K theory for C^* algebras

Algebraic K theory

A C^* algebra

trivial move = stabilizing A

$$M_n(A) \hookrightarrow M_{n+1}(A)$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \longmapsto \begin{bmatrix} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

This is a one-to-one $*$-homomorphism $\therefore$ This is norm preserving

$$M_\infty(A) = \varinjlim M_n(A)$$

$$= \left\{ \begin{matrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \end{matrix} \;\middle|\; \text{Almost all } a_{ij} \text{ are zero} \right\}$$

$$\dot{A} = \overline{M_\infty(A)}$$

$\dot{A}$ is the stabilization of A

$$K_j(\dot{A}) = K_j(A) \qquad j = 0, 1$$

EXAMPLE Let H be a separable (but not finite dimensional) Hilbert space.

i.e. H has a countable (but not finite) orthonormal basis

$$\dot{\mathbb{C}} = \mathcal{K} \subset \mathcal{L}(H)$$

$\mathcal{K}$ = THE compact operators on H

$$K_j \mathbb{C} = K_j \dot{\mathbb{C}}$$ C^* algebra K theory

$$K_j^{alg}(\dot{\mathbb{C}}) = \left\{ \begin{array}{ll} \mathbb{Z} & j \text{ even} \\ 0 & j \text{ odd} \end{array} \right\}$$ algebraic K theory

Summary

Within C^* algebra K-theory we have

Lemma. *Let A be a C^* algebra. Then*

$$K_j A = K_j \dot{A} \qquad\qquad j = 0, 1, 2, 3, \ldots$$

Proof. This is a straight-forward consequence of the definition of C^* algebra K-theory. $\square$

$\dot{A}$ is the stabilization of A.

KAROUBI CONJECTURE

Let A be a C^* algebra, then

$$K_j(\dot{A}) = K_j^{alg}(\dot{A})$$

C^* algebra K theory	Algebraic K theory

The Karoubi conjecture was proved by A. Suslin and M. Wodzicki

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Theorem (A. Suslin and M.Wodzicki). *Let A be a C^* algebra. Then*

$$K_j \mathring{A} = K_j^{\text{alg}} \mathring{A} \qquad\qquad j = 0, 1, 2, 3, \ldots$$

$\mathring{A}$ is the stabilization of A.

This theorem is the unity of K-theory. It says that C^* algebra K-theory is a pleasant sub-discipline of algebraic K-theory in which Bott periodicity is valid and certain basic examples are easy to calculate.