Equivariant K homology even case

A G-C^* algebra

$\mathcal{E}_G^0(A) = \{(H, \psi, T, \pi)\}$

(H, ψ, π) is a covariant representation of A

$T \in \mathcal{L}(H)$

$$\begin{Bmatrix} \pi(g)T - T\pi(g) \in \mathcal{K}(H) \quad \forall\, g \in G \\ \psi(a)T - T\psi(a) \in \mathcal{K}(H) \quad \forall\, a \in A \\ \psi(a)(I - T^*T) \in \mathcal{K}(H) \quad \forall\, a \in A \\ \psi(a)(I - TT^*) \in \mathcal{K}(H) \quad \forall\, a \in A \end{Bmatrix}$$

$A, B \quad G\text{-}C^*$ algebras

$\varphi : A \to B \quad G$-equivariant $\quad *$-homomorphism

$\varphi^* : \mathcal{E}_G^0(B) \to \mathcal{E}_G^0(A)$

$$(H, \psi, T, \pi) \mapsto (H, \psi \circ \varphi, T, \pi)$$

$KK_G^0(A, \mathbb{C}) := \mathcal{E}_G^0(A)/ \sim$

$\sim$ = "homotopy"

"homotopy" will be made precise later

addition in $KK_G^0(A, \mathbb{C})$ is direct sum

$$(H, \psi, T, \pi) + (H', \psi', T', \pi')$$
$$= (H \oplus H', \psi \oplus \psi', T \oplus T', \pi \oplus \pi')$$
$$-(H, \psi, T, \pi) = (H, \psi, T^*, \pi)$$

Topic 7: Hilbert modules

A $\qquad$ C^* algebra

Definition. $a \in A$ is positive (Notation: $a \geqq 0$) iff $\exists b \in A$ with $b^*b = a$.

Definition. A pre-Hilbert A-module is a right A-module with a given A-valued inner product $\langle\ ,\ \rangle$ such that:

$$\langle u, v_1 + v_2 \rangle = \langle u, v_1 \rangle + \langle u, v_2 \rangle \qquad u, v_1, v_2 \in \mathcal{H}$$

$$\langle u, va \rangle = \langle u, v \rangle a \qquad u, v \in \mathcal{H}$$

$$a \in A$$

$$\langle u, v \rangle = \langle v, u \rangle^* \qquad u, v \in \mathcal{H}$$

$$\langle u, u \rangle \geqq 0 \quad \forall\, u \in A$$

$$\langle u, u \rangle = 0 \Longleftrightarrow u = 0$$

Definition. A Hilbert A-module is a pre-Hilbert A-module $\mathcal{H}$ which is complete in the norm

$$\|u\| = \|\langle u, u \rangle\|^{1/2}$$

Example. A Hilbert $\mathbb{C}$-module is a Hilbert space (viewed as a right $\mathbb{C}$-module).

Remark. If $\mathcal{H}$ is a Hilbert A-module, and A has a unit 1_A, then $\mathcal{H}$ is a $\mathbb{C}$ vector space with

$$u\lambda = u(\lambda 1_A) \qquad \lambda \in \mathbb{C}$$

Moreover, even if A does not have a unit, then by using an "approximate identity" in A, it is a $\mathbb{C}$ vector space.

Example. A $\quad C^*$ algebra

$$n = 1, 2, 3, \ldots \qquad A^n = \overbrace{A \oplus A \oplus \cdots \oplus A}^{n}$$

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) =$$

$$(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \qquad a_j, b_j \in A$$

$$(a_1, a_2, \ldots, a_n)a = (a_1 a, a_2 a, \ldots, a_n a) \qquad a_j, a \in A$$

$$\langle (a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \rangle =$$

$$a_1^* b_1 + a_2^* b_2 + \cdots + a_n^* b_n$$

Example.

$$\mathcal{H} = \left\{ (a_1, a_2, a_3, \dots) \;\middle|\; \begin{matrix} a_j \in A \\ \text{and} \\ \displaystyle\sum_{j=1}^{\infty} a_j^* a_j \text{ is norm-convergent in } A \end{matrix} \right\}$$

$$(a_1, a_2, a_3, \dots) + (b_1, b_2, b_3, \dots) =$$

$$(a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots)$$

$$(a_1, a_2, a_3, \dots)a = (a_1 a, a_2 a, a_3 a, \dots)$$

$$\langle (a_1, a_2, a_3, \dots), (b_1, b_2, b_3, \dots) \rangle = \sum_{j=1}^{\infty} a_j^* b_j$$

Example. $L^2(G, A)$

G topological group: locally compact, Hausdorff, second countable

Fix a left-invariant Haar measure dg for G.

$A \quad G - C^*$ algebra

$$G \times A \longrightarrow A$$

$L^2(G, A) =$

$$\left\{ f : G \to A \,\middle|\, \int_G g^{-1}[f(g)^* f(g)]dg \text{ is norm-convergent in } A \right\}$$

$L^2(G, A)$ is a Hilbert A-module

$$(f + h)g = f(g) + h(g) \qquad g \in G$$

$$(fa)(g) = f(g)[ga] \qquad a \in A$$

$$\langle f, h \rangle = \int_G g^{-1}[f(g)^* h(g)]dg$$

Definition. An A-module map $T : \mathcal{H} \to \mathcal{H}$ is adjointable if $\exists$ an A-module map $T^* : \mathcal{H} \to \mathcal{H}$ with $\langle Tu, v\rangle = \langle u, T^* v\rangle \quad \forall u, v \in \mathcal{H}$.

If T^* exists, it is unique.

If T^* exists then $\displaystyle\operatorname*{Sup}_{||u||=1} ||Tu|| < \infty$.

Set $\mathcal{L}(H) = \{T : A \to A \mid T \text{ is adjointable}\}$

$\mathcal{L}(H)$ is a C^* algebra

$$(T + S)u = Tu + Su \qquad\qquad u \in \mathcal{H}$$

$$(ST)(u) = S(Tu)$$

$$(T\lambda)u = (Tu)\lambda \qquad\qquad \lambda \in \mathbb{C}$$

$$T^*$$

$$||T|| = \operatorname*{Sup}_{||u||=1} ||Tu||$$

Reduced crossed-product $C_r^*(G,A)$

Let A be a $G - C^*$ algebra

$$G \times A \longrightarrow A$$

$$C_c(G,A) = \left\{ f : G \longrightarrow A \;\middle|\; \begin{array}{c} f \text{ is continuous} \\ \text{and} \\ f \text{ has compact support} \end{array} \right\}$$

$C_c(G,A)$ is an algebra

$$(f+h)(g) = f(g) + h(g) \qquad g \in G$$

$$(f\lambda)(g) = f(g)\lambda \qquad \lambda \in \mathbb{C}$$

$$(f * h)(g_0) = \int_G f(g)[gh(g^{-1}g_0)]dg \qquad g_0 \in G$$

"twisted" convolution

Injection of algebras $C_c(G, A) \longrightarrow \mathcal{L}(L^2(G, A))$

$$f \longmapsto T_f$$

$$T_f(u) = f * u$$

$$(f * u)(g_0) = \int_G f(g)[g\, u(g^{-1}g_0)]dg$$

$C_r^*(G, A)$ is $C_c(G, A)$ completed with respect to the norm $\|f\| = \|T_f\| \qquad T_f \in \mathcal{L}(L^2(G, A))$

$C_r^*(G, A)$ is a C^* algebra.