a general explicit criterion for a chain complex to be $\mathfrak{I}$-decomposable.

Definition 73. Let $C_\bullet = (C_n, d_n)$ be a chain complex over $\mathfrak{T}$. For each $n \in \mathbb{N}$, embed d_n in an exact triangle

$$C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{f_n} X_n \xrightarrow{g_n} \Sigma C_n. \tag{74}$$

We call $C_\bullet$ $\mathfrak{I}$-*exact in degree* n if the map $X_n \xrightarrow{g_n} \Sigma C_n \xrightarrow{\Sigma f_{n+1}} \Sigma X_{n+1}$ belongs to $\mathfrak{I}(X_n, \Sigma X_{n+1})$. This does not depend on auxiliary choices because the exact triangles in (74) are unique up to (non-canonical) isomorphism.

We call $C_\bullet$ $\mathfrak{I}$-*exact* if it is $\mathfrak{I}$-exact in degree n for all $n \in \mathbb{Z}$.

This definition is designed to make the following lemma true:

Lemma 75. *Let $F\colon \mathfrak{T} \to \mathfrak{C}$ be a stable homological functor into a stable Abelian category $\mathfrak{C}$ with $\ker F = \mathfrak{I}$. A chain complex $C_\bullet$ over $\mathfrak{T}$ is $\mathfrak{I}$-exact in degree n if and only if*

$$F(C_{n+1}) \xrightarrow{F(d_{n+1})} F(C_n) \xrightarrow{F(d_n)} F(C_{n-1})$$

is exact at $F(C_n)$.

Proof. The complex $C_\bullet$ is $\mathfrak{I}$-exact in degree n if and only if the map

$$\Sigma^{-1}F(X_n) \xrightarrow{\Sigma^{-1}F(g_n)} F(C_n) \xrightarrow{F(f_{n+1})} F(X_{n+1})$$

vanishes. Equivalently, the range of $\Sigma^{-1}F(g_n)$ is contained in the kernel of $F(f_{n+1})$. The long exact sequences

$$\cdots \to \Sigma^{-1}F(X_n) \xrightarrow{\Sigma^{-1}F(g_n)} F(C_n) \xrightarrow{F(d_n)} F(C_{n-1}) \to \cdots,$$
$$\cdots \to F(C_{n+1}) \xrightarrow{F(d_{n+1})} F(C_n) \xrightarrow{F(f_{n+1})} F(X_{n+1}) \to \cdots$$

show that the range of $\Sigma^{-1}F(g_n)$ and the kernel of $F(f_{n+1})$ are equal to the kernel of $F(d_n)$ and the range of $F(d_{n+1})$, respectively. Hence $C_\bullet$ is $\mathfrak{I}$-exact in degree n if and only if $\ker F(d_n) \subseteq \operatorname{range} F(d_{n+1})$. Since $d_n \circ d_{n+1} = 0$, this is equivalent to $\ker F(d_n) = \operatorname{range} F(d_{n+1})$. □

Corollary 76. *$\mathfrak{I}$-decomposable chain complexes are $\mathfrak{I}$-exact.*

Proof. Let $F\colon \mathfrak{T} \to \mathfrak{C}$ be a stable homological functor with $\ker F = \mathfrak{I}$. If $C_\bullet$ is $\mathfrak{I}$-decomposable, then $F(C_\bullet)$ is obtained by splicing short exact sequences in $\mathfrak{C}$. This implies that $F(C_\bullet)$ is exact, so that $C_\bullet$ is $\mathfrak{I}$-exact by Lemma 75. □

Example 77. For the ideal $\mathfrak{I}_{\mathrm{K}} \subseteq \mathrm{KK}$, Lemma 75 yields that a chain complex $C_\bullet$ over KK is K-*exact* (in degree n) if and only if the chain complex

$$\cdots \to \mathrm{K}_*(C_{n+1}) \to \mathrm{K}_*(C_n) \to \mathrm{K}_*(C_{n-1}) \to \cdots$$

of $\mathbb{Z}/2$-graded Abelian groups is exact (in degree n). Similar remarks apply to the other ideals in §4.2 that are defined as kernels of stable homological functors.

As a trivial example, we consider the largest possible ideal $\mathfrak{I} = \mathfrak{T}$. This ideal is defined by the zero functor. Lemma 75 or the definition yield that *all* chain complexes are $\mathfrak{T}$-exact. In contrast, it seems hard to characterise the $\mathfrak{I}$-decomposable chain complexes, already for $\mathfrak{I} = \mathfrak{T}$.

Lemma 78. *A chain complex of length* 3

$$\cdots \to 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \to \cdots$$

is $\mathfrak{I}$-exact if and only if there are an $\mathfrak{I}$-exact exact triangle $A' \xrightarrow{f'} B' \xrightarrow{g'} C' \to \Sigma A'$ and a commuting diagram

$$\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "\alpha", "\sim"'] & B' \arrow[r, "g'"] \arrow[d, "\beta", "\sim"'] & C' \arrow[d, "\gamma", "\sim"'] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd} \tag{79}$$

where the vertical maps α, β, γ are $\mathfrak{I}$-equivalences. Furthermore, we can achieve that α and β are identity maps.

Proof. Let F be a stable homological functor with $\mathfrak{I} = \ker F$.

Suppose first that we are in the situation of (79). Lemma 68 yields that $F(\alpha)$, $F(\beta)$, and $F(\gamma)$ are invertible and that $0 \to F(A') \to F(B') \to F(C') \to 0$ is a short exact sequence. Hence so is $0 \to F(A) \to F(B) \to F(C) \to 0$. Now Lemma 75 yields that our given chain complex is $\mathfrak{I}$-exact.

Conversely, suppose that we have an $\mathfrak{I}$-exact chain complex. By Lemma 75, this means that $0 \to F(A) \to F(B) \to F(C) \to 0$ is a short exact sequence. Hence $f\colon A \to B$ is $\mathfrak{I}$-monic. Embed f in an exact triangle $A \to B \to C' \to \Sigma A$. Since f is $\mathfrak{I}$-monic, this triangle is $\mathfrak{I}$-exact. Let $\alpha = \mathrm{id}_A$ and $\beta = \mathrm{id}_B$. Since the functor $\mathfrak{T}(\text{\textvisiblespace}, C)$ is cohomological and $g \circ f = 0$, we can find a map $\gamma\colon C' \to C$ making (79) commute. The functor F maps the rows of (79) to short exact sequences by Lemmas 75 and 68. Now the Five Lemma yields that $F(\gamma)$ is invertible, so that γ is an $\mathfrak{I}$-equivalence. □

Remark 80. Lemma 78 implies that $\mathfrak{I}$-exact chain complexes of length 3 are $\mathfrak{I}$-decomposable. We do not expect this for chain complexes of length 4. But we have not searched for a counterexample.

Which chain complexes over $\mathfrak{T}$ are $\mathfrak{I}$-exact for $\mathfrak{I} = 0$ and hence for any homological ideal? The next definition provides the answer.

Definition 81. A chain complex $C_\bullet$ over a triangulated category is called *homologically exact* if $F(C_\bullet)$ is exact for any homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$.

Example 82. If $A \to B \to C \to \Sigma A$ is an exact triangle, then the chain complex

$$\cdots \to \Sigma^{-1}A \to \Sigma^{-1}B \to \Sigma^{-1}C \to A \to B \to C \to \Sigma A \to \Sigma B \to \Sigma C \to \cdots$$

is homologically exact by the definition of a homological functor.

Lemma 83. *Let $F\colon \mathfrak{T} \to \mathfrak{T}'$ be an exact functor between two triangulated categories. Let $C_\bullet$ be a chain complex over $\mathfrak{T}$. The following are equivalent:*

(1) $C_\bullet$ is $\ker F$*-exact in degree n;*

(2) $F(C_\bullet)$ is $\mathfrak{I}$-exact in degree n with respect to $\mathfrak{I} = 0$;

(3) the chain complex $\mathbb{Y} \circ F(C_\bullet)$ in $\mathfrak{Coh}(\mathfrak{T}')$ is exact in degree n;

(4) $F(C_\bullet)$ is homologically exact in degree n;

(5) the chain complexes of Abelian groups $\mathfrak{T}'\big(A, F(C_\bullet)\big)$ are exact in degree n for all $A \in\in \mathfrak{T}'$.

Proof. By Freyd's Theorem 62, $\mathbb{Y} \circ F\colon \mathfrak{T} \to \mathfrak{Coh}(\mathfrak{T}')$ is a stable homological functor with $\ker F = \ker(\mathbb{Y} \circ F)$. Hence Lemma 75 yields (1) $\iff$ (3). Similarly, we have (2) $\iff$ (3) because $\mathbb{Y}\colon \mathfrak{T}' \to \mathfrak{Coh}(\mathfrak{T}')$ is a stable homological functor with $\ker \mathbb{Y} = 0$. Freyd's Theorem 62 also asserts that any homological functor $F\colon \mathfrak{T}' \to$

$\mathfrak{C}'$ factors as $\bar{F} \circ \mathbb{Y}$ for an exact functor $\bar{F}$. Hence (3)$\Longrightarrow$(4). Proposition 57 yields (4)$\Longrightarrow$(5). Finally, (5) $\Longleftrightarrow$ (3) because kernels and cokernels in $\mathfrak{Coh}(\mathfrak{T}')$ are computed pointwise on objects of $\mathfrak{T}'$. □

Remark 84. More generally, consider a set of exact functors $F_i\colon \mathfrak{T} \to \mathfrak{T}'_i$. As in the proof of the equivalence (1) $\Longleftrightarrow$ (2) in Lemma 83, we see that a chain complex $C_\bullet$ is $\bigcap \ker F_i$-exact (in degree n) if and only if the chain complexes $F_i(C_\bullet)$ are exact (in degree n) for all i.

As a consequence, a chain complex $C_\bullet$ over KK^G for a locally compact quantum group G is $\mathcal{F}$-exact if and only if $\mathrm{Res}_G^H(C_\bullet)$ is homologically exact for all $H \in \mathcal{F}$.

Example 85. We exhibit an $\mathfrak{I}$-exact chain complex that is not $\mathfrak{I}$-decomposable for the ideal $\mathfrak{I} = 0$. By Lemma 71, any 0-exact triangle is split. Therefore, a chain complex is 0-decomposable if and only if it is a direct sum of chain complexes of the form $0 \to K_n \xrightarrow{\mathrm{id}} K_n \to 0$. Hence any decomposable chain complex is contractible and therefore mapped by any homological functor to a contractible chain complex. By the way, if idempotents in $\mathfrak{T}$ split then a chain complex is 0-decomposable if and only if it is contractible.

As we have remarked in Example 82, the chain complex

$$\cdots \to \Sigma^{-1}C \to A \to B \to C \to \Sigma A \to \Sigma B \to \Sigma C \to \Sigma^2 A \to \cdots$$

is homologically exact for any exact triangle $A \to B \to C \to \Sigma A$. But such chain complexes need not be contractible. A counterexample is the exact triangle $\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to \Sigma\mathbb{Z}/2$ in $\mathfrak{Der}(\mathfrak{Ab})$, which we have already used in the proof of Proposition 66. The resulting chain complex over $\mathfrak{Der}(\mathfrak{Ab})$ cannot be contractible because H_* maps it to a non-contractible chain complex.

5.2.1. More homological algebra with chain complexes. Using our notion of exactness for chain complexes, we can do homological algebra in the homotopy category $\mathrm{Ho}(\mathfrak{T})$. We briefly sketch some results in this direction, assuming some familiarity with more advanced notions from homological algebra. We will not use this later.

The $\mathfrak{I}$-exact chain complexes form a thick subcategory of $\mathrm{Ho}(\mathfrak{T})$ because of Lemma 75. We let $\mathfrak{Der} := \mathfrak{Der}(\mathfrak{T}, \mathfrak{I})$ be the localisation of $\mathrm{Ho}(\mathfrak{T})$ at this subcategory and call it the *derived category of $\mathfrak{T}$ with respect to $\mathfrak{I}$*.

We let $\mathfrak{Der}^{\geq n}$ and $\mathfrak{Der}^{\leq n}$ be the full subcategories of $\mathfrak{Der}$ consisting of chain complexes that are $\mathfrak{I}$-exact in degrees $< n$ and $> n$, respectively.

Theorem 86. *The pair of subcategories $\mathfrak{Der}^{\geq 0}$, $\mathfrak{Der}^{\leq 0}$ forms a* truncation structure *(t-structure) on $\mathfrak{Der}$ in the sense of* [2].

Proof. The main issue here is the truncation of chain complexes. Let $C_\bullet$ be a chain complex over $\mathfrak{T}$. We embed the map d_0 in an exact triangle $C_0 \to C_{-1} \to X \to \Sigma C_0$ and let $C_\bullet^{\geq 0}$ be the chain complex

$$\cdots \to C_2 \to C_1 \to C_0 \to C_{-1} \to X \to \Sigma C_0 \to \Sigma C_{-1} \to \Sigma X \to \Sigma^2 C_0 \to \cdots.$$

This chain complex is $\mathfrak{I}$-exact — even homologically exact — in negative degrees, that is, $C_\bullet^{\geq 0} \in \mathfrak{Der}^{\geq 0}$. The triangulated category structure allows us to construct a chain map $C_\bullet^{\geq 0} \to C_\bullet$ that is an isomorphism on C_n for $n \geq -1$. Hence its mapping cone $C_\bullet^{\leq -1}$ is $\mathfrak{I}$-exact — even contractible — in degrees ≥ 0, that is, $C_\bullet^{\leq -1} \Subset \mathfrak{Der}^{\leq -1}$. By construction, we have an exact triangle

$$C_\bullet^{\geq 0} \to C_\bullet \to C_\bullet^{\leq -1} \to \Sigma C_\bullet^{\geq 0}$$

in $\mathfrak{Der}$.

We also have to check that there is no non-zero morphism $C_\bullet \to D_\bullet$ in $\mathfrak{Der}$ if $C_\bullet \Subset \mathfrak{Der}^{\geq 0}$ and $D_\bullet \Subset \mathfrak{Der}^{\leq -1}$. Recall that morphisms in $\mathfrak{Der}$ are represented by diagrams $C_\bullet \xleftarrow{\sim} \tilde{C}_\bullet \to D_\bullet$ in $\mathrm{Ho}(\mathfrak{T})$, where the first map is an $\mathfrak{I}$-equivalence. Hence $\tilde{C}_\bullet \Subset \mathfrak{Der}^{\geq 0}$ as well. We claim that any chain map $f\colon \tilde{C}_\bullet^{\geq 0} \to D_\bullet^{\leq -1}$ is homotopic to 0. Since the maps $\tilde{C}_\bullet^{\geq 0} \to C_\bullet$ and $D_\bullet \to D_\bullet^{\leq -1}$ are $\mathfrak{I}$-equivalences, any morphism $C_\bullet \to D_\bullet$ vanishes in $\mathfrak{Der}$.

It remains to prove the claim. In a first step, we use that $D_\bullet^{\leq -1}$ is contractible in degrees ≥ 0 to replace f by a homotopic chain map supported in degrees < 0. In a second step, we use that $\tilde{C}_\bullet^{\geq 0}$ is homologically exact in the relevant degrees to recursively construct a chain homotopy between f and 0. □

Any truncation structure gives rise to an Abelian category, its *core*. In our case, we get the full subcategory $\mathfrak{C} \subseteq \mathfrak{Der}$ of all chain complexes that are $\mathfrak{I}$-exact except in degree 0. This is a stable Abelian category, and the standard embedding $\mathfrak{T} \to \mathrm{Ho}(\mathfrak{T})$ yields a stable homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$ with $\ker F = \mathfrak{I}$.

This functor is characterised uniquely by the following universal property: any (stable) homological functor $H\colon \mathfrak{T} \to \mathfrak{C}'$ with $\mathfrak{I} \subseteq \ker H$ factors uniquely as $H = \bar{H} \circ F$ for an exact functor $\bar{H}\colon \mathfrak{C} \to \mathfrak{C}'$. We construct $\bar{H}$ in three steps.

First, we lift H to an exact functor $\mathrm{Ho}(H)\colon \mathrm{Ho}(\mathfrak{T}, \mathfrak{I}) \to \mathrm{Ho}(\mathfrak{C}')$. Secondly, $\mathrm{Ho}(H)$ descends to a functor $\mathfrak{Der}(H)\colon \mathfrak{Der}(\mathfrak{T}, \mathfrak{I}) \to \mathfrak{Der}(\mathfrak{C}')$. Finally, $\mathfrak{Der}(H)$ restricts to a functor $\bar{H}\colon \mathfrak{C} \to \mathfrak{C}'$ between the cores. Since $\mathfrak{I} \subseteq \ker H$, an $\mathfrak{I}$-exact chain complex is also $\ker H$-exact. Hence $\mathrm{Ho}(H)$ preserves exactness of chain complexes by Lemma 75. This allows us to construct $\mathfrak{Der}(H)$ and shows that $\mathfrak{Der}(H)$ is compatible with truncation structures. This allows us to restrict it to an exact functor between the cores. Finally, we use that the core of the standard truncation structure on $\mathfrak{Der}(\mathfrak{C})$ is $\mathfrak{C}$. It is easy to see that we have $\bar{H} \circ F = H$.

Especially, we get an exact functor $\mathfrak{Der}(F)\colon \mathfrak{Der}(\mathfrak{T}, \mathfrak{I}) \to \mathfrak{Der}(\mathfrak{C})$, which restricts to the identity functor $\mathrm{id}_{\mathfrak{C}}$ between the cores. Hence $\mathfrak{Der}(F)$ is fully faithful on the thick subcategory generated by $\mathfrak{C} \subseteq \mathfrak{Der}(\mathfrak{T}, \mathfrak{I})$. It seems plausible that $\mathfrak{Der}(F)$ should be an equivalence of categories under some mild conditions on $\mathfrak{I}$ and $\mathfrak{T}$.

We will continue our study of the functor $\mathfrak{T} \to \mathfrak{C}$ in §5.7. The universal property determines it uniquely. Beligiannis ([3]) has another, simpler construction.

5.3. Projective objects. Let $\mathfrak{I}$ be a homological ideal in a triangulated category $\mathfrak{T}$.

Definition 87. A homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$ is called $\mathfrak{I}$-*exact* if $F(f) = 0$ for all $\mathfrak{I}$-phantom maps f or, equivalently, $\mathfrak{I} \subseteq \ker F$. An object $A \in\in \mathfrak{T}$ is called $\mathfrak{I}$-*projective* if the functor $\mathfrak{T}(A, \_)\colon \mathfrak{T} \to \mathfrak{Ab}$ is $\mathfrak{I}$-exact. Dually, an object $B \in\in \mathfrak{T}$ is called $\mathfrak{I}$-*injective* if the functor $\mathfrak{T}(\_, B)\colon \mathfrak{T} \to \mathfrak{Ab}^{\mathrm{op}}$ is $\mathfrak{I}$-exact.

We write $\mathfrak{P}_{\mathfrak{I}}$ for the class of $\mathfrak{I}$-projective objects in $\mathfrak{T}$.

The notions of projective and injective object are dual to each other: if we pass to the opposite category $\mathfrak{T}^{\mathrm{op}}$ with the canonical triangulated category structure and use the same ideal $\mathfrak{I}^{\mathrm{op}}$, then this exchanges the roles of projective and injective objects. Therefore, it suffices to discuss one of these two notions in the following. We will only treat projective objects because all the ideals in §4.2 have enough projective objects, but most of them do not have enough injective objects.

Notice that the functor F is $\mathfrak{I}$-exact if and only if the associated stable functor $F_*\colon \mathfrak{T} \to \mathfrak{C}^{\mathbb{Z}}$ is $\mathfrak{I}$-exact because $\mathfrak{I}$ is stable.

Since we require F to be homological, the long exact homology sequence and Lemma 75 yield that the following conditions are all equivalent to F being $\mathfrak{I}$-exact:

- F maps $\mathfrak{I}$-epimorphisms to epimorphisms in $\mathfrak{C}$;
- F maps $\mathfrak{I}$-monomorphisms to monomorphisms in $\mathfrak{C}$;
- $0 \to F(A) \to F(B) \to F(C) \to 0$ is a short exact sequence in $\mathfrak{C}$ for any $\mathfrak{I}$-exact triangle $A \to B \to C \to \Sigma A$;
- F maps $\mathfrak{I}$-exact chain complexes to exact chain complexes in $\mathfrak{C}$.

This specialises to equivalent definitions of $\mathfrak{I}$-projective objects.

Lemma 88. *An object $A \in\in \mathfrak{T}$ is $\mathfrak{I}$-projective if and only if $\mathfrak{I}(A, B) = 0$ for all $B \in\in \mathfrak{T}$.*

Proof. If $f \in \mathfrak{I}(A, B)$, then $f = f_*(\mathrm{id}_A)$. This has to vanish if A is $\mathfrak{I}$-projective. Suppose, conversely, that $\mathfrak{I}(A, B) = 0$ for all $B \in\in \mathfrak{T}$. If $f \in \mathfrak{I}(B, B')$, then $\mathfrak{T}(A, f)$ maps $\mathfrak{T}(A, B)$ to $\mathfrak{I}(A, B') = 0$, so that $\mathfrak{T}(A, f) = 0$. Hence A is $\mathfrak{I}$-projective. $\square$

An $\mathfrak{I}$-exact functor also has the following properties (which are strictly weaker than being $\mathfrak{I}$-exact):

- F maps $\mathfrak{I}$-equivalences to isomorphisms in $\mathfrak{C}$;
- F maps $\mathfrak{I}$-contractible objects to 0 in $\mathfrak{C}$.

Again we may specialise this to $\mathfrak{I}$-projective and objects.

Lemma 89. *The class of $\mathfrak{I}$-exact homological functors $\mathfrak{T} \to \mathfrak{Ab}$ or $\mathfrak{T} \to \mathfrak{Ab}^{\mathrm{op}}$ is closed under composition with $\Sigma^{\pm 1}\colon \mathfrak{T} \to \mathfrak{T}$, retracts, direct sums, and direct products. The class $\mathfrak{P}_{\mathfrak{I}}$ of $\mathfrak{I}$-projective objects is closed under (de)suspensions, retracts, and possibly infinite direct sums (as far as they exist in $\mathfrak{T}$).*

Proof. The first assertion follows because direct sums and products of Abelian groups are exact; the second one is a special case. $\square$

Notation 90. Let $\mathfrak{P} \subseteq \mathfrak{T}$ be a set of objects. We let $(\mathfrak{P})_\oplus$ be the smallest class of objects of $\mathfrak{T}$ that contains $\mathfrak{P}$ and is closed under retracts and direct sums (as far as they exist in $\mathfrak{T}$).

By Lemma 89, $(\mathfrak{P})_\oplus$ consists of $\mathfrak{I}$-projective objects if $\mathfrak{P}$ does. We say that $\mathfrak{P}$ *generates all $\mathfrak{I}$-projective objects* if $(\mathfrak{P})_\oplus = \mathfrak{P}_\mathfrak{I}$. In examples, it is usually easier to describe a class of generators in this sense.

Example 91. Suppose that G is discrete. Then the adjointness between induction and restriction functors implies that all compactly induced objects are projective for the ideal $\mathcal{VC}$. Even more, the techniques that we develop below show that $\mathfrak{P}_{\mathcal{VC}} = \mathcal{CI}$.

5.4. Projective resolutions.

Definition 92. Let $\mathfrak{I} \subseteq \mathfrak{T}$ be a homological ideal in a triangulated category and let $A \in\in \mathfrak{T}$. A *one-step $\mathfrak{I}$-projective resolution* is an $\mathfrak{I}$-epimorphism $\pi\colon P \to A$ with $P \in\in \mathfrak{P}_\mathfrak{I}$. An *$\mathfrak{I}$-projective resolution of A* is an $\mathfrak{I}$-exact chain complex

$$\cdots \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} P_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} A$$

with $P_n \in\in \mathfrak{P}_\mathfrak{I}$ for all $n \in \mathbb{N}$.

We say that $\mathfrak{I}$ has *enough projective objects* if each $A \in\in \mathfrak{T}$ has a one-step $\mathfrak{I}$-projective resolution.

The following proposition contains the basic properties of projective resolutions, which are familiar from the similar situation for Abelian categories.

Proposition 93. *If $\mathfrak{I}$ has enough projective objects, then any object of $\mathfrak{T}$ has an $\mathfrak{I}$-projective resolution (and vice versa).*

Let $P_\bullet \to A$ and $P'_\bullet \to A'$ be $\mathfrak{I}$-projective resolutions. Then any map $A \to A'$ may be lifted to a chain map $P_\bullet \to P'_\bullet$, and this lifting is unique up to chain homotopy. Two $\mathfrak{I}$-projective resolutions of the same object are chain homotopy equivalent. As a result, the construction of projective resolutions provides a functor

$$P\colon \mathfrak{T} \to \mathrm{Ho}(\mathfrak{T}).$$

Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ be an $\mathfrak{I}$-exact triangle. Then there exists a canonical map $\eta\colon P(C) \to P(A)[1]$ in $\mathrm{Ho}(\mathfrak{T})$ such that the triangle

$$P(A) \xrightarrow{P(f)} P(B) \xrightarrow{P(g)} P(C) \xrightarrow{\eta} P(A)[1]$$

in $\mathrm{Ho}(\mathfrak{T})$ is exact; here $[1]$ denotes the translation functor in $\mathrm{Ho}(\mathfrak{T})$, which has nothing to do with the suspension in $\mathfrak{T}$.

Proof. Let $A \in\in \mathfrak{T}$. By assumption, there is a one-step $\mathfrak{I}$-projective resolution $\delta_0\colon P_0 \to A$, which we embed in an exact triangle $A_1 \to P_0 \to A \to \Sigma A_1$. Since δ_0 is $\mathfrak{I}$-epic, this triangle is $\mathfrak{I}$-exact. By induction, we construct a sequence of such

$\mathfrak{I}$-exact triangles $A_{n+1} \to P_n \to A_n \to \Sigma A_{n+1}$ for $n \in \mathbb{N}$ with $P_n \in\in \mathfrak{P}$ and $A_0 = A$. By composition, we obtain maps $\delta_n \colon P_n \to P_{n-1}$ for $n \geq 1$, which satisfy $\delta_n \circ \delta_{n+1} = 0$ for all $n \geq 0$. The resulting chain complex

$$\dots \to P_n \xrightarrow{\delta_n} P_{n-1} \xrightarrow{\delta_{n-1}} P_{n-2} \to \dots \to P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} A \to 0$$

is $\mathfrak{I}$-decomposable by construction and therefore $\mathfrak{I}$-exact by Corollary 76.

The remaining assertions are proved exactly as their classical counterparts in homological algebra. We briefly sketch the arguments. Let $P_\bullet \to A$ and $P'_\bullet \to A'$ be $\mathfrak{I}$-projective resolutions and let $f \in \mathfrak{T}(A, A')$. We construct $f_n \in \mathfrak{T}(P_n, P'_n)$ by induction on n such that the diagrams

$$\begin{tikzcd} P_0 \arrow[r, "\delta_0"] \arrow[d, dotted, "f_0"'] \arrow[dr] & A \arrow[d, "f"] \\ P'_0 \arrow[r, "\delta'_0"'] & A', \end{tikzcd} \qquad \begin{tikzcd} P_n \arrow[r, "\delta_n"] \arrow[d, dotted, "f_n"'] \arrow[dr] & P_{n-1} \arrow[d, "f_{n-1}"] \\ P'_n \arrow[r, "\delta'_n"'] & P'_{n-1} \end{tikzcd}$$

for $n \geq 1$ commute. We must check that this is possible. Since the chain complex $P'_\bullet \to A$ is $\mathfrak{I}$-exact and P_n is $\mathfrak{I}$-projective for all $n \geq 0$, the chain complexes

$$\dots \to \mathfrak{T}(P_n, P'_m) \xrightarrow{(\delta'_m)_*} \mathfrak{T}(P_n, P'_{m-1}) \to \dots \to \mathfrak{T}(P_n, P'_0) \xrightarrow{(\delta'_0)_*} \mathfrak{T}(P_n, A) \to 0$$

are exact for all $n \in \mathbb{N}$. This allows us to find the needed maps f_n. By construction, these maps form a chain map lifting $f \colon A \to A'$. Its uniqueness up to chain homotopy is proved similarly. If we apply this unique lifting result to two $\mathfrak{I}$-projective resolutions of the same object, we get the uniqueness of $\mathfrak{I}$-projective resolutions up to chain homotopy equivalence. Hence we get a well-defined functor $P \colon \mathfrak{T} \to \mathrm{Ho}(\mathfrak{T})$.

Now consider an $\mathfrak{I}$-exact triangle $A \to B \to C \to \Sigma A$ as in the third paragraph of the lemma. Let $X_\bullet$ be the mapping cone of some chain map $P(A) \to P(B)$ in the homotopy class $P(f)$. This chain complex is supported in degrees ≥ 0 and has $\mathfrak{I}$-projective entries because $X_n = P(A)_{n-1} \oplus P(B)_n$. The map $X_0 = 0 \oplus P(B)_0 \to B \to C$ yields a chain map $X_\bullet \to C$, that is, the composite map $X_1 \to X_0 \to C$ vanishes. By construction, this chain map lifts the given map $B \to C$ and we have an exact triangle $P(A) \to P(B) \to X_\bullet \to P(A)[1]$ in $\mathrm{Ho}(\mathfrak{T})$. It remains to observe that $X_\bullet \to C$ is $\mathfrak{I}$-exact. Then $X_\bullet$ is an $\mathfrak{I}$-projective resolution of C. Since such resolutions are unique up to chain homotopy equivalence, we get a canonical isomorphism $X_\bullet \cong P(C)$ in $\mathrm{Ho}(\mathfrak{T})$ and hence the assertion in the third paragraph.

Let F be a stable homological functor with $\mathfrak{I} = \ker F$. We have to check that $F(X_\bullet) \to F(C)$ is a resolution. This reduces to a well-known diagram chase in Abelian categories, using that $F\bigl(P(A)\bigr) \to F(A)$ and $F\bigl(P(B)\bigr) \to F(B)$ are resolutions and that $F(A) \rightarrowtail F(B) \twoheadrightarrow F(C)$ is exact. $\square$

5.5. Derived functors. We only define derived functors if there are enough projective objects because this case is rather easy and suffices for our applications.

The general case can be reduced to the familiar case of Abelian categories using the results of §5.2.1.

Definition 94. Let $\mathfrak{I}$ be a homological ideal in a triangulated category $\mathfrak{T}$ with enough projective objects. Let $F\colon \mathfrak{T} \to \mathfrak{C}$ be an additive functor with values in an Abelian category $\mathfrak{C}$. It induces a functor $\mathrm{Ho}(F)\colon \mathrm{Ho}(\mathfrak{T}) \to \mathrm{Ho}(\mathfrak{C})$, applying F pointwise to chain complexes. Let $P\colon \mathfrak{T} \to \mathrm{Ho}(\mathfrak{T})$ be the projective resolution functor constructed in Proposition 93. Let $H_n\colon \mathrm{Ho}(\mathfrak{C}) \to \mathfrak{C}$ be the nth homology functor for some $n \in \mathbb{N}$. The composite functor

$$\mathbb{L}_n F\colon \mathfrak{T} \xrightarrow{P} \mathrm{Ho}(\mathfrak{T}) \xrightarrow{\mathrm{Ho}(F)} \mathrm{Ho}(\mathfrak{C}) \xrightarrow{H_n} \mathfrak{C}$$

is called the nth *left derived functor* of F. If $F\colon \mathfrak{T}^{\mathrm{op}} \to \mathfrak{C}$ is a contravariant additive functor, then the corresponding functor $H^n \circ \mathrm{Ho}(F) \circ P\colon \mathfrak{T}^{\mathrm{op}} \to \mathfrak{C}$ is denoted by $\mathbb{R}^n F$ and called the nth *right derived functor* of F.

More concretely, let $A \in\in \mathfrak{T}$ and let $(P_\bullet, \delta_\bullet)$ be an $\mathfrak{I}$-projective resolution of A. If F is covariant, then $\mathbb{L}_n F(A)$ is the homology at $F(P_n)$ of the chain complex

$$\cdots \to F(P_{n+1}) \xrightarrow{F(\delta_{n+1})} F(P_n) \xrightarrow{F(\delta_n)} F(P_{n-1}) \to \cdots \to F(P_0) \to 0.$$

If F is contravariant, then $\mathbb{R}^n F(A)$ is the cohomology at $F(P_n)$ of the cochain complex

$$\cdots \leftarrow F(P_{n+1}) \xleftarrow{F(\delta_{n+1})} F(P_n) \xleftarrow{F(\delta_n)} F(P_{n-1}) \leftarrow \cdots \leftarrow F(P_0) \leftarrow 0.$$

Lemma 95. *Let $A \to B \to C \to \Sigma A$ be an $\mathfrak{I}$-exact triangle. If $F\colon \mathfrak{T} \to \mathfrak{C}$ is a covariant additive functor, then there is a long exact sequence*

$$\begin{aligned}\cdots \to \mathbb{L}_n F(A) \to \mathbb{L}_n F(B) \to \mathbb{L}_n F(C) \to \mathbb{L}_{n-1} F(A)&\\ \to \cdots \to \mathbb{L}_1 F(C) \to \mathbb{L}_0 F(A) \to \mathbb{L}_0 F(B) \to \mathbb{L}_0 F(C) \to 0.&\end{aligned}$$

If $F\colon \mathfrak{T}^{\mathrm{op}} \to \mathfrak{C}$ is contravariant instead, then there is a long exact sequence

$$\begin{aligned}\cdots \leftarrow \mathbb{R}^n F(A) \leftarrow \mathbb{R}^n F(B) \leftarrow \mathbb{R}^n F(C) \leftarrow \mathbb{R}^{n-1} F(A)&\\ \leftarrow \cdots \leftarrow \mathbb{R}^1 F(C) \leftarrow \mathbb{R}^0 F(A) \leftarrow \mathbb{R}^0 F(B) \leftarrow \mathbb{R}^0 F(C) \leftarrow 0.&\end{aligned}$$

Proof. This follows from the third assertion of Proposition 93 together with the well-known long exact homology sequence for exact triangles in $\mathrm{Ho}(\mathfrak{C})$. □

Lemma 96. *Let $F\colon \mathfrak{T} \to \mathfrak{C}$ be a homological functor. The following assertions are equivalent:*

(1) F is $\mathfrak{I}$-exact;

(2) $\mathbb{L}_0 F(A) \cong F(A)$ and $\mathbb{L}_p F(A) = 0$ for all $p > 0$, $A \in\in \mathfrak{T}$;

(3) $\mathbb{L}_0 F(A) \cong F(A)$ for all $A \in\in \mathfrak{T}$.

The analogous assertions for contravariant functors are equivalent as well.

Proof. If F is $\mathfrak{I}$-exact, then F maps $\mathfrak{I}$-exact chain complexes in $\mathfrak{T}$ to exact chain complexes in $\mathfrak{C}$. This applies to $\mathfrak{I}$-projective resolutions, so that (1)$\Longrightarrow$(2)$\Longrightarrow$(3). It follows from (3) and Lemma 95 that F maps $\mathfrak{I}$-epimorphisms to epimorphisms. Since this characterises $\mathfrak{I}$-exact functors, we get (3)$\Longrightarrow$(1). $\square$

It can happen that $\mathbb{L}_p F = 0$ for all $p > 0$ although F is not $\mathfrak{I}$-exact.

We have a natural transformation $\mathbb{L}_0 F(A) \to F(A)$ (or $F(A) \to \mathbb{R}^0 F(A)$), which is induced by the augmentation map $P_\bullet \to A$ for an $\mathfrak{I}$-projective resolution. Lemma 96 shows that these maps are usually not bijective, although this happens frequently for derived functors on Abelian categories.

Definition 97. We let $\operatorname{Ext}^n_{\mathfrak{T},\mathfrak{I}}(A,B)$ be the nth right derived functor with respect to $\mathfrak{I}$ of the contravariant functor $A \mapsto \mathfrak{T}(A,B)$.

We have natural maps $\mathfrak{T}(A,B) \to \operatorname{Ext}^0_{\mathfrak{T},\mathfrak{I}}(A,B)$, which usually are not invertible. Lemma 95 yields long exact sequences

$$\begin{gathered}\cdots \leftarrow \operatorname{Ext}^n_{\mathfrak{T},\mathfrak{I}}(A,D) \leftarrow \operatorname{Ext}^n_{\mathfrak{T},\mathfrak{I}}(B,D) \leftarrow \operatorname{Ext}^n_{\mathfrak{T},\mathfrak{I}}(C,D) \leftarrow \operatorname{Ext}^{n-1}_{\mathfrak{T},\mathfrak{I}}(A,D) \leftarrow \\ \cdots \leftarrow \operatorname{Ext}^1_{\mathfrak{T},\mathfrak{I}}(C,D) \leftarrow \operatorname{Ext}^0_{\mathfrak{T},\mathfrak{I}}(A,D) \leftarrow \operatorname{Ext}^0_{\mathfrak{T},\mathfrak{I}}(B,D) \leftarrow \operatorname{Ext}^0_{\mathfrak{T},\mathfrak{I}}(C,D) \leftarrow 0\end{gathered}$$

for any $\mathfrak{I}$-exact exact triangle $A \to B \to C \to \Sigma A$ and any $D \Subset \mathfrak{T}$.

We claim that there are similar long exact sequences

$$\begin{gathered}0 \to \operatorname{Ext}^0_{\mathfrak{T},\mathfrak{I}}(D,A) \to \operatorname{Ext}^0_{\mathfrak{T},\mathfrak{I}}(D,B) \to \operatorname{Ext}^0_{\mathfrak{T},\mathfrak{I}}(D,C) \to \operatorname{Ext}^1_{\mathfrak{T},\mathfrak{I}}(D,A) \to \cdots \\ \to \operatorname{Ext}^{n-1}_{\mathfrak{T},\mathfrak{I}}(D,C) \to \operatorname{Ext}^n_{\mathfrak{T},\mathfrak{I}}(D,A) \to \operatorname{Ext}^n_{\mathfrak{T},\mathfrak{I}}(D,B) \to \operatorname{Ext}^n_{\mathfrak{T},\mathfrak{I}}(D,C) \to \cdots\end{gathered}$$

in the second variable. Since $P(D)_n$ is $\mathfrak{I}$-projective, the sequences

$$0 \to \mathfrak{T}(P(D)_n, A) \to \mathfrak{T}(P(D)_n, B) \to \mathfrak{T}(P(D)_n, C) \to 0$$

are exact for all $n \in \mathbb{N}$. This extension of chain complexes yields the desired long exact sequence.

We list a few more elementary properties of derived functors. We only spell things out for the left derived functors $\mathbb{L}_n F\colon \mathfrak{T} \to \mathfrak{C}$ of a covariant functor $F\colon \mathfrak{T} \to \mathfrak{C}$. Similar assertions hold for right derived functors of contravariant functors.

The derived functors $\mathbb{L}_n F$ satisfy $\mathfrak{I} \subseteq \ker \mathbb{L}_n F$ and hence descend to functors $\mathbb{L}_n F\colon \mathfrak{T}/\mathfrak{I} \to \mathfrak{C}$ because the zero map $P(A) \to P(B)$ is a chain map lifting of f if $f \in \mathfrak{I}(A,B)$. As a consequence, $\mathbb{L}_n F(A) \cong 0$ if A is $\mathfrak{I}$-contractible. The long exact homology sequences of Lemma 95 show that $\mathbb{L}_n F(f)\colon \mathbb{L}_n F(A) \to \mathbb{L}_n F(B)$ is invertible if $f \in \mathfrak{T}(A,B)$ is an $\mathfrak{I}$-equivalence.

Warning 98. The derived functors $\mathbb{L}_n F$ are *not homological* and therefore do not deserve to be called $\mathfrak{I}$-exact even though they vanish on $\mathfrak{I}$-phantom maps. Lemma 95 shows that these functors are only half-exact on $\mathfrak{I}$-exact triangles. Thus $\mathbb{L}_n F(f)$ need not be monic (or epic) if f is $\mathfrak{I}$-monic (or $\mathfrak{I}$-epic). The problem is that the $\mathfrak{I}$-projective resolution functor $P\colon \mathfrak{T} \to \operatorname{Ho}(\mathfrak{T})$ is not exact because it even fails to be stable.

The following remarks require a more advanced background in homological algebra and are not going to be used in the sequel.

Remark 99. The derived functors introduced above, especially the Ext functors, can be interepreted in terms of *derived categories*.

We have already observed in §5.2.1 that the $\mathfrak{I}$-exact chain complexes form a thick subcategory of $\mathrm{Ho}(\mathfrak{T})$. The augmentation map $P(A) \to A$ of an $\mathfrak{I}$-projective resolution of $A \in\in \mathfrak{T}$ is a quasi-isomorphism with respect to this thick subcategory. The chain complex $P(A)$ is projective (see [18]), that is, for any chain complex $C_\bullet$, the space of morphisms $A \to C_\bullet$ in the derived category $\mathfrak{Der}(\mathfrak{T}, \mathfrak{I})$ agrees with $[P(A), C_\bullet]$. Especially, $\mathrm{Ext}^n_{\mathfrak{T},\mathfrak{I}}(A, B)$ is the space of morphisms $A \to B[n]$ in $\mathfrak{Der}(\mathfrak{T}, \mathfrak{I})$.

Now let $F: \mathfrak{T} \to \mathfrak{C}$ be an additive covariant functor. Extend it to an exact functor $\bar{F}: \mathrm{Ho}(\mathfrak{T}) \to \mathrm{Ho}(\mathfrak{C})$. It has a total left derived functor

$$\mathbb{L}\bar{F}: \mathfrak{Der}(\mathfrak{T}, \mathfrak{I}) \to \mathfrak{Der}(\mathfrak{C}), \qquad A \mapsto \bar{F}\big(P(A)\big).$$

By definition, we have $\mathbb{L}_n F(A) := H_n\big(\mathbb{L}\bar{F}(A)\big)$.

Remark 100. In classical Abelian categories, the Ext groups form a graded ring, and the derived functors form graded modules over this graded ring. The same happens in our context. The most conceptual construction of these products uses the description of derived functors sketched in Remark 99.

Recall that we may view elements of $\mathrm{Ext}^n_{\mathfrak{T},\mathfrak{I}}(A, B)$ as morphisms $A \to B[n]$ in the derived category $\mathfrak{Der}(\mathfrak{T}, \mathfrak{I})$. Taking translations, we can also view them as morphisms $A[m] \to B[n + m]$ for any $m \in \mathbb{Z}$. The usual composition in the category $\mathfrak{Der}(\mathfrak{T}, \mathfrak{I})$ therefore yields an associative product

$$\mathrm{Ext}^n_{\mathfrak{T},\mathfrak{I}}(B, C) \otimes \mathrm{Ext}^m_{\mathfrak{T},\mathfrak{I}}(A, B) \to \mathrm{Ext}^{n+m}_{\mathfrak{T},\mathfrak{I}}(A, C).$$

Thus we get a graded additive category with morphism spaces $\big(\mathrm{Ext}^n_{\mathfrak{T},\mathfrak{I}}(A, B)\big)_{n\in\mathbb{N}}$.

Similarly, if $F: \mathfrak{T} \to \mathfrak{C}$ is an additive functor and $\mathbb{L}\bar{F}: \mathfrak{Der}(\mathfrak{T}, \mathfrak{I}) \to \mathfrak{Der}(\mathfrak{C})$ is as in Remark 100, then a morphism $A \to B[n]$ in $\mathfrak{Der}(\mathfrak{T}, \mathfrak{I})$ induces a morphism $\mathbb{L}\bar{F}(A) \to \mathbb{L}\bar{F}(B)[n]$ in $\mathfrak{Der}(\mathfrak{C})$. Passing to homology, we get canonical maps

$$\mathrm{Ext}^n_{\mathfrak{T},\mathfrak{I}}(A, B) \to \mathrm{Hom}_{\mathfrak{C}}\big(\mathbb{L}F_m(A), \mathbb{L}F_{m-n}(B)\big) \qquad \forall m \geq n,$$

which satisfy an appropriate associativity condition. For a contravariant functor, we get canonical maps

$$\mathrm{Ext}^n_{\mathfrak{T},\mathfrak{I}}(A, B) \to \mathrm{Hom}_{\mathfrak{C}}\big(\mathbb{R}F^m(B), \mathbb{R}F^{m+n}(A)\big) \qquad \forall m \geq 0.$$

5.6. Projective objects via adjointness. We develop a method for constructing enough projective objects. Let $\mathfrak{T}$ and $\mathfrak{C}$ be stable additive categories, let $F: \mathfrak{T} \to \mathfrak{C}$ be a stable additive functor, and let $\mathfrak{I} := \ker F$. In our applications, $\mathfrak{T}$ is triangulated and the functor F is either exact or stable and homological.

Recall that a covariant functor $R: \mathfrak{T} \to \mathfrak{Ab}$ is *(co)representable* if it is naturally isomorphic to $\mathfrak{T}(A, \text{\textvisiblespace})$ for some $A \in\in \mathfrak{T}$, which is then unique. If the functor

$B \mapsto \mathfrak{C}\bigl(A, F(B)\bigr)$ on $\mathfrak{T}$ is representable, we write $F^\dagger(A)$ for the representing object. By construction, we have natural isomorphisms

$$\mathfrak{T}(F^\dagger(A), B) \cong \mathfrak{C}\bigl(A, F(B)\bigr)$$

for all $B \in \mathfrak{T}$. Let $\mathfrak{C}' \subseteq \mathfrak{C}$ be the full subcategory of all objects $A \in\in \mathfrak{C}$ for which $F^\dagger(A)$ is defined. Then $F^\dagger$ is a functor $\mathfrak{C}' \to \mathfrak{T}$, which we call the *(partially defined) left adjoint* of F. Although one usually assumes $\mathfrak{C} = \mathfrak{C}'$, we shall also need $F^\dagger$ in cases where it is not defined everywhere.

The functor $B \mapsto \mathfrak{C}\bigl(A, F(B)\bigr)$ for $A \in\in \mathfrak{C}'$ vanishes on $\mathfrak{I} = \ker F$ for trivial reasons. Hence $F^\dagger(A) \in\in \mathfrak{T}$ is $\mathfrak{I}$-projective. This simple observation is surprisingly powerful: as we shall see, it often yields all $\mathfrak{I}$-projective objects.

Remark 101. We have $F^\dagger(\Sigma A) \cong \Sigma F^\dagger(A)$ for all $A \in\in \mathfrak{C}'$, so that $\Sigma(\mathfrak{C}') = \mathfrak{C}'$. Moreover, $F^\dagger$ commutes with infinite direct sums (as far as they exist in $\mathfrak{T}$) because

$$\mathfrak{T}\left(\bigoplus F^\dagger(A_i), B\right) \cong \prod \mathfrak{T}(F^\dagger(A_i), B) \cong \prod \mathfrak{C}\bigl(A_i, F(B)\bigr) \cong \mathfrak{C}\left(\bigoplus A_i, F(B)\right).$$

Example 102. Consider the functor $\mathrm{K}_*\colon \mathrm{KK} \to \mathfrak{Ab}^{\mathbb{Z}/2}$. Let $\mathbb{Z} \in\in \mathfrak{Ab}^{\mathbb{Z}/2}$ denote the trivially graded Abelian group $\mathbb{Z}$. Notice that

$$\begin{aligned} \mathrm{Hom}\bigl(\mathbb{Z}, \mathrm{K}_*(A)\bigr) &\cong \mathrm{K}_0(A) \cong \mathrm{KK}(\mathbb{C}, A), \\ \mathrm{Hom}\bigl(\mathbb{Z}[1], \mathrm{K}_*(A)\bigr) &\cong \mathrm{K}_1(A) \cong \mathrm{KK}(\mathcal{C}_0(\mathbb{R}), A), \end{aligned}$$

where $\mathbb{Z}[1]$ means $\mathbb{Z}$ in odd degree. Hence $\mathrm{K}_*^\dagger(\mathbb{Z}) = \mathbb{C}$ and $\mathrm{K}_*^\dagger(\mathbb{Z}[1]) = \mathcal{C}_0(\mathbb{R})$. More generally, Remark 101 shows that $\mathrm{K}_*^\dagger(A)$ is defined if both the even and odd parts of $A \in\in \mathfrak{Ab}^{\mathbb{Z}/2}$ are countable free Abelian groups: it is a direct sum of at most countably many copies of $\mathbb{C}$ and $\mathcal{C}_0(\mathbb{R})$. Hence all such countable direct sums are $\mathfrak{I}_\mathrm{K}$-projective (we briefly say K-*projective*). As we shall see, $\mathrm{K}_*^\dagger$ is not defined on all of $\mathfrak{Ab}^{\mathbb{Z}/2}$; this is typical of homological functors.

Example 103. Consider the functor $\mathrm{H}_*\colon \mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p) \to \mathfrak{C}^{\mathbb{Z}/p}$ of Example 54. Let $j\colon \mathfrak{C}^{\mathbb{Z}/p} \to \mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p)$ be the functor that views an object of $\mathfrak{C}^{\mathbb{Z}/p}$ as a p-periodic chain complex whose boundary map vanishes.

A chain map $j(A) \to B_\bullet$ for $A \in\in \mathfrak{C}^{\mathbb{Z}/p}$ and $B_\bullet \in\in \mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p)$ is a family of maps $\varphi_n\colon A_n \to \ker(d_n\colon B_n \to B_{n-1})$. Such a family is chain homotopic to 0 if and only if each φ_n lifts to a map $A_n \to B_{n+1}$. Suppose that A_n is projective for all $n \in \mathbb{Z}/p$. Then such a lifting exists if and only if $\varphi_n(A_n) \subseteq d_{n+1}(B_{n+1})$. Hence

$$[j(A), B_\bullet] \cong \prod_{n \in \mathbb{Z}/p} \mathfrak{C}\bigl(A_n, H_n(B_\bullet)\bigr) \cong \mathfrak{C}^{\mathbb{Z}/p}\bigl(A, \mathrm{H}_*(B_\bullet)\bigr).$$

As a result, the left adjoint of H_* is defined on the subcategory of projective objects $\mathfrak{P}(\mathfrak{C})^{\mathbb{Z}/p} \subseteq \mathfrak{C}^{\mathbb{Z}/p}$ and agrees there with the restriction of j. We will show in §5.8 that $\mathfrak{P}(\mathfrak{C})^{\mathbb{Z}/p}$ is equal to the domain of definition of $\mathrm{H}_*^\dagger$ and that all $\mathfrak{I}_\mathrm{H}$-projective objects are of the form $\mathrm{H}_*^\dagger(A)$ (provided $\mathfrak{C}$ has enough projective objects).

By duality, analogous results hold for injective objects: the domain of the *right* adjoint of H_* is the subcategory of injective objects of $\mathfrak{C}^{\mathbb{Z}/p}$, the right adjoint is equal to j on this subcategory, and this provides all H_*-injective objects of $\mathrm{Ho}(\mathfrak{C};\mathbb{Z}/p)$.

These examples show that $F^\dagger$ yields many $\ker F$-projective objects. We want to get *enough* $\ker F$-projective objects in this fashion, assuming that $F^\dagger$ is defined on enough of $\mathfrak{C}$. In order to treat ideals of the form $\bigcap F_i$, we now consider a more complicated setup. Let $\{\mathfrak{C}_i \mid i \in I\}$ be a set of stable homological or triangulated categories together with full subcategories $\mathfrak{P}\mathfrak{C}_i \subseteq \mathfrak{C}_i$ and stable homological or exact functors $F_i \colon \mathfrak{T} \to \mathfrak{C}_i$ for all $i \in I$. Assume that

- the left adjoint $F_i^\dagger$ is defined on $\mathfrak{P}\mathfrak{C}_i$ for all $i \in I$;
- there is an epimorphism $P \to F_i(A)$ in $\mathfrak{C}_i$ with $P \in\in \mathfrak{P}\mathfrak{C}_i$ for any $i \in I$, $A \in\in \mathfrak{T}$;
- the set of functors $F_i^\dagger \colon \mathfrak{P}\mathfrak{C}_i \to \mathfrak{T}$ is *cointegrable*, that is, $\bigoplus_{i\in I} F_i^\dagger(B_i)$ exists for all families of objects $B_i \in \mathfrak{P}\mathfrak{C}_i$, $i \in I$.

The reason for the notation $\mathfrak{P}\mathfrak{C}_i$ is that for a homological functor F_i we usually take $\mathfrak{P}\mathfrak{C}_i$ to be the class of projective objects of $\mathfrak{C}_i$; if F_i is exact, then we often take $\mathfrak{P}\mathfrak{C}_i = \mathfrak{C}_i$. But it may be useful to choose a smaller category, as long as it satisfies the second condition above.

Proposition 104. *In this situation, there are enough $\mathfrak{I}$-projective objects, and $\mathfrak{P}_\mathfrak{I}$ is generated by $\bigcup_{i\in I}\{F_i^\dagger(B) \mid B \in \mathfrak{P}\mathfrak{C}_i\}$. More precisely, an object of $\mathfrak{T}$ is $\mathfrak{I}$-projective if and only if it is a retract of $\bigoplus_{i\in I} F_i^\dagger(B_i)$ for a family of objects $B_i \in \mathfrak{P}\mathfrak{C}_i$.*

Proof. Let $\tilde{\mathfrak{P}}_0 := \bigcup_{i\in I}\{F_i^\dagger(B) \mid B \in \mathfrak{P}\mathfrak{C}_i\}$ and $\mathfrak{P}_0 := (\tilde{\mathfrak{P}}_0)_\oplus$. To begin with, we observe that any object of the form $F_i^\dagger(B)$ with $B \in\in \mathfrak{P}\mathfrak{C}_i$ is $\ker F_i$-projective and hence $\mathfrak{I}$-projective because $\mathfrak{I} \subseteq \ker F_i$. Hence $\mathfrak{P}_0$ consists of $\mathfrak{I}$-projective objects.

Let $A \in\in \mathfrak{T}$. For each $i \in I$, there is an epimorphism $p_i \colon B_i \to F_i(A)$ with $B_i \in \mathfrak{P}\mathfrak{C}_i$. The direct sum $B := \bigoplus_{i\in I} F_i^\dagger(B_i)$ exists. We have $B \in\in \mathfrak{P}_0$ by construction. We are going to construct an $\mathfrak{I}$-epimorphism $p\colon B \to A$. This shows that there are enough $\mathfrak{I}$-projective objects.

The maps $p_i \colon B_i \to F_i(A)$ provide maps $\hat{p}_i \colon F_i^\dagger(B_i) \to A$ via the adjointness isomorphisms $\mathfrak{T}(F_i^\dagger(B_i), A) \cong \mathfrak{C}_i\bigl(B_i, F_i(A)\bigr)$. We let $p := \sum \hat{p}_i \colon \bigoplus F_i^\dagger(B_i) \to A$. We must check that p is an $\mathfrak{I}$-epimorphism. Equivalently, p is $\ker F_i$-epic for all $i \in I$; this is, in turn equivalent to $F_i(p)$ being an epimorphism in $\mathfrak{C}_i$ for all $i \in I$, because of Lemma 68 or 71. This is what we are going to prove.

The identity map on $F_i^\dagger(B_i)$ yields a map $\alpha_i \colon B_i \to F_iF_i^\dagger(B_i)$ via the adjointness isomorphism $\mathfrak{T}\bigl(F_i^\dagger(B_i), F_i^\dagger(B_i)\bigr) \cong \mathfrak{C}_i\bigl(B_i, F_iF_i^\dagger(B_i)\bigr)$. Composing with the map

$$F_iF_i^\dagger(B_i) \to F_i\left(\bigoplus F_i^\dagger(B_i)\right) = F_i(B)$$

induced by the coordinate embedding $F_i^\dagger(B_i) \to B$, we get a map $\alpha_i' \colon B_i \to F_i(B)$. The naturality of the adjointness isomorphisms yields $F_i(\hat{p}_i) \circ \alpha_i = p_i$ and hence

$F_i(p) \circ \alpha_i' = p_i$. The map p_i is an epimorphism by assumption. Now we use a cancellation result for epimorphisms: if $f \circ g$ is an epimorphism, then so is f. Thus $F_i(p)$ is an epimorphism as desired.

If A is $\mathfrak{I}$-projective, then the $\mathfrak{I}$-epimorphism $p\colon B \to A$ splits; to see this, embed p in an exact triangle $N \to B \to A \to \Sigma N$ and observe that the map $A \to \Sigma N$ belongs to $\mathfrak{I}(A, \Sigma N) = 0$. Therefore, A is a retract of B. Since $\mathfrak{P}_0$ is closed under retracts and $B \in\in \mathfrak{P}_0$, we get $A \in\in \mathfrak{P}_0$. Hence $\tilde{\mathfrak{P}}_0$ generates all $\mathfrak{I}$-projective objects. □

5.7. The universal exact homological functor.

For the following results, it is essential to define an ideal by a single functor F instead of a family of functors as in Proposition 104.

Definition 105. Let $\mathfrak{I} \subseteq \mathfrak{T}$ be a homological ideal. An $\mathfrak{I}$-exact stable homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$ is called *universal* if any other $\mathfrak{I}$-exact stable homological functor $G\colon \mathfrak{T} \to \mathfrak{C}'$ factors as $\bar{G} = G \circ F$ for a stable exact functor $\bar{G}\colon \mathfrak{C} \to \mathfrak{C}'$ that is unique up to natural isomorphism.

This universal property characterises F uniquely up to natural isomorphism. We have constructed such a functor in §5.2.1. Beligiannis constructs it in [3, §3] using a localisation of the Abelian category $\mathfrak{Coh}(\mathfrak{T})$ at a suitable Serre subcategory; he calls this functor *projectivisation functor* and its target category *Steenrod category*. This notation is motivated by the special case of the Adams spectral sequence. The following theorem allows us to check whether a given functor is universal:

Theorem 106. *Let $\mathfrak{T}$ be a triangulated category, let $\mathfrak{I} \subseteq \mathfrak{T}$ be a homological ideal, and let $F\colon \mathfrak{T} \to \mathfrak{C}$ be an $\mathfrak{I}$-exact stable homological functor into a stable Abelian category $\mathfrak{C}$; let $\mathfrak{PC}$ be the class of projective objects in $\mathfrak{C}$. Suppose that idempotent morphisms in $\mathfrak{T}$ split.*

The functor F is the universal $\mathfrak{I}$-exact stable homological functor and there are enough $\mathfrak{I}$-projective objects in $\mathfrak{T}$ if and only if

- *$\mathfrak{C}$ has enough projective objects;*
- *the adjoint functor $F^\dagger$ is defined on $\mathfrak{PC}$;*
- *$F \circ F^\dagger(A) \cong A$ for all $A \in\in \mathfrak{PC}$.*

Proof. Suppose first that F is universal and that there are enough $\mathfrak{I}$-projective objects. Then F is equivalent to the projectivisation functor of [3]. The various properties of this functor listed in [3, Proposition 4.19] include the following:

- there are enough projective objects in $\mathfrak{C}$;
- F induces an equivalence of categories $\mathfrak{P}_{\mathfrak{I}} \cong \mathfrak{PC}$ ($\mathfrak{P}_{\mathfrak{I}}$ is the class of projective objects in $\mathfrak{T}$);
- $\mathfrak{C}\big(F(A), F(B)\big) \cong \mathfrak{T}(A, B)$ for all $A \in\in \mathfrak{P}_{\mathfrak{I}}$, $B \in\in \mathfrak{T}$.

Here we use the assumption that idempotents in $\mathfrak{T}$ split. The last property is equivalent to $F^\dagger \circ F(A) \cong A$ for all $A \in\in \mathfrak{P}_{\mathfrak{I}}$. Since $\mathfrak{P}_{\mathfrak{I}} \cong \mathfrak{PC}$ via F, this implies that $F^\dagger$ is defined on $\mathfrak{PC}$ and that $F \circ F^\dagger(A) \cong A$ for all $A \in\in \mathfrak{PC}$. Thus F has the properties listed in the statement of the theorem.

Now suppose conversely that F has these properties. Let $\mathfrak{P}'_{\mathfrak{I}} \subseteq \mathfrak{T}$ be the essential range of $F^\dagger\colon \mathfrak{PC} \to \mathfrak{T}$. We claim that $\mathfrak{P}'_{\mathfrak{I}}$ is the class of all $\mathfrak{I}$-projective objects in $\mathfrak{T}$. Since $F \circ F^\dagger$ is equivalent to the identity functor on $\mathfrak{PC}$ by assumption, $F|_{\mathfrak{P}'_{\mathfrak{I}}}$ and $F^\dagger$ provide an equivalence of categories $\mathfrak{P}'_{\mathfrak{I}} \cong \mathfrak{PC}$. Since $\mathfrak{C}$ is assumed to have enough projectives, the hypotheses of Proposition 104 are satisfied. Hence there are enough $\mathfrak{I}$-projective objects in $\mathfrak{T}$, and any object of $\mathfrak{P}_{\mathfrak{I}}$ is a retract of an object of $\mathfrak{P}'_{\mathfrak{I}}$. Idempotent morphisms in the category $\mathfrak{P}'_{\mathfrak{I}} \cong \mathfrak{PC}$ split because $\mathfrak{C}$ is Abelian and retracts of projective objects are again projective. Hence $\mathfrak{P}'_{\mathfrak{I}}$ is closed under retracts in $\mathfrak{T}$, so that $\mathfrak{P}'_{\mathfrak{I}} = \mathfrak{P}_{\mathfrak{I}}$. It also follows that F and $F^\dagger$ provide an equivalence of categories $\mathfrak{P}_{\mathfrak{I}} \cong \mathfrak{PC}$. Hence $F^\dagger \circ F(A) \cong A$ for all $A \in\in \mathfrak{P}_{\mathfrak{I}}$, so that we get $\mathfrak{C}\bigl(F(A), F(B)\bigr) \cong \mathfrak{T}(F^\dagger \circ F(A), B) \cong \mathfrak{T}(A, B)$ for all $A \in\in \mathfrak{P}_{\mathfrak{I}}$, $B \in\in \mathfrak{T}$.

Now let $G\colon \mathfrak{T} \to \mathfrak{C}'$ be a stable homological functor. We will later assume G to be $\mathfrak{I}$-exact, but the first part of the following argument works in general. Since F provides an equivalence of categories $\mathfrak{P}_{\mathfrak{I}} \cong \mathfrak{PC}$, the rule $\bar{G}\bigl(F(P)\bigr) := G(P)$ defines a functor $\bar{G}$ on $\mathfrak{PC}$. This yields a functor $\mathrm{Ho}(\bar{G})\colon \mathrm{Ho}(\mathfrak{PC}) \to \mathrm{Ho}(\mathfrak{C}')$. Since $\mathfrak{C}$ has enough projective objects, the construction of projective resolutions provides a functor $P\colon \mathfrak{C} \to \mathrm{Ho}(\mathfrak{PC})$. We let $\bar{G}$ be the composite functor

$$\bar{G}\colon \mathfrak{C} \xrightarrow{P} \mathrm{Ho}(\mathfrak{PC}) \xrightarrow{\mathrm{Ho}(\bar{G})} \mathrm{Ho}(\mathfrak{C}') \xrightarrow{H_0} \mathfrak{C}'.$$

This functor is right-exact and satisfies $\bar{G} \circ F = G$ on $\mathfrak{I}$-projective objects of $\mathfrak{T}$.

Now suppose that G is $\mathfrak{I}$-exact. Then we get $\bar{G} \circ F = G$ for all objects of $\mathfrak{T}$ because this holds for $\mathfrak{I}$-projective objects. We claim that $\bar{G}$ is exact. Let $A \in\in \mathfrak{C}$. Since $\mathfrak{C}$ has enough projective objects, we can find a projective resolution of A. We may assume this resolution to have the form $F(P_\bullet)$ with $P_\bullet \in\in \mathrm{Ho}(\mathfrak{P}_{\mathfrak{I}})$ because $F(\mathfrak{P}_{\mathfrak{I}}) \cong \mathfrak{PC}$. Lemma 75 yields that $P_\bullet$ is $\mathfrak{I}$-exact except in degree 0. Since $\mathfrak{I} \subseteq \ker G$, the chain complex $P_\bullet$ is $\ker G$-exact in positive degrees as well, so that $G(P_\bullet)$ is exact except in degree 0 by Lemma 75. As a consequence, $\mathbb{L}_p\bar{G}(A) = 0$ for all $p > 0$. We also have $\mathbb{L}_0\bar{G}(A) = \bar{G}(A)$ by construction. Thus $\bar{G}$ is exact.

As a result, G factors as $G = \bar{G} \circ F$ for an exact functor $\bar{G}\colon \mathfrak{C} \to \mathfrak{C}'$. It is clear that $\bar{G}$ is stable. Finally, since $\mathfrak{C}$ has enough projective objects, a functor on $\mathfrak{C}$ is determined up to natural equivalence by its restriction to projective objects. Therefore, our factorisation of G is unique up to natural equivalence. Thus F is the universal $\mathfrak{I}$-exact functor. $\square$

Remark 107. Let $\mathfrak{P}'\mathfrak{C} \subseteq \mathfrak{PC}$ be some subcategory such that any object of $\mathfrak{C}$ is a quotient of a direct sum of objects of $\mathfrak{P}'\mathfrak{C}$. Equivalently, $(\mathfrak{P}'\mathfrak{C})_\oplus = \mathfrak{PC}$. Theorem 106 remains valid if we only assume that $F^\dagger$ is defined on $\mathfrak{P}'\mathfrak{C}$ and that $F \circ F^\dagger(A) \cong A$ holds for $A \in\in \mathfrak{P}'\mathfrak{C}$ because both conditions are evidently hereditary for direct sums and retracts.

Theorem 108. *In the situation of Theorem 106, the domain of definition of the functor $F^\dagger$ is equal to $\mathfrak{PC}$, and its essential range is $\mathfrak{P}_\mathfrak{I}$. The functors F and $F^\dagger$ restrict to equivalences of categories $\mathfrak{P}_\mathfrak{I} \cong \mathfrak{PC}$ inverse to each other.*

An object $A \in\in \mathfrak{T}$ is $\mathfrak{I}$-projective if and only if $F(A)$ is projective and

$$\mathfrak{C}\big(F(A), F(B)\big) \cong \mathfrak{T}(A, B)$$

for all $B \in\in \mathfrak{T}$; following Ross Street [34]*, we call such objects F-*projective*. We have $F(A) \in\in \mathfrak{PC}$ if and only if there is an $\mathfrak{I}$-equivalence $P \to A$ with $P \in\in \mathfrak{P}_\mathfrak{I}$.*

The functors F and $F^\dagger$ induce bijections between isomorphism classes of projective resolutions of $F(A)$ in $\mathfrak{C}$ and isomorphism classes of $\mathfrak{I}$-projective resolutions of $A \in\in \mathfrak{T}$ in $\mathfrak{T}$.

If $G\colon \mathfrak{T} \to \mathfrak{C}'$ is any (stable) homological functor, then there is a unique right-exact (stable) functor $\bar{G}\colon \mathfrak{C} \to \mathfrak{C}'$ such that $\bar{G} \circ F(P) = G(P)$ for all $P \in\in \mathfrak{P}_\mathfrak{I}$.

The left derived functors of G with respect to $\mathfrak{I}$ and of $\bar{G}$ are related by natural isomorphisms $\mathbb{L}_n\bar{G} \circ F(A) = \mathbb{L}_n G(A)$ for all $A \in\in \mathfrak{T}$, $n \in \mathbb{N}$. There is a similar statement for cohomological functors, which specialises to natural isomorphisms

$$\mathrm{Ext}^n_{\mathfrak{T},\mathfrak{I}}(A, B) \cong \mathrm{Ext}^n_{\mathfrak{C}}\big(F(A), F(B)\big).$$

Proof. We have already seen during the proof of Theorem 106 that F restricts to an equivalence of categories $\mathfrak{P}_\mathfrak{I} \xrightarrow{\cong} \mathfrak{PC}$, whose inverse is the restriction of $F^\dagger$, and that $\mathfrak{C}\big(F(A), F(B)\big) \cong \mathfrak{T}(A, B)$ for all $A \in\in \mathfrak{P}_\mathfrak{I}$, $B \in\in \mathfrak{PC}$.

Conversely, if A is F-projective in the sense of Street, then A is $\mathfrak{I}$-projective because already $\mathfrak{T}(A, B) \cong \mathfrak{C}\big(F(A), F(B)\big)$ for all $B \in\in \mathfrak{T}$ yields $A \cong F^\dagger \circ F(A)$, so that A is $\mathfrak{I}$-projective; notice that the projectivity of $F(A)$ is automatic.

Since F maps $\mathfrak{I}$-equivalences to isomorphisms, $F(A)$ is projective whenever there is an $\mathfrak{I}$-equivalence $P \to A$ with $\mathfrak{I}$-projective P. Conversely, suppose that $F(A)$ is $\mathfrak{I}$-projective. Let $P_0 \to A$ be a one-step $\mathfrak{I}$-projective resolution. Since $F(A)$ is projective, the epimorphism $F(P_0) \to F(A)$ splits by some map $F(A) \to F(P_0)$. The resulting map $F(P_0) \to F(A) \to F(P_0)$ is idempotent and comes from an idempotent endomorphism of P_0 because F is fully faithful on $\mathfrak{P}_\mathfrak{I}$. Its range object P exists because we require idempotent morphisms in $\mathfrak{C}$ to split. It belongs again to $\mathfrak{P}_\mathfrak{I}$, and the induced map $F(P) \to F(A)$ is invertible by construction. Hence we get an $\mathfrak{I}$-equivalence $P \to A$.

If $C_\bullet$ is a chain complex over $\mathfrak{T}$, then we know already from Lemma 75 that $C_\bullet$ is $\mathfrak{I}$-exact if and only if $F(C_\bullet)$ is exact. Hence F maps an $\mathfrak{I}$-projective resolution of A to a projective resolution of $F(A)$. Conversely, if $P_\bullet \to F(A)$ is any projective resolution in $\mathfrak{C}$, then it is of the form $F(\hat{P}_\bullet) \to F(A)$ where $\hat{P}_\bullet := F^\dagger(P_\bullet)$ and where we get the map $\hat{P}_0 \to A$ by adjointness from the given map $P_0 \to F(A)$. This shows that F induces a bijection between isomorphism classes of $\mathfrak{I}$-projective resolutions of A and projective resolutions of $F(A)$.

We have seen during the proof of Theorem 106 how a stable homological functor $G\colon \mathfrak{T} \to \mathfrak{C}'$ gives rise to a unique right-exact functor $\bar{G}\colon \mathfrak{C} \to \mathfrak{C}'$ that satisfies $\bar{G} \circ F(P) = G(P)$ for all $P \in\in \mathfrak{P}_\mathfrak{I}$. The derived functors $\mathbb{L}_n\bar{G}\big(F(A)\big)$ for $A \in\in \mathfrak{T}$ are computed by applying $\bar{G}$ to a projective resolution of $F(A)$. Since such a projective resolution is of the form $F(P_\bullet)$ for an $\mathfrak{I}$-projective resolution $P_\bullet \to A$ and since $\bar{G} \circ F = G$ on $\mathfrak{I}$-projective objects, the derived functors $\mathbb{L}_n G(A)$ and $\mathbb{L}_n \bar{G}\bigl(F(A)\bigr)$ are computed by the same chain complex and agree. The same reasoning applies to cohomological functors and yields the assertion about Ext.

Finally, we check that $A \in\in \mathfrak{C}$ is projective if $F^\dagger(A)$ is defined. We prove $\mathrm{Ext}^1_{\mathfrak{C}}(A, B) = 0$ for all $B \in\in \mathfrak{C}$, from which the assertion follows. There is a projective resolution of the form $F(P_\bullet) \to B$, which we use to compute $\mathrm{Ext}^1_{\mathfrak{C}}(A, B)$. The adjointness of $F^\dagger$ and F yields that $F^\dagger(A) \in\in \mathfrak{T}$ is $\mathfrak{I}$-projective and that $\mathfrak{C}\bigl(A, F(P_\bullet)\bigr) \cong \mathfrak{T}(F^\dagger(A), P_\bullet)$. Since $P_\bullet$ is $\mathfrak{I}$-exact in positive degrees by Lemma 75 and $F^\dagger(A)$ is $\mathfrak{I}$-projective, we get $0 = H_1\bigl(\mathfrak{C}\bigl(A, F(P_\bullet)\bigr)\bigr) = \mathrm{Ext}^1_{\mathfrak{C}}(A, B)$. □

Remark 109. The assumption that idempotents split is only needed to check that the universal $\mathfrak{I}$-exact functor has the properties listed in Theorem 106. The converse directions of Theorem 106 and Theorem 108 do not need this assumption.

If $\mathfrak{T}$ has countable direct sums or countable direct products, then idempotents in $\mathfrak{T}$ automatically split by [28, §1.3]. This covers categories such as KK^G because they have countable direct sums.

5.8. Derived functors in homological algebra. Now we study the kernel $\mathfrak{I}_{\mathrm{H}}$ of the homology functor $\mathrm{H}_*\colon \mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p) \to \mathfrak{C}^{\mathbb{Z}/p}$ introduced Example 54. We get exactly the same statements if we replace the homotopy category by its derived category and study the kernel of $\mathrm{H}_*\colon \mathfrak{Der}(\mathfrak{C}; \mathbb{Z}/p) \to \mathfrak{C}^{\mathbb{Z}/p}$. We often abbreviate $\mathfrak{I}_{\mathrm{H}}$ to H and speak of H-epimorphisms, H-exact chain complexes, H-projective resolutions, and so on. We denote the full subcategory of H-projective objects in $\mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p)$ by $\mathfrak{P}_{\mathrm{H}}$.

We assume that the underlying Abelian category $\mathfrak{C}$ has enough projective objects. Then the same holds for $\mathfrak{C}^{\mathbb{Z}/p}$, and we have $\mathfrak{P}(\mathfrak{C}^{\mathbb{Z}/p}) \cong (\mathfrak{P}\mathfrak{C})^{\mathbb{Z}/p}$. That is, an object of $\mathfrak{C}^{\mathbb{Z}/p}$ is projective if and only if its homogeneous pieces are.

Theorem 110. *The category* $\mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p)$ *has enough* H*-projective objects, and the functor* $\mathrm{H}_*\colon \mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p) \to \mathfrak{C}^{\mathbb{Z}/p}$ *is the universal* H*-exact stable homological functor. Its restriction to* $\mathfrak{P}_{\mathrm{H}}$ *provides an equivalence of categories* $\mathfrak{P}_{\mathrm{H}} \cong \mathfrak{P}\mathfrak{C}^{\mathbb{Z}/p}$*. More concretely, a chain complex in* $\mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p)$ *is* H*-projective if and only if it is homotopy equivalent to one with vanishing boundary map and projective entries.*

The functor H_* *maps isomorphism classes of* H*-projective resolutions of* $A \in\in \mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p)$ *bijectively to isomorphism classes of projective resolutions of* $\mathrm{H}_*(A)$ *in* $\mathfrak{C}^{\mathbb{Z}/p}$*. We have*

$$\mathrm{Ext}^n_{\mathrm{Ho}(\mathfrak{C};\mathbb{Z}/p),\mathfrak{I}_{\mathrm{H}}}(A, B) \cong \mathrm{Ext}^n_{\mathfrak{C}}\bigl(\mathrm{H}_*(A), \mathrm{H}_*(B)\bigr).$$

Let $F\colon \mathfrak{C} \to \mathfrak{C}'$ *be some covariant additive functor and define*

$$\bar{F}\colon \mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p) \to \mathrm{Ho}(\mathfrak{C}'; \mathbb{Z}/p)$$

by applying F *entrywise. Then* $\mathbb{L}_n \bar{F}(A) \cong \mathbb{L}_n F\bigl(\mathrm{H}_*(A)\bigr)$ *for all* $n \in \mathbb{N}$*. Similarly, we have* $\mathbb{R}^n \bar{F}(A) \cong \mathbb{R}^n F\bigl(\mathrm{H}_*(A)\bigr)$ *if* F *is a contravariant functor.*

Proof. The category $\mathfrak{C}^{\mathbb{Z}/p}$ has enough projective objects by assumption. We have already seen in Example 103 that $\mathrm{H}_*^\dagger$ is defined on $\mathfrak{P}\mathfrak{C}^{\mathbb{Z}/p}$; this functor is denoted by j in Example 103. It is clear that $\mathrm{H}_* \circ j(A) \cong A$ for all $A \in\in \mathfrak{C}^{\mathbb{Z}/p}$. Now Theorem 106 shows that H_* is universal. We do not need idempotent morphisms in $\mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p)$ to split by Remark 109. □

Remark 111. Since the universal $\mathfrak{I}$-exact functor is essentially unique, the universality of $\mathrm{H}_*\colon \mathfrak{Der}(\mathfrak{C}; \mathbb{Z}/p) \to \mathfrak{C}^{\mathbb{Z}/p}$ means that we can recover this functor and hence the stable Abelian category $\mathfrak{C}^{\mathbb{Z}/p}$ from the ideal $\mathfrak{I}_\mathrm{H} \subseteq \mathfrak{Der}(\mathfrak{C}; \mathbb{Z}/p)$. That is, the ideal $\mathfrak{I}_\mathrm{H}$ and the functor $\mathrm{H}_*\colon \mathfrak{Der}(\mathfrak{C}; \mathbb{Z}/p) \to \mathfrak{C}^{\mathbb{Z}/p}$ contain exactly the same amount of information.

For instance, if we forget the precise category $\mathfrak{C}$ by composing H_* with some *faithful* functor $\mathfrak{C} \to \mathfrak{C}'$, then the resulting homology functor $\mathrm{Ho}(\mathfrak{C}; \mathbb{Z}/p) \to \mathfrak{C}'$ still has kernel $\mathfrak{I}_\mathrm{H}$. We can recover $\mathfrak{C}^{\mathbb{Z}/p}$ by passing to the universal $\mathfrak{I}$-exact functor.

We compare this with the situation for truncation structures ([2]). These cannot exist for periodic categories such as $\mathfrak{Der}(\mathfrak{C}; \mathbb{Z}/p)$ for $p \geq 1$. Given the standard truncation structure on $\mathfrak{Der}(\mathfrak{C})$, we can recover the Abelian category $\mathfrak{C}$ as its core; we also get back the homology functors $H_n\colon \mathfrak{Der}(\mathfrak{C}) \to \mathfrak{C}$ for all $n \in \mathbb{Z}$. Conversely, the functor $\mathrm{H}_*\colon \mathfrak{Der}(\mathfrak{C}) \to \mathfrak{C}^{\mathbb{Z}}$ together with the grading on $\mathfrak{C}^{\mathbb{Z}}$ tells us what it means for a chain complex to be exact in degrees ≥ 0 or ≤ 0 and thus determines the truncation structure. Hence the standard truncation structure on $\mathfrak{Der}(\mathfrak{C})$ contains the same amount of information as the functor $\mathrm{H}_*\colon \mathfrak{Der}(\mathfrak{C}) \to \mathfrak{C}^{\mathbb{Z}}$ together with the grading on $\mathfrak{C}^{\mathbb{Z}}$.

6. Universal Coefficient Theorems

First we study the ideal $\mathfrak{I}_\mathrm{K} := \ker \mathrm{K}_* \subseteq \mathrm{KK}$ of Example 51. We complete our analysis of this example and explain the Universal Coefficient Theorem for KK in our framework. We call $\mathfrak{I}_\mathrm{K}$-projective objects and $\mathfrak{I}_\mathrm{K}$-exact functors briefly K-*projective* and K-*exact* and let $\mathfrak{P}_\mathrm{K} \subseteq \mathrm{KK}$ be the class of K-projective objects.

Let $\mathfrak{Ab}_\mathrm{c}^{\mathbb{Z}/2} \subseteq \mathfrak{Ab}^{\mathbb{Z}/2}$ be the full subcategory of *countable* $\mathbb{Z}/2$-graded Abelian groups. Since the K-theory of a separable C^*-algebra is countable, we may view K_* as a stable homological functor $\mathrm{K}_*\colon \mathrm{KK} \to \mathfrak{Ab}_\mathrm{c}^{\mathbb{Z}/2}$.

Theorem 112. *There are enough* K*-projective objects in* KK*, and the universal* K*-exact functor is* $\mathrm{K}_*\colon \mathrm{KK} \to \mathfrak{Ab}_\mathrm{c}^{\mathbb{Z}/2}$*. It restricts to an equivalence of categories between* $\mathfrak{P}_\mathrm{K}$ *and the full subcategory* $\mathfrak{Ab}_\mathrm{fc}^{\mathbb{Z}/2} \subseteq \mathfrak{Ab}_\mathrm{c}^{\mathbb{Z}/2}$ *of* $\mathbb{Z}/2$*-graded countable free Abelian groups. A separable* C^**-algebra belongs to* $\mathfrak{P}_\mathrm{K}$ *if and only if it is* KK*-equivalent to* $\bigoplus_{i\in I_0} \mathbb{C} \oplus \bigoplus_{i\in I_1} C_0(\mathbb{R})$ *where the sets* I_0, I_1 *are at most countable.*

If $A \in\in \mathrm{KK}$*, then* K_* *maps isomorphism classes of* K*-projective resolutions of* A *in* $\mathfrak{T}$ *bijectively to isomorphism classes of free resolutions of* $\mathrm{K}_*(A)$*. We have*

$$\mathrm{Ext}^n_{\mathrm{KK},\mathfrak{I}_\mathrm{K}}(A,B) \cong \begin{cases} \mathrm{Hom}_{\mathfrak{Ab}^{\mathbb{Z}/2}}\big(\mathrm{K}_*(A), \mathrm{K}_*(B)\big) & \text{for } n = 0;\\ \mathrm{Ext}^1_{\mathfrak{Ab}^{\mathbb{Z}/2}}\big(\mathrm{K}_*(A), \mathrm{K}_*(B)\big) & \text{for } n = 1;\\ 0 & \text{for } n \geq 2.\end{cases}$$

Let $F\colon \mathrm{KK} \to \mathfrak{C}$ be some covariant additive functor; then there is a unique right-exact functor $\bar{F}\colon \mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{c}} \to \mathfrak{C}$ with $\bar{F} \circ \mathrm{K}_* = F$. We have $\mathbb{L}_n F = (\mathbb{L}_n \bar{F}) \circ \mathrm{K}_*$ for all $n \in \mathbb{N}$; this vanishes for $n \geq 2$. Similar assertions hold for contravariant functors.

Proof. Notice that $\mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{c}} \subseteq \mathfrak{Ab}^{\mathbb{Z}/2}$ is an Abelian category. We shall denote objects of $\mathfrak{Ab}^{\mathbb{Z}/2}$ by pairs (A_0, A_1) of Abelian groups. By definition, $(A_0, A_1) \in\in \mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{fc}}$ if and only if A_0 and A_1 are countable free Abelian groups, that is, they are of the form $A_0 = \mathbb{Z}[I_0]$ and $A_1 = \mathbb{Z}[I_1]$ for at most countable sets I_0, I_1. It is well-known that any Abelian group is a quotient of a free Abelian group and that subgroups of free Abelian groups are again free. Moreover, free Abelian groups are projective. Hence $\mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{fc}}$ is the subcategory of projective objects in $\mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{c}}$ and any object $G \in\in \mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{c}}$ has a projective resolution of the form $0 \to F_1 \to F_0 \twoheadrightarrow G$ with $F_0, F_1 \in\in \mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{fc}}$. This implies that derived functors on $\mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{c}}$ only occur in dimensions 1 and 0.

As in Example 102, we see that $\mathrm{K}^\dagger_*$ is defined on $\mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{fc}}$ and satisfies

$$\mathrm{K}^\dagger_*\bigl(\mathbb{Z}[I_0], \mathbb{Z}[I_1]\bigr) \cong \bigoplus_{i \in I_0} \mathbb{C} \oplus \bigoplus_{i \in I_1} \mathcal{C}_0(\mathbb{R})$$

if I_0, I_1 are countable. We also have $\mathrm{K}_* \circ \mathrm{K}^\dagger_*\bigl(\mathbb{Z}[I_0], \mathbb{Z}[I_1]\bigr) \cong \bigl(\mathbb{Z}[I_0], \mathbb{Z}[I_1]\bigr)$, so that the hypotheses of Theorem 106 are satisfied. Hence there are enough K-projective objects and K_* is universal. The remaining assertions follow from Theorem 108 and our detailed knowledge of the homological algebra in $\mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{c}}$. $\square$

Example 113. Consider the stable homological functor

$$F\colon \mathrm{KK} \to \mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{c}}, \qquad A \mapsto \mathrm{K}_*(A \otimes B)$$

for some $B \in\in \mathrm{KK}$, where $\otimes$ denotes, say, the spatial C^*-tensor product. We claim that the associated right-exact functor $\mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{c}} \to \mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{c}}$ is

$$\bar{F}\colon \mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{c}} \to \mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{c}}, \qquad G \mapsto G \otimes \mathrm{K}_*(B).$$

It is easy to check $F \circ \mathrm{K}^\dagger_*(G) \cong G \otimes \mathrm{K}_*(B) \cong \bar{F}(G)$ for $G \in\in \mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{fc}}$. Since the functor $G \mapsto G \otimes \mathrm{K}_*(B)$ is right-exact and agrees with $\bar{F}$ on projective objects, we get $\bar{F}(G) = G \otimes \mathrm{K}_*(B)$ for all $G \in\in \mathfrak{Ab}^{\mathbb{Z}/2}_{\mathrm{c}}$. Hence the derived functors of F are

$$\mathbb{L}_n F(A) \cong \begin{cases} \mathrm{K}_*(A) \otimes \mathrm{K}_*(B) & \text{for } n = 0; \\ \mathrm{Tor}^1\bigl(\mathrm{K}_*(A), \mathrm{K}_*(B)\bigr) & \text{for } n = 1; \\ 0 & \text{for } n \geq 2. \end{cases}$$

Here we use the same graded version of Tor as in the Künneth Theorem ([4]).

Example 114. Consider the stable homological functor

$$F\colon \mathrm{KK} \to \mathfrak{Ab}^{\mathbb{Z}/2}, \qquad B \mapsto \mathrm{KK}_*(A, B)$$

for some $A \in\in \mathrm{KK}$. We suppose that A is a *compact* object of KK, that is, the functor F commutes with direct sums. Then $\mathrm{KK}_*\bigl(A, \mathrm{K}_*^\dagger(G)\bigr) \cong \mathrm{KK}_*(A, \mathbb{C}) \otimes G$ for all $G \in\in \mathfrak{Ab}_{\mathrm{fc}}^{\mathbb{Z}/2}$ because this holds for $G = (\mathbb{Z}, 0)$ and is inherited by suspensions and direct sums. Now we get $\bar{F}(G) \cong \mathrm{KK}_*(A, \mathbb{C}) \otimes G$ for all $G \in\in \mathfrak{Ab}_{\mathrm{c}}^{\mathbb{Z}/2}$ as in Example 113. Therefore,

$$\mathbb{L}_n F(B) \cong \begin{cases} \mathrm{KK}_*(A, \mathbb{C}) \otimes \mathrm{K}_*(B) & \text{for } n = 0; \\ \mathrm{Tor}^1\bigl(\mathrm{KK}_*(A, \mathbb{C}), \mathrm{K}_*(B)\bigr) & \text{for } n = 1; \\ 0 & \text{for } n \geq 2. \end{cases}$$

Generalising Examples 113 and 114, we have $\bar{F}(G) \cong F(\mathbb{C}) \otimes G$ and hence

$$\mathbb{L}_n F(B) \cong \begin{cases} F(\mathbb{C}) \otimes \mathrm{K}_*(B) & \text{for } n = 0, \\ \mathrm{Tor}^1\bigl(F(\mathbb{C}), \mathrm{K}_*(B)\bigr) & \text{for } n = 1, \end{cases}$$

for any covariant functor $F\colon \mathrm{KK} \to \mathfrak{C}$ that commutes with direct sums.

Similarly, if $F\colon \mathrm{KK}^{\mathrm{op}} \to \mathfrak{C}$ is contravariant and maps direct sums to direct products, then $\bar{F}(G) \cong \mathrm{Hom}(G, F(\mathbb{C}))$ and

$$\mathbb{R}^n F(B) \cong \begin{cases} \mathrm{Hom}\bigl(\mathrm{K}_*(B), F(\mathbb{C})\bigr) & \text{for } n = 0, \\ \mathrm{Ext}^1\bigl(\mathrm{K}_*(B), F(\mathbb{C})\bigr) & \text{for } n = 1. \end{cases}$$

The description of $\mathrm{Ext}^n_{\mathrm{KK}, \mathfrak{I}_{\mathrm{K}}}$ in Theorem 112 is a special case of this.

6.1. Universal Coefficient Theorem in the hereditary case. In general, we need spectral sequences in order to relate the derived functors $\mathbb{L}_n F$ back to F. We will discuss this in a sequel to this article. Here we only treat the simple case where we have projective resolutions of length 1. The following universal coefficient theorem is very similar to but slightly more general than [3, Theorem 4.27] because we do not require *all* $\mathfrak{I}$-equivalences to be invertible.

Theorem 115. *Let $\mathfrak{T}$ be a triangulated category and let $\mathfrak{I} \subseteq \mathfrak{T}$ be a homological ideal. Let $A \in\in \mathfrak{T}$ have an $\mathfrak{I}$-projective resolution of length* 1*. Suppose also that $\mathfrak{T}(A, B) = 0$ for all $\mathfrak{I}$-contractible B. Let $F\colon \mathfrak{T} \to \mathfrak{C}$ be a homological functor, $\tilde{F}\colon \mathfrak{T}^{\mathrm{op}} \to \mathfrak{C}$ a cohomological functor, and $B \in\in \mathfrak{T}$. Then there are natural short exact sequences*

$$0 \to \mathbb{L}_0 F_*(A) \to F_*(A) \to \mathbb{L}_1 F_{*-1}(A) \to 0,$$
$$0 \to \mathbb{R}^1 \tilde{F}^{*-1}(A) \to \tilde{F}^*(A) \to \mathbb{R}^0 \tilde{F}^*(A) \to 0,$$
$$0 \to \mathrm{Ext}^1_{\mathfrak{T}, \mathfrak{I}}(\Sigma A, B) \to \mathfrak{T}(A, B) \to \mathrm{Ext}^0_{\mathfrak{T}, \mathfrak{I}}(A, B) \to 0.$$

Example 116. For the ideal $\mathfrak{I}_{\mathrm{K}} \subseteq \mathrm{KK}$, any object has a K-projective resolution of length 1 by Theorem 112. The other hypothesis of Theorem 115 holds if and only if A satisfies the Universal Coefficient Theorem (UCT). The UCT for $\mathrm{KK}(A, B)$

predicts $\mathrm{KK}(A,B)=0$ if $\mathrm{K}_*(B)=0$. Conversely, if this is the case, then Theorem 115 applies, and our description of $\mathrm{Ext}_{\mathrm{KK},\mathfrak{I}_\mathrm{K}}$ in Theorem 112 yields the UCT for $\mathrm{KK}(A,B)$ for all B. This yields our claim.

Thus the UCT for $\mathrm{KK}(A,B)$ is a special of Theorem 115. In the situations of Examples 113 and 114, we get the familiar Künneth Theorems for $\mathrm{K}_*(A\otimes B)$ and $\mathrm{KK}_*(A,B)$. These arguments are very similar to the original proofs (see [4]). Our machinery allows us to treat other situations in a similar fashion.

Proof of Theorem 115. We only write down the proof for homological functors. The cohomological case is dual and contains $\mathfrak{T}(\_,B)$ as a special case.

Let $0\to P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} A$ be an $\mathfrak{I}$-projective resolution of length 1 and view it as an $\mathfrak{I}$-exact chain complex of length 3. Lemma 78 yields a commuting diagram

$$\begin{tikzcd} P_1 \arrow[r, "\delta_1"] \arrow[d, equal] & P_0 \arrow[r, "\tilde\delta_0"] \arrow[d, equal] & \tilde A \arrow[d, dotted, "\alpha"] \\ P_1 \arrow[r, "\delta_1"] & P_0 \arrow[r, "\delta_0"] & A, \end{tikzcd}$$

such that the top row is part of an $\mathfrak{I}$-exact exact triangle $P_1\to P_0\to\tilde A\to\Sigma P_1$ and α is an $\mathfrak{I}$-equivalence. We claim that α is an isomorphism in $\mathfrak{T}$.

We embed α in an exact triangle $\Sigma^{-1}B\to\tilde A\xrightarrow{\alpha}A\xrightarrow{\beta}B$. Lemma 70 shows that B is $\mathfrak{I}$-contractible because α is an $\mathfrak{I}$-equivalence. Hence $\mathfrak{T}(A,B)=0$ by our assumption on A. This forces $\beta=0$, so that our exact triangle splits: $A\cong\tilde A\oplus B$. Then $\mathfrak{T}(B,B)\subseteq\mathfrak{T}(A,B)$ vanishes as well, so that $B\cong 0$. Thus α is invertible.

We get an exact triangle in $\mathfrak{T}$ of the form $P_1\xrightarrow{\delta_1}P_0\xrightarrow{\delta_0}A\to\Sigma P_1$ because any triangle isomorphic to an exact one is itself exact.

Now we apply F. Since F is homological, we get a long exact sequence

$$\cdots\to F_*(P_1)\xrightarrow{F_*(\delta_1)}F_*(P_0)\to F_*(A)\to F_{*-1}(P_1)\xrightarrow{F_{*-1}(\delta_1)}F_{*-1}(P_0)\to\cdots.$$

We cut this into short exact sequences of the form

$$\operatorname{coker}\bigl(F_*(\delta_1)\bigr)\rightarrowtail F_*(A)\twoheadrightarrow\ker\bigl(F_{*-1}(\delta_1)\bigr).$$

Since $\operatorname{coker}F_*(\delta_1)=\mathbb{L}_0F_*(A)$ and $\ker F_*(\delta_1)=\mathbb{L}_1F_*(A)$, we get the desired exact sequence. The map $\mathbb{L}_0F_*(A)\to F_*(A)$ is the canonical map induced by δ_0. The other map $F_*(A)\to\mathbb{L}_1F_{*-1}(A)$ is natural for all morphisms between objects with an $\mathfrak{I}$-projective resolution of length 1 by Proposition 93. □

The proof shows that — in the situation of Theorem 115 — we have

$$\mathrm{Ext}^0_{\mathfrak{T},\mathfrak{I}}(A,B)\cong\mathfrak{T}/\mathfrak{I}(A,B),\qquad \mathrm{Ext}^1_{\mathfrak{T},\mathfrak{I}}(A,B)\cong\mathfrak{I}(A,\Sigma B).$$

More generally, we can construct a natural map $\mathfrak{I}(A,\Sigma B)\to\mathrm{Ext}^1_{\mathfrak{T},\mathfrak{I}}(A,B)$ for any homological ideal, using the $\mathfrak{I}$-universal homological functor $F\colon\mathfrak{T}\to\mathfrak{C}$. We embed $f\in\mathfrak{I}(A,\Sigma B)$ in an exact triangle $B\to C\to A\to\Sigma B$. We get an extension

$$\bigl[F(B)\rightarrowtail F(C)\twoheadrightarrow F(A)\bigr]\Subset\mathrm{Ext}^1_{\mathfrak{C}}\bigl(F(A),F(B)\bigr)$$

because this triangle is $\mathfrak{I}$-exact. This class $\kappa(f)$ in $\mathrm{Ext}^1_{\mathfrak{C}}\big(F(A),F(B)\big)$ does not depend on auxiliary choices because the exact triangle $B \to C \to A \to \Sigma B$ is unique up to isomorphism. Theorem 108 yields $\mathrm{Ext}^1_{\mathfrak{T},\mathfrak{I}}(A,B) \cong \mathrm{Ext}^1_{\mathfrak{C}}\big(F(A),F(B)\big)$ because F is universal. Hence we get a natural map

$$\kappa\colon \mathfrak{I}(A,\Sigma B) \to \mathrm{Ext}^1_{\mathfrak{T},\mathfrak{I}}(A,B).$$

We may view κ as a secondary invariant generated by the canonical map

$$\mathfrak{T}(A,B) \to \mathrm{Ext}^0_{\mathfrak{T},\mathfrak{I}}(A,B).$$

For the ideal $\mathfrak{I}_{\mathrm{K}}$, we get the same map κ as in Example 51.

An Abelian category with enough projective objects is called *hereditary* if any subobject of a projective object is again projective. Equivalently, any object has a projective resolution of length 1. This motivates the following definition:

Definition 117. A homological ideal $\mathfrak{I}$ in a triangulated category $\mathfrak{T}$ is called *hereditary* if any object of $\mathfrak{T}$ has a projective resolution of length 1.

If $\mathfrak{I}$ is hereditary and if $\mathfrak{I}$-equivalences are invertible, then Theorem 115 applies to all $A \in\in \mathfrak{T}$ (and vice versa).

Example 118. As another example, consider the ideal $\mathcal{VC} \subseteq \mathrm{KK}^{\mathbb{Z}}$ for the group $\mathbb{Z}$. Here the family of subgroups only contains the trivial one. Theorem 36 shows that Theorem 115 applies to all objects of $\mathrm{KK}^{\mathbb{Z}}$. The resulting extensions are equivalent to the Pimsner–Voiculescu exact sequence. To see this, first cut the latter into two short exact sequences involving the kernel and cokernel of $\alpha_* - 1$. Then notice that the latter coincide with the group homology of the induced action of $\mathbb{Z}$ on $\mathrm{K}_*(A)$.

6.2. The Adams resolution.

Let $\mathfrak{I}$ be an ideal in a triangulated category and let $\mathfrak{P}$ be its class of projective objects. We assume that $\mathfrak{I}$ has enough projective objects. Let $A \in\in \mathfrak{T}$. Write $A = B_0$ and let $\Sigma B_1 \to P_0 \to B_0 \to B_1$ be a one-step $\mathfrak{I}$-projective resolution of $A = B_0$. Similarly, let $\Sigma B_2 \to P_1 \to B_1 \to B_2$ be a one-step $\mathfrak{I}$-projective resolution of B_1. Repeating this process we obtain objects $B_n \in\in \mathfrak{T}$, $P_n \in \mathfrak{P}$ for $n \in \mathbb{N}$ with $B_0 = A$ and morphisms $\beta_n^{n+1} \in \mathfrak{I}(B_n, B_{n+1})$, $\pi_n \in \mathfrak{T}(P_n, B_n)$, $\alpha_n \in \mathfrak{T}_1(B_{n+1}, P_n)$ that are part of distinguished triangles

$$\Sigma B_{n+1} \xrightarrow{\alpha_n} P_n \xrightarrow{\pi_n} B_n \xrightarrow{\beta_n^{n+1}} B_{n+1} \tag{119}$$

for all $n \in \mathbb{N}$. Thus the maps π_n are $\mathfrak{I}$-epic for all $n \in \mathbb{N}$. We can assemble these data in a diagram

$$\begin{tikzcd}
A = B_0 \arrow[rr, "\beta_0^1"] & & B_1 \arrow[rr, "\beta_1^2"] \arrow[dl, dotted, "\alpha_0"] & & B_2 \arrow[rr, "\beta_2^3"] \arrow[dl, dotted, "\alpha_1"] & & B_3 \arrow[rr] \arrow[dl, dotted, "\alpha_2"] & & \cdots \\
& P_0 \arrow[ul, "\pi_0"] & & P_1 \arrow[ul, "\pi_1"] & & P_2 \arrow[ul, "\pi_2"] & & \cdots \arrow[ul] &
\end{tikzcd}$$

called an *Adams resolution of A*. We also let

$$\beta_m^n := \beta_{n-1}^n \circ \cdots \circ \beta_m^{m+1} \colon B_m \to B_n$$

for all $n \geq m$ (by convention, $\beta_m^m = \mathrm{id}$). We have $\beta_m^n \in \mathfrak{I}^{n-m}(B_m, B_n)$, that is, β_m^n is a product of $n-m$ factors in $\mathfrak{I}$.

We are particularly interested in the maps $\beta^n := \beta_0^n \colon A \to B_n$ for $n \in \mathbb{N}$. Taking a mapping cone of β^n we obtain a distinguished triangle

$$\Sigma B_n \xrightarrow{\sigma_n} C_n \xrightarrow{\rho_n} A \xrightarrow{\beta^n} B_n \tag{120}$$

for each n, which is determined uniquely by β^n up to non-canonical isomorphism. Applying the octahedral axiom (TR4) of [35] or, equivalently, [28, Proposition 1.4.12], we get maps $\gamma_n^{n+1} \colon C_n \to C_{n+1}$ and $\nu_n \colon C_{n+1} \to P_n$ that are part of morphisms of distinguished triangles

$$\begin{tikzcd}
\Sigma B_n \arrow[r, "\sigma_n"] \arrow[d, "\Sigma\beta_n^{n+1}"] & C_n \arrow[r, "\rho_n"] \arrow[d, "\gamma_n^{n+1}"] & A \arrow[r, "\beta^n"] \arrow[d, equal] & B_n \arrow[d, "\beta_n^{n+1}"] \\
\Sigma B_{n+1} \arrow[r, "\sigma_{n+1}"] \arrow[d, equal] & C_{n+1} \arrow[r, "\rho_{n+1}"] \arrow[d, "\nu_n"] & A \arrow[r, "\beta^{n+1}"] \arrow[d, "\beta^n"] & B_{n+1} \arrow[d, equal] \\
\Sigma B_{n+1} \arrow[r, "\alpha_{n+1}"] & P_n \arrow[r, "\pi_n"] & B_n \arrow[r, "\beta_n^{n+1}"] & B_{n+1}
\end{tikzcd} \tag{121}$$

and of a distinguished triangle

$$\Sigma P_n \xrightarrow{\sigma_n \circ \Sigma\pi_n} C_n \xrightarrow{\gamma_n^{n+1}} C_{n+1} \xrightarrow{\nu_n} P_n. \tag{122}$$

It follows by induction on n that $C_n \in \mathfrak{P}_n$ for all $n \in \mathbb{N}$. Since $\beta^n \in \mathfrak{I}^n$ by construction, the distinguished triangle (120) shows that $\rho_n \colon C_n \to A$ is a one-step $\mathfrak{I}^n$-projective resolution.

6.3. Spectral sequences from the Adams resolution. The Adams resolution gives rise to an exact couple and thus to a spectral sequence in a canonical way (our reference for exact couples and spectral sequences is [20]). We let $F \colon \mathfrak{T} \to \mathfrak{Ab}$ be a contravariant cohomological functor and define $F^n(A) := F(\Sigma^n A)$ for $n \in \mathbb{Z}$. We define $\mathbb{Z} \times \mathbb{N}$-graded Abelian groups

$$D_1^{pq} := F^{p+q}(B_p), \qquad E_1^{pq} := F^{p+q}(P_p),$$

and homomorphisms

$$\begin{aligned}
i_1^{pq} &:= F^*(\beta_{p-1}^p) \colon D_1^{p,q} \to D_1^{p-1,q+1}, \\
j_1^{pq} &:= F^*(\pi_p) \colon D_1^{p,q} \to E_1^{p,q}, \\
k_1^{pq} &:= F^*(\alpha_p) \colon E_1^{p,q} \to D_1^{p+1,q}
\end{aligned}$$

of bidegree

$$\deg i_1 = (-1,1), \qquad \deg j_1 = (0,0), \qquad \deg k_1 = (1,0).$$

Since F is cohomological, we get long exact sequences for the distinguished triangles (119). This means that $(D_1, E_1, i_1, j_1, k_1)$ is an exact couple. As in [20, Section XI.5] we form the derived exact couples $(D_r, E_r, i_r, j_r, k_r)$ for $r \in \mathbb{N}_{\geq 2}$ and let $d_r = j_r k_r \colon E_r \to E_r$. The map d_r has bidegree $(r, 1-r)$ and the data (E_r, d_r) define a cohomological spectral sequence.

Now consider instead a covariant homological functor $F \colon \mathfrak{T} \to \mathfrak{Ab}$.

Let $F_n(A) := F(\Sigma^n A)$ for $n \in \mathbb{Z}$ and define $\mathbb{Z} \times \mathbb{N}$-graded ABELian groups

$$D^1_{pq} := F_{p+q}(B_p), \qquad E^1_{pq} := F_{p+q}(P_p)$$

and homomorphisms

$$\begin{aligned}
i^1_{pq} &:= F_*(\beta_p^{p+1}) \colon D^1_{p,q} \to D^1_{p+1,q-1},\\
j^1_{pq} &:= F_*(\alpha_p) \colon D^1_{p,q} \to E^1_{p-1,q}.\\
k^1_{pq} &:= F_*(\pi_p) \colon E^1_{p,q} \to D^1_{p,q}
\end{aligned}$$

of bidegree

$$\deg i^1 = (1,-1), \qquad \deg j^1 = (-1,0), \qquad \deg k^1 = (0,0).$$

This is an exact couple because F is homological. We form derived exact couples $(D^r, E^r, i^r, j^r, k^r)$ for $r \in \mathbb{N}_{\geq 2}$ and let $d^r = j^r k^r$. This map has bidegree $(-r, r-1)$, so that (E^r, d^r) is a homological spectral sequence.

The boundary maps d^1 and d_1 in the above spectral sequences are induced by the composition

$$\delta_n := \alpha_n \circ \Sigma\pi_{n+1} \colon \Sigma P_{n+1} \to P_n.$$

Letting $\delta_{-1} := \pi_0 \colon P_0 \to A$, we obtain a chain complex

$$A \xleftarrow{\delta_0} P_0 \xleftarrow{\delta_1} \Sigma P_1 \xleftarrow{\delta_2} \Sigma^2 P_2 \xleftarrow{\delta_3} \Sigma^3 P_3 \xleftarrow{\delta_4} \cdots$$

in $\mathfrak{T}$. This is an $\mathfrak{I}$-projective resolution of A.

Let $F \colon \mathfrak{T} \to \mathfrak{Ab}$ be a covariant functor. By construction, we have

$$E^2_{pq} \cong H_p\bigl(F_q(\Sigma^\bullet P_\bullet, \delta_\bullet)\bigr) \cong \mathbb{L}_p F_{p+q}(A).$$

Thus the second tableau of our spectral sequence comprises the derived functors of suspensions of F.

We do not analyse the convergence of the above spectral sequence here in detail. In general, we cannot hope for convergence towards $F(A)$ itself because the derived functors vanish if A is $\mathfrak{I}$-contractible, but $F(A)$ need not vanish. Thus we should replace F by $\mathbb{L}F$ right away. Under mild conditions, the spectral sequence converges towards $\mathbb{L}F$.

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