Γ (countable) discrete group

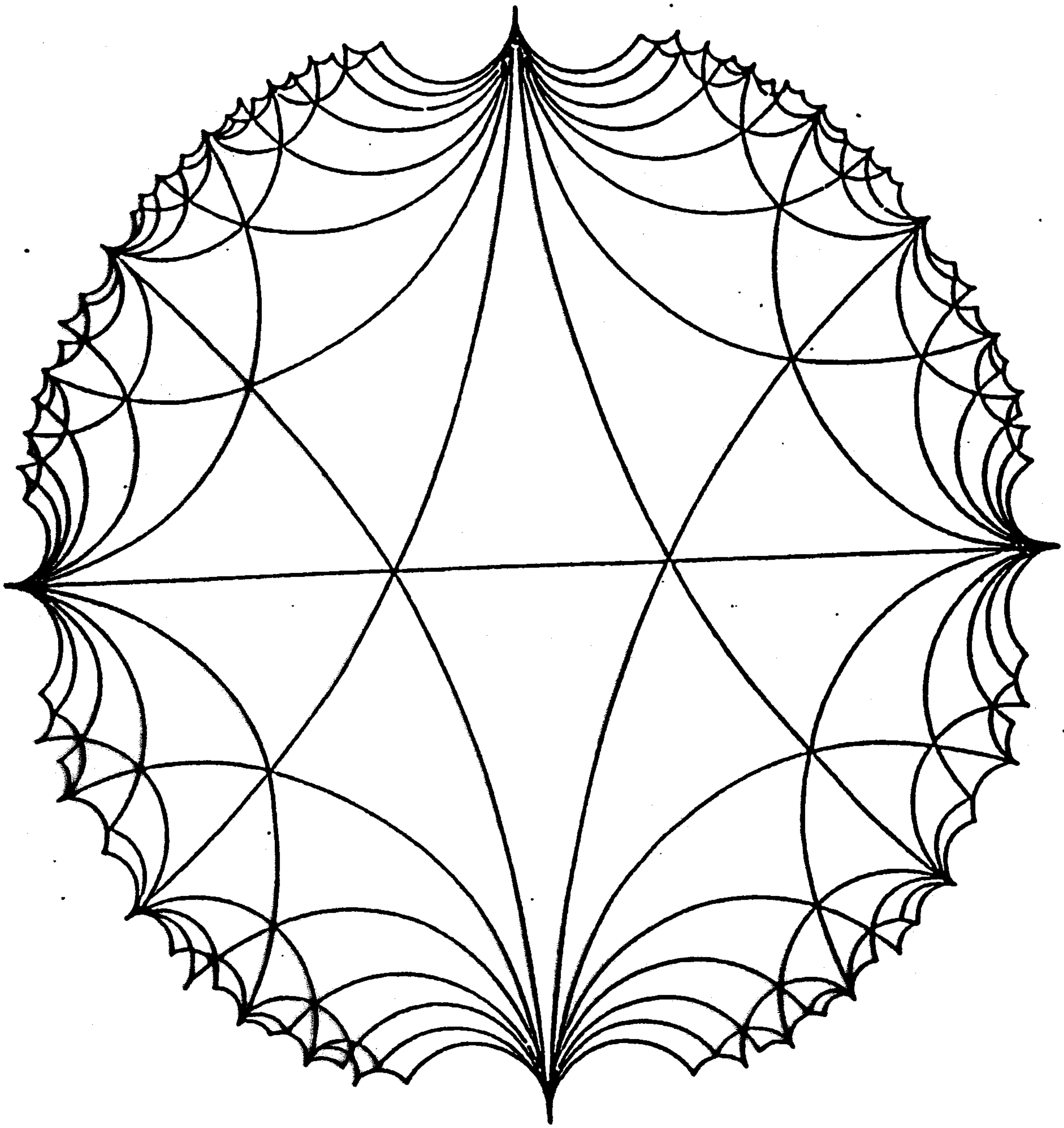
$$\underline{E}\Gamma = \left\{ f : \Gamma \longrightarrow [0,1] \;\middle|\; \begin{array}{l} \{\gamma \in \Gamma | f(\gamma) \neq 0\} \text{ is finite} \\ \sum_{\gamma \in \Gamma} f(\gamma) = 1 \end{array} \right\}$$

$$(\beta f)(\gamma) = f(\beta^{-1}\gamma) \qquad\qquad \beta, \gamma, \in \Gamma$$

$$f : \Gamma \longrightarrow [0,1]$$

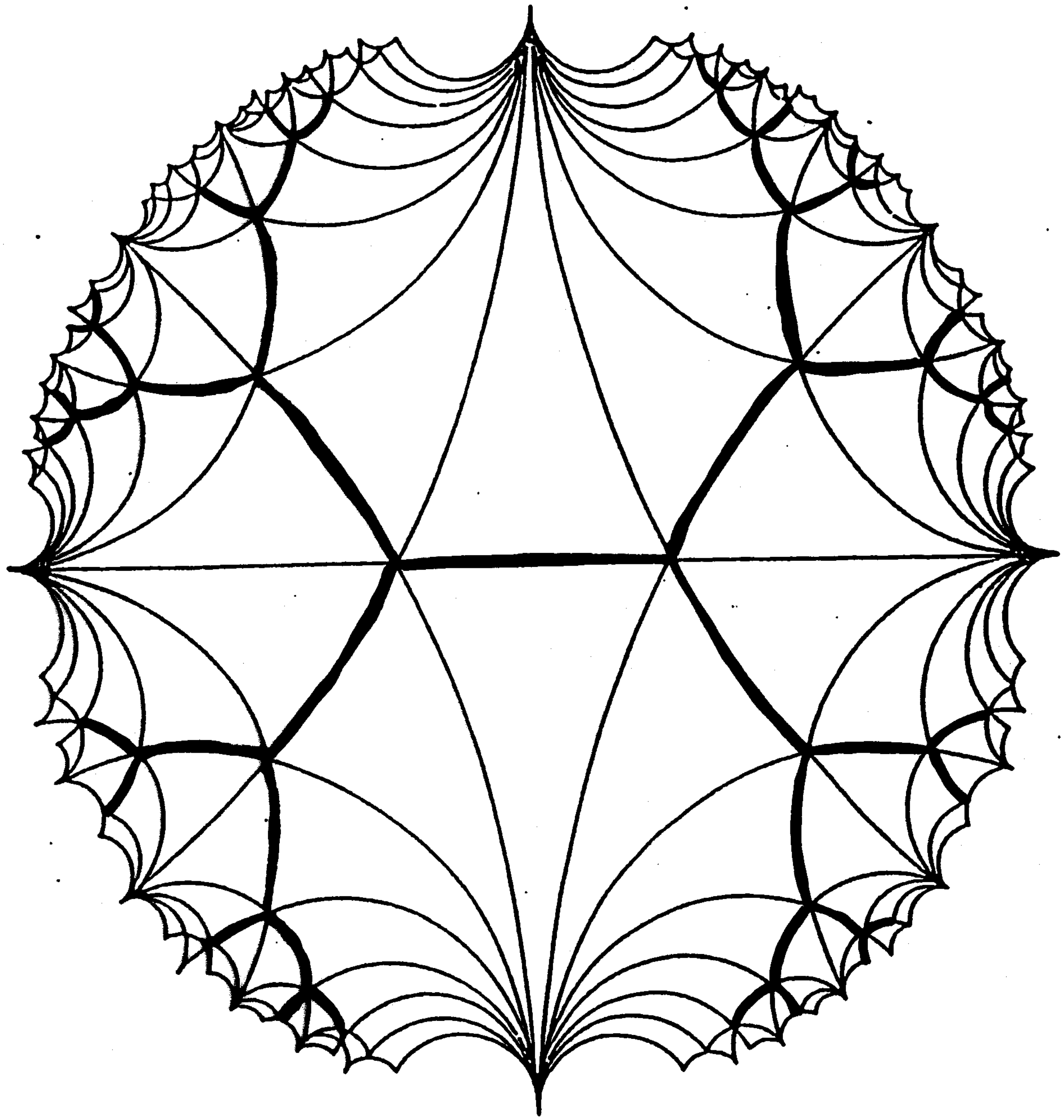
topologized by metric d

$$d(f,h) = \left(\sum_{\gamma \in \Gamma} |f(\gamma) - h(\gamma)|^2 \right)^{1/2}$$



Poincaré Disc Fundamental Domains for $SL(2, \mathbb{Z})$

Poincaré Disc $= SL(2, \mathbb{R})/SO(2)$



Poincaré Disc Fundamental Domains for $SL(2, \mathbb{Z})$

Topic 6: Equivariant K homology $\qquad K_*^G(X)$

G topological group — locally compact, Hausdorff, second countable

H separable Hilbert space

$$\mathcal{U}(H) = \{U \in \mathcal{L}(H) \mid UU^* = U^*U = I\}$$

Definition A unitary representation of G is a group homomorphism $\pi : G \longrightarrow \mathcal{U}(H)$ such that for each $v \in H$ the map $G \longrightarrow H$, $g \longmapsto \pi(g)v$, is a continuous map from G to H.

<u>Definition</u> A $G - C^*$ algebra is a C^* algebra A with a given continuous action

$$G \times A \longrightarrow A$$

of G on A.

G acts as C^* algebra automorphisms of A.

"continuous" means that for each $a \in A$, the map $G \longrightarrow A$, $g \longmapsto ga$ is a continuous map from G to A.

Let A be a (separable) $G - C^*$ algebra.

A *covariant representation* of A is a triple (H, ψ, π) such that:

H is a separable Hilbert space.

$\psi : A \to \mathcal{L}(H)$ is a $*$-homomorphism.

$\pi : G \to \mathcal{U}(H)$ is a unitary representation of G.

$$\psi(ga) = \pi(g)\psi(a)\pi(g^{-1}) \qquad \forall (g, a) \in G \times A.$$

Example Let X be a locally compact G-space.

G acts on $C_0(X)$

$$(g\alpha)(x) = \alpha(g^{-1}x) \qquad \begin{array}{l} g \in G \\ \alpha \in C_0(X) \\ x \in X \end{array}$$

$C_0(X)$ is a $G - C^*$ algebra.

Definition Let A be a $G - C^*$ algebra.

A covariant representation of A is a triple (H, ψ, π) with:

(i) H is a separable Hilbert space

(ii) $\psi : A \longrightarrow \mathcal{L}(H)$ is a $*$-homomorphism

(iii) $\pi : G \longrightarrow U(H)$ is a unitary representation of H

(iv) $\psi(ga) = \pi(g)\psi(a)\pi(g^{-1}) \quad \forall\ (g,a) \in G \times A$

Let X be a proper G-space with compact quotient space $G\backslash X$.

An equivariant (odd) K - cycle for X is a 4 - tuple (H, ψ, π, T) such that:

- (H, ψ, π) is a covariant representation of the $G - C^*$ algebra $C_0(X)$.

- $T \in \mathcal{L}(H)$

- $T = T^*$

- $\pi(g)T - T\pi(g) = 0 \qquad \forall g \in G$

- $\psi(\alpha)T - T\psi(\alpha) \in \mathcal{K}(H) \qquad \forall \alpha \in C_0(X)$

- $\psi(\alpha)(I - T^2) \in \mathcal{K}(H) \qquad \forall \alpha \in C_0(X)$

$$\mathcal{E}_1^G(X) = \{(H, \psi, \pi, T)\}$$

Example

$G = \mathbb{Z} \qquad X = \mathbb{R}$

$\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$

$(n, t) \mapsto n + t \qquad \mathbb{Z}\backslash\mathbb{R} = S^1$

$H = L^2(\mathbb{R}) \qquad \psi(\alpha)u = \alpha u$

$$\alpha u(t) = \alpha(t)u(t)$$

$$\alpha \in C_0(\mathbb{R}) \quad u \in L^2(\mathbb{R}) \quad t \in \mathbb{R}$$

$(\pi(n)u)(t) = u(t - n) \qquad n \in \mathbb{Z} \quad u \in L^2(\mathbb{R})$

$-i\frac{d}{dx}$

$(L^2(\mathbb{R}), \psi, \pi, -i\frac{d}{dx})$

$-i\frac{d}{dx}$ is not a bounded operator on $L^2(\mathbb{R})$