Kasparov descent map

A, B (separable) G-C^* algebras

$$KK_G^j(A,B) \longrightarrow KK^j(C_r^*(G,A), C_r^*(G,B))$$

$j = 0, 1$

$A \qquad C^*$ algebra

$$KK^j(\mathbb{C}, A) \cong K_j A \qquad j = 0, 1$$

i.e. $KK^*(\mathbb{C}, A)$ is the K-theory of A

Definition of $\mu : K_j^G(\underline{E}G) \to K_j C_r^* G$

Let X be a proper G-compact G-space

$\mu : KK_G^j(C_0(X), \mathbb{C}) \longrightarrow K_j C_r^* G$ is the composition of:

① Kasparov descent map
$$KK_G^j(C_0(X), \mathbb{C}) \longrightarrow KK^j(C_r^*(G,X), C_r^* G)$$

② Kasparov product with
$$\mathbb{1} \in K_0 C_r^*(G,X) = KK^0(\mathbb{C}, C_r^*(G,X))$$

$$K_j^G(\underline{E}G) := \varinjlim_{\substack{\Delta \subset \underline{E}G \\ \Delta \ G\text{-compact}}} KK_G^j(C_0(\Delta), \mathbb{C})$$

For each G-compact $\Delta \subset \underline{E}G$ have

$$\mu : KK_G^j(C_0(\Delta), \mathbb{C}) \longrightarrow K_j C_r^* G$$

If Δ, Ω are two G-compact subsets of $\underline{E}G$ with $\Delta \subset \Omega$, then the diagram

$$\begin{tikzcd} KK_G^j(C_0(\Delta), \mathbb{C}) \arrow[rr] \arrow[dr] & & KK_G^j(C_0(\Omega), \mathbb{C}) \arrow[dl] \\ & K_j C_r^* G & \end{tikzcd}$$

commutes

Thus obtain $\mu : K_j^G(\underline{E}G) \longrightarrow K_j C_r^* G$

A G-C^* algebra

$$K_j^G(\underline{E}G\,;A) := \varinjlim_{\substack{\Delta \subset \underline{E}G \\ \Delta \text{ G-compact}}} KK_G^j(C_0(\Delta), A)$$

Definition of

$$\mu : K_j^G(\underline{E}G\,;A) \longrightarrow K_j C_r^*(G, A)$$

X proper G-compact G-space

$$\mu : KK_G^j(C_0(X), A) \longrightarrow K_j C_r^*(G, A)$$

is the composition of

① Kasparov descent map

$$KK_G^j(C_0(X), A) \longrightarrow KK^j(C_r^*(G, X), C_r^*(G, A))$$

② Kasparov product with

$$\mathbb{1} \in K_0 C_r^*(G, X) = KK^0(\mathbb{C}, C_r^*(G, X))$$

For each G-compact $\Delta \subset \underline{E}G$ have

$$\mu : KK^j_G(C_0(\Delta), A) \longrightarrow K_j C^*_r(G, A)$$

If Δ, Ω are two G-compact subsets of $\underline{E}G$ with $\Delta \subset \Omega$,

then the diagram

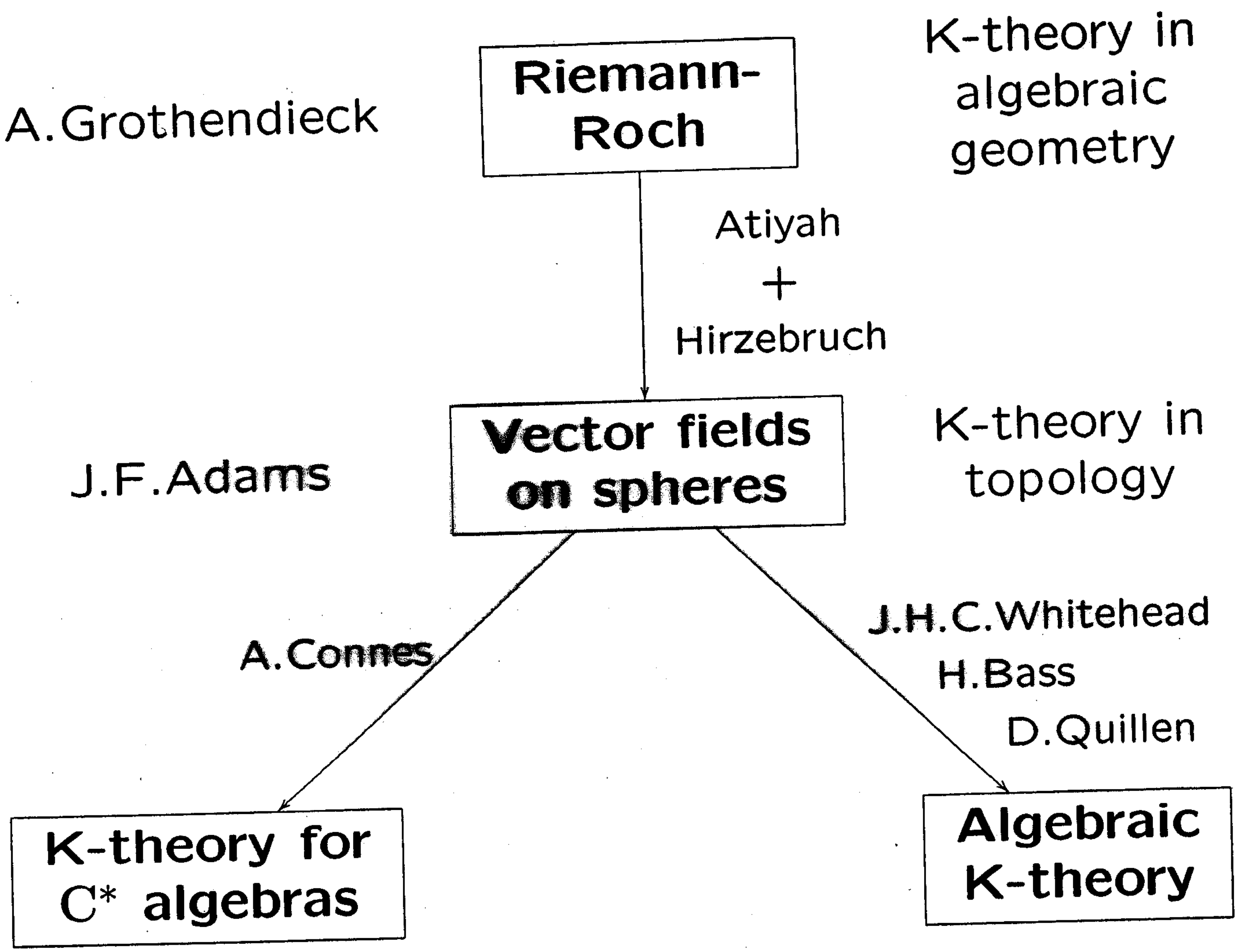
$$\begin{tikzcd} KK^j_G(C_0(\Delta), A) \arrow[rr] \arrow[dr] & & KK^j_G(C_0(\Omega), A) \arrow[dl] \\ & K_j C^*_r(G, A) & \end{tikzcd}$$

commutes

Thus obtain

$$\mu : K^G_j(\underline{E}G; A) \longrightarrow K_j C^*_r(G, A)$$

A brief history of K-theory



HIRZEBRUCH-RIEMANN-ROCH

M non-singular projective algebraic variety / $\mathbb{C}$

E an algebraic vector bundle on M

$\underline{E}$ = sheaf of germs of algebraic sections of E

$H^j(M, \underline{E}) :=$ j-th cohomology of M using $\underline{E}$

$j = 0, 1, 2, 3, \ldots$

LEMMA For all $j = 0, 1, 2, \ldots$ $\dim_{\mathbb{C}} H^j(M, \underline{E}) < \infty$

For $j > \dim_{\mathbb{C}}(M)$, $H^j(M, \underline{E}) = 0$

$$\chi(M, E) := \sum_{j=0}^{n} (-1)^j \dim_{\mathbb{C}} H^j(M, \underline{E})$$

$n = \dim_{\mathbb{C}}(M)$

THEOREM (HRR) Let M be a non-singular projective algebraic variety $/\mathbb{C}$ and let E be an algebraic vector bundle on M. Then:

$$\chi(M, E) = (ch(E) \cup Td(M))[M]$$