DEFINITION A topological space X is metrizable if $\exists$ a metric ρ for X whose underlying topology is the originally given topology of X

X metrizable $\Longrightarrow$ X Hausdorff and paracompact

X paracompact $\Longleftrightarrow$ whenever $\{U_\alpha\}$ $\alpha \in \mathcal{I}$ is an open cover of X, $\exists$ a locally finite partition of unity φ_α $\alpha \in \mathcal{I}$ sub-ordinate to $\{U_\alpha\}$

X a CW complex

$$\Downarrow$$

X is Hausdorff and paracompact

Definition. A G-space is a topological space X with a given continuous action of G.

$$G \times X \longrightarrow X$$

X, Y two G-spaces

Definition A G-map from X to Y is a continuous G-equivariant map $f : X \longrightarrow Y$.

$$f(gp) = gf(p) \qquad (g, p) \in G \times X$$

Definition Two G-maps $f_0, f_1 : X \longrightarrow Y$ are G-homotopic if they are homotopic through G-maps i.e. there exists a homotopy $\{f_t\}$ $0 \leqq t \leqq 1$ with each f_t a G-map.

Definition. A G-space X is proper if:

- X is paracompact and Hausdorff.

- The quotient space $G\backslash X$ (with the quotient topology) is paracompact and Hausdorff.

- For each $p \in X$ there exists a triple (U, H, ρ) such that :

  1. U is an open neighborhood of p in X with $gu \in U$ for all $(g, u) \in G \times U$.

  2. H is a compact subgroup of G.

  3. $\rho : U \longrightarrow G/H$ is a G-map from U to G/H.

Proposition (J. Chabert, S. Echterhoff, R. Meyer)

If X is a locally compact Hausdorff second countable G-space, then X is proper if and only if the map

$$G \times X \to X \times X$$
$$(g, x) \mapsto (gx, x)$$

is proper (*i.e.* the pre-image of any compact set in $X \times X$ is compact)

Definition. A universal example for proper actions of G, denoted $\underline{E}G$, is a proper G-space such that:

> If X is any proper G-space,
>
> then there exists a G-map $f : X \longrightarrow \underline{E}G$
>
> and any two G-maps from X to
>
> $\underline{E}G$ are G-homotopic.

Lemma. $\underline{E}G$ exists.

Uniqueness of $\underline{E}G$. Suppose that $\underline{E}G$ and $(\underline{E}G)'$ are both universal examples for proper actions of G. Then there exist G-maps

$$f : \underline{E}G \longrightarrow (\underline{E}G)'$$

$$f' : (\underline{E}G)' \longrightarrow \underline{E}G$$

with $f' \circ f$ and $f \circ f'$ G-homotopic to the identity maps of $\underline{E}G$ and $(\underline{E}G)'$ respectively. Moreover, f and f' are unique up to G-homotopy.

Axioms for $\underline{E}G$

(1) Y is a proper G-space.

(2) If H is any compact subgroup of G, $\exists\, p \in Y$ with $hp = p \quad \forall\, h \in H$.

(3) View $Y \times Y$ as a G-space

$$g(y_0, y_1) = (gy_0, gy_1)$$

$$\rho_0, \rho_1 : Y \times Y \longrightarrow Y$$

$$\rho_0(y_0, y_1) = y_0 \qquad \rho_1(y_0, y_1) = y_1$$

Axiom (3) : $\rho_0, \rho_1 : Y \times Y \longrightarrow Y$ are G-homotopic

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LEMMA If (1) (2) (3) are valid for Y, then Y is an $\underline{E}G$.

Examples.

G compact $\underline{E}G = \cdot$

G Lie group ($\pi_0 G$ finite) $\underline{E}G = G/H$

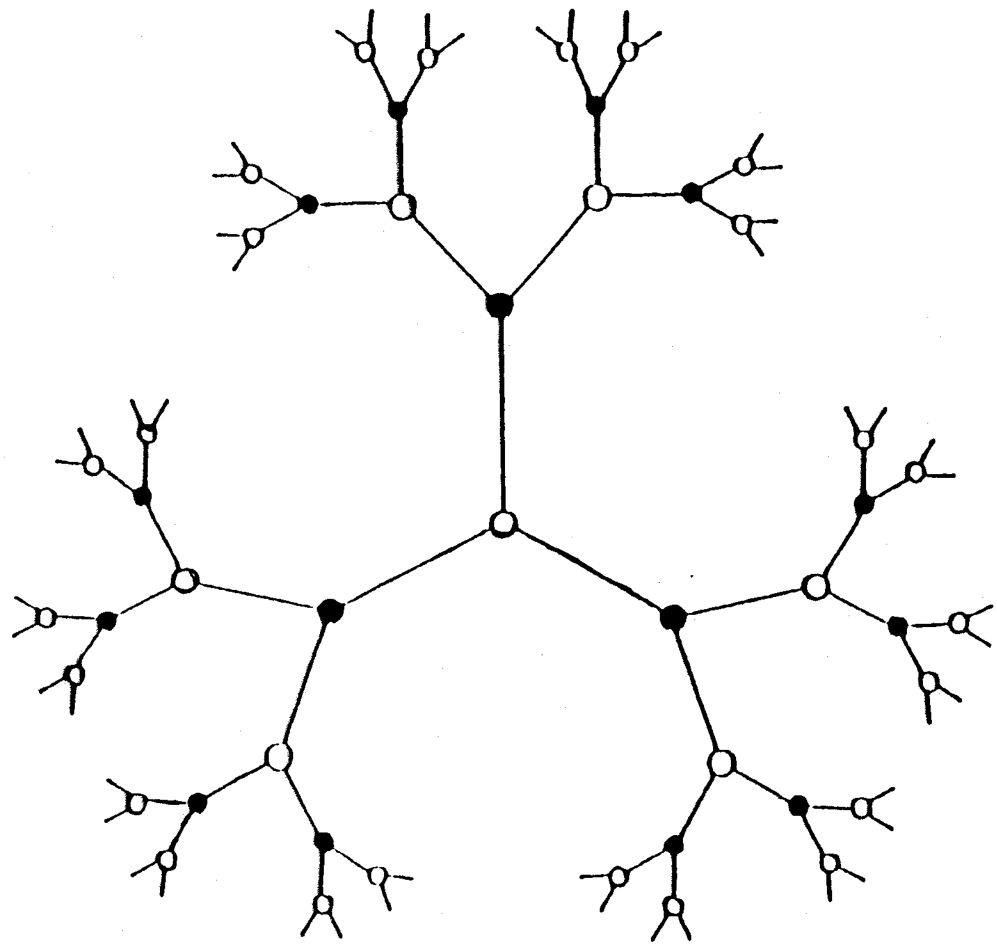
$H =$ maximal compact subgroup of G

G p-adic group $\underline{E}G = \beta G$

$=$ affine Bruhat-Tits building

for G

EXAMPLE. $\beta SL(2, \mathbb{Q}_p)$ is the $(p+1)$-regular tree. The $(p+1)$-regular tree is the unique tree with exactly $p+1$ edges at each vertex.



$\beta SL(2, \mathbb{Q}_2)$