Example. G a finite group

A a $G - C^*$ algebra

$dg =$ each $g \in G$ has mass 1.

$$C_r^*(G, A) = \left\{ \sum_{\gamma \in G} a_\gamma[\gamma] \;\middle|\; a_\gamma \in A \right\}$$

$$\left(\sum_{\gamma \in G} a_\gamma[\gamma] \right) + \left(\sum_{\gamma \in G} b_\gamma[\gamma] \right) = \sum_{\gamma \in G} (a_\gamma + b_\gamma)[\gamma]$$

$$(a_\alpha[\alpha])(b_\beta[\beta]) = a_\alpha(\alpha b_\beta)[\alpha\beta]$$

$$\left(\sum_{\gamma \in G} a_\gamma[\gamma] \right)^* = \sum_{\gamma \in G} (\gamma^{-1}(a_\gamma^*))[\gamma^{-1}]$$

$$\left(\sum_{\gamma \in G} a_\gamma[\gamma] \right) \lambda = \sum_{\gamma \in G} (a_\gamma \lambda)[\gamma] \qquad \lambda \in \mathbb{C}$$

Notation.

Let X be a locally compact G-space

$$G \times X \longrightarrow X$$

$C_0(X)$ is a $G - C^*$ algebra

$$(gf)(x) = f(g^{-1}x) \qquad \begin{array}{l} f \in C_0(X) \\ g \in G \\ x \in X \end{array}$$

$C_r^*(G, C_0(X))$ will be denoted $C_r^*(G, X)$.

$K_j C_r^*(G, X) = ?$

If G is compact $K_j C_r^*(G, X)$ is the Atiyah-Segal group $K_G^j(X)$ $j = 0, 1$. Hence for G non-compact $K_j C_r^*(G, X)$ is the natural extension of the Atiyah-Segal theory to the case when G is non-compact.

Notation If X is a locally compact Hausdorff G-space

$$C_r^*(G, C_0(X)) = C_r^*(G, X)$$

Terminology G-compact

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The quotient space $G\backslash X$ is compact

Remark Let X be a proper G-compact G-space. Then a G-equivariant $\mathbb{C}$ vector bundle E on X determines an element

$$[E] \in K_0 C_r^*(G,X)$$

Theorem (W. Lück and B. Oliver)

If Γ is a (countable) discrete group and X is a proper Γ-compact Γ-space, then

$$K_0 C_r^*(\Gamma, X) = \text{Grothendieck group of } \Gamma\text{-equivariant } \mathbb{C} \text{ vector bundles on } X$$

Remark Let X be a proper G-compact G-space. Set $\mathbb{1} = X \times \mathbb{C}$

$$\mathbb{1} = X \times \mathbb{C}$$

$$g(x,\lambda) = (gx, \lambda) \qquad \begin{array}{l} g \in G \\ x \in X \\ \lambda \in \mathbb{C} \end{array}$$

Then $[\mathbb{1}] \in K_0 C_r^*(G, X)$

Push-forward of Hilbert modules

A, B $\quad$ C^* algebra

$\varphi : A \longrightarrow B$ $*$-homomorphism

$\mathcal{H}$ $\quad$ Hilbert A-module

Shall define $\mathcal{H} \underset{A}{\otimes} B$ which will be a Hilbert B-module

First form the algebraic tensor product $\mathcal{H} \underset{A}{\odot} B$

$$\mathcal{H} \underset{A}{\odot} B = \mathcal{H} \overset{(\text{alg})}{\underset{A}{\otimes}} B$$

$\mathcal{H} \underset{A}{\odot} B$ is a (right) B-module

$$(h \otimes b)b' = h \otimes b\,b' \qquad \begin{array}{l} h \in \mathcal{H} \\ b, b' \in B \end{array}$$

Define a B-valued inner product $\langle\ ,\ \rangle$ on $\mathcal{H} \underset{A}{\odot} B$ by:

$$\langle h \otimes b,\ h' \otimes b' \rangle = b^* \varphi \langle h, h' \rangle b'$$

Set $\mathcal{N} = \{\xi \in \mathcal{H} \underset{A}{\odot} B \mid \langle \xi, \xi \rangle = 0\}$

$\mathcal{N}$ is a B-sub-module of $\mathcal{H} \underset{A}{\odot} B$.

$(\mathcal{H} \underset{A}{\odot} B)/\mathcal{N}$ is a pre-Hilbert B-module.

Definition. $\mathcal{H} \underset{A}{\otimes} B$ is the Hilbert B-module obtained by completing $\mathcal{H} \underset{A}{\odot} B/\mathcal{N}$.

Topic 8: Homotopy made precise and KK

A $\quad$ C^* algebra

$\mathcal{H}$ $\quad$ Hilbert A-module

$\mathcal{L}(\mathcal{H})$

$u, v \in \mathcal{H}$ $\qquad$ $\theta_{u,v} \in \mathcal{L}(\mathcal{H})$ $\qquad$ $\theta_{u,v}(\xi) = u\langle v, \xi\rangle$

$$\theta^*_{u,v} = \theta_{v,u}$$

The $\theta_{u,v}$ are the "rank one" operators on $\mathcal{H}$

A "finite rank" operator on $\mathcal{H}$ is any $T \in \mathcal{L}(H)$ such that T is a finite sum of $\theta_{u,v}$.

$$T = \theta_{u_1,v_1} + \theta_{u_2,v_2} + \cdots + \theta_{u_n,v_n}$$

$\mathcal{K}(\mathcal{H}) = \overline{\{\text{Finite rank operators}\}}$

Closure is taken in the norm of $\mathcal{L}(\mathcal{H})$

$\mathcal{K}(\mathcal{H})$ is an ideal in $\mathcal{L}(\mathcal{H})$

$\mathcal{H}$ is countably generated if in $\mathcal{H}$ there is a countable (or finite) set such that the A -module generated by this set is dense in $\mathcal{H}$.

Let $\mathcal{H}_0, \mathcal{H}_1$ be two Hilbert A-modules. $\mathcal{H}_0$ and $\mathcal{H}_1$ are isomorphic if $\exists$ an A-module isomorphism $\Phi : \mathcal{H}_0 \to H_1$ with

$$\langle u, v \rangle_0 = \langle \Phi u, \Phi v \rangle_1 \qquad \forall\, u, v \in \mathcal{H}_0$$