X a proper G-space with compact quotient space $G\backslash X$

There is a map of abelian groups

$$K_j^G(X) \longrightarrow K_j C_r^* G$$

$$(H, \psi, \pi, T) \longmapsto \text{Index}(T)$$

j = 0, 1

This map

$$
\begin{aligned}
K_j^G(X) &\longrightarrow K_j C_r^* G \\
(H, \psi, \pi, T) &\longmapsto \text{Index}(T)
\end{aligned}
$$

is natural, i.e. If X, Y are proper G-spaces with compact quotient spaces $G \backslash X$, $G \backslash Y$ and if

$f : X \longrightarrow Y$ is a continuous G-equivariant map.

Then there is commutativity in the diagram

$$
\begin{array}{ccc}
K_j^G(X) & \xrightarrow{\quad f_* \quad} & K_j^G(Y) \\
& \searrow \quad \swarrow & \\
& K_j C_r^* G &
\end{array}
$$

$\underline{E}G$ universal example for proper actions of G

Definition. $\Delta \subseteq \underline{E}G$ is G-compact if

1. $gx \in \Delta$ for all $(g, x) \in G \times \Delta$.

2. The quotient space $G \backslash \Delta$ is compact.

Set $K_j^G(\underline{E}G) = \lim_{\Delta \subseteq \underline{E}G | \Delta \text{ is } G\text{-compact}} K_j^G(\Delta)$.

The direct limit is taken over all G-compact subsets Δ of $\underline{E}G$.

$K_j^G(\underline{E}G)$ is the equivariant K-homology of $\underline{E}G$ with G-compact supports.

$$\mu : K_j^G(\underline{E}G) \longrightarrow K_j C_r^* G$$

$$(H, \psi, \pi, T) \longmapsto \text{Index}(T)$$

Conjecture (P.B. and A.Connes , 1980) Let G be a locally compact Hausdorff second countable topological group. then

$$\mu : K_j^G(\underline{E}G) \longrightarrow K_j C_r^* G$$

is an isomorphism.

j=0, 1

Γ (countable) discrete group

$$F\Gamma := \left\{ \sum_{\gamma\in\Gamma} \lambda_\gamma[\gamma] \;\middle|\; \begin{array}{l} \text{order } (\gamma) < \infty \\ \lambda_\gamma \in \mathbb{C} \end{array} \right\}$$

Finite Formal Sums

$F\Gamma$ is a vector space over $\mathbb{C}$

$$\left(\sum_{\gamma\in\Gamma} \lambda_\gamma[\gamma]\right) + \left(\sum_{\gamma\in\Gamma} \mu_\gamma[\gamma]\right) = \sum_{\gamma\in\Gamma} (\lambda_\gamma + \mu_\gamma)[\gamma]$$

$$\lambda \left(\sum_{\gamma\in\Gamma} \lambda_\gamma[\gamma]\right) = \sum_{\gamma\in\Gamma} \lambda\lambda_\gamma[\gamma] \qquad \lambda \in \mathbb{C}$$

$F\Gamma$ is a Γ-module

$\Gamma \times F\Gamma \to F\Gamma$

$g \in \Gamma \qquad \sum_{\gamma\in\Gamma} \lambda_\gamma[\gamma] \in F\Gamma$

$$g \left(\sum_{\gamma\in\Gamma} \lambda_\gamma[\gamma]\right) = \sum_{\gamma\in\Gamma} \lambda_\gamma[g\,\gamma\,g^{-1}]$$

$H_j(\Gamma; F\Gamma) :=$

the j-th homology group of Γ with coefficients
the Γ-module $F\Gamma$
$j = 0, 1, 2, \ldots$

Remark. This is standard homological algebra, and is pure algebra (*i.e.* Γ is a discrete group and $F\Gamma$ is a non-topologized module over Γ).

Γ (countable) discrete group

Notation $\quad K_j^{top}(\Gamma) := K_j^{\Gamma}(\underline{E}\Gamma)$

$j = 0, 1$

$$\mathrm{ch} : K_*^{\mathrm{top}}(\Gamma) \to H_*(\Gamma; F\Gamma)$$

$$\mathrm{ch} : K_j^{\mathrm{top}}(\Gamma) \to \bigoplus_{\ell} H_{j+2\ell}(\Gamma; F\Gamma)$$

j = 0,1

Proposition.

$$K_j^{\mathrm{top}}(\Gamma) \underset{\mathbb{Z}}{\otimes} \mathbb{C} \to \bigoplus_{\ell} H_{j+2\ell}(\Gamma; F\Gamma)$$

is an isomorphism of vector spaces over $\mathbb{C}$.

Γ (countable) discrete group

<u>Proposition</u> If BC is valid for Γ, then

$$\mathbb{C} \otimes_{\mathbb{Z}} K_j C_r^* \Gamma \cong \bigoplus_{\ell} H_{j+2\ell}(\Gamma; F\Gamma)$$

$j = 0, 1$

Theorem (N. Higson + G. Kasparov).

If Γ is a discrete group which is amenable, (or a-t-menable) then BC is true for Γ.

Theorem (I. Mineyev + G. Yu) (V. Lafforgue).

If Γ is a discrete group which is hyperbolic (in Gromov's sense), then BC is true for Γ.

$$BC = \text{Baum-Connes}$$