Shall "normalize" $-i\frac{d}{dx}$ to obtain a bounded operator T

Since $-i\frac{d}{dx}$ is self-adjoint there is functional calculus,

and T can be taken to be the function $\frac{x}{\sqrt{1+x^2}}$

applied to $-i\frac{d}{dx}$

$$T = \left(\frac{x}{\sqrt{1+x^2}}\right)\left(-i\frac{d}{dx}\right)$$

Equivalently, T can be constructed using Fourier transform

Let $\mathcal{M}_x$ be the operator "multiplication by x"

$$\mathcal{M}_x\Big(f(x)\Big) = xf(x)$$

Fourier transform converts $-i\frac{d}{dx}$ to $\mathcal{M}_x$

i.e. there is commutativity in the diagram

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}) \\ {\scriptstyle -i\frac{d}{dx}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{M}_x} \\ L^2(\mathbb{R}) & \xrightarrow[\mathcal{F}]{} & L^2(\mathbb{R}) \end{array}$$

$\mathcal{F}$ = Fourier transform

Let $\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}$ be the operator "multiplication by $\frac{x}{\sqrt{1+x^2}}$"

$$\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}\Big(f(x)\Big) = \frac{x}{\sqrt{1+x^2}} f(x)$$

$\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}$ is a bounded operator

$$\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

T is the unique bounded operator

$$T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

such that there is commutativity in the

diagram

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}} \\ L^2(\mathbb{R}) & \xrightarrow[\mathcal{F}]{} & L^2(\mathbb{R}) \end{array}$$

$\mathcal{F}$ = Fourier transform

$(L^2(\mathbb{R}), \psi, \pi, T) \in \mathcal{E}_1^{\mathbb{Z}}(\mathbb{R})$

X a proper G-space with compact quotient space $G\backslash X$.

$$\mathcal{E}_1^G(X) = \{(H, \psi, \pi, T)\}$$

$$K_1^G(X) := \mathcal{E}_1^G(X)/\sim$$

$\sim\ =$ "homotopy"

"homotopy" will be made precise later.

$K_1^G(X) = \mathcal{E}_1^G(X)/\sim$ is an abelian group.

$$(H, \psi, \pi, T) + (H', \psi', \pi', T') =$$

$$(H \oplus H', \psi \oplus \psi', \pi \oplus \pi', T \oplus T')$$

$$-(H, \psi, \pi, T) = (H, \psi, \pi, -T)$$

Let X be a proper G-space with compact quotient space $G\backslash X$.

An equivariant (even) K - cycle for X is a 4 - tuple (H, ψ, π, T) such that:

- (H, ψ, π) is a covariant representation of the $G - C^*$ algebra $C_0(X)$.

- $T \in \mathcal{L}(H)$

- ~~$T = T^*$~~

- $\pi(g)T - T\pi(g) = 0 \qquad \forall g \in G$

- $\psi(\alpha)T - T\psi(\alpha) \in \mathcal{K}(H) \qquad \forall \alpha \in C_0(X)$

- $\psi(\alpha)(I - T^*T) \in \mathcal{K}(H) \qquad \forall \alpha \in C_0(X)$

- $\psi(\alpha)(I - TT^*) \in \mathcal{K}(H) \qquad \forall \alpha \in C_0(X)$

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$$-(H, \psi, \pi, T) = (H, \psi, \pi, T^*)$$

X, Y proper G-spaces with compact quotient spaces $G \backslash X$, $G \backslash Y$.

$f : X \longrightarrow Y$ continuous G-equivariant map.

$$\tilde{f} : C_0(X) \longleftarrow C_0(Y)$$

$$\tilde{f}(\alpha) = \alpha \circ f \qquad\qquad \alpha \in C_0(Y)$$

$$K_j^G(X) \longrightarrow K_j^G(Y)$$

homomorphism of abelian groups $j = 0, 1$

$$(H, \psi, \pi, T) \longmapsto (H, \psi \circ \tilde{f}, \pi, T)$$

$$(H, \psi, \pi, T) \in \mathcal{E}_j^G(X)$$